A Heuristic Theory of the Zonally-Averaged State of the Atmosphere

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Abstract

A two level, quasi-nondivergent model is used to investigate the distribution of wind and temperature in the meridional plane under influence of the exchange processes between the eddies and the zonally averaged state. The model incorporates heating and friction, and the transports of potential vorticity and sensible heat by the eddies are parameterized using exchange coefficients which depend on the vertical and the meridional coordinates. The steady state problem is solved using low order representations of the streamfunction. The beta plane case and the spherical case are treated separately.

In both cases it is found that the models can account for the main characteristics of the zonally averaged state of the atmosphere. The region of validity of the parameterization schemes is determined by experiment in each case. It turns out that the validity is determined by the intensity of the heating.

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1. Introduction

About 300 years ago the famous British astronomer Edmund Halley (1686) made the first detailed survey of what was then known about the winds in the lower latitudes in three separate oceans. Halley is remembered today mainly because his name was given to a comet which visited our part of the solar system a short time ago, but in the meterological community he stands as the first scientist who systematically mapped the winds and sought a common cause for them. He was the first to point out that the northeasterly trade winds north of the equator and the southeasterly trades on the south side blow as they do because the equator is the most strongly heated region. The maximum heat source at the equator creates a general rising motion and for continuity reasons the upgoing air must be replaced by essentially horizontal currents streaming in toward the equator. These winds do not come straight from the north and the south in the two hemispheres because of the deflecting force due to the earth rotation which in the northern part deflects to the right resulting in a northeasterly wind while the deflection is to the left in the southern hemisphere resulting in the southeasterly trade winds.

Half a century later George Hadley (1735) returned to the subject in a famous paper where he discussed the cause of the general trade winds. His opening words are: "I think the causes of the General Trade-Winds have not been fully explained by any of those who have wrote on that subject..." As late as two decades ago Edward N. Lorenz (1967) stated in his book on the nature and theory of the general atmospheric circulation that these words "seem to afford an apt description of the state of the same subject today." He states further: "Yet not in any of the thousand or more excellent works which have appeared since that time, nor in any combination of the works, is a full explanation of the distribution of easterly and westerly winds to be found." The author of the present contribution to the subject can agree with Lorenz's statement although new ideas and theories have appeared in recent years. As we shall see, these theories do not answer all questions. In spite of the incomplete nature of the new approaches it seems worthwhile to present an account of present thinking with the purpose of formulating strengths and weaknesses which may result in further improvements.

The classical problem is that an explanation of the typical wind distribution with easterlies in the low latitudes, westerlies in the middle latitudes and weak easterlies in the high latitudes is required. The present formulation is broader. We seek an answer to the above question which is in agreement with what we know now about the windsystems in the whole atmosphere and the transport processes which are required to maintain the winds and are observed in the atmosphere. In a sense it is much more difficult to produce an acceptable theory today simply because we have increased our knowledge of the temperature and wind distributions in the major part of the atmosphere through the global network of surface and upper air observations which have been developed since World War II. It is as a matter of fact relatively easy to discard some earlier theories since they do not agree with present knowledge of the transport processes. In this connection it is illustrating to review briefly the classical approach to the problem.

Common to all investigations before this century are that solutions were sought within the framework of the axi-symmetrical circulations or solutions which can be represented in a meridional plane from the South Pole to the North Pole of the earth. It is not the purpose of this paper to review all the various proposals which have been made of this kind. They have mainly historical interest and are further described in an excellent manner by Lorenz (1967) in his chapter on former theories of the general circulation. However, it is pertinent to mention that they consisted of a number of mean meridional cells of circulation. The most common scheme has a thermally direct tropical cell, the Hadley circulation, with rising motion at and near to the equator and descending motion in the area of the subtropics, say 30°N. The middle latitudes have a neighboring cell with the just mentioned sinking motion, but connected to a rising motion in the higher middle latitudes, say 60°N for simplicity. This cell, normally called the Ferrel cell after the American scientist, is thermally indirect because the warmer air is sinking and the cooler air is rising. Finally, the polar cell is again thermally direct with sinking motion in the region of the pole. With this arrangement one can account for the surface winds since the deflecting force, hereafter called the Coriolis force, will give easterlies and westerlies in the correct latitudes at the surface, but the description will fail miserably aloft where it will give easterlies when westerlies are observed. In addition, the existence of these meridional circulations are postulates and not the results of physical reasoning.

An additional test to which one should put any proposed scheme is to investigate whether or not the circulations can satisfy the general balance requirements for the atmosphere. It is of course known that the earth for astronomical reasons is heated at the equator and cooled at the poles. Since the temperatures in these regions are not steadily increasing and decreasing, it is required that a heat transport takes place from equator to pole in each hemisphere. Similarly, due to the friction between the atmosphere and the surface of the earth, the westerly momentum will decrease in a region of surface westerlies, while the easterly momentum will decrease, *i.e.* the westerly momentum will increase, in a region of surface easterlies. The requirement for balance is therefore that westerly momentum is transported from the low and high latitudes into the middle latitudes where westerly momentum is lost. Regardless of whether or not a proposed circulation scheme is correct it must satisfy such balance requirements. A further balance requirement exists for the moisture in the atmosphere. It requires that moisture is transported into a latitudinal zone when the precipitation exceeds the evaporation, but the later description of the present calculations will not include moisture.

As stated before, the former theories, *i.e.* before this century, attempted to solve the problem by using axi-symmetrical models. The required transports are then to be carried out by the proposed mean meridional circulations. A first question is, naturally, if this is a realistic approach, or, in other words: Are the transports in the real atmosphere carried out by mean meridional circulations or by other mechanisms? A novel attempt to break the deadlock concerning the old theories was made by Albert Defant (1921). At that time one had drawn weather maps for quite a number of years and was familiar with the motion from day to day of the atmospheric waves, the low and high pressure centers, etc. Defant's idea was to consider all these disturbances on the zonal current as macro-turbulence elements knowing of course that the empirical description of turbulence, especially the concept of exchange coefficients, was developed for turbulence elements on a minute scale. The concept of the exchange (or Austausch) coefficient is based on the so-called mixing length description and says that the transport of a conservative quantity is from a region of high values to a region of low values. Mathematically the principle says that:

$$\overline{A\mathbf{V}} = -K\nabla\overline{A} \tag{1.1}$$

where the overbar means an average value.

Applied to the northward transport of heat (1.1) may be written in the form:

$$\overline{Tv} = -K\frac{\partial\overline{T}}{\partial y} \tag{1.2}$$

when T is the absolute temperature, v the meridional velocity component, y is the ordinate pointing northward and K the exchange coefficient. Defant found that the exchange coefficient was about $7 \times 10^6 m^2 s^{-1}$ applying the mixing length description. Considering a column from the surface to the top of the atmosphere we have from the steady-state thermodynamic equation that:

$$\frac{d\,(\overline{Tv})}{dy} = \frac{1}{C_p}\overline{Q} \tag{1.3}$$

or using (1.2) that:

$$\frac{d^2\overline{T}}{dy^2} = -\frac{\overline{Q}}{C_pK} \tag{1.4}$$

$\overline{Q}$ is the average heating per unit mass and unit time. Assuming that

$$\overline{Q} = Q_A \cos\left(\pi\frac{y}{W}\right) \tag{1.5}$$

where W is the distance from equator to pole, *i.e.* $W = 10^7 m$ we may solve (1.4) and obtain:

$$\Delta T = \frac{W^2 Q_A}{K C_p \pi^2} \cos\left(\pi\frac{y}{W}\right) \tag{1.6}$$

Using $Q_A = 10^{-2}\mathrm{kj}t^{-1}s^{-1}$ we find that ΔT_{max} is about 14°K or a temperature difference from pole to equator of 28°K in reasonable agreement with observations. Based on calculations of this kind Defant concluded that the large-scale circulation could be considered as a form of turbulence.

The idea of considering the atmospheric waves as macro-turbulence was not taken up again for many years. Defant is, however, the first to suggest that the required heat transport may be carried out by the atmospheric waves rather than by the mean meriodional circulations. If this is so, it follows also that the zonally averaged circulation cannot be explained without paying attention to the interaction between the zonally averaged quantities and the waves.

A similar conclusion was reached by Jeffreys (1926) with respect to angular momentum. He found that the balance requirements for momentum could not be satisfied by a mean-meridional circulation because the amounts were too small, and he concluded the cyclones and anti-cyclones gave a major contribution to the total transport.

At the time of the investigations by Defant and Jeffreys it was not possible to test their ideas by a direct calculation of the transports of sensible heat and angular momentum since a global observation network did not exist, and since observations from the upper part of the troposphere were not yet available. The direct calculations became possible around 1950 when upper air observations became available from the radiosonde network which had sufficient coverage to permit calculations over the greater part of the Northern Hemisphere. Many investigations have given valuable contributions through such diagnostic calculations based on the standard data over the last few decades, but the original proposals and the basic framework were provided by Starr (1948), Bjerknes (1948) and Priestley (1949). The main results of all the investigations have once again been summerized by Lorenz (1967). For our purposes it will suffice to say that the results of the diagnostic studies confirm that the transport carried out by the eddies in all cases (*i.e.* for momentum, heat and moisture) give a very significant contribution to satisfying the required balance requirements. On the other hand, it would be incorrect to say that the contributions from the mean meridional circulation in all cases and at all latitudes are insignificant. In particular, it seems as if the mean meridional circulations play a larger role in the tropics than in middle latitudes.

We may in any case conclude that the questions posed earlier in this section cannot be answered without considering the interaction between the zonally-averaged state and the eddies. The crucial question is how one can incorporate the interactions in a given model. One way to do this experimentally is to formulate a model of the total atmosphere incorporating properly formulated heat sources and sinks as well as dissipations, secure a sufficient horizontal and vertical resolution of the atmospheric parameters, formulate a stable and accurate time-integration system, and finally use the model to simulate the general circulation of the atmosphere. The total results of such simulations may subsequently be used to investigate how the zonally-averaged circulation and other budgets are maintained in the model.

Very interesting and illuminating experiments of this kind have been carried out by a large number of investigators following the pioneering work by Phillips (1956) and Smagorinsky (1963). All experiments have shown that the simulations are in essential agreement with the observed behavior of the atmosphere with respect to the zonally-averaged structure. One may, however, say that these experiments confirm that we know the equations applicable to the atmosphere and that our knowledge of the driving and dissipating forces for the atmosphere is good enough to reproduce some essential features of the general circulation. On the other hand, experiments do not replace theory, and a theory is still wanted.

In the following sections an attempt will be made to present a theory for the zonally-averaged state of the atmosphere. It will be based on the assumption that the larger-scale eddies in the atmosphere in their interaction with the zonally-averaged state will behave as macroscopic turbulence elements. It will be assumed that conservative properties are transported in agreement with an exchange coefficient relation of the type (1.2). The theory may thus be considered as an expansion of the ideas of Defant (1921) and is similar to, but not identical with ideas expressed by Green (1970) and White and Green (1984). The theory will in contrast to the just quoted references be formulated as a steady-state theory and will furthermore differ with respect to the treatment of the heat transports and the transport of potential vorticity.

2. On larger-scale exchange and transport processes

The modeling of the large-scale processes in terms of exchange processes is far from obvious. One wants a description of the transport processes expressed in terms of the zonally-averaged quantities, an example of a so-called parameterization prescription. Severe difficulties appear immediately if we want to use prescriptions of the type (1.2), the exchange coefficient approach. While this approach is reasonable with respect to the transport of heat as shown by Defant (1921) and confirmed later by all tropospheric, but not stratospheric, diagnostic calculations, it is clear from the diagnostic calculations of the momentum transports that these are by and large from the low to the high regions of the momentum itself. An exchange coefficient approach is therefore out of the question for the momentum transport unless one wants to use such unphysical quantities as negative diffusion coefficients. We shall abstain although not everyone has done so. A different solution must be found for the momentum transport.

The author was working on this problem in the late 1960's (Wiin-Nielsen, 1968) when a brief note was presented, not even by the author himself, at a symposium in Tokyo, Japan (Green, 1968). The main idea by Green was that the exchange coefficient approach is physically justified only for quantities which are conservative for a particle or at least conserved in an approximate way if other processes which may change them work on a much larger time-scale. For the larger-scale processes two conservative quantities exist, namely the potential temperature, which is influenced by the slowly working diabatic heating, and the quasi-nondivergent potential vorticity, which is influenced also by the slowly working dissipation. For the first quantity we may write the first law of thermodynamics in the form:

$$\frac{d(\ell n\Theta)}{dt} = \frac{1}{C_pT}Q \tag{2.1}$$

in which Θ is the potential vorticity, C_p the specific heat for constant pressure, T the temperature and Q the heating per unit mass and unit time. The potential temperature Θ is defined by:

$$\Theta = T\left(\frac{p}{p_0}\right)^{-R/C_p} ; p_0 = 100cb \tag{2.2}$$

The quasi-nondivergent potential vorticity is a quantity which is applicable to a class of atmospheric models which rest on the following assumptions:

- the horizontal wind can be considered as non-divergent for advection purposes, *i.e.* $\mathbf{V} = \mathbf{K} \times \nabla\ \psi$ where ψ is the stream function
- the vertical advection of momentum is negligible as compared to the horizontal advection
- the relative vorticity is negligible compared to the earth's vorticity when appearing undifferentiated

With these approximations which may be justified for the larger-scale flow by a detailed scale-analysis (Phillips, 1963) we may write the vorticity equation for the earth's atmosphere in the form:

$$\frac{\partial \varsigma}{\partial t} + \mathbf{V} \cdot \nabla (f + \varsigma) = f_0 \frac{\partial \omega}{\partial p} + g\mathbf{K} \cdot \nabla \times \frac{\partial \tau}{\partial p} \tag{2.3}$$

in which $\varsigma = \nabla^2 \psi$ is the vorticity, $f = 2\Omega \sin\varphi$ is the Coriolis parameter, Ω the angular velocity of the earth, φ latitude, $\omega = dp/dt$ is the vertical velocity in a system with pressure (p) as a vertical velocity, g is the gravity, $\mathbf{K}$ a vertical unit vector and τ the frictional stress. We shall now combine (2.1) and (2.3) by eliminating the vertical velocity ω. Writing out (2.1) we obtain:

$$\frac{\partial \ell n \Theta}{\partial t} + \mathbf{V} \cdot \nabla \ell n \Theta + \omega \frac{\partial \ell n \Theta}{\partial p} = \frac{1}{C_p T} Q \tag{2.4}$$

Using (2.2) and the gas equation

$$p\alpha = RT \tag{2.5}$$

in which α is the specific volume we obtain

$$\frac{\partial \alpha}{\partial t} + \mathbf{V} \cdot \nabla \alpha + \omega \alpha \frac{\partial \ell n \Theta}{\partial p} = \frac{R}{C_p} \frac{1}{p} Q \tag{2.6}$$

In this equation we introduce the hydrostatic equation

$$\alpha = -\frac{\partial \Phi}{\partial p} = -f_0 \frac{\partial \psi}{\partial p} \text{ and } \sigma = -\alpha \frac{\partial \ell n \Theta}{\partial p} \tag{2.7}$$

with the result that

$$\frac{\partial}{\partial t}\left(\frac{\partial \psi}{\partial p}\right) + \mathbf{V}\cdot\nabla\left(\frac{\partial \psi}{\partial p}\right) + \frac{\sigma}{f_0}\,\omega = -\frac{R}{C_p}\,\frac{Q}{f_0 p} \tag{2.8}$$

It is now a straightforward matter to solve (2.8) for ω and introduce the result in (2.3). After some rearrangements we find:

$$\frac{d\xi}{dt} = \frac{\partial \xi}{\partial t} + \mathbf{V}\cdot\nabla\xi = g\mathbf{K}\cdot\nabla\times\frac{\partial \tau}{\partial p} - \frac{R}{C_p} f \frac{\partial}{\partial p}\left(\frac{Q}{\sigma p}\right) \tag{2.9}$$

in which:

$$\xi = f + \nabla^2\psi + \frac{\partial}{\partial p}\left(\frac{f_0^2}{\sigma}\,\frac{\partial \psi}{\partial p}\right) \tag{2.10}$$

is the quasi-nondivergent potential vorticity which from (2.9) is seen to be conserved in the horizontal flow in the adiabatic, nonviscous case.

(2.9), (2.10) may also be written in a different form. Going back to (2.4) and multiplying by a standard value of α, say α_0 we obtain after rearrangement:

$$\omega = \frac{\partial}{\partial t}\left(\frac{\alpha_0}{\sigma}\ell n\Theta\right) + \mathbf{V}\cdot\nabla\left(\frac{\alpha_0}{\sigma}\ell n\Theta\right) - \frac{R}{C_p}\,\frac{Q}{\sigma p} \tag{2.11}$$

Differentiation with respect to pressure and substitution in (2.3) yields again (2.9) where the potential vorticity is:

$$\xi = f + \nabla^2\psi - \frac{\partial}{\partial p}\left(\frac{\alpha_0 f_0}{\sigma}\ell n\Theta\right) \tag{2.12}$$

The reason for this results is that potential vorticity appears only differentiated with respect to time or the horizontal coordinates. We may for example obtain the result directly from the thermal wind relation. Starting from:

$$\mathbf{V} = \mathbf{K}\times\nabla\psi \tag{2.13}$$

we find:

$$\frac{\partial \mathbf{V}}{\partial p} = \mathbf{K}\times\nabla\frac{\partial \psi}{\partial p} = -\frac{1}{f_0}\mathbf{K}\times\nabla\alpha = -\frac{\alpha}{f_0}\mathbf{K}\times\nabla\ell n\Theta \approx \mathbf{K}\times\nabla\left(\frac{-\alpha_0}{f_0}\ell n\Theta\right) \tag{2.14}$$

showing that:

$$\frac{\partial\psi}{\partial p} \sim -\frac{\alpha_0}{f_0}\ell n\Theta \qquad (2.15)$$

The idea put forward by Green (1968, 1970) can now be explained. As mentioned it rests on the facts that the potential temperature is conserved for a particle in its three dimensional motion while the potential vorticity is conserved for a particle in its horizontal (or more strictly, its isobaric) motion. The assumption is that the transport of such quantities may be realistically approximated by a diffusion approximation.

Let us first explore the formal aspects of this assumption. For the potential vorticity we obtain:

$$\overline{\xi v} = -K(y,p)\frac{\partial\overline{\xi}}{\partial y} \qquad (2.16)$$

From (2.10) we obtain:

$$\frac{\partial\overline{\xi}}{\partial y} = \beta - \frac{\partial^2\overline{u}}{\partial y^2} - \frac{\partial}{\partial p}\left(\frac{f_0^2}{\sigma}\frac{\partial\overline{u}}{\partial p}\right) \qquad (2.17)$$

while (2.12) after multiplication by v, the meridional velocity, and averaging in the x– or west-east direction:

$$\overline{\xi v} = \overline{\zeta v} - \frac{\partial}{\partial p}\left(\frac{\alpha_0 f_0}{\sigma}\overline{\ell n\Theta v}\right) \qquad (2.18)$$

where $\overline{\zeta v}$ is the transport of relative vorticity. (2.18) gives therefore a relation between the meridional transports of potential vorticity, relative vorticity and the heat transport. It is thus already clear that (2.18) gives a possibility for parameterization of the relative vorticity transports because the remaining transports in (2.18) may be approximated by exchange coefficients. For the heat transport we write:

$$\overline{\ell n\Theta\cdot\mathbf{V}_{2,3}} = -L(y,p)\frac{\partial\overline{\ell n\Theta}}{\partial y} + \frac{p_0}{W}L_3(y,p)\frac{\partial\overline{\ell n\Theta}}{\partial p} \qquad (2.19)$$

The factor $\frac{p_0}{W}$ where $p_0 = 100$cb and $W = 10^7 m$, the distance from equator to pole, is introduced to obtain the same dimensions for L and L_3.

In (2.19) we introduce:

$$\frac{\partial \overline{u}}{\partial p} = \frac{\alpha_0}{f_0}\frac{\partial \ell n \overline{\Theta}}{\partial y} \;;\; \sigma = -\alpha_0 \frac{\partial \overline{\ell n \Theta}}{\partial p} \tag{2.20}$$

and obtain:

$$\overline{\ell n \Theta \cdot v} = -\frac{f_0}{\alpha_0} L \frac{\partial \overline{u}}{\partial p} - L_3 \frac{p_0 \sigma}{W \cdot \alpha_0} \tag{2.21}$$

Introducing (2.21) in (2.18) we find finally:

$$-K\frac{\partial \overline{\xi}}{\partial y} = V + \frac{\partial}{\partial p}\left[L\frac{f_0^2}{\sigma}\frac{\partial \overline{u}}{\partial p} + L_3 \frac{f_0 p_0}{W_0}\right] \tag{2.22}$$

where $V = \overline{\varsigma v}$ is the transport of relative vorticity. (2.22) contains the key to the parameterization of the momentum transport because:

$$V = \overline{\varsigma v} = \overline{\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) v} = -\overline{v\frac{\partial u}{\partial y}} = -\frac{\partial \overline{uv}}{\partial y} + \overline{u\frac{\partial v}{\partial y}} = -\frac{\partial \overline{uv}}{\partial y} - \overline{u\frac{\partial u}{\partial x}} = -\frac{\partial \overline{uv}}{\partial y} \tag{2.23}$$

In deriving (2.23) we have used the assumption that the horizontal wind is nondivergent, *i.e.*:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{2.24}$$

The use of (2.22) in our calculations to be reported later are eased by introducing nondimensional coordinates. We use:

$$y = W\eta, \;\; t = \varepsilon^{-1}\tau, \;\; p = p_0 p_* \tag{2.25}$$

where ε is a dissipation time to be introducted later. The nondimensional quantities will be denoted by an accent ($\hat{}$). In this way we get:

$$\hat{V} = -\hat{K}\left(\hat{\beta} - \frac{\partial^2 \hat{u}}{\partial \eta^2}\right) + \hat{K}\frac{\partial}{\partial p_*}\left(\lambda^2 \frac{\partial \hat{u}}{\partial p_*}\right) - \frac{\partial}{\partial p_*}\left[\hat{L}\,\lambda^2 \frac{\partial \hat{u}}{\partial p_*} + \hat{L}_3 \frac{f_0}{\varepsilon}\right] \;;\; \lambda^2 = \frac{f_0^2}{\sigma\, p_0^2} W^2 \tag{2.26}$$

(2.26) is the key expression in the parameterization prescription because it relates the transport of relative vorticity, and thus also the momentum transport, to the zonally-averaged flow. The derived expression is slightly more general than the equivalent expression used by Green (1970) and White and Green (1984) because we have not assumed from the beginning that exchange coefficients for potential vorticity and for heat are identical. If they were, (2.26) would become somewhat simpler because of cancellations. (2.26) would be changed to:

$$\hat{V} = -\hat{K}\left(\hat{\beta} - \frac{\partial^2 \hat{u}}{\partial \eta_2}\right) - \lambda^2 \frac{\partial \hat{u}}{\partial p_*}\frac{\partial \hat{K}}{\partial p_*} - \frac{f_0}{\varepsilon}\frac{\partial \hat{L}_3}{\partial p_*} \tag{2.27}$$

As has been mentioned before it is a fact that the transport of sensible heat in the troposphere is from south to north or from the warmer to colder regions and thus satisfying a necessary condition for the use of our assumption. The transport of potential vorticity have been investigated by Wiin-Nielsen and Sela (1971) using atmospheric data. It was found that the horizontal transport is from north to south in the troposphere above the lowest layers say above 80 cb, where the potential vorticity below this level may be less conservative due to the strong dissipation in the atmospheric boundary layer. The negative sign of the transport is in good agreement with the exchange coefficient approach because $d\bar{\xi}/dy$ in general is positive due to the contribution from the beta term, $\beta = df/dy$. It was then possible to calculate numerical values of K as a function of latitude and height. $K(y,p)$ is positive in the troposphere, has a maximum in middle latitudes and decreases in general with decreasing pressure. The same investigation includes an investigation of the transports of sensible heat and a similar evaluation of the exchange coefficient (L). This coefficient has also a maximum in middle latitudes and gradually decreasing values with decreasing pressures. No evaluation of L_3 from data is known to the author. The parameterizations formulated above have been used in various ways. They were used by Sela and Wiin-Nielsen (1971) to simulate the annual energy cycle, by Wiin-Nielsen (1971) to formulate a simplified theory of the annual variation of the general circulation, by Wiin-Nielsen (1972) to investigate the annual variation of the zonally-averaged state of the atmosphere, and by Fuenzalida and Wiin-Nielsen (1975) to simulate the axisymmetric circulation. Furthermore, the concept has naturally been used by Green (1970), White and Green (1983, 1984) and by Wu and White (1986). In spite of these applications it is still worthwhile to investigate the concepts partly because we want to test the present somewhat broader formulation, partly due to the emphasis which Green and White (1982) put on the integral constraints, and partly because we want to solve the steady-state problem.

The integral constraint just mentioned refers to the fact that:

$$\int_0^1 \hat{V}(\eta)d\eta = -\int_0^1 \frac{\partial \hat{M}}{\partial \eta}d\eta = 0 \tag{2.28}$$

when $\hat{M}$ is the nondimensional momentum transport, *i.e.* $\hat{M} = \overline{\hat{u}\hat{v}}$. While (2.28) is not new, but has been used since the invention of the parameterization scheme, (see Wiin-Nielsen, 1971) it turns out that White and Green (1982) have proposed a new way of satisfying it.

It may be of interest before we apply the results of this section to a full model to give some simple examples of how the indirect momentum transport prescription (2.26) with the constraint (2.28) may work in practice. For this purpose and for simplicity we divide the atmosphere in an upper part, (subscript 1), and lower part (subscript 3). We find from (2.26):

$$\hat{V}_1 = -\hat{K}_1 \left(\hat{\beta} - \frac{\partial^2 \hat{u}_1}{\partial \eta^2}\right) + 16\lambda^2 \hat{L}_\tau \hat{u}_\tau + 2\frac{f_0}{\varepsilon} \hat{L}_3 \tag{2.29}$$

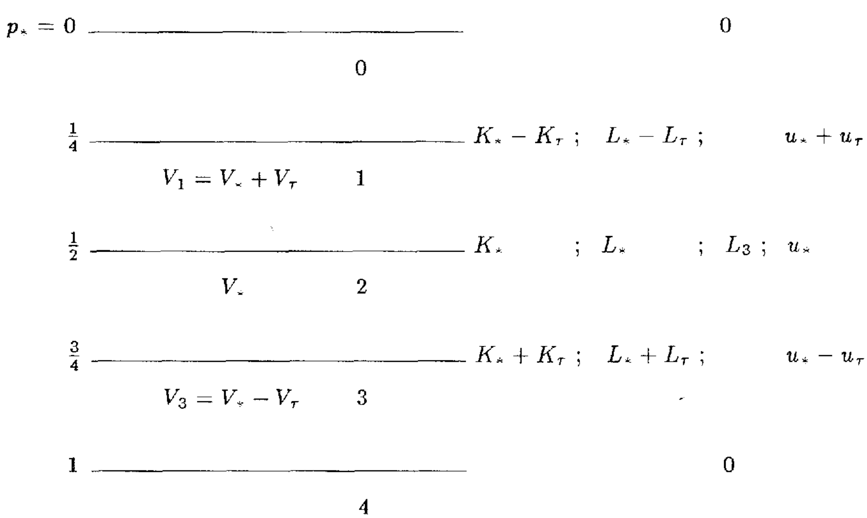
and:

$$\hat{V}_3 = -\hat{K}_3 \left(\hat{\beta} - \frac{\partial^2 u_3}{d\eta^2}\right) + 16\lambda^2 \hat{L}_\tau \hat{u}_\tau - 2\frac{f_0}{\varepsilon} \hat{L}_3 \tag{2.30}$$

in which $\hat{L}_\tau$ are $\hat{u}_\tau$ are defined as:

$$\frac{\partial \hat{L}}{\partial p_*} = \frac{\hat{L}_3 - \hat{L}_1}{\frac{1}{2}} = 4\hat{L}_\tau \; ; \; \frac{\partial \hat{u}}{\partial p_*} = \frac{\hat{u}_3 - \hat{u}_1}{\frac{1}{2}} = -4\hat{u}_\tau \tag{2.31}$$

(see Fig. 1)



Adding and subtracting (2.29) and (2.30) and using again Fig. 1 we find introducing the momentum transport the following equations:

$$\frac{\partial \hat{M}_*}{\partial \eta} = \hat{K}_* \left(\hat{\beta} - \frac{\partial^2 \hat{u}_*}{\partial \eta^2}\right) + \hat{K}_\tau \frac{\partial^2 \hat{u}_\tau}{\partial \eta^2} - 16\ \lambda^2\ \hat{L}_\tau\ \hat{u}_\tau \tag{2.32}$$

$$\frac{\partial \hat{M}_\tau}{\partial \eta} = -\hat{K}_\tau \left(\hat{\beta} - \frac{\partial^2 \hat{u}_*}{\partial \eta^2}\right) - \hat{K}_* \frac{\partial^2 \hat{u}_\tau}{\partial \eta^2} + 2\frac{f_0}{\varepsilon}\ \hat{L}_3 \tag{2.33}$$

For simplicity we assume:

$$\hat{u}_* = U_*(0) - U_*(2)\ \cos\ 2\pi\eta$$

$$\hat{u}_\tau = U_\tau(0) - U_\tau(2)\ \cos\ 2\pi\eta \tag{2.34}$$

giving:

$$\frac{\partial \hat{M}_*}{\partial \eta} = \hat{\beta}\hat{K}_* - 16\ \lambda^2\ \hat{L}_\tau U_\tau(0)$$

$$- \left(4\pi^2 \left(\hat{K}_* U_*(2) - \hat{K}_\tau U_\tau(2)\right) - 16\ \lambda^2\ \hat{L}_\tau \hat{U}_\tau(2)\right) \cos\ 2\pi\eta \tag{2.35}$$

$$\frac{\partial \hat{M}_\tau}{\partial \eta} = -\hat{\beta}\hat{K}_\tau + 2\frac{f_0}{\varepsilon}\ \hat{L}_3 + 4\pi^2 \left(\hat{K}_\tau U_*(2) - \hat{K}_* U_\tau(2)\right) \cos\ 2\pi\eta \tag{2.36}$$

To secure that the integral constraints are satisfied it is necessary that:

$$\hat{\beta}\hat{K}_* - 16\lambda_2 \hat{L}_\tau U_\tau(0) = 0$$

$$-\hat{\beta}\hat{K}_\tau + 2\frac{f_0}{\varepsilon}\hat{L}_3 = 0 \tag{2.37}$$

The two equations in (2.37) can be used to calculate $\hat{L}_\tau$ and $\hat{L}_3$ from $\hat{K}_*$ and $\hat{K}_\tau$ and the other parameters. By integrations of (2.35) and (2.36) we find:

$$\hat{M}_* = \left[\frac{16\lambda_2}{2\pi}\hat{L}_\tau U_\tau(2) + 2\pi(\hat{K}_\tau U_\tau(2) - \hat{K}_* U_*(2)\right] \sin(2\pi\eta) \tag{2.38}$$

$$\hat{M}_\tau = 2\pi\left[\hat{K}_\tau U_*(2) - \hat{K}_* U_\tau(2)\right] \sin(2\pi\eta) \tag{2.39}$$

If (2.38) and (2.39) shall correspond to conditions in the atmosphere it is necessary that the quantities in the factors to the sine-functions are positive. These considerations lead to the inequalities:

$$\frac{\hat{K}_*}{\hat{K}_\tau} U_\tau(2) < U_*(2) < \frac{\hat{K}_\tau}{\hat{K}_*} U_\tau(2) + \frac{\hat{\beta}}{4\pi^2}\frac{1}{U_\tau(0)} \tag{2.40}$$

The momentum parameterization is thus such that it may or may not result in an agreement with atmospheric conditions. On the other hand if realistic values of $U_*(2)$ and $U_\tau(2)$ are obtained there is a rather large interval for $U_*(2)$. If we select reasonable mean values for $U_\tau(0)$ and $U_\tau(2)$ being 0.45 ($\sim 9ms^{-1}$) and 0.3 ($\sim 6ms^{-1}$) respectively we find that:

$$0.7 < U_*(2) < 4.57 \tag{2.41}$$

or, in ms^{-1}:

$$14 < U_*(2) < 91.46 \tag{2.42}$$

We stress, however, after this illustration that we are not free to select the values of the wind components. They should be determind from a properly formulated model.

3. A two-level model

In this section we shall formulate a closed model of the atmosphere including heating and dissipation. We shall form the zonally-averaged equations in the steady state, apply the parameterizations for the momentum transports developed in section 2 and seek solutions to these equations. The purpose will then be to see if these solutions bear any resemblance to the observed flow in the atmosphere. Deliberately we make the model as simple as possible to explore the solutions in this case. Under the assumption that the dissipation in the planetary boundary layer is larger than above this layer we shall use the surface stress only.

The zonally-averaged first equation of motion is:

$$\frac{\partial M}{\partial y} = f\overline{v} + g\frac{\partial \overline{\tau}_x}{\partial p};\ M = \overline{uv}\ ;\ V = -\frac{\partial M}{\partial y} \tag{3.1}$$

Applying this equation at levels 1 and 3 as described in section 2 we find under the assumption made above:

$$\begin{aligned} -V_1 &= +f\overline{v}_1 \\ -V_3 &= +f\overline{v}_3 + 2\frac{g}{p_0}\overline{\tau}_{4,x} \end{aligned} \tag{3.2}$$

From the continuity equation applied at the same levels we find:

$$\begin{aligned} \frac{\partial \overline{v}_1}{\partial y} &= -\frac{2}{p_0}\overline{\omega}_2 \\ \frac{\partial \overline{v}_3}{\partial y} &= +\frac{2}{p_0}\overline{\omega}_2 \end{aligned} \tag{3.3}$$

Adding the two equations in (3.3) and noting that:

$$\overline{v}_1 = \overline{v}_3 = 0 \tag{3.4}$$

at the boundaries to the north and to the south we see that:

$$\overline{v}_1 + \overline{v}_3 = 0 \tag{3.4}$$

and we need only compute one of them. The procedure is then to obtain $\overline{\omega}_2$ from the thermodynamic equation and find, say $\overline{v}_1$, by integrating the first equation in (3.3). From (2.21) we know that:

$$\frac{\alpha_0}{f_0}\overline{\ln\theta v} = -L\frac{\partial \overline{u}}{\partial p} - L_3\frac{p_0\sigma}{Wf_0} \tag{3.5}$$

and the zonally-averaged thermodynamic equation gives:

$$+\frac{\partial}{\partial y}\left[L\left(\frac{\partial \overline{u}}{\partial p}\right)_2 + L_3\frac{p_0\sigma}{Wf_0}\right] + \frac{\sigma}{f_0}\overline{\omega}_2 = -\frac{R}{C_p}\frac{1}{p_2 f_0}Q_2 \tag{3.6}$$

When (3.6) is evaluated and made nondimensional we obtain:

$$\hat{\omega}_2 = -\frac{\varepsilon}{f_0}\lambda^2\left[\frac{\partial}{\partial\eta}\left\{-4\hat{L}_*\hat{U}_T + \frac{f_0}{\varepsilon}\frac{1}{\lambda^2}\hat{L}_3\right\} - 2\hat{Q}_2\right] \tag{3.7}$$

where:

$$Q_2 = \frac{C_p}{R}\cdot f_0\,\varepsilon^2 W^2 \hat{Q}_2 \tag{3.8}$$

(3.7) is then integrated to obtain $\hat{v}$, with the result that:

$$\hat{v}_1 = 2\frac{\varepsilon}{f_0}\lambda^2\left[-4\hat{L}_*\hat{U}_T + \frac{f_0}{\varepsilon}\frac{1}{\lambda^2}\hat{L}_3 - 2\int_0^\eta \hat{Q}_2\,d\eta\right] \tag{3.9}$$

In (3.2) we write in the usual way:

$$2\frac{g}{p_0}\overline{\tau}_{4,x} = -2\varepsilon\overline{u}_4 \tag{3.10}$$

and (3.2) becomes then by addition and subtraction;

$$\begin{aligned} -\hat{V}_* &= -\hat{u}_4 \\ -\hat{V}_T &= \frac{f_0}{\varepsilon}\hat{v}_1 + \hat{u}_4 \end{aligned} \tag{3.11}$$

After all these preparations it is possible to write the final equations using (2.32) and (2.33). We find:

$$\hat{K}_*\left(\hat{\beta} - \frac{\partial^2\hat{U}_*}{\partial\eta^2}\right) + \hat{K}_T\frac{\partial^2\hat{U}_T}{\partial\eta^2} - 16\lambda^2\hat{L}_T\hat{U}_T = -\left(\hat{U}_* - 2\hat{U}_T\right) \tag{3.12}$$

$$-\hat{K}_T\left(\hat{\beta} - \frac{\partial^2\hat{U}_*}{\partial\eta^2}\right) - \hat{K}_*\frac{\partial^2\hat{U}_T}{\partial\eta^2} + 2\frac{f_0}{\varepsilon}\hat{L}_3 = \tag{3.13}$$

$$-8\lambda^2\hat{L}_*\hat{U}_T + 2\frac{f_0}{\varepsilon}\hat{L}_3 - 4\lambda^2\int_0^\eta \hat{Q}_2\,d\eta + \left(\hat{U}_* - 2\hat{U}_T\right)$$

In the derivations leading to this system we have made one additional assumption. We have assumed that the expression in the parenthesis after $\partial/\partial\eta$ in (3.7) is zero for $\eta = 0$. This assumption is definitely satisfied if $\hat{L}_*$ and $\hat{L}_3$ are zero for $\eta = 0$, but can satisfied also if $\hat{U}_\tau$ and $\hat{L}_3$ are zero for $\eta = 0$. The system (3.12) (3.13) is in general nonlinear because the coefficient, $\hat{K}_*$, $\hat{K}_\tau$,$\hat{L}_*$,$\hat{L}_\tau$,and $\hat{L}_3$, are functions of η. Diagnostic studies show that they normally have a maximum in middle latitudes (Wiin-Nielsen and Sela, 1971) with smaller values at the North Pole and towards the Equator. The system of equations is written in such a way that the left hand sides are the derivatives $d\,\hat{M}_*/d\eta$ and $d\hat{M}_\tau/d\eta$. The integral constraints are therefore that these expressions when integrated from wall to wall, *i.e.* from 0 to 1 with respect to η, should vanish.

To satisfy the assumptions made in the derivations we shall assume that all the exchange coefficients are zero at the two boundaries and that they have a maximum in the middle of the channel. Many specifications of this kind are possible, but since we later on shall solve the equations in the spectral domain it is an advantage to use trigonometric functions. We adopt:

$$\left(\hat{K},\, \hat{L}\right) = (K,\, L)\sin^2(\pi\eta) \;=\; \frac{1}{2}\,(K,\, L)\,(1 - \cos 2\pi\eta) \tag{3.14}$$

We shall further expand the velocities in Fourier series of the form:

$$(\hat{U}_*,\, \hat{U}_T,\, \hat{U}_4) = \sum_n \,(U_*(n),\, U_T(n),\, U_4(n))\sin(\pi n\eta) \tag{3.15}$$

The functions $\sin(\pi p\eta)$ are orthogonal over the interval $0 \le \eta \le 1$ because:

$$\int_0^1 \sin(\pi p\eta)\,\sin(\pi q\eta)\,d\eta = \begin{cases} 0 & p \neq q \\ \frac{1}{2} & p = q \end{cases} \tag{3.16}$$

The expressions (3.14) and (3.15) are introduced in (3.12) and (3.13) whereafter the spectral equations are derived in the usual way. We must, however, pay special attention to the integral constraint (2.28) which applies to V_* as well as V_τ. This means that the left hand sides of (3.12) and (3.13) should integrate to zero across the channel. It follows then that so must the right hand sides of the same equations.

In this part of the investigation we shall use a low order system restricting the investigation to the largest scales. On the other hand, it will be necessary to include a sufficient number of components to allow the typical wind changes with latitude which we are looking for. It is easily seen that restriction of n in (3.15) to the values

1 and 3 is a minimum system. If it is adopted, it is evident also that we shall obtain solutions which are simple and schematic. On balance it would seem desirable to investigate if the theory works under these conditions.

We decide therefore to restrict the components to the two mentioned above. When (3.14) and (3.15) are inserted in the left hand sides of (3.12) and (3.13) we obtain after integration from 0 to 1 with respect to η the two integral constraints which guarantee that the transport of relative vorticity has a vanshing mean value. We obtain:

$$F = K_* \left(\hat{\beta} + \frac{8\pi}{15} \left(5U_*(1) - 9U_*(3) \right) - K_T \frac{8\pi}{15} \left(5U_T(1) - 9U_T(3) \right) \right)$$

$$- \frac{16}{15\pi} \lambda^2 L_T \left(5U_T(1) - U_T(3) \right) = 0 \qquad \text{(I.C.1)}$$

$$G = K_T \left(\hat{\beta} + \frac{8\pi}{15} \left(5U_*(1) - 9U_*(3) \right) - K_* \frac{8\pi}{15} \left(5U_T(1) - 9U_T(3) \right) \right)$$

$$- \frac{f_0}{\varepsilon} L_3 = 0 \qquad \text{(I.C.2)}$$

Since the left hand sides of (3.12) and (3.13) integrate to zero as described above the same must hold for the right sides. We obtain from the right hand sides:

$$U_*(1) - 2U_T(1) + \frac{1}{3} \left(U_*(3) - 2U_T(3) \right) = 0 \qquad \text{(I)}$$

$$U_T(1) + \frac{1}{3} U_T(3) = \frac{\pi}{2\lambda^2 L_*} \left(\frac{f_0}{\varepsilon} L_3 + 2 \frac{\lambda^2}{\pi^2} A_H \right) \qquad \text{(II)}$$

In the derivation of (II) we have assumed that the heating is specified in the simple form that $\hat{Q} = A_H \cos(\pi\eta)$. The next two equations are obtained by multiplying (3.12) and (3.13) by $\sin\pi\eta$ and thereafter integrating from 0 to 1 with respect to η. If additional components are included we would go on multiplying by $\sin 3\pi\eta$ and integrating and so on. We find:

$$-\left(1+\frac{3\pi^2}{4}K_*\right)U_*(1)+\frac{9\pi^2}{4}K_*U_*(3)+\left(2+\frac{3\pi^2}{4}K_\tau+\frac{3}{2}\lambda^2 L_\tau\right)U_T(1)$$

$$-\left(\frac{9\pi^2}{4}K_\tau+\frac{1}{2}\lambda^2 L_\tau\right)U_T(3)=\frac{8}{3\pi}\hat{\beta}K_* \qquad \text{(III)}$$

$$-\left(1+\frac{3\pi^2}{4}K_\tau\right)U_*(1)+\frac{9\pi^2}{4}K_\tau U_*(3)+\left(2+\frac{3\pi^2}{4}K_*+\lambda^2 L_*\right)U_T(1)$$

$$-\frac{9\pi^2}{4}K_*U_T(3)$$

$$=\frac{\lambda^2}{\pi^2}A_H+\frac{8}{3\pi}\left(\hat{\beta}K_\tau-\frac{f_0}{\varepsilon}L_3\right) \qquad \text{(IV)}$$

The system (I.C.1), (I.C.2), (I), (II), (III) and (IV) describes the selected low-order system. It is seen that it is a nonlinear system because of the products of the exchange coefficients and the velocity components. The solution is in each case obtained by assuming fixed tropospheric values for A_H,K_*,$K_\tau$andL_*. The following iterative procedure is adopted. We make a guess of the quantities L_τ and L_3. The system (I) - (IV) is solved for$U_*(1)$,$U_*(3)$,$U_\tau(1)$,and$U_\tau(3)$. The values F and G are computed from (I.C.1) and (I.C.2). If these functions are different from zero we calculate new values of L_τ and L_3 from the formulas:

$$L_\tau^{(n+1)}=L_\tau^{(n)}+\delta L_\tau \qquad (3.17)$$

$$L_3^{(n+1)}=L_3^{(n)}+\delta L_3 \qquad (3.18)$$

where:

$$\delta L_\tau=\frac{F^{(n)}}{\frac{16}{15\pi}\lambda^2\left(5U_\tau(1)^{(n)}-U_\tau(3)^{(n)}\right)} \qquad (3.19)$$

$$\delta L_3=\frac{G^{(n)}}{\frac{f_0}{\varepsilon}} \qquad (3.20)$$

With the new values of $L_\tau^{(n+1)}$, $L_3^{(n+1)}$ we solve the system (I)-(IV) obtaining the values $U_*(1)^{(n+1)}, U_*(3)^{(n+1)}, U_\tau(1)^{(n+1)}, U_\tau(3)^{(n+1)}$, which in turn lead to new values $L_\tau^{(n+2)}, L_3^{(n+2)}$ and so on. The iterations are continued until both F and G satisfy the inequalities:

$$|F| < \varepsilon_\tau \, , \; |G| < \varepsilon_\tau \tag{3.21}$$

where ε_τ is specified as a sufficiently small number.

The iterative procedure converges for realistic values of A_H, K_*, K_τ and L_*, and the results will be described below. It was also found that no convergence could be obtained for small values af A_H for fixed values of K_*, K_τ and L_*. We may interpret this finding in the following way. For sufficiently small values of A_H, we obtain a meridional temperature gradient or, equivalently, a vertical windshear which is too small for baroclinic instability to occur. The present parameterization of the meridional transports of heat and potential vorticity assumes that waves are present, but this assumption will not be satisfied if the heating contrast is too small.

The diagnostic calculations carried out by Lawniczak (1969) lead to:

$$Q \simeq 4 \times 10^{-3} \cos(\pi\eta) \tag{3.22}$$

where Q is measured in $kjt^{-1}s^{-1}$. Using (3.8) we find that $A_H \simeq 0.03$. White and Green (1984) use a value which corresponds to $A_H \simeq 0.05$.

Regarding the transfer coefficients Wiin-Nielsen (1971) finds average values of K_* and K_τ. In nondimensional form they correspond to $K_* = 0.006$ and $K_\tau = 0.0025$. The coefficients entering the equations should then in view of (3.14) have the values 0.012 and 0.005, respectively. From the same source we find $L_* = 0.014$ while the values of L_τ and L_3 are determined by the solutions. We present first the details of the solution for this standard case.

Fig. 1 shows the wind distributions at the upper level (25 cb) and the lower level (75 cb). Both distributions are characterized by very flat maxima of $40ms^{-1}$ and $15ms^{-1}$, respectively. The wind at 100 cb, or approximately the surface wind, is also shown in Fig. 1. It has a maximum of $2ms^{-1}$ in the middle of the channel with easterlies in the polar and equatorial regions. The maximum easterlies is about $1.2ms^{-1}$. We may thus conclude that the parameterizations of the meridional transports of quasi-nondivergent potential vorticity and sensible haet, and thus indirectly of the meridional transport of momentum, can give solutions which in terms of the wind distributions describe the gross-features of the observed circulations with reasonable magnitudes.

As can be seen from a combination of (2.32) and (3.12) we have:

$$\frac{\partial \hat{M}_*}{\partial \eta} = -\hat{U}_4 \tag{3.23}$$

indicating that the 100cb wind profile is determined by the divergence of the vertically intregrated momentum transport. It is thus possible to obtain an essentially correct profile $U_4 = U_4(\eta)$ without having a correct distribution of $M = M(\eta, p_*)$. It is thus of importance to investigate how the transports are distributed vertically.

Fig. 2 shows the transport of sensible heat as a function of the north-south coordinate. We find again a broad maximum with a maximum value of 3000 $kjm^{-1}s^{-1}cb^{-1}$. Fig. 2 shows the transport of sensible heat as a function of the north-south coordinate. We find again a broad maximum with a maximum value af 3000 $kjm^{-1}s\text{-}^{1}cb^{-1}$. In comparison we note that Wiin-Nielsen *et al.* (1963) found values which may be converted to about 2200 $kjm^{-1}s^{-1}cb^{-1}$. The momentum transports M_1 and M_3 are also shown in Fig. 2 in the units m^2s^{-2}. They show at both levels a northward transport in the region $0 < \eta < 0.5$ and a southward transport for $0.5 < \eta < 1$. At both levels we find therefore a convergence in the middle of the channel and divergence in the region close to the boundary.

The zonally-averaged vertical velocity is shown in Fig. 3. It is converted into $\overline{w}$ and measured in mms^{-1}. It shows the typical three cell pattern with two direct (Hadley) cells surrounding the direct (Ferrell) cell in the middle. The maximum value is about 2.7 mms^{-1}.

The results shown in Fig. 2 and Fig. 3 indicate that the adopted parameterizations give qualitatively correct results for both the heat and momentum transports and the meridional circulation. Quantitatively, they are all of the correct order of magnitude except the transports of momentum which are considerably smaller than observed. One may speculate on the reasons for this failure of the model. One of the causes could be the low order nature of the model in both horizontal and vertical directions. In the present model we have a linear variation of the momentum transport because:

$$M_1 = M_* + M_\tau \; ; \; M_3 = M_* - M_\tau \tag{3.24}$$

while such a linear variation is absent in observational studies. Similarly, the asymmetric distribution around the center of the channel is not observed in the atmosphere where the northward momentum transport extends to much higher latitudes. Both of these considerations would call for increased resolution in the vertical and horizontal directions.

It is of interest to investigate the sensitivity of the model to the intensity of the heating. The solutions were therefore obtained for a series of heating values expressed this time in the unit: deg day^{-1} which can be obtained from the previously used non-dimensional unit. Various quantities are shown in Fig. 4 as a function of the heating. The maxima of the heat flux and of the zonal winds increase with the intensity of the heating. The same is true for the 100 cb wind, but it is seen that the increase is much smaller, and the $\overline{u}_{4,max}$ seems to reach an asymptotic value of about $2.6ms^{-1}$. A similar tendency is observed for the calculated values of M_1, max and M_3, max. For the sake of completeness we mention that those experiments were carried out with the same values of the exchange coefficients as in the main experiment, *i.e.* $K_* = 0.012, K_\tau = 0.005, L_* = 0.014$. Fig. 5 shows how L_τ and L_3 vary with the heating intensity in the experiments. Both of these decrease as the heating increases, but L_τ decreases at a larger rate than L_3. It looks as if L_τ might go to infinity for a fixed value of H. When no solution is found for very small values of H it could be because a sufficiently large value of L_τ could not be found to satisfy the heat transport requirement.

A series of experiments with smaller values of the specified exchange coefficients were also carried out. In this set of calculations we used $K_* = 0.008, K_\tau = 0.0035$ and $L_* = 0.0093$. The distributions with respect to η are similar to those shown in Fig. 1 and Fig. 2 for the values of the heating included in the various calculations. The results are therefore shown in Fig. 6 in the same form as in Fig. 4. A comparison between these figures show that the solutions included in Fig. 6 are more unrealistic because the maximum zonal velocities are too large for realistic values of the heating. We notice also somewhat smaller maximum values of U_4 and of M_1 and M_3.

As mentioned before no solution can be obtained for small values of the heating. Fig. 7 shows the type of solution which is obtained when the heating is slightly above critical value. The solution displayed in Fig. 7 is calculated for $A_H = 0.9$ deg day^{-1}, $K_* = 0.008$, $K_\tau = 0.0035$ and $L_* = 0.0093$. The zonal velocities, $\overline{U}_1$ and $\overline{U}_3$, have double maxima of $21ms^{-1}$ and $7ms^{-1}$. $\overline{U}_4$ is extremely small, just a few cms^{-1}, with westerlies close to the boundaries and weak easterlies in the middle of the channel. The heat transport is small, and the momentum transports are both extremely small and of the opposite sign compared to the other solutions shown so far.

The model employed so far is the so-called quasi-nondivergent two level model. A similar, vertically integrated, two parameter model can also be used as done by White and Green (1984). The final equations for such a model can be compared with the system (I.C.1, 2) (I-IV). The two systems differ only in the numerical values of some coefficients.

We shall not go through the two-parameter model at this point. It will suffice to show a single example. The parameters are: $A_H = 0.5$degday^{-1},$\hat{\beta} = 80, f_o = 50, \lambda_2 =$

$100, K_* = 0.012, K_\tau = 0.005, L_* = 0.014, L_\tau = 0.0506, L_3 = 0.00397$. Fig. 8 shows the wind profiles $\overline{U}_1, \overline{U}_3$, and $\overline{U}_4$, in ms^{-1}. It is seen that the easterlies in this case extend to high levels, and that $\overline{U}_4$ is somewhat stronger than in the two-level case. The mean meridional circulation, seen in Fig. 9, is also stronger, particularly in the Ferrel cell. The transport of sensible heat, shown in the same figure, is concentrated in the middle of the channel in agreement with the strong temperature gradient, shown to the right in Fig. 9.

The momentum transports M_1 and M_3 are larger than in the previous cases as one would expect from the larger surface velocities. In this case it is also true that $| M_1 | > | M_3 |$, but the difference is so small that we have drawn M_1 only.

The difference between this and the previous cases may be explained by the fact that the transport of relative vorticity with the present parameterization is a relatively small difference between the transports of potential vorticity and heat as they enter the formulas. The numerical coefficients are determined by the assumptions made for the functions which enter the specifications of the type:

$$\mathbf{V} = \mathbf{V}_* + A(p_*)\,\mathbf{V}_\tau$$

in the parameter model.

4. The spherical case

The model treated in Sections 2 and 3 is on the so-called beta plane. Experience shows that such models in general contain the main mechanisms, but the solutions are somewhat schematic. For this reason it may be worthwhile to make a similar model on the sphere. In this case we treat the Coriolis parameter correctly, avoid the unrealistic boundary conditions and shall work with the Legendre polynominals which are the natural set of functions for the zonally-averaged case. We shall introduce a new mechanism of internal dissipation, following Charney (1959), as compared with the model used earlier in this paper.

As in the previous case we start with the first equation of motion in the zonally-averaged form for the levels 1 and 3:

$$-V_1 = f\,\overline{v}_1 + 2\,\frac{g}{p_0}\,\overline{\tau}_{x,2}$$

$$-V_3 = f\,\overline{v}_1 - 2\,\frac{g}{p_0}\,\overline{\tau}_{x,2} + 2\,\frac{g}{p_0}\,\overline{\tau}_{x,4} \tag{4.1}$$

V_1 and V_3 are the vorticity transports at the two-levels. It is easy to show that

$$V = -\frac{1}{a\cos^2\varphi}\frac{\partial M\cos^2\varphi}{\partial\varphi} \tag{4.2}$$

where $M = \overline{(u^1v^1)}$, a is the radius of the earth and φ is latitude. Note also that we have already introduced the fact that $\overline{v_3} = -\overline{v_1}$ in (4.1). This relation is obtained from the continuity equation which at the upper level (1) is

$$\frac{\partial\,\overline{v_1}\cos\varphi}{a\cos\varphi\,\partial\varphi} + \frac{\overline{\omega}_2}{P} = 0\ ;\ P = 50\text{cb} \tag{4.3}$$

At level 3 we get

$$\frac{\partial\overline{v_3}\cos\varphi}{a\cos\varphi\,\partial\varphi} - \frac{\overline{\omega}_2}{P} = 0 \tag{4.4}$$

We introduce

$$\overline{v}_1 = \overline{v}_* + \overline{v}_T$$

$$\overline{v}_3 = \overline{v}_* - \overline{v}_T \tag{4.5}$$

Adding and subtracting (4.3) and (4.4) we find:

$$\frac{\partial\,\overline{v}_*\cos\varphi}{a\cos\varphi\,\partial\varphi} = 0 \tag{4.6}$$

and

$$\frac{\partial\,\overline{v}_T\cos\varphi}{a\cos\varphi\,\partial\varphi} + \frac{\overline{\omega}_2}{P} = 0 \tag{4.7}$$

It follows from (4.6) that $\overline{v}_* = 0$ and thus $\overline{v}_1 = -\overline{v}_3$. In (4.1) we introduce also following Charney (1959) that

$$2\frac{g}{p_0}\,\overline{\tau}_{x,2} = -A\,\overline{U}_T$$

$$2\frac{g}{p_0}\,\overline{\tau}_{x,4} = -2\varepsilon\,\overline{U}_4 \tag{4.8}$$

The standard estimates are that $A = 6 \times 10^{-7} s^{-1}$ and $\varepsilon = 2 \times 10^{-6} s^{-1}$. Finally, adding and subtracting the two equations in (4.1) we obtain

$$-\overline{V}_* = -\varepsilon \overline{u}_4$$

$$-\overline{V}_\tau = f \overline{v}_\tau - A \overline{u}_\tau + \varepsilon \overline{u}_4 \tag{4.9}$$

The further derivations follow the previous model. $\overline{V_*}$ and $\overline{V_\tau}$ are obtained from the parameterization prescriptions, $\overline{\omega}_2$ is obtained from the thermo-dynamic equation, $\overline{v}_\tau$ from (4.7), and we have then a closed system.

Since the difference between the spherical model and the beta plane model, according to experience, does not produce any new physical insight as long as the physical processes are the same it would hardly be justified to repeat the complete calculation without changing the description of the physical processes which enter the model. We have already introduced an internal friction by specifying the stress at level 2 in terms of the vertical windshear, $\overline{u}_\tau$ while this process was neglected in the beta-plane model. We shall also reconsider the parameterizations of the transports of quasi-nondivergent potential vorticity and sensible heat.

In the beta-plane model we formulated the prescriptions in such a way that the processes in the two-level formulations were characterized by the exchange coefficients K_* and K_τ for the potential vorticity and L_*, L_τ and L_3 for the sensible heat. The introduction of L_τ and L_3 is not really consistent with the two-level formulation as used here although the formulation is correct in principle for a vertically integrated model or a model with higher vertical resolution. The fact is that the two-level model in the present formulation contains only one temperature, T_2 at level 2, in each vertical column. Strictly speaking we are therefore only capable of calculating the horizontal transport of sensible heat at this level. The introduction of L_τ goes therefore beyond the normal formulation, but could be justified in a different formulation carrying temperatures at levels 1 and 3. Furthermore, the two-level formulation with only one temperature at level 2 does not permit a vertical stability which varies with pressure and time. The static stability is therefore in the two-level formulation characterized by a constant value of σ which is defined as follows:

$$\sigma = -\alpha \frac{\partial \ell n \, \theta}{\partial p} = \text{const.} \tag{4.10}$$

The introduction of L_3, see (2.19), which is used to describe the vertical eddy transport of heat, is thus only consistent with the model structure if σ can vary with pressure and time as in a model which goes beyond the quasi-geostrophic theory. In the model used in Section 3 we can see the role played by L_3 by an inspection of the equations, I.C. 1, 2, I - IV where L_3 appears with a constant factor everywhere. This means that the vertical transport of sensible heat by the eddies is completely prescribed by the assumed specifications of L_3 as a function of latitude, and it is not in any way related to the distribution of temperature or the winds in the model. One may conclude that the vertical heat transport in the two level, quasi-nondivergent model is somewhat artificial, and that a more realistic description requires a model based on the primitive equations. A truly consistent model with two-levels and based on quasi-geostrophic theory permits therefore only the exchange coefficients K_*, K_τ and L_*. When we nevertheless have used L_τ and L_3 in sections 2 and 3 it is due to the fact that the two level model is very similar to and can be considered as a special case of the vertically integrated two-parameter model.

Retaining only K_*, K_τ and L_* and recalling that two of these are determined by the integral constraints we decrease the number of parameters which can be determined numerically in advance. The empirical elements in the theory are thus reduced.

The quasi-conserved potential vorticities in the two-level model at levels 1 and 3 are

$$\xi_1 = f + \zeta_1 - 8q^2 \psi_\tau$$

$$q^2 = \frac{\Omega^2}{\sigma p_0^2} \tag{4.11}$$

$$\xi_3 = f + \zeta_3 + 8q^2 \psi_\tau$$

from which we get:

$$\overline{\xi_1 v_1} = V_1 - 8\, q^2\, \overline{\psi_\tau\, v_2}$$

$$\overline{\xi_3 v_3} = V_3 + 8\, q^2\, \overline{\psi_\tau\, v_2} \tag{4.12}$$

According to our assumptions we have

$$\overline{\xi_1 v_1} = -K_1 \frac{\partial \overline{\xi_1}}{a\,\partial \varphi} = -K_1 \Big[\frac{2\,\Omega}{a}\cos\varphi + \frac{1}{a^3}\frac{\partial}{\partial \varphi}\Big\{\frac{1}{\cos\varphi}\frac{\partial}{\partial \varphi}\Big(\cos\varphi \frac{\partial \overline{\psi}_1}{\partial \varphi}\Big)\Big\} - 8\,q^2 \frac{\partial \overline{\psi}_\tau}{a\,\partial \varphi}\Big]$$

$$\overline{\xi_3 v_3} = -K_3 \frac{\partial \overline{\xi_3}}{a\,\partial \varphi} = -K_3 \Big[\frac{2\,\Omega}{a}\cos\varphi + \frac{1}{a^3}\frac{\partial}{\partial \varphi}\Big\{\frac{1}{\cos\varphi}\frac{\partial}{\partial \varphi}\Big(\cos\varphi \frac{\partial \overline{\psi}_1}{\partial \varphi}\Big)\Big\} + 8\,q^2 \frac{\partial \overline{\psi}_1}{a\,\partial \varphi}\Big]$$

$$\overline{\psi_\tau v_2} = -\,L_* \frac{\partial \overline{\psi}_\tau}{a\,\partial \varphi} = L_* \overline{u}_\tau$$

(4.13)

Introducing $\mu = \sin\,\varphi$ we obtain by combining (4.12) and (4.13)

$$V_1 \cos\varphi = -\,K_1 \Big[\frac{2\,\Omega}{a}(1-\mu^2) + \frac{1}{a^3}(1-\mu^2)\frac{\partial}{\partial \mu}\Big\{\frac{\partial}{\partial \mu}\Big[(1-\mu^2)\frac{\partial \overline{\psi}_1}{\partial \mu}\Big]\Big\}$$
$$-\,8\,q^2 \frac{1}{a}(1-\mu^2)\frac{\partial \overline{\psi}_\tau}{\partial \mu}\Big] - 8\,q^2\,L_*\,(1-\mu^2)\frac{\partial \overline{\psi}_\tau}{\partial \mu}$$

$$V_3 \cos\varphi = -\,K_3 \Big[\frac{2\,\Omega}{a}(1-\mu^2) + \frac{1}{a^3}(1-\mu^2)\frac{\partial}{\partial \mu}\Big\{\frac{\partial}{\partial \mu}\Big[(1-\mu^2)\frac{\partial \overline{\psi}_3}{\partial \mu}\Big]\Big\}$$
$$+\,8\,q^2 \frac{1}{a}(1-\mu^2)\frac{\partial \overline{\psi}_\tau}{\partial \mu}\Big] - 8\,q^2\,L_*\,(1-\mu^2)\frac{\partial \overline{\psi}_\tau}{\partial \mu} \qquad (4.14)$$

In these equations we use the following transformations to obtain nondimensional quantities

$$V = a\,\Omega^2\,\hat{V},\ K = a^2\,\Omega\,\hat{K},\ \psi = a^2\Omega\hat{\psi} \qquad (4.15)$$

The two equations in (4.14) are added and subtracted using the notations:

$$\hat{\psi}_1 = \hat{\psi}_* + \hat{\psi}_\tau \qquad \hat{K}_1 = \hat{K}_* - K_\tau$$
$$\hat{\psi}_3 = \hat{\psi}_* - \hat{\psi}_\tau \qquad \hat{K}_3 = \hat{K}_* + K_\tau \qquad (4.16)$$

The result is:

$$\hat{V}_* \cos\varphi = -\; 2\,\hat{K}_*\,(1-\mu^2) - \hat{K}_*\,(1-\mu^2)\,\frac{\partial}{\partial\mu}\left\{\frac{\partial}{\partial\mu}\left[(1-\mu^2)\,\frac{\partial\,\hat{\psi}_*}{\partial\mu}\right]\right\}$$

$$+\,\hat{K}_\tau\,(1-\mu^2)\,\frac{\partial}{\partial\mu}\left\{\frac{\partial}{\partial\mu}\left[(1-\mu^2)\,\frac{\partial\,\hat{\psi}_\tau}{\partial\mu}\right]\right\}$$

$$\hat{V}_\tau \cos\varphi = 2\,\hat{K}_\tau\,(1-\mu^2)\; + \hat{K}_\tau\,(1-\mu^2)\,\frac{\partial}{\partial\mu}\left\{\frac{\partial}{\partial\mu}\left[(1-\mu^2)\,\frac{\partial\,\hat{\psi}_*}{\partial\mu}\right]\right\}$$

$$-\,\hat{K}_*\,(1-\mu^2)\,\frac{\partial}{\partial\mu}\left\{\frac{\partial}{\partial\mu}\left[(1-\mu^2)\,\frac{\partial\,\hat{\psi}_\tau}{\partial\mu}\right]\right\}$$

$$-\,8\,\lambda^2\,\hat{L}_*\,(1-\mu^2)\,\frac{\partial\,\hat{\psi}_\tau}{\partial\mu} \tag{4.17}$$

where:

$$\lambda^2 = a^2\,q^2 \tag{4.18}$$

The remaining step is to obtain the mean meridional circulation. From the thermodynamic equation in its steady state form we find using the parameterization of the horizontal heat transport

$$\hat{\omega}_2 = -4\lambda^2\,\frac{\partial}{\partial\mu}\left[\,\hat{L}_*\,(1-\mu^2)\,\frac{\partial\,\hat{\psi}_\tau}{\partial\mu}\right] - \hat{Q}_2 \tag{4.19}$$

in which we have used

$$\overline{\omega}_2 = p_0\,\Omega\,\hat{\omega}_2, \quad \overline{Q}_2 = \frac{1}{2}\,\frac{C_p}{R}\,\Omega\,\sigma\,p_0^2\,\hat{Q}_2 \tag{4.20}$$

Using the continuity equation

$$\frac{\partial\,\hat{v}_\tau\,\cos\varphi}{\partial\mu} = -2\hat{\omega}_2 \tag{4.21}$$

we obtain by integration from $\mu = -1$ to μ

$$\hat{v}_\tau \cos\varphi = 8\lambda^2 \hat{L}_* (1-\mu^2) \frac{\partial \hat{\psi}_\tau}{\partial \mu} + 2H(\mu) \tag{4.22}$$

where:

$$H(\mu) = \int_{-1}^{\mu} \hat{Q}_2 \, d\mu \tag{4.23}$$

We have now expressed all processes in terms of the zonally averaged quantities and have consequently closed the system. The basis equations are those given in (4.9). We may write these equations in the form

$$\begin{aligned} \hat{V}_* \cos\varphi &= -E(1-\mu^2)\frac{\partial(\hat{\psi}_* - 2\hat{\psi}_\tau)}{\partial\mu} \\ \hat{V}_\tau \cos\varphi &= -8\lambda^2 \hat{L}_* (1-\mu^2)\frac{\partial\hat{\psi}_\tau}{\partial\mu} - 2\mu H(\mu) - k(1-\mu^2)\frac{\partial\hat{\psi}_\tau}{\partial\mu} \\ &\quad + E(1-\mu^2)\frac{\partial\hat{\psi}_4}{\partial\mu} \end{aligned} \tag{4.24}$$

The left hand sides of (4.24) are given by the system (4.18) We have furthermore introduced the notations:

$$E = \frac{\epsilon}{\Omega} \; ; \; k = \frac{A}{\Omega} \tag{4.25}$$

The remaining part of this section will deal with the solution of the nonlinear system (4.18), (4.25). The system is nonlinear because the unknown stream functions $\hat{\psi}_*(\mu)$ and $\hat{\psi}_\tau(\mu)$ appear multiplied with the unknown exchange coefficients $\hat{K}_\tau$ and $\hat{L}_*$.

There are several ways in which one might seek solutions to (4.18), (4.25) but the form of the various terms suggest immediately that a natural way is to use a spectral respresentation of the stream function, which is written as a series of Legendre polynomials. In the preliminary study we shall restrict ourselves to low order systems. We want to study the case of symmetry around the equator, meaning that the heating shall consist of a sum of even Legendre functions. To obtain such a solution it is known that the streamfunction must be expressed as a sum of odd Legendre functions. We start therefore by setting

$$\hat{\psi}_* (\mu) = \Psi_*(1)\, P_1 (\mu) + \Psi_*(3)\, P_3 (\mu) \tag{4.26}$$

and

$$\hat{\psi}_T (\mu) = \Psi_T(1)\, P_1 (\mu) + \Psi_T(3)\, P_3 (\mu) \tag{4.27}$$

Using the properties of the Legendre functions it is straightforward to express the terms containing the stream functions in the system. However, we need also to specify the exchange coefficients. Guided once again by the diagnostic studies based on data we select the form

$$\left\{\hat{K} (\mu),\ \hat{L} (\mu)\right\} = \left\{A,\, B\right\} G_* (\mu) \tag{4.28}$$

where

$$G_* = \sin^2 \varphi \cos^2 \varphi = \mu^2 (1 - \mu^2) \tag{4.29}$$

Due to the form of $G_* (\mu)$ it is seen that when (4.27)–(4.30) are substituted in the system (4.18), or (4.25) we shall obtain terms of the form

$$\mu^2 P_n (\mu) \ ; \ \mu^4 P_n (\mu) \tag{4.30}$$

These terms can be handled in at least two ways. One may either, particularly in a low-order system, use the specific expressions for $P_1 (\mu)$ and $P_3 (\mu)$ or one may make repeated use of the relation

$$\mu P_n (\mu) = \frac{1}{2n + 1} \left((n + 1) P_{n+1} (\mu) + n\, P_{n-1} (\mu)\right) \tag{4.31}$$

deriving formulas where $\mu^2\ P_m$ and $\mu^4\ P_m$ are expressed as sums of Legendre functions. A third possibility is to express $G_* (\mu)$ as a sum of Legendre functions.

The final system consists in our case of six nonlinear equations. Two of them are derived from (4.18) by using the integral constraints that

$$\int_{-1}^{+1} \hat{V}_* \cos \varphi \, d\mu = 0 \ ; \ \int_{-1}^{+1} \hat{V}_T \cos \varphi \, d\mu = 0 \tag{4.32}$$

The resulting equations are:

$$F = A_* \left(1 - \Psi_*(1) - 6\Psi_*(3)\right)$$

$$+ A_\tau \left[(1 + 8\lambda^2)\,\Psi_\tau(1) + (6 + 8\lambda^2)\,\Psi_\tau(3)\right] = 0 \qquad \text{(I.C.1')}$$

$$G = A_\tau \left(1 - \Psi_*(1) - 6\Psi_*(3)\right)$$

$$+ A_* \left[(1 + 8\lambda^2)\,\Psi_\tau(1) + (6 + 8\lambda^2)\,\Psi_\tau(3)\right]$$
$$- 8\,\lambda^2\,B_H\,[\Psi_T(1) + \Psi_T(3)] = 0 \qquad \text{(I.C.2')}$$

The next two equations come from the fact that the right hand side of the equations in (4.25) must also integrate to zero. The final two equations in the system are obtained by multiplying both sides of the equations (4.25) by $P_2(\mu)$ followed by an integration from -1 to $+1$ with respect to μ. The right hand side of the first equation in (4.25) gives a particularly simple contribution

$$\Psi_4(1) = \Psi_*(1) - 2\,\Psi_\tau(1) = 0 \qquad (\text{I}^1)$$

The remaining three equations may be written in the form

$$a_{11}\,\Psi_T(1) + a_{12}\,\Psi_*(3) + a_{13}\,\Psi_T(3) = b_1 \qquad (\text{II}^1)$$

$$a_{21}\,\Psi_T(1) + a_{22}\,\Psi_*(3) + a_{23}\,\Psi_T(3) = b_2 \qquad (\text{III}^1)$$

$$a_{31}\,\Psi_T(1) + a_{32}\,\Psi_*(3) + a_{33}\,\Psi_T(3) = b_3 \qquad (\text{IV}^1)$$

where:

$a_{11} = 35k + 64\lambda^2 B_H \quad a_{12} = 0 \quad a_{13} = 64\lambda^2 B_H$

$a_{21} = 0 \quad a_{22} = 99E + 120A_* \quad a_{23} = -(198E + 120A_T + 160\lambda^2 A_T)$

$$a_{31} = -77k \qquad a_{32} = -(198E + 240A_T) \quad a_{33} = 396E + 198k + (240 + 320\lambda^2)A_*$$

$$b_1 = 7H(2) \qquad b_2 = 0 \qquad b_3 = 11\,(H(2) + H(4)) \tag{4.33}$$

We should explain that the heating is specified in such a way that

$$\hat{Q}_2 = H(2)\,P_2\,(\mu) + H(4)\,P_4(\mu) \tag{4.34}$$

The coefficients H(2) and H(4) are calculated in such a way that the heating at the equator is numerically equal to the cooling at the poles. If $\hat{Q}_2(0)$ is the heating at the equator we find

$$\begin{aligned} H(2) &= -\frac{11}{7}\,\hat{Q}_2(0) \\ H(4) &= \frac{4}{7}\,\hat{Q}_2(0) \end{aligned} \tag{4.35}$$

The system (I.C.1^1 – 2^1, I^1 – IV^1) is similar to the analogous system treated in Section 3, but it is more difficult to locate the steady state solution for the four stream function amplitudes and the exchange coefficients A_T and B_H supposedly because the coefficients to A_T and B_H in (I.C.1^1 – 2^1) both depend on the stream-function amplitudes. A more primitive method of obtaining the steady state solutions, if they exist, were therefore adopted.

We may consider A_T and B_H as the major unknows. Selecting a set (A_T,B_H) we may solve (I^1–IV^1) in the usual way and then compute F and G from (I.C.1^1–2^1). The goal is, of course, to select (A_T,B_H) in such a way that $F = G = 0$. A first indication of the solution for (A_T,B_H) is obtained simply by calculating the F and G values in a two-dimensional grid in the (A_T,B_H) - plane restricting the region to $0 \leq A_T \leq A_*$ and $0 \leq B_H \leq 2A_*$. From the calculated values of F and G one may draw the curves $F = 0$ and $G = 0$ in the diagram. The intersection of the two curves is a good guess of the solution. Having thus obtained guesses on A_T and B_H we calculate the corresponding values of the stream function amplitutes from (I^1–IV^1). These six values are finally used as input to a general computer program which can find zeros of a set of nonlinear equations if the first guess is good.

In the following section we shall describe the solutions so obtained in a number of cases.

5. Results from the spherical model

As a first example we have selected a case with the following parameters: $\lambda^2 = 6.25$, $k = 0.0082$, $E = 0.0274$, $A_* = 0.003554$, $H(2) = -1.375\times10^{-3}$ and $H(4) = 5\times10^{-4}$. The values of the six unknowns are:

$$A_T = 0.003294$$

$$B_H = 0.0005283$$

$$\Psi_*(1) = -0.03166$$

$$\Psi_\tau(1) = -0.01583$$

$$\Psi_*(3) = -0.02386$$

$$\Psi_\tau(3) = -0.008218$$

On the basis of this solution we have computed a number of interesting quantities related to the zonal state and to the eddies.

Fig. 11 shows the velocities at level 1 and level 3 as a function of $\mu = \sin\varphi$. The plotted quantities are $\bar{u}_1 \cos\varphi$ and $\bar{u}_3 \cos\varphi$ in ms^{-1}. The maxima are at about 45° N corresponding to annual mean conditions. The order of magnitude, giving $\bar{u}_1$ max $\approx 41ms^{-1}$, is in good agreement with the various estimates shown by Lorenz (1967). To the right in Fig. 11 we find the curve $\bar{u}_4 \cos\varphi$ which shows the easterlies in the low latitudes with a strength of about $5ms^{-1}$ and middle latitude westerlies of a little more than $4ms^{-1}$. There is no indication of any polar easterlies. The heating and the zonally averaged vertical velocity are seen in Fig. 12. The vertical velocity is quite small but indicates the three cell configuration.

In Fig. 13 is shown the momentum transports at levels 1 and 3 converted into the unit: $10^{25} gcm^2s^{-2}$/100mb to make comparisons easy with the various estimates collected by Lorenz (loc. cib.).

The maximum in M_1 is at about 25°N, with the observed maxima in the various estimates seem to be slightly more to the north. The computed maximum in our example is $8 \times 10^{25} gcm^2s^{-2}$/100mb which is in excellent agreement with the results obtained by Mintz (1955) and somewhat larger than the estimates presented by Holopainen (1966) and Buch (1954).

The northward transport of sensible heat is given also Fig. 13 in the unit of 10^{14} Watts. Compared to Peixoto's (1960) estimate we find that both of them have maxima around 50°N, but that the computed maximum is about half of the observed.

The value of $\hat{Q}_2$ used in this calculation $\hat{Q}_{2,\max} = 8.75 \times 10^{-4}$ corresponding to

$Q_2 \sim 3.6 \times 10^{-3} kJt^{-1}s^{-1}$. This value is slightly smaller than the value quoted in (3.22) and obtained from a diagnostic study. We may conclude that the theory proposed for the zonally averaged circulation and based on the parameterization of the potential vorticity transport and the transport of sensible heat as exchange processes give results which are in good qualitative agreement with the results obtained from observations. The main difference is in the transport of sensible heat which is smaller in the model than in the atmosphere.

While there are similarities between this theory and the one presented by White and Green (1984), because both theories are based on a common parameterization idea, there are also differences. The calculation in this section is first of all on the sphere. Secondly, our model is characterized by just three exchange coefficients A_*, A_T and A_H of which only $\hat{A}_*$ is given, while $\hat{A}_T$ and $\hat{A}_H$ are determined by the integral constraints. Thirdly, our model has a basis resolution just enough to give the required profiles, and, fourthly, we solve a steady-state problem, while their calculation is based on a time-integration with possibilities for a "climatic drift" as is often observed in long-term climate integrations.

It is obviously quite important to investigate the range of validity af the theory. Such an investigation can be done in various ways. We have selected to explore the range of $\hat{Q}_2$ which gives physically acceptable solutions. Based on the experience with the betaplane model we would expect that no solution can be found when $\hat{Q}_2$ is sufficiently small. Similarly, a very large value of the heating may give solutions which do not agree with observed conditions.

Fig. 14 shows the isolines for F and G in the above experiment ($\hat{Q}_2 = 8.75 \times 10^{-14}$) as functions of A_T and B_H. The intersection of the two zero lines gives the values of A_T and B_H satisfying the integral constraints (points on the diagram) from which we find the numerical values. The first part of the experiment was to increase $\hat{Q}_2$ from the above value and examine the solution in each case. It turns out that $m_T \cdot (1 - \mu^2)$ is the most sensitive part of the solution. By experiment we determined the value of $\hat{Q}_2$ for which $m_* \; (1 - \mu^2)$ and $m_T \; (1 - \mu^2)$ were about equally large and found:

$$\hat{Q}_2 = 1.3 \times 10^{-3}$$

If $\hat{Q}_2$ were larger than this value there would still be a solution, but the momentum transport at the lower level would be from north to south.

The next series of experiments consisted of gradually decreasing the value of the heating. The solution will then show decreasing values of the sensible heat transport, *i.e.* decreasing values of B_H. When $B_H = 0$ there is a vanishing heat transport by the eddies, and this level has been taken as the lower limit for the validity of the parameterization scheme.

It happens when

$$\hat{Q}_2 \sim 3.7 \times 10^{-4}$$

In summary, we may then say that the model gives physically acceptable results for the following values

$$3.7 \times 10^{-4} < \hat{Q}_2 < 13 \times 10^{-4}$$

which corresponds to

$$1.5 \times 10^{-3} kjt^{-1}s^{-1} < Q_2 < 5.4 \times 10^{-3} kjt^{-1}s^{-1}$$

Considering that the value of Q_2 obtained from Lawniczak's (1970) diagnostic study is about $4 \times 10^{-3} kjt^{-1}s^{-1}$ we may conclude that the region of validity of the theory is satisfactory.

We mention finally that the numerical values given above are obtained for a value of the exchange coefficient K_* which corresponds to a meridional average $\tilde{K}_*$ of

$$\tilde{K}_* = 1.3 \times 10^6 m^2 s^{-1}$$

corresponding to the mean value obtained from diagnostic studies. This value of $\tilde{K}_*$ is the best guess available from data studies at the moment, but the solution will change if $\tilde{K}_*$ changes.

Fig. 15 summarizes the main results of these experiments. In this figure we have shown the maximum values of U_1,U_3, the maximum surface westerlies U_4,W and the maximum surface easterlies U_4,E. Only a very small variation in the location is found in these wind maxima. It is seen that the wind maxima are rather insensitive to the intensity of the heating. On the other hand, the maximum heat transport and, above all, the momentum transports show a large change as the heating changes. It is the larger sensitivity of the momentum transport to the heating which limits the interval, in which the parameterizations give physically meaningful results.

6. Conclusions

The main purpose of the investigation has been to explore if one can account for the major features of the zonally-averaged wind and temperature field by a theory which parameterizes the transport of potential vorticity and sensible heat in terms of zonally averaged quantities. The prescriptions for the two transports and their parameterization give an indirect way of calculating the divergence of the momentum transports and thus the transport itself.

The parameterization idea was first proposed by Green (1970) and has later been used by him and his co-workers to explore many questions, including an investigation of the zonally-averaged winds and temperature fields. Since the first proposal of the parameterization scheme it has been realized that a constraint on the expression for the divergence transport should be that it integrates to zero across the meridional plane or across the channel if the beta-plane geometry is used. (Wiin-Nielsen, 1971).

The present investigation differs from the study by White and Green (1982) in several important ways. The first is that our study applies to a steady-state while they treat the timedependent case. They used a beta-plane geometry while we include also the spherical case, but the major difference is probably in the models adopted for the study and in the way in which the parameterization prescription is carried out.

One may conclude from the investigation that the parameterization can account for the typical surface wind distribution. In fact, since the surface wind in the adopted models depends on the vertically integrated divergence of the momentum transport alone, it is obvious that one may obtain an essentially correct surface wind distribution with an erroneous vertical distribution of the momentum transport. It is this property of the model which makes studies such as the one carried out by White and Wu (1986) rather unsatisfactory because they investigate the surface wind in isolation without telling the reader anything about the corresponding vertical distribution of the momentum transport.

The major conclusion from this study is that one can account for the major aspect of the wind and temperature distributions in the meridional plane from the present theory, and, in addition, that the momentum transports and the heat transports as computed from the parameterization scheme are in reasonable agreement with observations although the heat transport is too small. This statement is true provided the heating intensity is within an interval containing the present estimate of the heating. The lower limit of the interval is determined by the heating intensity which results in a zero-value of the exchange coefficient for the heat transport. The upper limit on the other hand indicates the heating intensity for which the momentum transport at the lower level reverses its sign. The momentum transport in the troposphere below 50 cb is small compared with the momentum transport in the higher levels below the troposphere.

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Figures

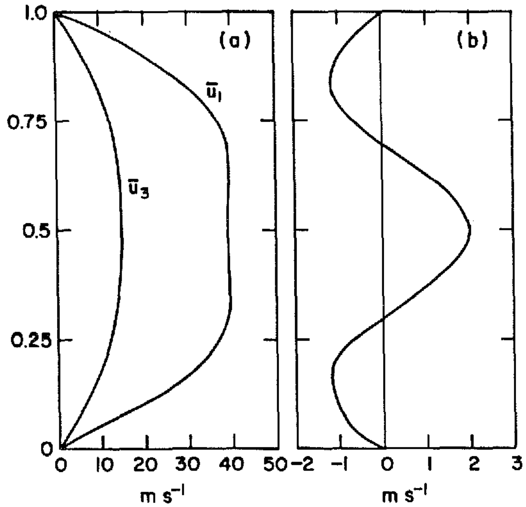

Fig. 1. The winds at level 1 (25 cb), level 3 (75 cb) and level 4 (100 cb) as functions of the south-north coordinate. Unit: ms^{-1}.

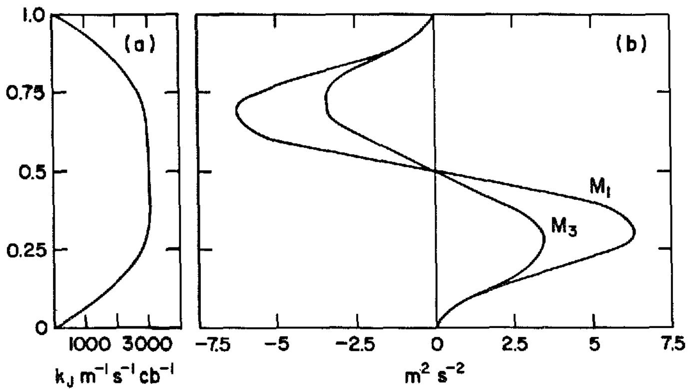

Fig. 2. The sensible heat transport by the eddies (left) as a function of the south-north coordinate. Unit: $kJm^{-1}s^{-1}cb^{-1}$; and the momentum transport at the upper level M_1 and the lower level M_3. Unit: m^2s^{-2}.

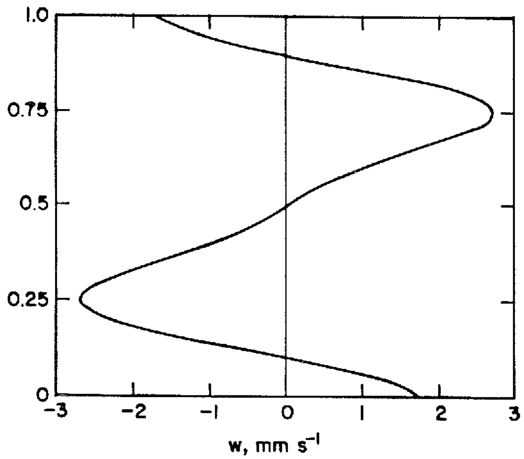

Fig. 3. The vertical velocity at the middle level (50 cb) as a function of the south-north coordinate. Unit: mms^{-1}.

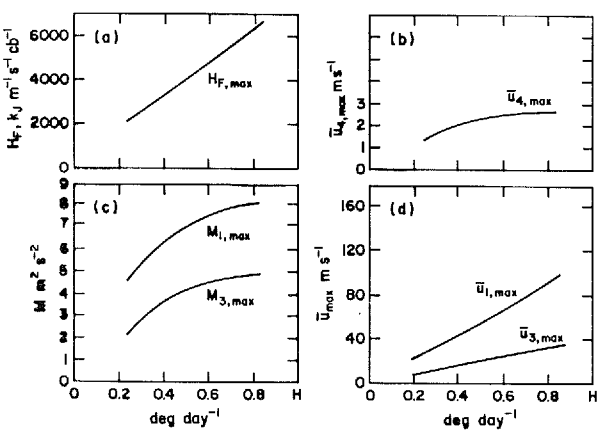

Fig. 4. The maximum transport of sensible heat as a function of the intensity of the heating. The unit for the heat transport is: $kJm^{-1}s^{-1}cb^{-1}$, while the unit for the heating is degday^{-1} (upper left). The momentum transport maxima at the upper and lower levels (m^2s^{-2}) as a function of the heating (lower left). The maximum surface wind in ms^{-1} as a function of the heating (upper right). The maximum winds at levels 1 and 3 in ms^{-1} as a function of the heating (lower right).

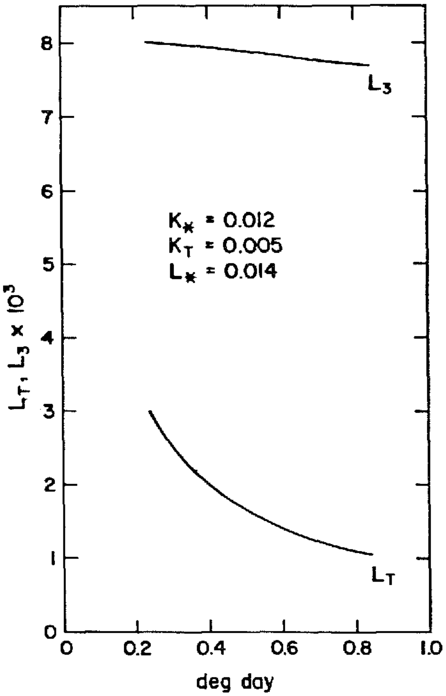

Fig. 5. The exchange coefficients L_τ and L_3 as a function of the heating. L_τ and L_3 are expressed in non-dimensional units times 10^3. The heating unit is degday^{-1}.

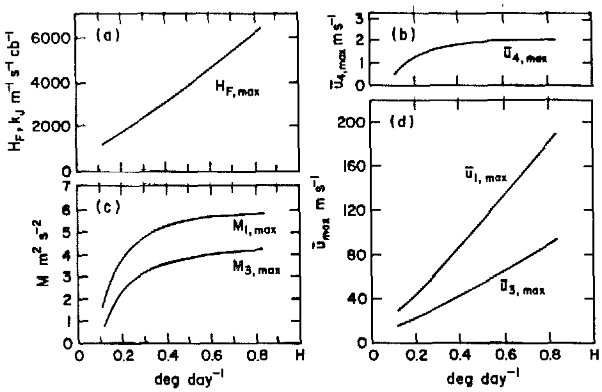

Fig. 6. Similar to Fig. 4 and with the same arrangement. Non-dimensional values were: $K_* = 0.008$, $K_\tau = 0.0035$, $L_* = 0.0093$. Note the much larger values of $\bar{u}_{1,max}$ and $\bar{u}_{3,max}$.

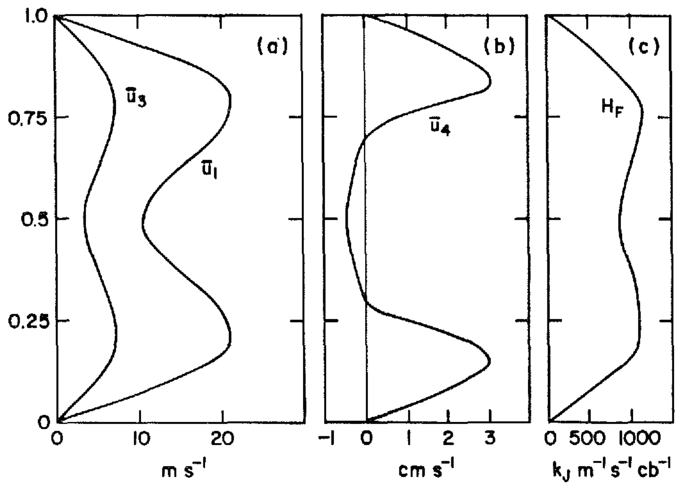

Fig. 7. A solution obtained for a slightly supercritical value of the heating (0.9 deg day^{-1}). $\bar{u}_1$, $\bar{u}_3$, and $\bar{u}_4$ have double maxima, and $\bar{u}_4$ is very small.

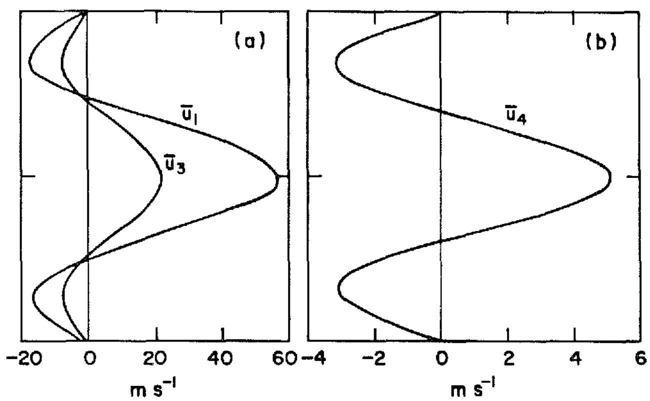

Fig. 8. The wind profiles of $\bar{u}_1$, $\bar{u}_3$, and $\bar{u}_4$ in a vertically integrated model with $K_* = 0.012$, $K_\tau = 0.005$ and $L_* = 0.014$. Unit: ms^{-1}.

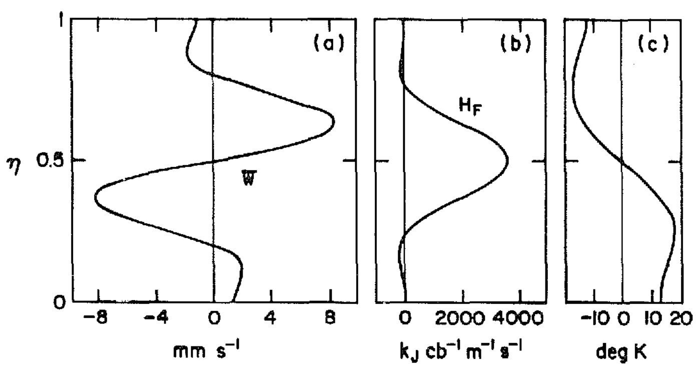

Fig. 9. The vertical velocity, the heat transport and the heating for the case displayed in Fig. 8. Unit: mms^{-1}.

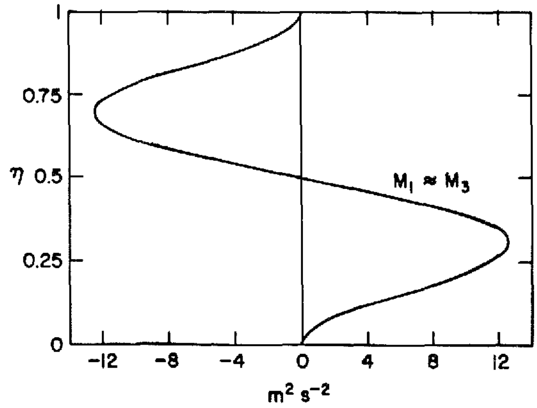

Fig. 10. The momentum transports $M_1 \simeq M_3$.. Same case as Fig. 8 and Fig. 9. Unit: m^2s^{-2}.

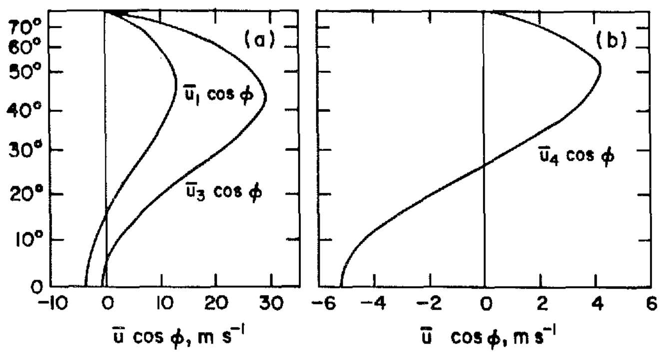

Fig. 11. The wind profiles $\bar{u}_1 \cos\varphi$, $\bar{u}_3 \cos\varphi$, and $\bar{u}_4 \cos\varphi$ for spherical case. Unit: ms^{-1}.

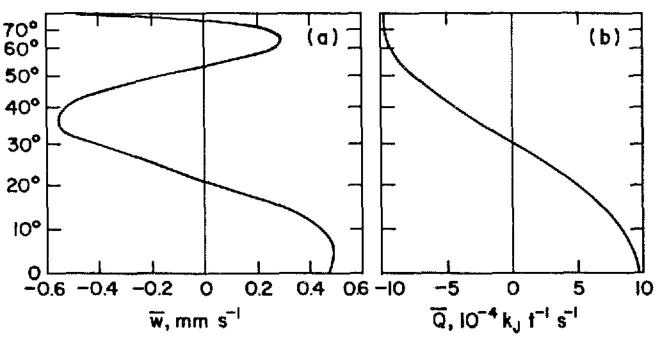

Fig. 12. The vertical velocity and the heating. The same case as in Fig. 11. Units: mms^{-1} and $10^{-4}kJt^{-1}s^{-1}$.

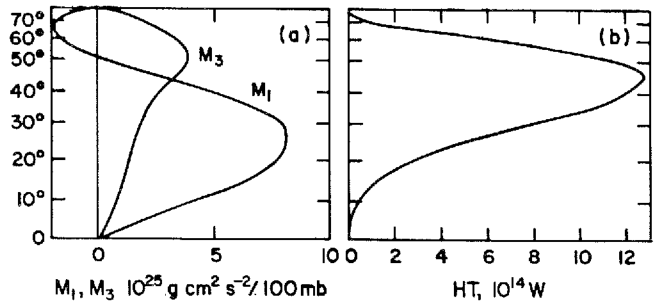

Fig. 13. The angular momentum transports M_1 and M_3 (Unit: $10^{25} gcm^2s^{-2}$/10cb) and the heat transport (Unit: $10^{14} W$). Same case as Fig. 11 and Fig. 12.

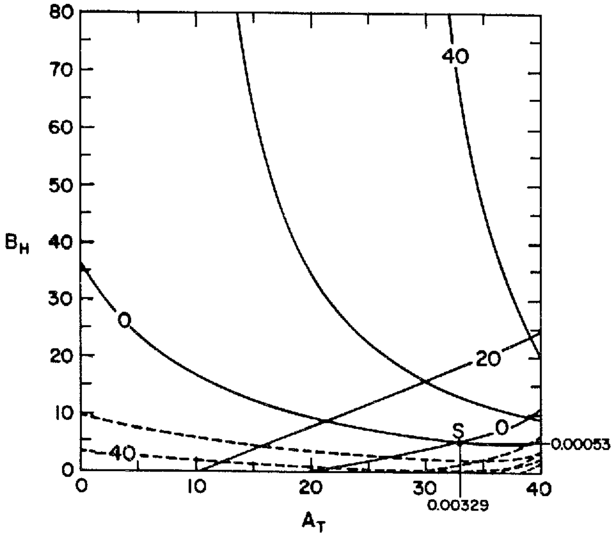

Fig. 14. The isolines for the two functions F and G as functions of A_T and B_H. S denotes the steady states.

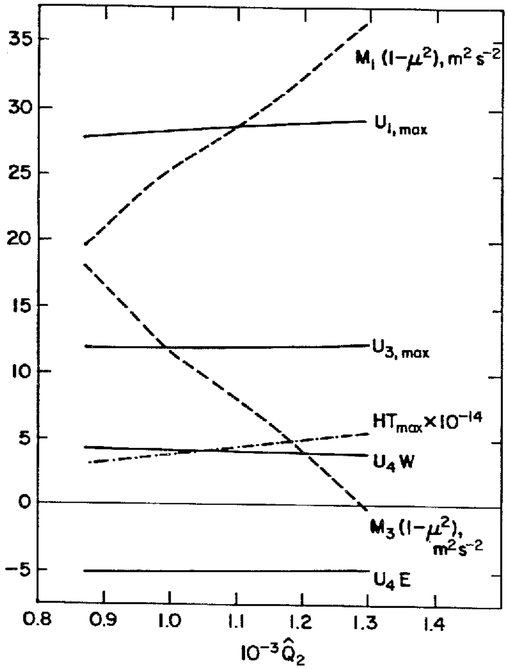

Fig. 15. The maximum values of $\bar{u}_1$, $\bar{u}_3$, and $\bar{u}_4$ (divided in maximum values for westerlies and easterlies), the heat transport HT_{max}, and the angular momentum transports $M_1(1-\mu^2)$ and $M_3(1-\mu^2)$. The latter quantities are those showing the strongest dependence on the heating.

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