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A STUDY IN GRAVITATIONAL COLLAPSE

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CONTENTS

Synopsis

In order to study the nature of the essentiel singularities occurring in Einstein's theory of gravitation the collapse of an arbitrary spherical distribution of incoherent matter is investigated in detail. The treatment is characterized by the use of one global Gaussian system of coordinates in which the matter is constantly at rest and which is free of coordinate singularities both inside and outside the matter. These coordinates have a simple physical interpretation and in the empty space outside the matter they represent a substitute for the rather formal Kruskal coordinates. The metric tensor is expressable in terms of simple well-known functions of the coordinates and the radial motion of light signals and the line shift of spectral lines are given by simple formulae.

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Introduction

One of the most surprising and disturbing discoveries in later years is that of the occurrence of essential singularities in the solutions of Einstein's gravitational field equations. Unlike the "coordinate singularities", as for instance the well-known Schwarzschild singularity in empty space, essential singularities cannot be removed by any change of coordinates. The occurrence of non-trivial singularities in a physical theory may generally be taken as a sign that the theory has been applied to a phenomenon that lies outside the domain of applicability of the theory. Thus in the gravitational case one would be inclined to conclude that Einstein's theory breaks down in regions of space-time close to the singularities i.e. for extremely strong gravitational fields – a thought that was not unfamiliar to Einstein himself.[1]

The most general proofs of the inevitability of singularities in Einstein's theory were given by PENROSE (1965), GEROCH (1966) and by HAWKING and ELLIS (1968),[2] who showed that this phenomenon is independent of the special form of the energy-momentum tensor of the matter provided the equation of state is such that

$$\mu^0 c^2 + p > 0 \qquad \text{I}$$

everywhere inside the matter. Here μ° is the proper mass density regarded as a scalar, p is the pressure and c is the usual universal constant.

In order to study the nature of the singularities a little more closely we shall in this paper reconsider the simple problem treated by OPPENHEIMER and SNYDER in 1939.[3] These authors considered a model consisting of a spherical distribution of incoherent matter which initially (at $t = 0$) is at rest and fills a sphere of finite radius with constant density. Due to the mutual gravitational attraction the matter will start contracting and after some time the appropriate solutions of Einstein's field equations develop singularities both inside and outside the matter. We shall here consider the somewhat more general case where the proper mass density initially is an arbitrarily given function of the distance from the centre. Of course, this does not make the model much more realistic, but the justification for treating this model in more detail is that the solution of the field equations, also in this more general case, can be expressed by simple well-known mathematical functions and that the Hawking-Penrose condition I obviously is satisfied in this case, since $p = 0$ for incoherent matter and μ° is positive. Thus the case considered represents the simplest imaginable illustration of the general theorems.

1. A Global System of Comoving Gaussian Coordinates

Since the physical system considered is spherically symmetric we can introduce a system of coordinates

$$x^i = \{r, \theta, \varphi, t\} \tag{1.1}$$

in which the line element in 4-space

$$ds^2 = g_{ik}dx^i dx^k \tag{1.2}$$

is of the form*

$$\left.\begin{aligned} ds^2 &= a(r,t)\, dr^2 + R(r,t)^2 d\Omega^2 - b(r,t)dt^2 \\ d\Omega^2 &= d\theta + \sin^2\theta\, d\varphi^2. \end{aligned}\right\} \tag{1.3}$$

The functions a, R and b are determined by Einstein's field equations

$$G_i^k = -\varkappa T_i^k \tag{1.4}$$

which have the "conservation laws"

$$T_{i;\,k}^k = 0 \tag{1.5}$$

as a consequence.

For incoherent matter T_i^k has the form

$$T_i^k = h^0 U_i U^k = \mu^0 c^2 U_i U^k \tag{1.6}$$

where U^i is the four-velocity of the matter (divided by c) and the scalar function

$$h^0 = \mu^0 c^2 \tag{1.7}$$

is the proper energy density as measured in a local rest system of inertia. From (1.6) and (1.5) we obtain the law of conservation of proper energy in the form**

$$(h^0 U^k)_{;k} = \frac{1}{\sqrt{-g}}(\sqrt{-g}\, h^0 U^k)_{,k} = 0. \tag{1.8}$$

In our case $h^0(r, t)$ and $U^i(r, t)$ are functions of r and t only, and the determinant g is (by 1.3)

*) In this paper the variables t, T, τ, τ_0 etc, denote time variables multiplied by the universal constant c.

**) Semicolon and comma denote covariant and usual partial derivatives respectively.

$$g = - abR^4 \sin^2\theta . \tag{1.9}$$

Since g must be negative in any physically meaningful case we must have

$$ab > 0 . \tag{1.10}$$

In order to secure a simple and unique physical interpretation of the solutions of the field equations we shall in this paper preferably use coordinates (1.1) for which a and b are positive:

$$a > 0, \qquad b > 0. \tag{1.11}$$

Only in this case does the system S of coordinates in 4-space correspond to a uniquely defined system of reference R in the 3-dimensional physical space with reference points (r, θ, φ) = (constants) that are moving with subluminar velocities, so that real measuring instruments can be attached to the reference points. The distance σ between two reference points p_1 and p_2 on the same "radius vector", as measured by means of standard measuring sticks at rest in R, is then at the time t given by

$$\sigma = \int_{r_1}^{r_2} \sqrt{a(r, t)}\, dr . \tag{1.12}$$

Similarly, the time τ_0 between two events P_1 and P_2 at a fixed reference point p, as measured by a standard clock at rest in p, is

$$\tau_0 = \int_{t_1}^{t_2} \sqrt{b(r, t)}\, dt . \tag{1.13}$$

In their treatment of the problem OPPENHEIMER and SNYDER used different systems of coordinates (and corresponding different systems of reference) inside and outside the matter. Inside the matter sphere they used a *comoving* system of reference of the type first introduced by TOLMAN, relative to which each matter particle is constantly at rest. Since the particles of incoherent matter are freely falling in the gravitational field, it follows that b is a function of t only; for the acceleration of a free particle momentarily at rest is quite generally proportional to $b' \equiv \frac{\partial b}{\partial r}$. Therefore b' must be zero in a comoving system of coordinates, and by a suitable transformation of the time variable it is then always possible to make

$$b = 1 . \tag{1.14}$$

In a comoving "Gaussian" system of coordinates of this type the components of the four-velocity are

$$U^i = \delta_{i4}, \qquad U_i = -\delta_{i4} \tag{1.15}$$

and the matter tensor (1.6) has only one non-vanishing component:

$$T_i^k = -h^0(r,t)\,\delta_{i4}\delta_{k4}. \tag{1.16}$$

In this case the conservation law (1.8) reduces to

$$(\sqrt{-g}\,h^0)_{,4} = 0. \tag{1.17}$$

In the empty space outside the matter OPPENHEIMER and SNYDER used so called "curvature coordinates"

$$X^i = \{R, \theta, \varphi, T\} \tag{1.18}$$

in which the line element is of the form

$$ds^2 = A\,dR^2 + R^2 d\Omega^2 - B\,dT^2. \tag{1.19}$$

As shown first by BIRKHOFF the metric is in these coordinates outside matter in arbitrary radial motion given by the static Schwarzschild metric. Thus

$$A = \frac{1}{1-\alpha/R}, \qquad B = 1 - \alpha/R. \tag{1.20}$$

where α is the Schwarzschild constant

$$\alpha = \frac{\varkappa M c^2}{4\pi} = \frac{\varkappa H}{4\pi}, \tag{1.21}$$

M is the total gravitational mass of the system and $H = Mc^2$ is the total energy of matter plus gravitational field.

By matching the internal and external expressions for the metric at the boundary of the matter one then obtains the motion of the boundary relative to the static system of reference of the system of curvature coordinates. However, the line element (1.19), (1.20) in the latter coordinates has the well-known drawback that the quantities A and B are singular for $R = \alpha$ and that the conditions (1.11) are violated for $R < \alpha$. This makes the physical interpretation of the solution somewhat obscure for $R < \alpha$, and as we shall see it may easily lead to a wrong physical picture of the contraction process. On the other hand we cannot simply exclude the region $R \leq \alpha$; for the determinant (1.9) is here $g = -R^4 \sin^2\theta$, i.e. it remains finite and negative in the whole domain $R > 0$. In fact the Schwarzschild singularity is only a coordinate singularity which can be removed by suitable coordinate transformations. This was shown already in 1933 by LEMAITRE[4] who introduced a Gaussian system of coordinates which has a singularity at $R = 0$ only. Contrary to the Schwarzschild system the Lemaitre system of coordinates is not stationary. It has even been shown by SERINI and by EINSTEIN and PAULI[5] that no non-singular solutions of the field equations for empty space exist that are stationary and for which $g_{44} \to -1 + \alpha/r$ for $r \to \infty$.

Now one could think of repeating the Oppenheimer-Snyder considerations with the curvature coordinates replaced by the Lemaître coordinates in the outer space, in which case the reference points would be moving like freely falling particles both in the external and in the internal system of reference. Still this would not be quite practical; for the initial velocities of the reference points in the Lemaître system are not adapted to the initial conditions of our problem. Therefore we shall now try to introduce one global system of coordinates in which the matter is constantly at rest and where b is given by (1.14) throughout space-time.

The initial distribution of matter at $t = 0$ is described by

$$h^0(r, 0) = \mu^0(r, 0)c^2 \tag{1.22}$$

which may be regarded as a given function of r that vanishes for large values of r. Since the matter is initially at rest we have for $t = 0$ and arbitrary r

$$\dot{h}^0(r, 0) = \frac{\partial h^0}{\partial t}(r, 0) = 0. \tag{1.23}$$

In our system of coordinates the metric (1.3) is throughout of the form

$$ds^2 = a(r, t)\, dr^2 + R(r, t)^2 d\Omega^2 - dt^2. \tag{1.24}$$

The reference points $(r, \theta, \varphi) = $ (constants) are moving as freely falling particles and inside the matter the numbers (r, θ, φ) are simply fixed labels of the different matter particles, $r = 0$ corresponding to the centre of the matter. According to (1.13) with $b = 1$ the time variable t is equal to the time τ_0 of standard clocks at rest in the reference points, and inside the matter t is simply equal to the proper time of the matter particles. The form (1.24) of the metric is unchanged under arbitrary transformations

$$\bar{r} = f(r) \tag{1.25}$$

of the radial coordinate.

The matter tensor T_i^k is again of the form (1.16) and the field equations (1.4) consist of three independent equations only:

$$G_1^4 = 0, \quad G_1^1 = 0, \quad G_4^4 = \varkappa h^0. \tag{1.26}$$

For given initial conditions they are just sufficient to determine the three unknown functions $a(r, t)$, $R(r, t)$ and $h^\circ(r, t)$. In view of (1.23) we are looking for solutions a and R that are stationary at $t = 0$, i.e. for which the partial time derivatives of first order are zero at $t = 0$. In particular we require

$$\dot{R}(r, 0) = 0. \tag{1.27}$$

By a suitable transformation of the type (1.25) we can always arrange it so that

$$R(r, 0) = r \tag{1.28}$$

i.e.

$$R'(r, 0) = 1. \tag{1.29}$$

With the expressions for G_{ik} following from (1.24) the field equations (1.26) are

$$-G_1^4 \equiv \frac{2\dot{R}'}{R} - \frac{R'}{R}\frac{\dot{a}}{a} = 0 \tag{1.30}$$

$$G_1^1 \equiv \frac{1}{R^2}\left(2R\ddot{R} + \dot{R}^2 + 1 - \frac{R'^2}{a}\right) = 0 \tag{1.31}$$

$$G_4^4 \equiv -\frac{1}{aR^2}\left(2RR'' + R'^2 - \frac{a'RR'}{a}\right) + \frac{1}{R^2} + \frac{1}{R^2}\left(\dot{R}^2 + \frac{\dot{a}R\dot{R}}{a}\right) = \varkappa h^0, \tag{1.32}$$

where dot and dash denote partial derivatives with respect to t and r respectively.

Multiplication of (1.30) by RR'/a yields

$$\frac{\partial}{\partial t}(R'^2/a) = 0 \tag{1.33}$$

which shows that R'^2/a is a function of r only. If we denote this function of integration by $1 - \psi(r)$ we obtain

$$a = \frac{R'^2}{1 - \psi(r)}. \tag{1.34}$$

Introduction of this expression for a into (1.31) gives

$$2R\ddot{R} + \dot{R}^2 + \psi(r) = 0. \tag{1.35}$$

Thus

$$\frac{\partial}{\partial t}\left\{R\left[\dot{R}^2-\psi\left(\frac{r}{R}-1\right)\right]\right\} = \dot{R}[2R\ddot{R}+\dot{R}^2+\psi(r)] = 0$$

which shows that $R\left[\dot{R}^2-\psi\left(\frac{r}{R}-1\right)\right]$ is a function of r only; but this function must be zero for all r on account of the initial conditions (1.27), (1.28). Hence

$$\dot{R}^2+\psi = r\psi/R \tag{1.36}$$

and from (1.35)

$$\ddot{R} = -r\psi/2R^2. \tag{1.37}$$

By means of (1.34) and (1.36) the field equation (1.32) may be written

$$G_4^4 = \frac{[R(\psi+\dot{R}^2)]'}{R^2R'} = \frac{(r\psi)'}{R^2R'} = \varkappa h^0(r,t). \tag{1.38}$$

Since $(r\psi)'$ is time-independent it follows from this equation that $\varkappa h^0(r,t)R^2R'$ is a function of r only, say

$$\varkappa h^0(r,t)\,R^2R' = 3r^2\lambda(r). \tag{1.39}$$

This is in accordance with the conservation equation (1.17) since

$$\sqrt{-g} = R^2\sqrt{ab}\sin\theta = R^2R'\sin\theta/\sqrt{1-\psi(r)} \tag{1.40}$$

from (1.24) and (1.34). The function $\lambda(r)$ is obtained by putting $t=0$ in (1.39) and by using the initial conditions (1.28), (1.29) which gives

$$\lambda(r) = \frac{\varkappa h^0(r,0)}{3}, \tag{1.41}$$

Thus $\lambda(r)$ may be regarded as a known function of r given by the initial distribution of matter. The energy density $h^0(r,t)$ at any time is, by (1.39) and (1.41),

$$h^0(r,t) = \frac{3r^2\lambda(r)}{\varkappa R^2R'} = \frac{r^2h^0(r,0)}{R^2R'} = \mu^0(r,t)c^2. \tag{1.42}$$

If we introduce (1.39) into (1.38) we get the following differential equation for the function $\psi(r)$:

$$(r\psi)' = 3r^2\lambda(r), \tag{1.43}$$

From (1.10), (1.14) and (1.34) it follows that $\psi(r)$ in any physically meaningful case must lie in the interval

$$\psi(r) < 1\,. \tag{1.44}$$

Thus $r\psi = 0$ for $r = 0$ and by solving (1.43) we obtain

$$\psi(r) = \frac{3}{r}\int_0^r r^2\lambda(r)\,dr = r^2\lambda(r) - \frac{1}{r}\int_0^r r^3\lambda'(r)\,dr \tag{1.45}$$

which determines $\psi(r)$ uniquely for a given initial distribution of matter.

When $\psi(r)$ has been determined by (1.45) we can solve the differential equation (1.36) for $R(r, t)$. It is easily verified that the solution corresponding to the initial condition (1.28) is given by

$$R(r, t) = rC(u) \tag{1.46}$$

$$u = t\sqrt{\psi/r^2} \tag{1.47}$$

where the function $C(u)$ of the variable u is a solution of the differentiel equation

$$\left(\frac{dC}{du}\right)^2 = C'(u)^2 = \frac{1}{C(u)} - 1 \tag{1.48}$$

with

$$C(0) = 1\,. \tag{1.49}$$

From (1.48) we obtain

$$C'(u) = \pm\sqrt{\frac{1 - C(u)}{C(u)}} \tag{1.50}$$

and by differentiation

$$C''(u) = -\frac{1}{2C(u)^2}\,. \tag{1.51}$$

The solutions of (1.50), (1.49) are

$$\mp u = -\int_1^C \sqrt{\frac{C}{1-C}}\,dC = \sqrt{C(1-C)} + \tan^{-1}\sqrt{\frac{1-C}{C}} \tag{1.52}$$

or

$$\mp u = \sqrt{C(1-C)} + \cos^{-1}\sqrt{C} = \sqrt{C(1-C)} + \frac{\pi}{2} - \sin^{-1}\sqrt{C}\,, \tag{1.53}$$

where the signs here correspond to the signs in (1.50).

The graphical picture of the function $C(u)$ of u is a cycloid with the parametric representation

$$C = \tfrac{1}{2}(1 + \cos\eta), \quad u = \tfrac{1}{2}(\eta + \sin\eta). \tag{1.54}$$

We have now obtained the complete solution of our problem in a global system of coordinates S_M that is Gaussian and in which all parts of the matter are constantly at rest in the corresponding system of reference R_M. In S_M the metric in 4-space is

$$ds^2 = \frac{R'(r,t)^2}{1-\psi(r)}dr^2 + R(r,t)^2 d\Omega^2 - dt^2. \tag{1.55}$$

Here $\psi(r)$ is determined by the given initial distribution of the matter through (1.45) and $R(r, t)$ is given by (1.46) – (1.54). Thus

$$\left.\begin{aligned} R &= rC(u), \quad u(r,t) = t\sqrt{\psi}/r, \quad \dot{u} = \sqrt{\psi}/r, \\ u' &= \frac{u}{r}\frac{(r\psi)' - 3\psi}{2\psi} = \frac{3u}{2r^2\psi}\int_0^r r^3\lambda'(r)\,dr, \\ \dot{R} &= r\,C'(u)\dot{u} = \sqrt{\psi}\,C'(u), \\ R' &= C(u) + rC'(u)u'. \end{aligned}\right\} \tag{1.56}$$

For simplicity we shall assume that $\lambda(r)$ is a never increasing function of r. Then

$$\int_0^r r^3\lambda'(r)\,dr < 0 \quad \text{and} \quad R' > 0. \tag{1.57}$$

The matter density at any time is given by (1.42), and the naturally measured distance from the centre of a fixed point in the matter is from (1.12)

$$\sigma(r,t) = \int_0^r \frac{R'(r,t)}{\sqrt{1-\psi(r)}}\,dr. \tag{1.58}$$

2. Discussion of the Solution

In a contraction process starting from a state of rest at $t = 0$ the solution (1.55) is regular everywhere in a region of r and t for which u in (1.56) lies in the interval

$$0 \leq u < \frac{\pi}{2}. \tag{2.1}$$

The corresponding values of the parameter η in (1.54) and of $C(u)$ lie in the intervals

$$0 \leq \eta < \pi, \quad 1 \geqslant C(u) > 0. \tag{2.2}$$

In this region

$$C'(u) = -\sqrt{\frac{1-C}{C}} \tag{2.3}$$

and we have to use the lower signs in (1.52), (1.53). Further

$$R'(r, t) > 0, \quad \dot{R} \leq 0 \tag{2.4}$$

from (1.56) and (1.57), and

$$R = rC \leq r. \tag{2.5}$$

However for $u \to \frac{\pi}{2}$ we have

$$C(u) \to 0, \quad C'(u) \to -\infty, \quad R \to 0. \tag{2.6}$$

Thus for any fixed value of r in the interval $0 \leq r < \infty$ the metric becomes singular after a finite time $t_s(r)$ measured on a standard clock at rest in R_M. This time is given by

$$u(r, t_s) = \frac{\pi}{2} \tag{2.7}$$

or from (1.47)

$$t_s(r) = \frac{\pi r}{2\sqrt{\psi(r)}} \tag{2.8}$$

which goes to infinity for $r \to \infty$ (comp. (2.13)). According to (1.40) the quantity $\sqrt{-g}$ is proportional to R^2R' and from (1.56), (1.57) and (2.3) we have

$$R^2R' = r^2\left[C^3 + C\sqrt{C(1-C)}\frac{3u}{2r\psi}\left(-\int_0^r r^3\lambda'(r)dr\right)\right], \tag{2.9}$$

which shows that $\sqrt{-g}$ goes to zero as $C^{3/2}$ for $u \to \frac{\pi}{2}$. Therefore we are dealing with essential singularities in this limit, and it has no physical meaning to extend space-time beyond the region defined by (2.1). From (1.42) it follows that the proper mass density at any point inside the matter with constant r goes to infinity for $t \to t_s(r)$. This is accompanied by a singularity in the metric that spreads outward into the empty space outside the matter according to the equation

$$u(r, t) = \sqrt{\psi}\, t / r = \frac{\pi}{2} \tag{2.10}$$

or

$$R(r, t) = 0 . \tag{2.11}$$

By differentiation of (2.10) we get for the velocity with which the singularity propagates

$$V_s = \left.\frac{dr}{dt}\right|_{u=\frac{\pi}{2}} = -\frac{\dot{u}}{u'} = \frac{4r\psi^{3/2}}{3\pi} \Bigg/ \left(- \int\limits_0^r r^3 \lambda'(r) dr \right) > 0 \tag{2.12}$$

from (1.56) and 1(1.57). $V_s \to 0$ for $r \to \infty$ (comp. (2.21)). At any time $t < t_s(r)$ the metric is regular for all r. There are no coordinate singularities in S_M.

For an insular system of the type considered here the function $h^0(r, 0)$ or $\lambda(r)$ is zero in the empty space outside the matter, say for $r \geq r_b$. In this region we get from (1.45) and (1.41)

$$\psi(r) = \alpha / r \tag{2.13}$$

where

$$\alpha = 3 \int\limits_0^\infty r^2 \lambda(r) dr = \frac{\varkappa}{4\pi} \iiint h^0(r, 0) r^2 \sin\theta dr d\theta d\varphi \tag{2.14}$$

is a constant which has a simple physical meaning. For a real physical system we must have

$$r_b < \alpha \tag{2.15}$$

on account of (1.44). The naturally measured spatial volume element is

$$dV = \sqrt{\gamma}\, dr d\theta d\varphi = \sqrt{ab}\, R^2 \sin\theta\, dr d\theta d\varphi = \frac{R^2 R' \sin\theta}{\sqrt{1 - \psi(r)}} dr d\theta d\varphi \tag{2.16}$$

which for constant (r, θ, φ) and $(dr, d\theta, d\varphi)$ goes to zero for $t \to t_s$. However, from (1.42) the total proper matter energy is

$$H^0 = \iiint h^0(r, t) dV = \iiint \frac{h^0(r, 0)}{\sqrt{1 - \psi(r)}} r^2 \sin\theta dr d\theta d\varphi \tag{2.17}$$

which is constant in time. As we shall see now the constant α, which also may be written

$$\alpha = \frac{\varkappa}{4\pi} \iiint h^0(r, t) \sqrt{1 - \psi(r)}\, dV, \tag{2.18}$$

represents the total energy of the system, i.e. the total energy of matter plus gravitational field.

In any asymptotically Lorentzian system of coordinates the total four-momentum P_i of an insular system is given by[6]

$$P_i = \lim_{r \to \infty} \frac{1}{c} \int_f \psi_i^{4\lambda} n_\lambda r^2 \sin\theta \, d\theta d\varphi \tag{2.19}$$

where the integration is extended over a large sphere f of radius r_0. n_λ is a normal unit vector in the outward direction and ψ_i^{kl} is the v. Freud super-potential.[7] For $r \geq r_b$ we get from (1.56), (2.13), (2.3)

$$\left.\begin{aligned} R &= r\, C(u), \quad u = t\sqrt{\alpha/r^3}, \quad \dot{u} = \sqrt{\alpha/r^3} \\ u' &= -3u/2r, \quad R' = C(u) + \frac{3u}{2}\sqrt{\frac{1-C}{C}}, \\ \dot{R} &= -\sqrt{\alpha/r}\sqrt{\frac{1}{C} - 1} = -\sqrt{(\alpha/R) - \alpha/r} \end{aligned}\right\} \tag{2.20}$$

and from (2.12)

$$V_s = -\dot{u}/u' = \frac{2}{3u}\sqrt{\alpha/r} = \frac{4}{3\pi}\sqrt{\alpha/r}. \tag{2.21}$$

It is seen that $u \to 0$ in the limit $r \to \infty$ for any constant t, and a Taylor expansion of the function $C(u)$ for small u gives by (1.49)—(1.51)

$$\left.\begin{aligned} C(u) &= 1 - \tfrac{1}{4}u^2 + O(u^4) \\ R'(u) &= 1 + \tfrac{1}{2}u^2 + O(u^4). \end{aligned}\right\} \tag{2.22}$$

In calculating P_i in (2.19) we shall only need the asymptotic expression of g_{ik} entering in $\psi_i^{4\lambda}$ in which terms of order $u^2 = t^2\alpha/r^3$ can be neglected. In this approximation we have $C = R' = 1$, $R = r$ and we get for the symptotic form of (1.55)

$$ds^2 = \frac{dr^2}{1 - \alpha/r} + r^2 d\Omega^2 - dt^2 \tag{2.23}$$

which is time-independent. For $r \to \infty$ this goes over into the Minhowski line element written in polar coordinates. Therefore introducing spatial coordinates (x, y, z) that are connected with (r, θ, φ) in the same way as Cartesian and polar coordinates in a Euclidian space, we obtain an asymptotically Lorentzian system of space-time coordinates

$$x^i = \{x, y, z, t\} \tag{2.24}$$

in which the expression (2.19) can be safely applied. In these coordinates the metric tensor of the line element (2.23) takes the form

$$g_{ik} = \eta_{ik} + \frac{\alpha}{r} n_i n_k + O\left(\frac{1}{r^2}\right) \tag{2.25}$$

where η_{ik} is the Minhowski tensor and

$$n_i = \frac{\partial r}{\partial x^i} = \{n_\iota, 0\}. \tag{2.26}$$

$O(1/r^2)$ is a term of order $1/r^2$ which does not give any contribution to the integral (2.19) in the limit $r \to \infty$.

With g_{ik} given by (2.25) the superpotential is easily calculated. Neglecting terms of order $1/r^3$ we obtain

$$\psi_i^{4\lambda} = -\delta_{i4} \frac{\alpha}{\varkappa r^2} n_\lambda \tag{2.27}$$

and from (2.19)

$$P_i = -\delta_{i4} \frac{4\pi\alpha}{\varkappa c} = \{\boldsymbol{P}, -H/c\}. \tag{2.28}$$

Thus the total momentum $\boldsymbol{P}$ of the system is zero and the total energy H is

$$H = \frac{4\pi\alpha}{\varkappa} = Mc^2 \tag{2.29}$$

A comparison of this equation with (1.21) shows that the constant α in (2.13), (2.14) or (2.18) is identical with the Schwarzschild constant.

So far we have not made any assumption about the initial distribution of the matter, except that $\lambda'(r) \leq 0$ and $\lambda(r) = 0$ for $r \geq r_b$. Let us now consider the case where $\lambda(r)$ is equal to a constant λ_0 for $r \leq r_c < r_b$, i.e.

$$\lambda(r) = \begin{cases} \lambda_0 & \text{for} \quad 0 \leq r \leq r_c \\ 0 & \text{for} \quad r \geq r_b \end{cases} . \tag{2.30}$$

In the inner region $r \leq r_c$ we obtain from (1.45), (1.56)

$$\left.\begin{aligned} &\psi = \lambda_0 r^2, \quad R = rS(t), \quad S(t) = C(u), \\ &u = \sqrt{\lambda_0}\, t, \quad \dot{u} = \sqrt{\lambda_0}, \quad u' = 0, \\ &\dot{S} = \sqrt{\lambda_0}\, C' = -\sqrt{\lambda_0}\sqrt{\frac{1}{S} - 1}, \quad \dot{R} = r\dot{S}, \quad R' = C(u) = S(t) \end{aligned}\right\} \tag{2.31}$$

and the line element (1.55) takes the simple form

$$ds^2 = S(t)^2 \left[\frac{dr^2}{1 - \lambda_0 r^2} + r^2 d\Omega^2 \right] - dt^2. \tag{2.32}$$

In the same region we get from (1.42)

$$h^0(r, t) = \frac{3r^2\lambda_0}{\varkappa r^2 S^3} = \frac{h^0(r, 0)}{S(t)^3} \tag{2.33}$$

and the singularity occurs simultaneously at all points inside r_c at the time

$$t_s = \pi / 2\sqrt{\lambda_0}. \tag{2.34}$$

From (1.58) we get for the naturally measured distance

$$\sigma(r, t) = S(t) \int_0^r \frac{dr}{\sqrt{1 - \lambda_0 r^2}} = \frac{C(\sqrt{\lambda_0}\, t)}{\sqrt{\lambda_0}} \sin^{-1}(\sqrt{\lambda_0}\, r). \tag{2.35}$$

For constant r this distance decreases steadily in our case to the value zero for $t \to t_s$, which shows that we have contraction of the matter when $C'(u)$ is negative.

If we make the transition from the constant value λ_0 of $\lambda(r)$ for $r \leq r_c$ to the value $\lambda(r) = 0$ for $r \geq r_b$ sufficiently smooth in the interval $r_c \leq r \leq r_b$, the components of the metric tensor in (1.55) will be continuous and differentiable in the whole region of space-time with $t < t_s(r)$, even if $r_b - r_c$ is very small. However, it should be remarked that in the limit $r_c \to r_b$, where $\lambda(r)$ is a step function at $r = r_b$ and therefore

$$\lambda_0 \, r_b^2 = \alpha / r_b, \tag{2.36}$$

the quantities u' in (1.56) and hence R' and a in (1.34) are discontinuous at $r = r_b$. According to (1.38) this is directly connected with the discontinuity of h° at this point. The relation (2.36) between λ_0 and r_b is a good approximation also for finite $r_b - r_c$ provided that

$$(r_b - r_c) / r_b \ll 1. \tag{2.37}$$

3. Curvature Coordinates

According to (1.55) the area of a sphere of constant r and t is equal to $4\pi R^2$ and the "curvature radius" R is a function of r and t given by

$$R = R(r, t) = r\,C(u) = r\,C\left(t\sqrt{\psi/r^2}\right). \tag{3.1}$$

We are now looking for a system S_c of curvature coordinates

$$X^i = \{R, \theta, \varphi, T\} \tag{3.2}$$

in which ds^2 is of the form (1.19):

$$ds^2 = AdR^2 + R^2 d\Omega^2 - BdT^2. \tag{3.3}$$

To this end we have to find a transformation

$$T = \varphi(r, t) \tag{3.4}$$

which together with (3.1) brings (1.55) into the form (3.3). This is brought about by choosing $\varphi(r, t)$ as a solution of the partial differential equation

$$(1 - \psi(r))\varphi'(r, t) - \dot{R}R'\dot{\varphi}(r, t) = 0. \tag{3.5}$$

In fact, from (3.1) and (3.4) we obtain by differentiation and by solving for dr and dt

$$\left.\begin{aligned} dr &= (\dot{\varphi}dR - \dot{R}dT)\,/\,(R'\dot{\varphi} - \dot{R}\varphi') \\ dt &= (R'dT - \varphi'dR)\,/\,(R'\dot{\varphi} - \dot{R}\varphi'). \end{aligned}\right\} \tag{3.6}$$

When this is introduced into (1.55) it is seen that the terms containing $dRdT$ cancel, on account of (3.5), and ds^2 takes the form (3.3) with

$$\left.\begin{aligned} A &= \frac{R'^2\dot{\varphi}^2 - \varphi'^2(1-\psi)}{(1-\psi)(R'\dot{\varphi} - \dot{R}\varphi')^2} \\ B &= \frac{R'^2(1 - \psi - \dot{R}^2)}{(1-\psi)(R'\dot{\varphi} - \dot{R}\varphi')^2}. \end{aligned}\right\} \tag{3.7}$$

Eliminating φ' by means of (3.5) and using (1.36) we obtain

$$A = \frac{1}{1 - r\psi/R}, \quad B = \frac{1-\psi}{\dot{\varphi}^2(1 - r\psi/R)}. \tag{3.8}$$

In the system S_c the metric tensor is more singular than in the system S_M of section 1 and 2. Besides for

$$R = 0 \tag{3.9}$$

which is the essential singularity (2.11), A and B are singular also at

$$R = r\psi \tag{3.10}$$

and for $R < r\psi$, the conditions (1.11) are violated since

$$A < 0, \quad B < 0. \tag{3.11}$$

In empty space, where (2.13) holds, the singularity (3.10) is identical with the Schwarzschild singularity at

$$R = \alpha. \tag{3.12}$$

From the form (3.3) of ds^2 in S_c one would be inclined to draw the wrong conclusion that the essential singularity (3.9) occurs at the centre of the matter only. It is true that $R = 0$ at the centre $r = 0$; but from (1.46) it follows that R is also zero for $r \neq 0$ whenever $C(u) = 0$, i.e. along the whole curve (2.7) which also concerns points outside the matter. If we put $t = 0$ in (3.5) we obtain by (1.27)

$$\varphi'(r, 0) = 0 \tag{3.13}$$

or

$$\varphi(r, 0) = \text{constant}$$

It is convenient to choose the value of this constant equal to zero, i.e.

$$\varphi(r, 0) = 0 \tag{3.14}$$

for then the time variable T in the system S_c is zero for all events at $t = 0$. the time coordinates T and t coincide at the origin $T = t = 0$.

In the empty space outside the matter, where $\psi = \alpha/r$, the solution of the differential equation (3.5) with the initial condition (3.14) is given by

$$\left.\begin{aligned} \varphi(r, t) = t\sqrt{1 - \alpha/r} + 2\sqrt{\alpha(r - \alpha)}\tan^{-1}\sqrt{\frac{r - R}{R}} \\ + 2\alpha\log\frac{\sqrt{R(r - \alpha)} + \sqrt{\alpha(r - R)}}{\sqrt{r}\,|R - \alpha|} \end{aligned}\right\} \tag{3.15}$$

where $R(r, t)$ is the function of r and t given by (3.1) and $|R - \alpha|$ is the absolute value of $R - \alpha$, i.e.

$$|R - \alpha| = \begin{cases} R - \alpha & \text{for} \quad R > \alpha \\ \alpha - R & \text{for} \quad R < \alpha. \end{cases} \tag{3.16}$$

Partial differentiation of (3.15) with respect to t and r gives after a somewhat lengthy calculation using (2.20) and (1.52)

$$\dot{\varphi} = \frac{\sqrt{1-\alpha/r}}{1-\alpha/R}, \quad \varphi' = \frac{\dot{R}R'}{\sqrt{1-\alpha/r}\,(1-\alpha/R)} \tag{3.17}$$

which shows that (3.15) is a solution of (3.5) with $\psi = \alpha/r$. Further since $R = r$ for $t = 0$ it also satisfies the initial condition (3.14). Since $r \geq r_b > \alpha$ in the outer space and $R \leq r$ the function $T = \varphi(r, t)$ is real and positive for $t > 0$ and arbitrary r. For $R \to \alpha$ this function diverges logarithmically. It is this singularity in the time transformation (3.4) that causes the coordinate singularity of the metric in S_c at $R = \alpha$.

With $\dot{\varphi}$ given by (3.17) and $r\psi = \alpha$ the quantities A and B in (3.8) reduce to the Schwarzschild expressions

$$B = \frac{1}{A} = 1 - \alpha/R \tag{3.18}$$

in accordance with Birkhoff's theorem. In a region of space-time where $R > \alpha$ and A and B positive, the system S_c furnishes the simplest and most convenient description of the motion of particles and light signals. This is above all due to the fact that the system of reference R_c corresponding to S_c is rigid. According to (1.12) the naturally measured radial distance σ is

$$\sigma = \int_{R_1}^{R_2} \frac{dR}{\sqrt{1-\alpha/R}} \tag{3.19}$$

which is time independent and by (1.13) the time τ_0 of a standard clock at a fixed reference point in R_c is

$$\tau_0 = T\sqrt{1-\alpha/R}. \tag{3.20}$$

However in a region where $R < \alpha$ the expressions (3.19), (3.20) have no physical meaning. The reason for this becomes clear when we consider the motion of a point of constant (R, θ, φ) relative to the 'rest system' S_M of the matter which is described by the equation (3.1) with constant R. Thus it moves radially outward with the velocity

$$\left.\frac{dr}{dt}\right|_R = -\frac{\dot{R}}{R'} = \frac{\sqrt{(r\psi/R)-\psi}}{R'} = \frac{\sqrt{(\alpha/R)-\alpha/r}}{R'} \tag{3.21}$$

on account of (1.56) and (2.20). On the other hand we get from (1.55) for an outward moving light signal for which $ds^2 = 0$

$$\left(\frac{dr}{dt}\right)_L = \frac{\sqrt{1-\psi}}{R'} = \frac{\sqrt{1-\alpha/r}}{R'} \tag{3.22}$$

which is smaller than $\left.\frac{dr}{dt}\right|_R$ for $R < \alpha$. Thus in the region $R < \alpha$ the reference points of R_c are moving with super light velocities so that it is impossible to attach real measuring instruments to these points. This explains why the expressions (3.19), (3.20) are meaningless in this domain. On the other hand, if r is very large compared with α the quantity $u^2 = t^2\alpha/r^3$ will during a long period of time be small compared with 1. In this region

$$\alpha/r \ll 1, \qquad u^2 = t^2\alpha/r^3 \ll 1 \tag{3.23}$$

and we have according to (2.20), (2.22) and (3.21)

$$C = R' = 1, \qquad R = r \gg \alpha, \quad \left.\frac{dr}{dt}\right|_R \ll 1. \tag{3.24}$$

Thus, in the region of space-time (3.23) which covers the larger part of the outer space during a long time the systems of reference R_c and R_M coincide.

4. Radial Motion of Free Particles and Light Signals

On account of the just mentioned "unphysical" motion of the system of reference R_c for $R < \alpha$ it is preferable to describe the motion of free particles and light signals in the system S_M with the metric (1.55). The simplest solution of the equations of motion for a free particle is given by

$$(r, \theta, \varphi) = (\text{constants})\,. \tag{4.1}$$

In terms of the radial curvature coordinate the motion is described by (1.46) with constant r:

$$R = r\,C(u) = r\,C\left(t\sqrt{\psi/r^2}\right). \tag{4.2}$$

Thus, R is a steadily decreasing function of the time t measured on a standard clock following the particle. For $dr = 0$ we get from (3.1) and (3.4)

$$dR = \dot{R}dt, \qquad dT = \dot{\varphi}dt. \tag{4.3}$$

In empty space this gives by (2.20) and (3.17)

$$dR = -\sqrt{(\alpha/R) - \alpha/r}\,dt, \qquad dT = \frac{\sqrt{1-\alpha/r}}{1-\alpha/R}\,dt, \tag{4.4}$$

$$\frac{dR}{dT} = -\frac{\sqrt{(\alpha/R) - \alpha/r}}{\sqrt{1 - \alpha/r}} \cdot (1 - \alpha/R). \tag{4.5}$$

Thus, the "velocity" dR / dT of the particle in S_c is positive for $R < \alpha$ in spite of the fact that R is steadily decreasing. This is due to the circumstance that T is decreasing along the time-track of the particle for $R < \alpha$ as is seen from (4.4). Further since $T \to \infty$ for $R \to \alpha$ observers in S_c might come to the conclusion that the value $R = \alpha$ never can be reached if R initially is larger than α, although we know that this happens in a finite time measured on the standard clocks.

For $r = r_b$ where $\psi = \alpha / r_b$ we get from (4.2)

$$R_b = r_b\ C(u_b) = r_b C\left(t\sqrt{\alpha / r_b^3}\right) \tag{4.6}$$

which describes the motion of the boundary of the matter. R_b decreases from the value $r_b > \alpha$ at $t = 0$ to the value $R_b = \alpha$ in a finite time $t_\alpha(r_b)$ determined by the equation

$$C\left(t_\alpha(r_b)\sqrt{\alpha / r_b^3}\right) = \alpha / r_b. \tag{4.7}$$

Introducing this value for C into (1.53) we get for this time

$$t_\alpha(r_b) = r_b\sqrt{1 - \alpha / r_b} + r_b\sqrt{r_b / \alpha}\left(\frac{\pi}{2} - \sin^{-1}\sqrt{\alpha / r_b}\right). \tag{4.8}$$

Somewhat later at the time

$$t_s(r_s) = r_b\sqrt{r_b / \alpha}\,\frac{\pi}{2} \tag{4.9}$$

where $C = 0$ the surface of the matter runs into the singularity (2.11) which then spreads into the outer space with the velocity (2.21). The time interval $t_{\alpha s}$ in which R_b of the surface decreases from the Schwarzschild value α to the value zero is

$$t_{\alpha s} = t_s(r_s) - t_\alpha(r_s) = r_b\sqrt{r_b / \alpha}\sin^{-1}\left(\sqrt{\alpha / r_b}\right) - r_b\sqrt{1 - \alpha / r_b}. \tag{4.10}$$

As an example we consider a spherical system of incoherent matter with the mass and radius of a typical galaxy, say $M = 10^{45} gm$ and $r_b = 10^{23}$ cm. Then $\alpha \simeq 10^{17}\ cm$ from (1.21). According to (4.8) such a system would collapse through the Schwarzschild radius after a time

$$t_\alpha(r_b)/c \simeq \tfrac{1}{2}\, 10^{16} \text{ sec} \approx 160 \text{ million years}. \tag{4.11}$$

At this time the individual stars in the galaxy would still be far from touching each other, so that the approximation of incoherent matter would seem not to be too unrealistic. Provided the galaxy is non-rotating this phenomenon should actually occur after the comparatively short time (4.11). After a further very short time $t_{\alpha s}/c$ our model predicts a total collapse of the matter into the singularity. For $\alpha/r_b \ll 1$ (4.10) reduces to

$$t_{\alpha s}/c = \alpha/2c \approx 19 \text{ days} \tag{4.12}$$

in our case; but in the later part of this process the assumption of zero pressure is of course highly unrealistic. During the first 10 million years of the contraction process the quantity $u^2 = t^2\alpha/r_b^3$ is only about 0.01. Thus, we are in the region (3.23), where the systems R_c and R_M coincide for the whole outside space $r > r_b$. During the time interval

$$t_\alpha(r_b) < t < t_s(r_b) \tag{4.13}$$

the system represents a "black hole", i.e. no information can be transferred from the surface of the matter to regions of space-time where $R > \alpha$. In order to study this phenomenon a little more closely we need only to consider the motion of light signals through empty space; for no real signal can move faster than light. From (1.55) and (2.13) we obtain for the radial velocity of light, since $ds^2 = 0$ for such signals,

$$\left(\frac{dr}{dt}\right)_L^2 = \frac{1 - \alpha/r}{R'^2} \tag{4.14}$$

with R' given by (2.20). Hence

$$\left(\frac{dr}{dt}\right)_L = \pm \frac{\sqrt{1 - \alpha/r}}{R'} \tag{4.15}$$

where the upper and lower signs hold for signals that are moving in the outward and inward directions, respectively, relative to S_M, i.e. relative to the matter. The changes of the curvature coordinates R and T along the time-tracks of the signals are by (4.15), (3.4) and (3.17)

$$\frac{dR}{dt} = R'\left(\frac{dr}{dt}\right)_L + \dot{R} = \pm\sqrt{1 - \alpha/r} + \dot{R} \tag{4.16}$$

$$\frac{dT}{dt} = \varphi'\left(\frac{dr}{dt}\right)_L + \dot{\varphi} = \left(\pm \dot{R} + \sqrt{1 - \alpha/r}\right)/(1 - \alpha/R), \tag{4.17}$$

or using (2.20)

$$\frac{dR}{dt} = \pm\sqrt{1 - \alpha/r} - \sqrt{(\alpha/R) - \alpha/r} \tag{4.18}$$

$$\frac{dT}{dt} = \left(\sqrt{1-\alpha/r} \mp \sqrt{(\alpha/R)-\alpha/r}\right)/(1-\alpha/R). \tag{4.19}$$

The solutions of the two differential equations (4.15) are given implicitly by the equations

$$F_\pm(r,t) = C_\pm \tag{4.20}$$

where C_+ and C_- are constants of integration and F_+ and F_- are the following functions of r and t:

$$F_\pm(r,t) = \pm\varphi(r,t) - R - \alpha\log\frac{|R-\alpha|}{\alpha}. \tag{4.21}$$

In fact, by differentiation of these functions we get

$$\frac{dF_\pm}{dt} = F'_\pm\left(\frac{dr}{dt}\right)_L + \dot{F}_\pm(r,t) = \pm\frac{\sqrt{1-\alpha/r}}{R'}F'_\pm + \dot{F}_\pm \tag{4.22}$$

and by (3.17)

$$\left.\begin{aligned} F'_\pm &= \pm\varphi' - \left(1+\frac{\alpha}{R-\alpha}\right)R' = \frac{R'}{\sqrt{1-\alpha/r}}\,\frac{\pm\dot{R}-\sqrt{1-\alpha/r}}{1-\alpha/R} \\ \dot{F}_\pm &= \pm\dot{\varphi} - \frac{\dot{R}}{1-\alpha/R} = \frac{\pm\sqrt{1-\alpha/r}-\dot{R}}{1-\alpha/R}. \end{aligned}\right\} \tag{4.23}$$

Hence

$$\frac{dF_\pm}{dt} = \frac{\dot{R}\mp\sqrt{1-\alpha/r}}{1-\alpha/R} + \frac{\pm\sqrt{1-\alpha/r}-\dot{R}}{1-\alpha/R} = 0,$$

which shows that the functions $F_\pm(r,t)$ are integrals of the motion of the light signals.

When the signals start at the point r_0 at the time t_0 the motion of the outward and inward going signals are described by the two equations

$$F_\pm(r,t) = F_\pm(r_0,t_0) \tag{4.24}$$

that also may be written

$$\pm(T-T_0) - (R-R_0) - \alpha\log\frac{|R-\alpha|}{|R_0-\alpha|} = 0 \tag{4.25}$$

from (4.21) and (3.4). Two signals starting at the times t_0/c and $(t_0+dt_0)/c$ from the point r_0 will arrive in the fixed point r at the times t/c and $(t+dt)/c$ respectively, where the relation between dt and dt_0 is obtained by differentiation of (4.24):

$$\dot{F}_\pm(r, t)dt = \dot{F}_\pm(r_0, t_0)dt_0. \tag{4.26}$$

If the light at the start from r_0 has the (standard) frequency ν_0 and the wave length $\lambda_0 = c/\nu_0$, the interval dt_0 between the emission of successvie wave crests is $dt_0 = c/\nu_0 = \lambda_0$ and the corresponding interval for their arrival in r is $dt = c/\nu = \lambda$ where λ is the observed wave length at r.
Then, we obtain the relative shift of the spectral lines

$$z = \frac{\lambda - \lambda_0}{\lambda_0} = \frac{\lambda}{\lambda_0} - 1 \tag{4.27}$$

from (4.26) and (4.23) which give

$$\lambda/\lambda_0 = \frac{\dot{F}_\pm(r_0, t_0)}{\dot{F}_\pm(r, t)} = \frac{\pm\sqrt{1 - \alpha/r_0} - \dot{R}_0}{1 - \alpha/R_0} \cdot \frac{1 - \alpha/R}{\pm\sqrt{1 - \alpha/r} - \dot{R}} \tag{4.28}$$

The function F_+ given by (4.21) and (3.15) contains a term $-2\alpha\log\frac{|R - \alpha|}{\alpha}$ so that

$$F_+ \to \infty \quad \text{for} \quad R \to \alpha \tag{4.29}$$

while F_- remains finite in this limit, since the logarithmic terms in F_- cancel.

Let us now first consider the case of an outward moving signal, where the upper signs hold in the preceding formulae. If t_0 lies in the interval (4.13) and r_0 is equal to r_b we have $R_0 = R(r_0, t_0) < \alpha$ and $F_+(r_0, t_0)$ is a finite constant. Then it follows from (4.24) and (4.29) that R can never become equal to α during the motion, i.e. the signal can never penetrate into a region where $R > 0$. At the first moment this is somewhat surprising, since $\left(\frac{dr}{dt}\right)_L$ according to (4.15) with the plus sign never becomes negative. However the outward velocity is zero for $R' = \infty$ which happens when $u = t\sqrt{\alpha/r^3} = \pi/2$. Thus the signal does not stop before it runs into the singularity (2.7) where r and t have values (r^*, t^*) connected by the equation

$$u^* = t^*\sqrt{\alpha/r^{*3}} = \pi/2. \tag{4.30}$$

For these values also

$$R^* = R(r^*, t^*) = 0 \tag{4.31}$$

which shows that R must decrease along the time-track of the signal from the value R_0 to the value zero while r increases from r_0 to r^*. This is in accordance with (4.18) since dR/dt is negative all the time for $R < \alpha$. Also it follows from (4.19) that $dT/dt > 0$ in this case, i.e. T increases from the value T_0 to the value

$$T^* = \varphi(r^*, t^*) = t^* \sqrt{1 - \alpha / r^*} + \pi \sqrt{\alpha(r^* - \alpha)} \tag{4.32}$$

obtained from (3.15) by putting $R = R^* = 0$. Further we have by (4.25) with $R = R^* = 0$

$$T^* = T_0 - R_0 + \alpha \log \frac{\alpha}{\alpha - R_0} \tag{4.33}$$

which is indeed larger than T_0 for $R_0 < \alpha$. The equations (4.30) and (4.33) with (4.32) determine the place r^* and the time t^* at which the light ray runs into the singularity.

For $r_0 = r_b$ and $t_0 < t_\alpha(r_b)$ we have $R_0 > \alpha$ and the outward going light signal will proceed to arbitrarily large values of r and of $R = r\, C(t \sqrt{\alpha / r^3})$. This follows at once from (4.18) and (4.15) which show that dR/dt and $(dr/dt)_L$ are positive all the way. An observer sitting at a point with

$$r \gg \alpha, \qquad R \gg \alpha. \tag{4.34}$$

will observe a line shift given by (4.28). With the value for $\dot{R}$ given by (2.20) and with (4.34) we obtain

$$\lambda / \lambda_0 = \frac{\sqrt{1 - \alpha / r_0} + \sqrt{(\alpha / R_0) - \alpha / r_0}}{1 - \alpha / R_0}. \tag{4.35}$$

The light is shifted towards the red. For $t_0 \ll t_\alpha(r_b)$ so that $u_0^2 = t_0^2 \alpha / r_0^3 \ll 1$, i.e. in the region (3.23) we have from (3.24) $C_0 = 1$, $R_0 = r_0$ and the formula reduces to the well-known red shift formula

$$\lambda = \lambda_0 / \sqrt{1 - \alpha / R_0} \tag{4.36}$$

for a source at rest in the Schwarzschild system of coordinates. In general $\lambda > \lambda_0 \sqrt{1 - \alpha / r_0} / (1 - \alpha / R_0) > \lambda_0 / \sqrt{1 - \alpha / R_0}$ which shows that the light is always redshifted in this case.

Let us now consider the case of ingoing light where we have to use the lower signs in the equations (4.17)–(4.28). Since F_- is regular everywhere in the physical region $t < t_s(r)$, there is nothing to stop the signal from going from a place with $R_0 > \alpha$ right through to the surface of the matter sphere even if R_b is smaller than α at the time of arrival. This is also seen from (4.18) which shows $dR / dt < 0$ for all R in this case. Thus the Schwarzschild wall $R = \alpha$ separates space-time into two regions I and II with $R < \alpha$ and $R > \alpha$ respectively. While information can pass freely from II to I, no information about happenings in I can ever reach the region II.

Now consider an observer on the boundary of the sphere $r = r_b$ which at the time $t < t_s(r_b)$ receives light from a distant star with $r_0 \gg \alpha$ and $R_0 \gg \alpha$. Then we get from (4.28) and (2.20)

$$\frac{\lambda}{\lambda_0} = \frac{1 - \alpha/R_b}{\sqrt{1-\alpha/r_b} - \sqrt{(\alpha/R_b) - \alpha/r_b}} \tag{4.37}$$

where R_b is the value of R at the time t of reception at r_b. When this time is small compared with $t_\alpha(r_b)$ in (4.8), i.e. in the region (3.23), we have $R_b = r_b$ and (4.37) reduces to

$$\lambda = \lambda_0 \sqrt{1 - \alpha/R_b}, \tag{4.38}$$

the light is shifted towards blue. In the limit $t \to t_\alpha(r_b)$ where $R_b \to \alpha$ we get from (4.37)

$$\lambda \to 2\lambda_0 \sqrt{1 - \alpha/r_b}. \tag{4.39}$$

For $\alpha/r_b \ll 1$, as in the example on p. 21, this corresponds to a redshift. Under the same assumption we have for $t_\alpha(r_b) < t < t_s(r_b)$, where $R_b < \alpha$, the redshift formula

$$\lambda = \lambda_0 \frac{\alpha/R_b - 1}{\sqrt{\alpha/R_b} - 1} = \lambda_0\left(1 + \sqrt{\alpha/R_b}\right). \tag{4.40}$$

5. Continuation of the Solution of Section 1 to $t < 0$

In the preceding sections we have considered a contraction process starting from a state of rest at $t = 0$ corresponding to the initial conditions (1.23), (1.27). However, it is clear that (1.55) with (1.56) is a regular solution of Einstein's field equations in the whole region

$$- t_s(r) < t < t_s(r) \tag{5.1}$$

with $t_s(r)$ given by (2.8). In this region the parameters u and η in (1.54) take on all values in the intervals

$$-\frac{\pi}{2} < u < \frac{\pi}{2}, \qquad -\pi < \eta < \eta. \tag{5.2}$$

During the time interval

$$- t_s(r) < t \leq 0 \tag{5.3}$$

the quantity

$$u = \dot{t}\sqrt{\varphi/r^2} \tag{5.4}$$

goes from $-\pi/2$ to 0 and $C(u)$ increases from 0 to 1. Thus in the interval (5.3) we have instead of (2.3)

$$C'(u) = \sqrt{\frac{1-C}{C}} \tag{5.5}$$

corresponding to an expansion process. From (1.56), (5.5) it follows that R' is still positive in the interval (5.3), but

$$\dot{R} \geq 0 \tag{5.6}$$

in contrast with (2.4). More precisely we have

$$\left.\begin{aligned} R(r,-t) &= R(r,t), \quad R'(r,-t) = R'(r,t) \\ \dot{R}(r,-t) &= -\dot{R}(r,t). \end{aligned}\right\} \tag{5.7}$$

In empty space we have in particular from (5.7) and (2.20)

$$\dot{R}(r,t) = \sqrt{(\alpha/R) - \alpha/r} \tag{5.8}$$

for t in the interval (5.3).

The system described by this solution corresponds to a spherical distribution of incoherent matter which jumps out of a singularity at $t = -t_s(r)$ for which $u = -\pi/2$, $R = 0$ and expands with decreasing speed until it comes to rest at $t = 0$, after which it performs the contraction process described in the preceding sections.

For $t > 0$ the function $\varphi(r,t)$ was defined by (3.15). We extend the definition to negative t by requiring that $\varphi(r,t)$ is an uneven function of t, i.e.

$$\varphi(r,-t) = -\varphi(r,t). \tag{5.9}$$

Since R is an even function of t this gives

$$\left.\begin{aligned} \varphi(r,t) = t\sqrt{1-\alpha/r} - 2\sqrt{\alpha(r-\alpha)}\tan^{-1}\sqrt{\frac{r-R}{R}} \\ - 2\alpha \log \frac{\sqrt{R(r-\alpha)} + \sqrt{\alpha(r-R}}{\sqrt{r|R-\alpha|}} \end{aligned}\right\} \tag{5.10}$$

for t in the interval (5.3). From (5.9) we obtain

$$\dot{\varphi}(r,-t) = \varphi(r,t), \quad \varphi'(r,-t) = -\varphi'(r,t). \tag{5.11}$$

Since also $\dot{R}$ is an uneven function of t it follows that the expressions (3.17) are valid also for negative t. Therefore, in empty space the transition to curvature coordinates is also in the region (5.3) effected by the transformations (3.1, (3.4) with φ given by (5.10), and the metric in S_c is the same as in (3.3) (3.18). Since $T \geq 0$ for $t \geq 0$ and arbitrary r we have $T \leq 0$ for $t \leq 0$ from (5.9).

All the considerations performed in the preceding sections for $t > 0$ can now be repeated for the region (5.3). The motion of the boundary relative to S_c is again given by (4.6) with negative t. It starts at $t = -t_s(r_b)$ with $R_b = 0$ and increases to the value $R_b = \alpha$ for $t = -t_s(r_b)$ after which it increases further to the value $R_b = r_b$ at $t = 0$.

The motions of outward and inward going light signals are still determined by (4.15) and (4.16), but in view of (5.8) we have instead of (4.18)

$$\frac{dR}{dt} = \pm\sqrt{1-\alpha/r} + \sqrt{(\alpha/R)-\alpha/r} \tag{5.12}$$

for $t < 0$. The solutions of the equations (4.15) are again given by (4.24), (4.25) where the functions $F_{\pm}(r, t)$ are determined by (4.21) and (5.10) for $t < 0$, but in this region $F_{+}(r, t)$ is everywhere regular, while

$$F_{-} \to \infty \quad \text{for} \quad R \to \alpha \tag{5.13}$$

Therefore, light emitted from a point $r_0 = r_b$ on the surface of the matter sphere at the time $t_0 < 0$ can freely move to the outside region with $R > \alpha$ even if $R_b < \alpha$ at the time of emission t_0. This also follows from (5.12) with the upper sign since dR / dt is positive for all R. On the other hand, ingoing light starting at an event point (r_0, t_0) with $R_0 > \alpha$, where $F_{-}(r_0, t_0)$ has a finite value, can never penetrate into a region where $R < \alpha$ on account of (5.13). This is also seen from (5.12) with the lower sign, since $dR / dt = 0$ for $R = \alpha$ and $dR / dt > 0$ for $R < \alpha$.

Thus, during the time interval

$$-t_s(r_b) < t < -t_{\alpha}(r_b) \tag{5.14}$$

where $R_b < \alpha$, the system is a "white hole". It can emit light into the outside world, where r and R are large compared with α, but an observer on the sphere cannot during the period (5.14) receive any light from a distant star with $r_0 \gg \alpha$, $R_0 \gg \alpha$. On the other hand for $t > -t_{\alpha}(r_b)$ an observer on the sphere can receive any message from the outside world since $R > \alpha$ and $dR/dt < 0$ all the way along the time-tracks of ingoing light signals.

The relative line shift z is in the whole region (5.1) given by (4.27), (4.28), but for negative t the quantity $\dot{R}$ in (4.28), as given by (5.8), has the opposite sign of the expression in (2.20) valid for $t > 0$. Therefore, light of wave length λ_0 emitted at $r_0 = r_b$ at a time t_0 in the interval (5.3) will be observed by a distant observer at $r \approx R \gg \alpha$ with the frequency λ given by

$$\frac{\lambda}{\lambda_0} = \frac{\sqrt{1 - \alpha/r_0} - \sqrt{(\alpha/R_0) - \alpha/r_0}}{1 - \alpha/R_0} \tag{5.15}$$

which is the reciprocal of (4.37). Thus, if the light received at a place $r = r_b$ at a time $t > 0$ from a distant star is redshifted, the light emitted from the time-inversed point $(r_b, -t)$ as observed by a distant observer will be blueshifted and vice versa.

Similarly, for light from a distant star with $r_0 \approx R_0 \gg \alpha$ that is received at a point $r = r_b$ at a time in the interval

$$-t_\alpha(r_b) < t < 0 \tag{5.16}$$

where $R_b > \alpha$, we have

$$\frac{\lambda}{\lambda_0} = \frac{1 - \alpha/R_b}{\sqrt{1 - \alpha/r_b} + \sqrt{(\alpha/R_b) - \alpha/r_b}}. \tag{5.17}$$

This expression is the reciprocal of (4.35), i.e. the light received from a distant star at the surface of the sphere is blueshifted.

If the matter inside the sphere is uniformly distributed, the metric is given by (2.32) for $r < r_e$, which is identical with the Friedman solution for a spatially closed universe with constant positive curvature. According to the conventional cosmological ideas there is nothing outside this closed world, but the question now arises if the observable part of the universe in reality could be the inner part of a "meta galaxy" immersed in a much larger closed or open universe. In this respect the usually assumed values for the radius and average mass density of the universe are strangely suggestive. For a model of the type considered in this section with a radius of 10^{10} light years and density $10^{-29}\,\mathrm{gm/cm^3}$ the Schwarzschild constant would be of the order of magnitude of the radius, and it is conceivable that the observable universe at the present time is a "white hole", so that no information from distant stars outside the meta galaxy can penetrate into the interior. However, we shall not enlarge upon this picture here.

Conclusion

In this paper we have reconsidered in more detail the problem of the collapse of incoherent matter under the influence of its own gravitational field. It serves as a simple illustration of the general theorem of Hawking and Penrose according to which singularities will develop both inside and outside the matter after a comparatively short time as measured on standard clocks at rest in a system S_M in which the matter is constantly at rest (compare the example mentioned on p. 21). Not only does the density of matter go to infinity, as would be the case also in Newton's theory of gravitation, but in Einstein's theory the metric of space-time itself becomes singular at the finite time $t_s(r)$. In section 2 it was shown that the coefficients of dr^2 and $d\Omega^2$ in ds^2 in general have the following limiting values for $t \to t_s(r)$ and constant r:

$$a \to \infty, \quad R^2 \to 0$$

and for the determinant g we found

$$g \to 0 \text{ for } t \to t_s(r).$$

The singularities in question are essential singularities that cannot be removed by any coordinate transformation.

In section 1 it was emphasized that the determinant g must be negative in any case which has a physical meaning. Thus the occurrence of the just mentioned essential singularities means that the system according to Einstein's theory after a finite time runs into an unphysical state — a kind af nirvana where the time stops and the notions of space and time lose their meaning. It is hard for a physicist to accept this and one would rather conclude that Einstein's theory, which so admirably accounts for all phenomena in the case of normal gravitational fields, breaks down in cases where the components of the curvature tensor of space-time are extremely large.

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