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# HYDROGEN-BURNING MODELS FOR STARS IN THE HORIZONTAL BRANCH OF GLOBULAR CLUSTERS

BY

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## Synopsis

The problem of the structure of stars in the horizontal branch of globular clusters is discussed, and hydrogen-burning models, assumed to be mixed by the helium flash, are studied in detail in order to investigate whether such model sequences can provide an accurate description of the observed properties of horizontal branch stars.

It is shown that homogeneous models with reasonable masses and chemical compositions correspond satisfactorily to the hottest stars of the observed horizontal branches, but in the later, fast stages their evolutionary tracks fall more than one magnitude above the horizontal branch region. Models containing an outer zone of the original chemical composition are shown to be placed along the horizontal branch, the position being determined by the mass fraction of the outer zone. Also their evolutionary tracks will in general fall above the branch when the fast evolution after the hydrogen burning in the central region starts. Hence, according to this interpretation the horizontal branch is not an evolutionary sequence, but rather a main-sequence-like phenomenon.

The model sequences are compared with observed colour-magnitude diagrams for M5 and M15, and it is concluded that such sequences can explain several features of the observed diagrams very satisfactorily. It is emphasized that the time scale for the evolution agrees reasonably well with the empirical value determined by Sandage for M3.

These results may be understood if the helium flash gives rise to mixing of the stars, and this mixing is more violent for stars poor in metal than for stars rich in metal. Finally a few other predictions based on this picture of the evolution of horizontal branch stars are mentioned.

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## 1. Introduction

Investigations of the evolution of population II stars have given a very satisfactory explanation of the observed main-sequences and red giant-sequences of globular clusters (HOYLE and SCHWARZSCHILD 1955, KIPPENHAHN, TEMESVARY, and BIERMANN 1958, HOYLE 1959, DEMARQUE and GEISLER 1963). The observed properties of stars in the horizontal branches of globular clusters, on the other hand, are not understood in a corresponding convincing way. For instance, it seems that the empirically determined time scale (SANDAGE 1957) is considerably longer than the time scale of helium-burning model sequences (HAYASHI, HOSHI, and SUGIMOTO 1962, NISHIDA and SUGIMOTO 1962), and the observed characteristic differences between colour-magnitude diagrams of globular clusters with different content of heavy elements have not been explained in detail.

The interpretation of the horizontal branches as loci for helium-burning sequences is based on the results of SCHWARZSCHILD and HÄRM (1962), HÄRM and SCHWARZSCHILD (1964), HAYASHI, HOSHI, and SUGIMOTO (1962), and SUGIMOTO (1964), suggesting a helium-burning stage to follow after the red giant stage.

If stars, however, are mixed by the helium flash we might expect that the subsequent evolutionary phase, with hydrogen burning in the core, would last considerably longer than the helium-burning phase following a helium flash without mixing.

We shall here first discuss the general problem of the structure of horizontal branch stars; we shall then consider models which are assumed to be mixed by the helium flash, and investigate whether such hydrogen-burning models can provide a reasonable interpretation of the horizontal branch stars, especially if it is possible to understand the observed colour-magnitude diagrams of globular clusters assuming mixing.

In Section 2 a summary of the observational results important for our investigation is given, and in Section 3 the possible structures of horizontal branch models are discussed. The last Sections, 4 and 5, contain the computed evolutionary sequences and comparisons with observed colour-magnitude diagrams, together with a discussion of the results. The computational technique is briefly described in an appendix.

## 2. Observational Material

### Colour-Magnitude Diagrams

Discussions of colour-magnitude diagrams of globular clusters have been offered by many authors. An important result of comparisons of clusters with different contents of heavy elements is that several characteristic properties are found to vary systematically with the strength of metal lines (Sandage 1953, Arp 1955, Sandage and Wallerstein 1960, Wildey 1961). The following Table 1 shows a summary of observational results important for a discussion of horizontal branch stars, for some of the well-observed clusters.

Table 1.

| Catalogue no. | | Composition | | $M_V$ | Age ($10^9$ years) | Percentage of | | |
|---|---|---|---|---|---|---|---|---|
| NGC | Messier | $[M/H]$ | Morgan type | | | Blue | RR Lyrae | Yellow |
| 6341 | 92 | | I | | | 88 | 10 | 2 |
| 7078 | 15 | | I | 1.0 | | 65 | 25 | 10 |
| 7089 | 2 | −2.3 | II | 1.0 | 24 | 94 | 4 | 2 |
| 5272 | 3 | −1.6 | II | 1.0 | 24 | 36 | 37 | 27 |
| 6205 | 13 | −1.4 | III | 0.3 | 19 | 97 | 3 | 0 |
| 5904 | 5 | −1.3 | II | 0.8 | 22 | 50 | 30 | 20 |
| 6254 | 10 | | IV | | | 91 | 0 | 9 |
| 6171 | | | V | | | 3 | 5 | 92 |
| 6356 | | −0.3 | VI | | | 0 | 0 | 100 |
| 104 | 47 Tuc | | | 1.0 | | 0 | 2 | 98 |

The objects are arranged according to increasing content of heavy elements as given by $[M/H]$, defined by Wallerstein and Carlson (1960) as the logarithm of the metals-to-hydrogen ratio minus the same quantity in the sun. Besides the values of $[M/H]$ (from Arp 1962a) the Morgan type of the clusters (Morgan 1959) are given. $M_V$ is the absolute magnitude for the RR Lyrae stars as given by Sandage (1962a), and the ages are those obtained by Arp (1962a) by means of the method given by Sandage (1962b). Corrections to the $B$ and $V$ magnitudes for interstellar reddening are probably important (ARP 1962c), causing corrections of the values of $M_V$ and the ages. It seems likely that the ages mentioned are systematically too large, not only because of reddening corrections, but also because recent evolutionary sequences using accurate formulae for the energy-generation rates of hydrogen-burning processes indicate smaller ages than earlier sequences based on the approximate temperature-power law $e_{pp} \approx T^4$ (DEMARQUE and LARSON 1964).

Following WOOLF (1964) we shall call stars in a horizontal branch, with higher temperatures than the RR Lyrae stars, "blue" stars of the horizontal branch and stars with lower temperatures than the RR Lyrae stars correspondingly "yellow" stars. The last three columns of Table 1 give the distribution of the horizontal branch stars in the three groups estimated from published colour-magnitude diagrams (ARP 1955, SANDAGE and WALLERSTEIN 1960, WILDEY 1961, WOOLF 1964).

The clusters can clearly be divided into three groups: Group 1 with $[M/H] < -2$, Group 2 with $[M/H] \simeq -1.5$ and Group 3 having a more normal population I chemical composition. As shown by SANDAGE and WALLERSTEIN (1960), the quantity $\Delta V$ (the difference in visual magnitude between the horizontal branch and the giant branch of a cluster read at $(B-V)_0 = +1.4$), the periods of the RR Lyrae stars and some characteristic properties of the horizontal branch are all well-correlated with the metal abundance. Of special interest for our investigation is the distribution of stars along the horizontal branch.

Group 1 with very weak metal lines have most of the horizontal branch stars situated in the blue part of the horizontal branch, while the clusters in Group 2 with weak metal lines show a more uniform distribution of the stars between the three parts of the horizontal branch, and Group 3 clusters with strong metal lines have practically all their horizontal branch stars placed in the yellow part of the branch.

The general rules described, however, are violated by several clusters, exceptions being especially conspicuous for M10 and M13. They have both practically all stars placed in the blue part of their horizontal branches, even though they belong to Group 2 and M10 has no RR Lyrae stars at all. Furthermore, a closer inspection of the colour-magnitude diagrams shows that many clusters display special individual features. Therefore differences in metal content is probably not the only reason for the observed differences between the three cluster groups. Other important causes could be found in the hydrogen content and the distribution of hydrogen throughout the stars, or in different masses for the horizontal branch stars in different clusters. In this connection Sandage and Wallerstein mention that the abnormal distribution of the stars in M 13 may be due to higher masses of the horizontal branch stars in M 13, compared with the other clusters. M 13 is the youngest of the clusters considered here and therefore probably contains the most massive stars.

### Lifetime of Evolutionary Stages

Sandage (1957) has applied a semiempirical method, utilizing observed luminosity functions and colour-magnitude diagrams and assumptions for the absolute magnitude of RR Lyrae stars to determine: the evolutionary tracks for stars in the clusters M 67 and M 3, the time scale of the evolution, and the fraction of hydrogen exhausted at each evolutionary stage. Of special interest for us is the time taken to evolve along the horizontal branch which for M 3 was found to be $2.3 \times 10^8$ years, the RR Lyrae stage lasting about one third of this time. Further, the analysis indicated that the stars in M 3 have burned nearly all their hydrogen supply at the top of the red giant branch just before the horizontal branch stage starts. On the other hand, the stars in M 67 were found to have used only about half of their hydrogen supply at the top of the red giant branch.

The basis of Sandage's analysis was Schönberg-Chandrasekhar models (1942) for the early evolution. Woolf (1962) studied the evolutionary tracks in M 3 using the same method, but a model sequence by Hoyle (1959) for the early evolution. Woolf found the same time for evolution along the horizontal branch as Sandage, $2.3 \times 10^8$ years, but emphasized an apparent discrepancy: The stars seemed to emit more energy in the computed lifetime than burning of nuclear fuel could supply. As mentioned above, reliable ages of globular clusters probably cannot be given yet; first a satisfactory agreement between the observed colour-magnitude diagrams and the computed evolutionary sequences must be accomplished. Therefore the estimate of the lifetime of the horizontal branch stage is still considerably uncertain.

### Mass Loss

Small differences in mass between different types of cluster stars can be determined from observations of differences in distribution of the types, provided that equipartition of energy between all stars in the cluster has been reached by gravitational interaction. Since heavier stars evolve faster than less massive, we should expect the most evolved stars of a cluster to be concentrated near the cluster center, and since horizontal branch stars probably correspond to a later evolutionary phase than red giants do, we should expect the horizontal branch stars to be the most concentrated group of stars in globular clusters.

Oort and van Herk (1959) and Woolf (1964) studied the concentration of stellar groups in the globular clusters M 3, M 5, M 15 and $\omega$ Centauri, and found the above-mentioned expectations not to be fulfilled. Oort and van Herk found the RR Lyrae stars less concentrated than the red giants and Woolf showed that the degree of concentration in M 3 decreases in the following order:

(a) red giants
(b) blue stars of the horizontal branch
(c) RR Lyrae stars
(d) yellow stars of the horizontal branch.

These results can be interpreted as due to mass loss taking place between the red giant stage and the horizontal branch stage. The quantitative analysis showed that the mass loss is about 20 % of the total mass.

Woolf proposed the following evolutionary scheme in order to understand the observations. A mixing was assumed at the top of the red giant branch, quickly transferring the star to the blue end point of the horizontal branch, and afterwards a slow evolution through the blue part of the horizontal branch, the RR Lyrae region, and the second giant branch somewhat above the red giant branch was supposed. The observed concentrations of the four stellar types can then be understood if mass loss of about 20 % occurs in the later part of the red giant phase, or, as preferred by Woolf, at the RR Lyrae stage. After the massloss equipartition of energy will be established on a time scale not very different from the time scale for the evolution along the horizontal branch, giving rise to the observed differences in concentration of the stellar types.

## 3. Structure of Models for Horizontal Branch Stars

### Helium-Burning Sequences

The first model sequence constructed in order to explain the horizontal branches of globular clusters was published by Hoyle and Schwarzschild (1955). The models, assumed to result from a helium flash, were built on a helium core containing about half the total mass (1.2 $M_\odot$) and an envelope rich in hydrogen, with the original chemical composition. They were found to burn helium in a small convective core and hydrogen in a thin shell, and HOYLE and SCHWARZSCHILD showed that the gradual conversion of hydrogen to helium in the shell could explain an evolution from right to left in the colour-magnitude diagram somewhat above the horizontal branch.

However, as already stated by HOYLE and SCHWARZSCHILD, this idea is hardly realistic, because the helium burning at the center of the star will proceed much faster than the hydrogen burning in the shell. When the helium in the convective core is exhausted, the central temperature will rise and further nuclear burnings will start, quickly resulting in models with more complicated central regions than those considered by HOYLE and SCHWARZSCHILD. This will probably result in evolutionary tracks more complicated than the simple evolution from right to left in the colour-magnitude diagram.

The most promising attempt known to the author, of constructing an evolutionary sequence for the horizontal branch is due to HAYASHI, HOSHI, and SUGIMOTO (1962). They treated the evolution of a population II star of mass 0.7 $M_\odot$ in the helium-burning phase. The model consisted of a helium core with 76 % of the total mass and an outer envelope with the original chemical composition $X = 0.90$, $Y = 0.10$, $Z = 0.001$. In the first evolutionary phase helium was found to burn in a small convective core, and in the following phase the helium burning proceeds in a shell surrounding a core consisting of carbon and oxygen. The evolution was followed through most of the horizontal branch from left to right up to the tip of the second giant branch somewhat above the red giant branch, where, according to the computations, 90 % of the helium core were converted into carbon and oxygen. HAYASHI, HOSHI, and SUGIMOTO could correlate the length of the red giant phase, the horizontal branch phase, and the phase where the star moved through the second giant branch, respectively, with the relative numbers of stars observed in the corresponding regions of the colour-luminosity diagram of M 5 by ARP (1962b). The comparison turned out to be quite satisfactory.

The length of the horizontal branch phase according to this evolutionary sequence is $3 \times 10^7$ years, while SANDAGE, as mentioned above, for M 3 determined the same quantity at $2.3 \times 10^8$ years; and it does not seem likely that this value should be in error by one order of magnitude.

Also the investigations of NISHIDA and SUGIMOTO (1962) showed that helium-burning sequences for the horizontal branch seem to give rather short time scales. They computed evolutionary sequences for population II stars of mass 1.2 $M_\odot$ with an inner helium region containing about 50 %

of the total mass and an envelope rich in hydrogen, and showed that the stage in which helium was burned in a small convective core would last $2.3 \times 10^7$ years and could correspond to the RR Lyrae stars.

However, both the evolutionary sequences and the comparison with observations are still so uncertain that it seems quite possible that further investigations could show that sequences burning helium at the centre of the stars were able to describe the horizontal branch stage. The serious problem of the too short time scale may be solved if a considerable part of the hydrogen content could be burned in models with energy production in two regions.

**Mass and Chemical Composition**

A major difficulty in the construction of evolutionary sequences for the horizontal branch is that the parameters determining the structure of the first model are little known.

For the mass of the heaviest stars in globular clusters 1.2 $M_\odot$ has often been used. But, as mentioned above, some clusters (M 13, for instance) may be younger than the average, hence containing more massive stars; and, on the other hand, a mass loss of about 20 % may occur between the red giant stage and the horizontal branch stage. Probably 1.5 $M_\odot$ will be an upper limit and 0.7 $M_\odot$ a lower limit for the masses.

The chemical composition also presents serious uncertainties. Theoretical evolutionary tracks show that the helium core contains about half of the mass when the helium flash starts for population II stars, and somewhat less (40 %) for population I stars (HAYASHI, HOSHI, and SUGIMOTO 1962), while, according to the method of semiempirical evolutionary tracks (SANDAGE 1957), practically all the hydrogen should be converted into helium for the stars in M 3 before the flash starts. If mass loss occurs, it seems likely that the mass which escapes, is nearly pure hydrogen from the outer layers, thus mass loss is also reducing the hydrogen content. For the average hydrogen content $X$ of horizontal branch stars we can therefore assume $X < 0.50$.

Before the helium flash is finished, a few per cent. of the mass of the star are probably burned to carbon, and if hydrogen somehow mixes into the hot central regions several nuclear processes will occur (WALLERSTEIN and GREENSTEIN 1964). These processes may give rise to a chemical composition of the horizontal branch stars, which neither corresponds to usual population II stars nor to population I stars. WALLERSTEIN and GREENSTEIN for two CH stars poor in metal, HD 26 and HD 201626, found a high carbon-to-iron ratio and an excess of Ba, La, Ce, and Nd with respect to Fe with factors about 20, which they interpreted as the result of a helium flash followed by neutron addition to the iron-peak elements, caused by a mixing of a small amount of hydrogen into the edge of the hot core of the star. Afterwards a convective mixing of the edge of the core with the envelope was postulated to transport the resulting nuclei to the surface of the star.

For the average content of elements heavier than helium $Z$ it therefore seems reasonable to assume $Z < 0.05$, remembering that the chemical composition of horizontal branch stars may be abnormal.

### Mixing

The structure of a horizontal branch star will, however, depend much more on the distribution of hydrogen throughout the star than on the exact mass-value or the average chemical composition. If hydrogen is mixed into the central region by the helium flash, the structure and evolution of the star will be very different from the case where no mixing occurs, the star then being in a hydrogen-burning main-sequence-like stage instead of a helium-burning one. Therefore the question of mixing by the helium flash is important.

Härm and Schwarzschild (1964) investigated this question by detailed computations by means of the Henyey method. The evolution of three stellar models, differing in mass and chemical composition, was followed through the helium flash. The convective core round the centre was found to grow until, in the cooling stage after the peak of the flash, it extended to 99 % of the helium core. But it never reached the layers rich in hydrogen, so these computations indicated that mixing by the helium flash will not occur. However, Härm and Schwarzschild emphasized that in view of the narrowness of the margin by which the convective helium core missed the hydrogen layers, and in view of the fact that a number of physical items were not yet properly taken into account, they had to conclude that no decisive answer to the question of mixing by the helium flash could be given. Three crucial assumptions were made: The adiabatic approximation for the temperature gradient in the convective core was used, also in the phases around the peak of the flash, infinite conductivity was assumed in the degenerate region, and hydrodynamic effects were neglected in all phases.

The influence of these complications was considered by Sugimoto (1964), who attempted an extraction of the essential features of the physical processes by a more general investigation, using reasonable simplifications in order to reduce the computational work. With respect to mixing, SUGIMOTO's results were almost identical with those already described. The convective core was found to extend to 99 % of the mass of the helium core in the cooling phase, but failed to reach the envelope rich in hydrogen. Hydrodynamic effects caused a core more massive than 0.7 $M_\odot$ to explode, but cores as massive as this probably do not occur; and the effect of finite efficiency of convective heat transport, investigated by means of the mixing length theory was also shown to cause an explosion of the core, if $\alpha$, the mixing length in units of pressure scale height, was taken to be less than 0.2. It is, however, by no means evident that the mixing length theory provides a realistic description of the energy-transport mechanisms under these extreme conditions.

The processes involved in the helium flash seem to be so complicated that realistic theoretical computations probably will not be obtained in the near future. In this connection we should like to mention the recent discovery (SCHWARZSCHILD and HÄRM 1965) that instabilities may occur in certain models. Furthermore, it seems likely that a fast rotation of the dense core might develop while the contraction proceeds just prior to the helium flash.

Besides the possible role of the helium flash for mixing, such an effect could also, at least theoretically, be caused by the outer convective zone, the question being whether this zone will extend into the helium core.

SCHWARZSCHILD and SELBERG (1962) have published a detailed model for a population II star of mass 1.3 $M_\odot$ just prior to the flash, and this model shows that convection stops before the helium core is reached. HAYASHI, HOSHI, and SUGIMOTO (1962) also arrived at the conclusion that the outer convective zone does not cause mixing before the flash. And when the surface temperature of the star increases after the flash, it seems improbable that the outer convective zone should move downwards.

Observationally, as stressed by WOOLF (1964), mixing would be indicated if determinations of element abundances for horizontal branch stars showed helium and carbon enrichment. As mentioned above, WALLERSTEIN and GREENSTEIN found carbon enrichment for two CH stars poor in metal, but because observed colours and magnitudes $(B-V, M_V) \simeq (1^m, 0^m)$ will place the CH stars somewhat to the right of the horizontal branches in globular clusters, there is no clear relationship between them and horizontal branch stars.

Let us consider a star with a certain mass and chemical composition before the helium flash. Its hydrogen content $X$ as a function of relative mass $q = M(r)/M$ has approximately the form shown in Figure 1a. We shall now

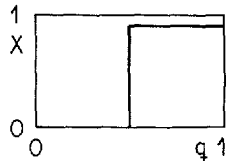


Fig. 1a. – Sketch of hydrogen content $X$ as a function of relative mass $q$ before the helium flash.

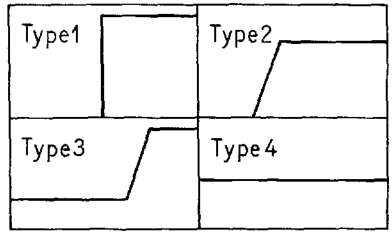


Fig. 1b – $X(q)$ for 4 possible model-types after the helium flash.

classify the different possibilities for models after the helium flash according to the mixing type, and distinguish between four main types. For each type the function $X(q)$ is sketched in Figure 1b. We do not, of course, know the exact shape of the curve $X(q)$ in the region of varying $X$ in the mixing types 2 and 3; but this will not cause serious difficulties in our discussion.

*Type 1: No mixing.*

Models of Type 1 have been investigated by Hoyle and Schwarzschild (1955), Hayashi, Hoshi, and Sugimoto (1962) and by Nishida and Sugimoto (1962), as mentioned in the first part of this section.

*Type 2: Partial mixing from the surface.*

This type of models might be produced if the outer convective zone reaches the helium core. As this seems an unrealistic possibility, and because the properties of this kind of models must be very similar to those of Type 1 models, we shall not consider Type 2 any further.

*Type 3: Partial mixing from the centre.*

Type 3 models will appear if the central convective core reaches the layers rich in hydrogen, mixing not being effective enough in transporting helium to the surface-layers. It is well-known that models with low hydrogen concentration in the inner regions and envelopes rich in hydrogen have larger radii than main-sequence stars. Table 1 shows that horizontal branch stars in clusters with high $Z$-values all have rather large radii; most of them are yellow and very few blue. Therefore one might suspect that stars in clusters rich in metal after a helium flash would appear as Type 3 stars, and it might be interesting to investigate whether the evolution of Type 3 stars will explain the colour-magnitude diagrams observed for clusters with high metal content.

*Type 4: Complete mixing.*

This type has been suggested by WOOLF (1964). Such homogeneous stars will start their evolution near the usual main-sequence, but they will, of course, have an unusually high luminosity for their mass ($X < 0.5$).

Using EDDINGTON's mass-luminosity relation

$$L = CM^{5.5}(\mu\beta)^{7.5}R^{-0.5}$$

we can easily estimate the absolute bolometric magnitudes. If we consider two stars with the same mass, but different hydrogen content, we find, neglecting the dependence on the radius $R$ and assuming for the relative gas pressure $\beta = 1$, which will be good approximations here,

$$\Delta M_{\text{bol}} = -2.5 \log \frac{L'}{L} \simeq -2.5 \log \left(\frac{\mu'}{\mu}\right)^{7.5} \simeq -19 \log \frac{2X + 0.75\,Y}{2X' + 0.75\,Y'}$$

Let us compare the horizontal branch models with "normal" population II main-sequence stars of mass 1.2 $M_\odot$, hydrogen content $X = 0.90$ and $M_{\text{bol}} \simeq 4^m$. For $X = 0.20$ the formula gives: $\Delta M_{\text{bol}} = -5^m.0$ and for $X = 0.30$: $\Delta M_{\text{bol}} = -3^m.8$. Hence our horizontal branch models will have $M_{\text{bol}} \simeq 0^m$ and will thus start their evolution near the left terminal point of the horizontal branch.

In the globular clusters poor in metal most stars are found in the blue part of the horizontal branch (Table 1). Therefore it is tempting to assume that the stars of these clusters are all completely mixed by the helium flash and evolve along the horizontal branch from left to right as proposed by WOOLF.

In the next section the evolution of Types 3 and 4 models will be considered in detail.

## 4. Model Sequences

The technique used in the computations is essentially the one developed by REIZ and PETERSEN (1964); it is briefly described in the Appendix.

### Reddening and Blanketing Corrections

The evolutionary tracks will be compared with the observations of globular cluster stars by ARP (1955). Later ARP (1962b) gave a very detailed analysis of the colour-magnitude diagram of M5, extending the earlier results. M5, being poor in metal by a factor of 17 relative to the sun, belongs to the Group 2 clusters. Besides M5 we shall especially consider M15 as a representative of the Group 1 clusters extremely poor in metal, because the colour-magnitude diagram for M15 shows some very characteristic features: The blue part of the horizontal branch is very narrow, and about $0^m.7$ below its blue terminal point a separate group of stars is located.

By means of the relations between $B-V$, $B.C.$ and $T_e$ given by D. L. Harris (1963) the theoretical evolutionary tracks in the $(\log T_e, M_{bol})$ diagram can be transformed into tracks in the colour-magnitude diagram for Hyades-like stars. As usual the intrinsic magnitudes obtained in this way will be denoted by $(B-V)_c$ and $V_c$.

The observed magnitudes, however, will in general need corrections for interstellar reddening and blanketing effects before a comparison with theoretical results is possible.

Reddening corrections of magnitudes of globular cluster stars have been a rather unsafe undertaking based on doubtful assumptions. Probably the best thing which can be done is, as proposed by Arp (1962c), to use the reddening laws:

$$E_{B-V} = 0.058 \operatorname{cosec} b\,, \quad A_V = 3.0\, E_{B-V}$$

for clusters with not too low galactic latitudes. Then we find for M5: $E_{B-V} = 0.08$, $A_V = 0.24$ and for M15: $E_{B-V} = 0.12$, $A_V = 0.36$.

Let us consider a globular cluster star and a Hyadeslike star with the same radius and luminosity, in other words with the same $T_e$. The much greater strength of metallic absorption lines in the atmosphere of the Hyades-like star will cause different distributions of the emergent radiation in the $U$, $B$, and $V$ bands in the two stars. For main-sequence stars the metallic absorption lines block more light in $B$ wavelengths than in $V$, and still more in $U$.

Wildey, E. M. Burbidge, Sandage, and G. R. Burbidge (1962) have published corrections $\Delta(B-V)$, $\Delta(U-B)$ and $\Delta V$ for these blanketing effects. The blanketing corrections must be applied to the observed $(B-V, V)$ values to give the corresponding $((B-V)_c, V_c)$, which are needed for a comparison with the theoretical evolutionary tracks.

The table, however, can only be used for main-sequence stars and only for $0.30 < B-V < 0.80$, thus this reduction is necessarily rather uncertain for horizontal branch stars with $B-V \simeq 0$. But for the main-sequence stars in M5 Arp determined the blanketing corrections $\Delta(B-V) = 0.16$, $\Delta V = -0.08$ in the region $0.44 < B-V < 0.64$, and it seems likely that the corrections will decrease with higher temperature. Because the reddening correction for M5 is 0.08, it is reasonable for M5 to use: $(B-V)_c = B-V$ in the blue part of the horizontal branch and $(B-V)_c = B-V+0.08$ in the yellow part. Furthermore, $\Delta V = -0.08$ will be adopted. By means of these relations we can determine the absolute magnitudes in $B_c$ and $V_c$ for the observed stars in M 5, when $M_V$ for the RR Lyrae stars are known; and for this quantity we shall adopt the value 0.2 obtained by taking reddening corrections into account, according to the cosec reddening law (ARP 1962c).

For M 15 the situation is somewhat more complicated, partly because the observations (ARP 1955) were given in the $(CI, m_{pv})$ system, and partly because even less is known about reddening and blanketing effects here than for M 5. Comparing ARP's tables for M 5 in the $(CI, m_{pv})$ system and the same transformed to the $(B-V, V)$ system (ARP 1962b) it is seen that the following relations are valid within a few hundredths of a magnitude. For the blue part of the horizontal branch: $B-V = CI$, $V = m_{pv}+0.20-0.5\,CI$ and for the yellow part: $B-V = CI+0.10$, $V = m_{pv}+0.12$. As both reddening corrections and blanketing corrections in $B-V$ for M 15 are a few hundredths of a magnitude larger than for M 5 their combined effect will be nearly the same for the two clusters, and we can use for M 15: $(B-V)_c = CI$ in the blue part of the horizontal branch and $(B-V)_c = CI+0.18$ in the yellow part. With the above-mentioned reddening correction for M 15 $A_V = 0.36$ and an estimated blanketing correction $\Delta V = -0.11$ we get: $V_c = m_{pv}-0.27+0.5\,CI$ in the blue part, and $V_c = m_{pv}-0.35$ in the yellow part of the horizontal branch.

This procedure clearly is not entirely satisfactory. A reliable comparison of colour-magnitude diagrams with evolutionary sequences probably cannot be made before the $B$ and $V$ magnitudes can be directly determined from sufficiently accurate model atmospheres computed for the specific models of the evolutionary sequences.

### Homogeneous Models

We shall first consider completely homogeneous models, Type 4, and investigate the evolution of three stars with the following masses and chemical compositions:

| Model-sequence | $M/M_\odot$ | $X$ | $Y$ | $Z$ |
|---|---|---|---|---|
| h1 | 1.0 | 0.200 | 0.799 | 0.001 |
| h2 | 1.2 | 0.300 | 0.699 | 0.001 |
| h3 | 1.3 | 0.300 | 0.699 | 0.001 |

TABLE 2. Evolutionary tracks of homogeneous models.

| Series | Age ($10^6$ years) | $M_{bol}$ | $\log T_e$ | $(B-V)_c$ | $M_{Vc}$ |
|---|---|---|---|---|---|
| h1 | 0 | 0.52 | 4.230 | −0.17 | 2.00 |
| | 20 | 0.44 | 4.230 | −0.17 | 1.92 |
| | 35 | 0.37 | 4.229 | −0.17 | 1.84 |
| | 50 | 0.30 | 4.229 | −0.17 | 1.77 |
| | 60 | 0.24 | 4.230 | −0.17 | 1.72 |
| | 67 | 0.19 | 4.233 | −0.17 | 1.69 |
| | 71 | 0.15 | 4.238 | −0.18 | 1.68 |
| | 73 | 0.13 | 4.243 | −0.18 | 1.69 |
| | 75 | 0.10 | 4.251 | −0.19 | 1.72 |
| | 76 | 0.08 | 4.260 | −0.19 | 1.76 |
| | 76.4 | 0.05 | 4.267 | −0.20 | 1.78 |
| | 76.6 | 0.03 | 4.273 | −0.20 | 1.80 |
| | 76.7 | 0.00 | 4.278 | −0.20 | 1.80 |
| | 76.8 | −0.10 | 4.282 | −0.20 | 1.73 |
| | 76.85 | −0.17 | 4.283 | −0.21 | 1.66 |
| | 76.92 | −0.24 | 4.279 | −0.20 | 1.57 |
| | 77.5 | −0.25 | 4.281 | −0.20 | 1.57 |
| | 78.0 | −0.27 | 4.281 | −0.20 | 1.55 |
| | 78.5 | −0.28 | 4.281 | −0.20 | 1.54 |
| | 80.0 | −0.32 | 4.281 | −0.20 | 1.50 |
| | 82.0 | −0.37 | 4.282 | −0.20 | 1.46 |
| | 84 | −0.42 | 4.283 | −0.21 | 1.41 |
| | 88 | −0.55 | 4.286 | −0.21 | 1.30 |
| | 92 | −0.74 | 4.284 | −0.21 | 1.10 |
| | 94 | −0.96 | 4.269 | −0.20 | 0.78 |
| h2 | 0 | 0.51 | 4.210 | −0.16 | 1.86 |
| | 20 | 0.45 | 4.209 | −0.16 | 1.79 |
| | 40 | 0.38 | 4.207 | −0.15 | 1.71 |
| | 60 | 0.31 | 4.205 | −0.15 | 1.63 |
| | 75 | 0.26 | 4.204 | −0.15 | 1.57 |
| | 90 | 0.20 | 4.201 | −0.15 | 1.49 |
| | 100 | 0.15 | 4,201 | −0.15 | 1.44 |
| h3 | 0 | 0.19 | 4.232 | −0.17 | 1.68 |
| | 20 | 0.11 | 4.231 | −0.17 | 1.60 |
| | 40 | 0.04 | 4.228 | −0.17 | 1.51 |
| | 60 | −0.05 | 4.226 | −0.17 | 1.40 |
| | 80 | −0.16 | 4.222 | −0.16 | 1.27 |
| | 85 | −0.20 | 4.222 | −0.16 | 1.23 |
| | 90 | −0.22 | 4.222 | −0.16 | 1.21 |
| | 95 | −0.26 | 4.225 | −0.17 | 1.19 |
| | 98 | −0.30 | 4.230 | −0.17 | 1.18 |
| h4 | 0 | 1.41 | 4.033 | 0.00 | 1.81 |
| | 20 | 1.38 | 4.031 | 0.00 | 1.77 |
| | 40 | 1.35 | 4.028 | 0.00 | 1.73 |
| | 60 | 1.31 | 4.026 | 0.01 | 1.68 |
| | 80 | 1.27 | 4.024 | 0.01 | 1.64 |
| | 100 | 1.22 | 4.024 | 0.01 | 1.59 |
| | 108 | 1.19 | 4.026 | 0.01 | 1.56 |
| | 114 | 1.15 | 4.031 | 0.00 | 1.54 |
| | 118 | 1.11 | 4.037 | −0.01 | 1.52 |
| | 121 | 1.06 | 4.046 | −0.02 | 1.51 |
| | 123 | 0.99 | 4.058 | −0.03 | 1.49 |
| h5 | 0 | 1.06 | 4,037 | −0.01 | 1.47 |
| | 20 | 1.04 | 4.034 | 0.00 | 1.44 |
| | 80 | 0.96 | 4.025 | 0.01 | 1.33 |
| | 110 | 0.91 | 4.019 | 0.02 | 1.26 |
| | 130 | 0.87 | 4.015 | 0.02 | 1.20 |
| | 150 | 0.81 | 4.016 | 0.02 | 1.15 |
| | 160 | 0.76 | 4.022 | 0.01 | 1.12 |
| | 165 | 0.69 | 4.034 | 0.00 | 1.09 |
| | 167 | 0.63 | 4.047 | −0.02 | 1.08 |
| | 168 | 0.41 | 4.057 | −0.03 | 0.90 |
| | 169 | 0.40 | 4.057 | −0.03 | 0.89 |
| | 171 | 0.39 | 4.057 | −0.03 | 0.88 |
| | 175 | 0.36 | 4.056 | −0.03 | 0.85 |
| | 180 | 0.32 | 4.055 | −0.02 | 0.81 |
| | 185 | 0.28 | 4.054 | −0.02 | 0.76 |
| | 193 | 0.21 | 4.052 | −0.02 | 0.68 |
| | 200 | 0.14 | 4.047 | −0.02 | 0.59 |
| | 207 | 0.04 | 4.039 | −0.01 | 0.46 |

TABLE 3. Characteristics of sequence h1.

| Age ($10^6$ years) | $X_c$ | $q_c$ | $f_c$ | $\log R/R_\odot$ | $\log L/L_\odot$ | $\log T_c$ | $\log \varrho_c$ |
|---|---|---|---|---|---|---|---|
| 0 | 0.200 | 0.214 | 0.977 | −0.118 | 1.637 | 7.454 | 2.106 |
| 20 | 0.160 | 0.204 | 0.980 | −0.101 | 1.668 | 7.464 | 2.119 |
| 35 | 0.125 | 0.193 | 0.980 | −0.086 | 1.696 | 7.473 | 2.131 |
| 60 | 0.058 | 0.173 | 0.977 | −0.062 | 1.747 | 7.497 | 2.182 |
| 67 | 0.035 | 0.167 | 0.972 | −0.057 | 1.769 | 7.511 | 2.215 |
| 71 | 0.021 | 0.161 | 0.962 | −0.059 | 1.784 | 7.525 | 2.249 |
| 73 | 0.014 | 0.159 | 0.949 | −0.064 | 1.793 | 7.535 | 2.279 |
| 75 | 0.007 | 0.157 | 0.911 | −0.077 | 1.804 | 7.552 | 2.331 |
| 76 | 0.0035 | 0.150 | 0.851 | −0.090 | 1.812 | 7.567 | 2.375 |
| 76.4 | 0.0020 | 0.128 | 0.748 | −0.099 | 1.822 | 7.578 | 2.415 |
| 76.6 | 0.0013 | 0.108 | 0.622 | −0.106 | 1.833 | 7.586 | 2.456 |
| 76.7 | 0.0009 | 0.088 | 0.493 | −0.109 | 1.845 | 7.590 | 2.493 |
| 76.8 | 0.0005 | 0.043 | 0.187 | −0.099 | 1.884 | 7.583 | 2.585 |
| 76.85 | 0.0003 | 0.016 | 0.047 | −0.084 | 1.913 | 7.566 | 2.679 |
| 76.92 | 0.0001 | 0 | 0 | −0.063 | 1.941 | 7.508 | 2.830 |
| 77.5 | | 0.016 | | −0.065 | 1.943 | 7.478 | 2.889 |
| 78.0 | | 0.030 | | −0.062 | 1.950 | 7.475 | 2.908 |
| 78.5 | | 0.050 | | −0.059 | 1.955 | 7.474 | 2.922 |
| 80.0 | | 0.080 | | −0.052 | 1.971 | 7.471 | 2.958 |
| 82 | | 0.100 | | −0.044 | 1.991 | 7.473 | 2.994 |
| 84 | | 0.120 | | −0.037 | 2.011 | 7.476 | 3.033 |
| 88 | | 0.143 | | −0.015 | 2.063 | 7.483 | 3.136 |
| 92 | | 0.193 | | 0.026 | 2.140 | 7.492 | 3.344 |
| 94 | | 0.224 | | 0.099 | 2.226 | 7.507 | 3.666 |

in order to get an idea of the question how changes in mass and hydrogen content influence the evolutionary tracks. As expected from the analysis of Section 3, the three models start their evolution near the main-sequence band and in the bluest region of the observed horizontal branches. They all have a convective core containing 21 % of the total mass, and at the first evolutionary stage with hydrogen burning in the slowly receding core, the evolution is proceeding in nearly the same way as the well-known evolution of heavy population I stars. The effective temperature $T_e$, however, decreases very little.

The evolution of the less massive star was studied also at the following stage, where the luminosity is supplied by hydrogen burning in a shell, until a helium core containing 22 % of the mass had developed, while the computations for the heavier models only covered the "main-sequence" phase. The results are given in Tables 2 and 3, and shown in a $(\log T_e, M_{\text{bol}})$

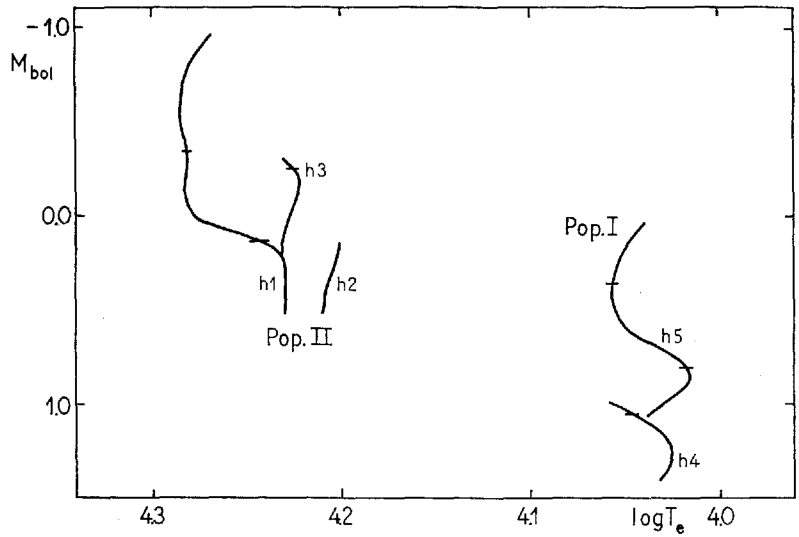


Figure 2. – Evolutionary tracks of homogeneous models.

diagram in Figure 2, where similar tracks for population I stars, which will be discussed below, are also drawn.

In order to facilitate the comparison with observations we shall distinguish between three parts of the evolutionary sequence for the sequence h1, Table 2. In Part 1, lasting $73 \times 10^6$ years, the luminosity increases from $M_{\text{bol}} = 0.52$ to $M_{\text{bol}} = 0.13$, while $T_e$ is nearly constantly $T_e \simeq 17000^0$. Part 2 is a very fast evolutionary phase, corresponding to the development of a small helium core round the centre. In $7 \times 10^6$ years the luminosity rises by 0.5, while $T_e$ increases to $19000^0$. The evolution in Part 3, lasting $14 \times 10^6$ years, is also fast, though not so fast as in Part 2, and the energy is now produced in a hydrogen-burning shell round the inert helium core. While the mass fraction of the helium core $q_c$ grows from 0.08 to 0.22, the luminosity increases by 0.6 , $T_e$ being first nearly constant. Only in the last models of the computed sequence, for $q_c > 0.20$ the evolutionary track in the colour-magnitude diagram turns towards lower temperatures. In Figure 2 the three parts of the sequence are indicated.

Using the tables of Harris we find that this evolutionary track starts at $((B-V)_c, M_{Vc}) = (-0.17, 2.0)$. At the first stage it moves about 0.4 upwards while the hydrogen is burned in the convective core. During the following fast evolution the luminosity increases from $M_{Vc} = 1.7$ to $M_{Vc} = 0.8$ with nearly constant $(B-V)_c \simeq -0.20$.

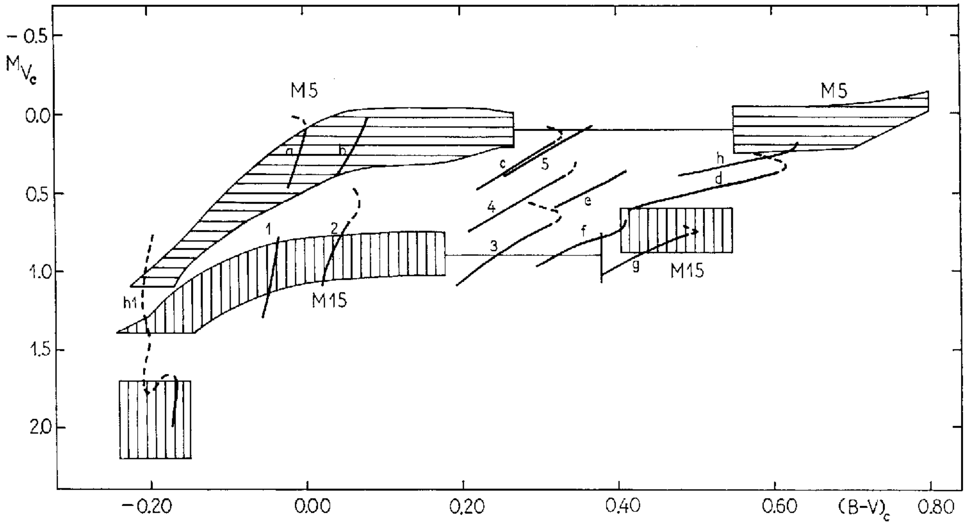


Figure 3. – Comparison of the evolutionary tracks with ARP's observations. The vertically, respectively horizontally, hatched regions are richly populated in M15 and M5. The dashed part of the tracks correspond to fast stages in the evolution.

The evolution following the stages given here must be fast, and probably the evolutionary track will move upwards and to the right in the colour-magnitude diagram. Even if $M_{bol}$ did not change from $-1.0$ (model age $94 \times 10^6$ years) $M_{Vc}$ would rise to $-0.1$ for $(B-V)_c = -0.1$ and to $-0.6$ for $(B-V)_c = 0.0$, simply because $B.C.$ in this region of the diagram changes strongly with $(B-V)_c$. The star will then move to the right in the diagram, 1–2 magnitudes above the horizontal branch.

Part 1 of the evolutionary track till age $73 \times 10^6$ years corresponds nicely to the groups of stars observed in M2 and M15 with $CI \simeq -0.20$ and $m_{pv} \simeq 16.7$ These groups are evidently separated from the rest of the horizontal branch by a gap; they and the rest of the horizontal branch do not form a continuous sequence. From ARP's diagram it is seen that the group can be described by means of a slow evolution, during which the luminosity increases by half a magnitude (corresponding to a rise from $m_{pv} = 16.9$ to $16.4$ at $CI = -0.20$). If we use SANDAGE's value (Table 1) for $M_V$ for the RR Lyrae stars in M15: $M_V = 1.0$ or $M_{Vc} = 0.9$, it is seen that most of the stars in the group belonging to M15 cluster round a vertical line with $(B-V)_c = -0.20$ from $M_{Vc} = 2.2$ to 1.7. The theoretical track corresponds to a line with $(B-V)_c = -0.17$ from $M_{Vc} = 2.0$ to 1.6. In Figure 3 the $((B-V)_c, V_c)$ diagram is sketched for M15 and M5 by means of ARP's colour-magnitude diagrams and the above-mentioned transformation formulae, and computed evolutionary tracks are drawn.

It may be fortuitous that the values of the quantities derived from observations and from the theoretical computations for the separate group of stars in M15 are almost identical, because both the theoretically and the observationally determined data are rather uncertain. In particular, knowledge of the accurate mass value and chemical composition for the stars of the group is lacking, and the absolute magnitudes are based on the little known magnitude of the RR Lyrae stars. But it is satisfactory that we can understand the group separated from the rest of the horizontal branch, the gap being caused by the second very fast phase of evolution following the hydrogen exhaustion at the centre of the stars.

M10 and M92 are in many respects similar to M2 and M15. Therefore we should expect also these clusters to have a separate group of completely mixed stars. As Arp's diagrams show, this is not the case, although many stars are found in the region where we should look for these groups. This may be understood as a result of larger variations in mass and chemical composition of the stars in M10 and M92, a necessary condition for the appearence of a well-separated group clearly being that all stars in the group must have very nearly the same mass and chemical composition (immediately after the helium flash). Perhaps also a smaller mass for the group stars than for the rest of the horizontal branch stars is necessary to explain the separation.

As mentioned above, the star of 1 $M_\odot$ in the late evolutionary phase with its energy produced in a thin shell surrounding an inert helium core, will move quickly to the right in the colour-magnitude diagram 1–2 magnitudes above the horizontal branch. Therefore, we should expect a few stars in this region of the diagrams of M2, M10, M15, and M92, and we find, in fact, a few of them in each cluster.

The globular clusters with relatively high metal content have few or no blue stars in the horizontal branch; hence they include no completely homogeneous stars. In order to investigate this matter more quantitatively, two evolutionary sequences for extreme population I stars with the following parameters were computed:

| Model-sequence | $M/M_\odot$ | $X$ | $Y$ | $Z$ |
|---|---|---|---|---|
| h4 | 1.0 | 0.200 | 0.756 | 0.044 |
| h5 | 1.3 | 0.301 | 0.655 | 0.044 |

The evolutionary tracks, displayed in Figure 2, show general properties very similar to those mentioned above for the extreme population II models: After a slow phase with hydrogen burning in the central regions follows a fast phase with an energy-producing shell round a growing helium core. The model of 1.3 $M_\odot$ was followed until the core contained about 16 % of the total mass. In the slow phase (lasting $110 \times 10^6$ years for the less massive and $155 \times 10^6$ years for the heavier star) we found:

$$4.037 > \log T_e > 4.015; \quad 0.76 < M_{\mathrm{bol}} < 1.41$$

or

$$0.00 < (B-V)_c < 0.02 \quad \text{and} \quad 1.1 < M_{Vc} < 1.8 .$$

As the globular cluster NGC 6356 (Sandage and Wallerstein 1960) has no stars bluer than $(B-V)_c \simeq 0.25$, and 47 Tuc (Tifft 1963) apart from two RR Lyrae stars and one extremely blue star contains none bluer than $(B-V)_c \simeq 0.60$, we can conclude that these clusters do not contain homogeneous horizontal branch stars. Even extreme population I stars of this kind would have $(B-V)_c \simeq 0.00$.

On the other hand the rich galactic cluster M 67 has a horizontal branch extending into the blue region (Eggen and Sandage 1964). At its blue end three stars are found with $(B-V)_0 \simeq 0.05$ and $10.5 < V_0 < 11.1$. As blanketing effects for M 67 are very small and its distance modulus is determined very accurately: $(m-M)_0 = 9.38$, we can safely conclude that the three blue stars are placed at $(B-V)_c \simeq 0.05$ and have $1.1 < M_{Vc} < 1.8$. This is exactly the region corresponding to the slow phases of the computed evolutionary sequences for the population I stars of 1.0 and 1.3 solar masses. Of course, this is no crucial test of our assumptions; it shows, however, that the blue part of the horizontal branch in M 67 can be explained by completely mixed models of reasonable masses and chemical compositions. It is emphasized that we do not understand the reason why the horizontal branch of M 67 differs so noticeably from the horizontal branches of globular clusters rich in metal.

**Partially Mixed Models**

The homogeneous models just considered will not evolve along the horizontal branch from left to right. Hence, if our interpretation of horizontal branches as loci for hydrogen-burning sequences is correct, most horizontal branch stars must correspond to inhomogeneous models of Type 3 (cf. Section 3), and thus contain more hydrogen in the outer layers than in the central regions.

TABLE 4. Evolutionary tracks of partially mixed sequences.

| Series | Age ($10^6$ years) | $M_{bol}$ | $\log T_e$ | $(B-V)_c$ | $M_{Vc}$ |
|---|---|---|---|---|---|
| 1 | 0 | 0.67 | 4.087 | −0.06 | 1.30 |
| | 20 | 0.62 | 4.085 | −0.05 | 1.24 |
| | 40 | 0.56 | 4.082 | −0.05 | 1.16 |
| | 60 | 0.51 | 4.078 | −0.05 | 1.09 |
| | 80 | 0.43 | 4.074 | −0.04 | 1.00 |
| | 100 | 0.37 | 4.070 | −0.04 | 0.92 |
| | 110 | 0.33 | 4.067 | −0.04 | 0.87 |
| | 115 | 0.30 | 4.067 | −0.04 | 0.84 |
| | 120 | 0.28 | 4.066 | −0.04 | 0.81 |
| | 125 | 0.26 | 4.066 | −0.04 | 0.79 |
| 2 | 0 | 0.74 | 4.015 | 0.02 | 1.08 |
| | 20 | 0.69 | 4.011 | 0.03 | 1.01 |
| | 40 | 0.63 | 4.007 | 0.03 | 0.94 |
| | 60 | 0.58 | 4.004 | 0.04 | 0.88 |
| | 80 | 0.52 | 3.997 | 0.05 | 0.80 |
| | 95 | 0.47 | 3.992 | 0.05 | 0.73 |
| | 110 | 0.41 | 3.987 | 0.06 | 0.66 |
| | 120 | 0.38 | 3.984 | 0.07 | 0.62 |
| | 130 | 0.32 | 3.981 | 0.07 | 0.55 |
| | 136 | 0.29 | 3.983 | 0.07 | 0.53 |
| 3 | 0 | 0.98 | 3.916 | 0.19 | 1.10 |
| | 30 | 0.92 | 3.908 | 0.21 | 1.03 |
| | 60 | 0.85 | 3.899 | 0.23 | 0.95 |
| | 90 | 0.78 | 3.889 | 0.26 | 0.87 |
| | 105 | 0.73 | 3.881 | 0.28 | 0.82 |
| | 115 | 0.71 | 3.877 | 0.29 | 0.79 |
| | 125 | 0.68 | 3.873 | 0.29 | 0.76 |
| | 141.33 | 0.62 | 3.864 | 0.32 | 0.69 |
| | 149 | 0.60 | 3.862 | 0.32 | 0.67 |
| | 155 | 0.58 | 3.860 | 0.33 | 0.65 |
| | 159 | 0.56 | 3.861 | 0.33 | 0.63 |
| | 162 | 0.54 | 3.862 | 0.32 | 0.61 |
| | 165 | 0.52 | 3.865 | 0.32 | 0.59 |
| | 167 | 0.50 | 3.869 | 0.31 | 0.57 |
| | 168.5 | 0.48 | 3.875 | 0.29 | 0.56 |
| 4 | 0 | 0.64 | 3.907 | 0.21 | 0.75 |
| | 20 | 0.58 | 3.898 | 0.23 | 0.68 |
| | 40 | 0.52 | 3.888 | 0.26 | 0.61 |
| | 60 | 0.45 | 3.876 | 0.29 | 0.53 |
| | 70 | 0.41 | 3.870 | 0.30 | 0.49 |
| | 80 | 0.38 | 3.864 | 0.32 | 0.45 |
| | 90 | 0.33 | 3.858 | 0.33 | 0.40 |
| | 97 | 0.29 | 3.854 | 0.34 | 0.35 |
| 5 | 0 | 0.29 | 3.890 | 0.25 | 0.39 |
| | 10 | 0.24 | 3.883 | 0.27 | 0.33 |
| | 20 | 0.20 | 3.874 | 0.29 | 0.28 |
| | 24.84 | 0.17 | 3.869 | 0.31 | 0.24 |
| | 28.5 | 0.15 | 3.867 | 0.31 | 0.22 |
| | 35 | 0.13 | 3.863 | 0.32 | 0.20 |
| | 42 | 0.10 | 3.856 | 0.33 | 0.17 |
| | 49 | 0.06 | 3.850 | 0.36 | 0.12 |
| | 54 | 0.02 | 3.844 | 0.37 | 0.08 |
| *a* | 0 | 0.00 | 4.053 | −0.02 | 0.48 |
| | 10 | −0.06 | 4.050 | −0.02 | 0.40 |
| | 20 | −0.12 | 4.045 | −0.01 | 0.32 |
| | 30 | −0.18 | 4.040 | −0.01 | 0.24 |
| | 38 | −0.24 | 4.036 | 0.00 | 0.17 |
| | 43 | −0.28 | 4.034 | 0.00 | 0.12 |
| | 47 | −0.32 | 4.033 | 0.00 | 0.08 |
| | 51 | −0.35 | 4.034 | 0.00 | 0.05 |
| | 53 | −0.37 | 4.037 | 0.00 | 0.04 |
| | 54.2 | −0.39 | 4.040 | −0.01 | 0.03 |
| | 55 | −0.41 | 4.043 | −0.01 | 0.03 |
| *b* | 0 | 0.11 | 4.001 | 0.04 | 0.40 |
| | 10 | 0.06 | 3.996 | 0.05 | 0.33 |
| | 20 | 0.01 | 3.991 | 0.06 | 0.27 |
| | 30 | −0.05 | 3.985 | 0.06 | 0.19 |
| | 38 | −0.10 | 3.980 | 0.07 | 0.13 |
| | 45 | −0.15 | 3.976 | 0.08 | 0.07 |
| | 50 | −0.19 | 3.973 | 0.08 | 0.03 |
| *c* | 0 | 0.38 | 3.904 | 0.22 | 0.49 |
| | 10 | 0.33 | 3.898 | 0.23 | 0.43 |
| | 20 | 0.29 | 3.891 | 0.25 | 0.39 |
| | 30 | 0.25 | 3.883 | 0.27 | 0.34 |
| | 38 | 0.21 | 3.877 | 0.28 | 0.29 |

TABLE 4 (continued).

| Series | Age ($10^6$ years) | $M_{bol}$ | $\log T_e$ | $(B-V)_c$ | $M_{Vc}$ |
|---|---|---|---|---|---|
| | 45 | 0.18 | 3.872 | 0.30 | 0.26 |
| | 51 | 0.15 | 3.867 | 0.31 | 0.22 |
| | 55 | 0.13 | 3.865 | 0.32 | 0.20 |
| | 59 | 0.10 | 3.863 | 0.32 | 0.17 |
| | 62 | 0.08 | 3.861 | 0.33 | 0.15 |
| | 65 | 0.05 | 3.861 | 0.33 | 0.12 |
| | 67 | 0.04 | 3.862 | 0.32 | 0.11 |
| | 69 | 0.01 | 3.866 | 0.31 | 0.08 |
| *d* | 0 | 0.57 | 3.828 | 0.41 | 0.62 |
| | 10 | 0.54 | 3.820 | 0.44 | 0.59 |
| | 25 | 0.49 | 3.809 | 0.47 | 0.53 |
| | 35 | 0.45 | 3.800 | 0.49 | 0.50 |
| | 45 | 0.41 | 3.790 | 0.53 | 0.46 |
| | 52 | 0.38 | 3.785 | 0.55 | 0.44 |
| | 59 | 0.35 | 3.778 | 0.58 | 0.41 |
| | 64 | 0.33 | 3.774 | 0.59 | 0.39 |
| | 68 | 0.30 | 3.770 | 0.60 | 0.37 |
| | 72 | 0.28 | 3.766 | 0.62 | 0.35 |
| | 75 | 0.26 | 3.765 | 0.62 | 0.33 |
| | 77 | 0.25 | 3.766 | 0.62 | 0.32 |
| | 79 | 0.22 | 3.774 | 0.59 | 0.28 |
| | 80 | 0.20 | 3.777 | 0.58 | 0.26 |
| *e* | 0 | 0.52 | 3.863 | 0.32 | 0.59 |
| | 10 | 0.48 | 3.856 | 0.34 | 0.55 |
| | 20 | 0.44 | 3.848 | 0.36 | 0.50 |
| | 30 | 0.40 | 3.841 | 0.38 | 0.46 |
| | 37 | 0.37 | 3.836 | 0.39 | 0.42 |
| | 42 | 0.34 | 3.833 | 0.40 | 0.39 |
| | 46 | 0.32 | 3.832 | 0.40 | 0.37 |
| | 50 | 0.30 | 3.831 | 0.41 | 0.35 |
| *f* | 0 | 0.88 | 3.871 | 0.30 | 0.96 |
| | 20 | 0.83 | 3.862 | 0.32 | 0.90 |

| Series | Age ($10^6$ years) | $M_{bol}$ | $\log T_e$ | $(B-V)_c$ | $M_{Vc}$ |
|---|---|---|---|---|---|
| | 40 | 0.78 | 3.852 | 0.35 | 0.84 |
| | 55 | 0.74 | 3.844 | 0.37 | 0.80 |
| | 65 | 0.71 | 3.838 | 0.39 | 0.76 |
| | 73 | 0.69 | 3.834 | 0.40 | 0.74 |
| | 78 | 0.67 | 3.831 | 0.41 | 0.72 |
| | 83 | 0.65 | 3.828 | 0.41 | 0.70 |
| | 87 | 0.63 | 3.828 | 0.41 | 0.68 |
| | 89 | 0.62 | 3.828 | 0.41 | 0.67 |
| | 91 | 0.61 | 3.828 | 0.41 | 0.66 |
| *g* | 0 | 0.97 | 3.841 | 0.38 | 1.03 |
| | 20 | 0.92 | 3.833 | 0.40 | 0.97 |
| | 35 | 0.88 | 3.826 | 0.42 | 0.93 |
| | 45 | 0.86 | 3.822 | 0.43 | 0.91 |
| | 55 | 0.83 | 3.818 | 0.44 | 0.87 |
| | 65 | 0.80 | 3.816 | 0.45 | 0.84 |
| | 73 | 0.78 | 3.813 | 0.46 | 0.82 |
| | 80 | 0.76 | 3.809 | 0.47 | 0.80 |
| | 85 | 0.74 | 3.806 | 0.47 | 0.78 |
| | 90 | 0.72 | 3.801 | 0.49 | 0.76 |
| | 94 | 0.71 | 3.799 | 0.50 | 0.76 |
| | 97 | 0.69 | 3.800 | 0.49 | 0.73 |
| | 99 | 0.67 | 3.801 | 0.49 | 0.71 |
| *h* | 0 | 0.35 | 3.806 | 0.48 | 0.39 |
| | 10 | 0.31 | 3.797 | 0.51 | 0.36 |
| | 18 | 0.27 | 3.788 | 0.54 | 0.32 |
| | 24 | 0.24 | 3.782 | 0.56 | 0.30 |
| | 29 | 0.22 | 3.779 | 0.57 | 0.28 |
| | 33 | 0.20 | 3.774 | 0.59 | 0.26 |
| | 37 | 0.17 | 3.768 | 0.61 | 0.24 |
| | 40 | 0.16 | 3.765 | 0.62 | 0.23 |
| | 42.5 | 0.14 | 3.763 | 0.63 | 0.22 |
| | 44.5 | 0.13 | 3.763 | 0.63 | 0.21 |
| | 46.3 | 0.11 | 3.762 | 0.63 | 0.19 |

TABLE 5. Characteristics of five partially mixed series.

| Series | Age ($10^6$ years) | $X_c$ | $q_c$ | $f_c$ | $\log R/R_\odot$ | $\log L/L_\odot$ | $\log T_c$ | $\log \varrho_c$ |
|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 0.300 | 0.215 | 0.978 | 0.138 | 1.573 | 7.402 | 1.958 |
| | 20 | 0.271 | 0.205 | 0.978 | 0.153 | 1.596 | 7.407 | 1.962 |
| | 40 | 0.240 | 0.195 | 0.979 | 0.171 | 1.619 | 7.413 | 1.968 |
| | 60 | 0.204 | 0.182 | 0.979 | 0.188 | 1.641 | 7.419 | 1.977 |
| | 80 | 0.165 | 0.171 | 0.980 | 0.212 | 1.670 | 7.428 | 1.988 |
| | 100 | 0.120 | 0.161 | 0.980 | 0.233 | 1.697 | 7.439 | 2.009 |
| | 110 | 0.094 | 0.154 | 0.980 | 0.246 | 1.713 | 7.447 | 2.024 |
| | 115 | 0.080 | 0.150 | 0.979 | 0.252 | 1.722 | 7.451 | 2.035 |
| | 120 | 0.066 | 0.145 | 0.977 | 0.257 | 1.731 | 7.457 | 2.048 |
| | 125 | 0.051 | 0.141 | 0.975 | 0.262 | 1.741 | 7.464 | 2.067 |
| 3 | 0 | 0.300 | 0.193 | 0.972 | 0.417 | 1.450 | 7.394 | 1.998 |
| | 30 | 0.264 | 0.182 | 0.973 | 0.446 | 1.475 | 7.400 | 2.005 |
| | 60 | 0.224 | 0.170 | 0.974 | 0.477 | 1.503 | 7.407 | 2.014 |
| | 90 | 0.178 | 0.157 | 0.974 | 0.512 | 1.532 | 7.416 | 2.027 |
| | 105 | 0.151 | 0.151 | 0.976 | 0.537 | 1.552 | 7.423 | 2.037 |
| | 115 | 0.132 | 0.145 | 0.975 | 0.551 | 1.561 | 7.427 | 2.046 |
| | 125 | 0.112 | 0.142 | 0.976 | 0.565 | 1.572 | 7.432 | 2.058 |
| | 141.33 | 0.077 | 0.135 | 0.975 | 0.593 | 1.595 | 7.444 | 2.083 |
| | 149 | 0.058 | 0.130 | 0.972 | 0.602 | 1.603 | 7.452 | 2.102 |
| | 155 | 0.043 | 0.126 | 0.970 | 0.611 | 1.613 | 7.460 | 2.123 |
| | 159 | 0.033 | 0.122 | 0.966 | 0.613 | 1.619 | 7.467 | 2.143 |
| | 162 | 0.025 | 0.119 | 0.959 | 0.614 | 1.625 | 7.474 | 2.163 |
| | 165 | 0.016 | 0.114 | 0.944 | 0.612 | 1.634 | 7.484 | 2.192 |
| | 167 | 0.0103 | 0.110 | 0.920 | 0.608 | 1.642 | 7.495 | 2.224 |
| | 168.5 | 0.0057 | 0.102 | 0.867 | 0.600 | 1.652 | 7.508 | 2.266 |
| $a$ | 0 | 0.200 | 0.224 | 0.992 | 0.338 | 1.843 | 7.433 | 1.913 |
| | 10 | 0.174 | 0.214 | 0.991 | 0.358 | 1.866 | 7.438 | 1.918 |
| | 20 | 0.144 | 0.210 | 0.991 | 0.379 | 1.890 | 7.445 | 1.927 |
| | 30 | 0.113 | 0.197 | 0.990 | 0.402 | 1.915 | 7.454 | 1.939 |
| | 38 | 0.085 | 0.187 | 0.990 | 0.422 | 1.940 | 7.463 | 1.955 |
| | 43 | 0.066 | 0.180 | 0.989 | 0.434 | 1.955 | 7.471 | 1.971 |
| | 47 | 0.049 | 0.173 | 0.987 | 0.444 | 1.970 | 7.479 | 1.990 |
| | 51 | 0.031 | 0.166 | 0.981 | 0.448 | 1.984 | 7.492 | 2.021 |
| | 53 | 0.021 | 0.161 | 0.975 | 0.448 | 1.993 | 7.501 | 2.047 |
| | 54.2 | 0.015 | 0.158 | 0.966 | 0.445 | 2.000 | 7.510 | 2.070 |
| | 55 | 0.0107 | 0.156 | 0.954 | 0.441 | 2.006 | 7.518 | 2.092 |
| $c$ | 0 | 0.200 | 0.198 | 0.989 | 0.563 | 1.693 | 7.422 | 1.963 |
| | 10 | 0.179 | 0.191 | 0.989 | 0.584 | 1.710 | 7.426 | 1.967 |
| | 20 | 0.157 | 0.183 | 0.989 | 0.607 | 1.727 | 7.431 | 1.974 |

TABLE 5 (continued).

| Series | Age ($10^6$ years) | $X_c$ | $q_c$ | $f_c$ | $\log R/R_\odot$ | $\log L/L_\odot$ | $\log T_c$ | $\log \varrho_c$ |
|---|---|---|---|---|---|---|---|---|
| | 30 | 0.133 | 0.177 | 0.989 | 0.630 | 1.745 | 7.437 | 1.983 |
| | 38 | 0.112 | 0.174 | 0.988 | 0.650 | 1.759 | 7.442 | 1.993 |
| | 45 | 0.092 | 0.167 | 0.988 | 0.666 | 1.772 | 7.448 | 2.005 |
| | 51 | 0.075 | 0.160 | 0.987 | 0.682 | 1.785 | 7.454 | 2.018 |
| | 55 | 0.062 | 0.156 | 0.986 | 0.690 | 1.792 | 7.460 | 2.030 |
| | 59 | 0.049 | 0.153 | 0.984 | 0.700 | 1.803 | 7.466 | 2.046 |
| | 62 | 0.038 | 0.149 | 0.981 | 0.708 | 1.812 | 7.472 | 2.061 |
| | 65 | 0.028 | 0.145 | 0.977 | 0.714 | 1.822 | 7.481 | 2.083 |
| | 67 | 0.020 | 0.141 | 0.970 | 0.715 | 1.829 | 7.489 | 2.104 |
| | 69 | 0.012 | 0.135 | 0.951 | 0.711 | 1.838 | 7.500 | 2.139 |
| $g$ | 0 | 0.200 | 0.162 | 0.982 | 0.570 | 1.457 | 7.406 | 2.041 |
| | 20 | 0.171 | 0.154 | 0.982 | 0.598 | 1.477 | 7.412 | 2.050 |
| | 35 | 0.146 | 0.147 | 0.982 | 0.617 | 1.490 | 7.417 | 2.060 |
| | 45 | 0.129 | 0.143 | 0.982 | 0.631 | 1.501 | 7.421 | 2.068 |
| | 55 | 0.111 | 0.138 | 0.982 | 0.644 | 1.511 | 7.426 | 2.078 |
| | 65 | 0.092 | 0.135 | 0.981 | 0.654 | 1.523 | 7.431 | 2.090 |
| | 73 | 0.075 | 0.133 | 0.981 | 0.665 | 1.532 | 7.437 | 2.103 |
| | 80 | 0.061 | 0.129 | 0.979 | 0.676 | 1.541 | 7.443 | 2.118 |
| | 85 | 0.049 | 0.125 | 0.978 | 0.686 | 1.548 | 7.448 | 2.131 |
| | 90 | 0.038 | 0.122 | 0.974 | 0.699 | 1.553 | 7.454 | 2.150 |
| | 94 | 0.028 | 0.118 | 0.971 | 0.706 | 1.561 | 7.462 | 2.169 |
| | 97 | 0.021 | 0.115 | 0.964 | 0.708 | 1.568 | 7.469 | 2.190 |
| | 99 | 0.016 | 0.112 | 0.950 | 0.709 | 1.574 | 7.476 | 2.211 |

In order to investigate the evolution of such inhomogeneous models we shall construct evolutionary sequences for the mass $M = 1.2\ M_\odot$ and the content of heavy elements $Z = 0.005$. The outer zone, which contains at least 1 % of the total mass, has the following chemical composition:

$$X = 0.900, \quad Y = 0.095, \quad Z = 0.005.$$

The hydrogen content $X$ as a function of relative mass $q$ for the first model of the sequences computed is shown in Figures 4 and 5, and the evolutionary tracks are drawn in the $((B-V)_c, V_c)$ diagram in Figure 3. Additional relevant information about the properties of the sequences is given in Tables 4 and 5.

The series 1–5 consist of two homogeneous zones. 1–3 have 1 %, 5 %,

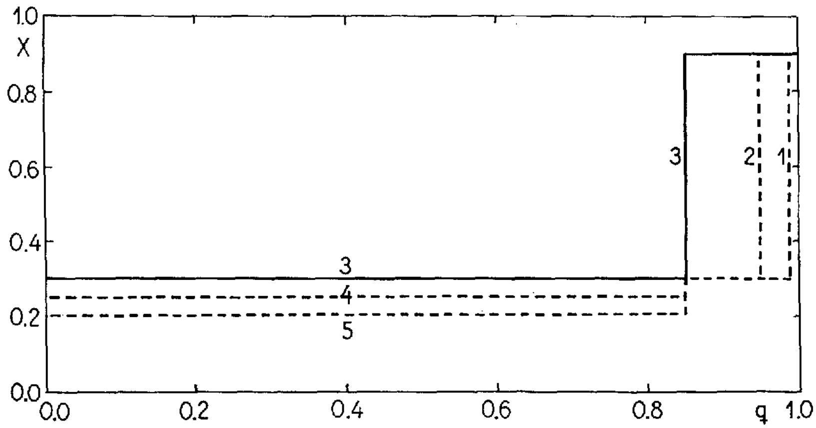


Figure 4. – $X(q)$ for the first model of the series 1–5.

and 15 %, respectively, of the mass in the outer zone, and an inner zone with the composition:

$$X = 0.300, \quad Y = 0.695, \quad Z = 0.005.$$

As was to be expected, the radius of the models was found to increase and the luminosity to decrease in the order 1, 2, 3. The series 3, 4, 5 all have an outer zone containing 15 % of the mass, and the hydrogen content $X$ of the inner zone is in the order 0.30, 0.25, and 0.20. As was to be expected, the luminosity was found to increase in the order 3, 4, 5.

It does not seem quite realistic to assume a sudden jump in $X(q)$ at a certain relative mass. Therefore, we shall consider the model sequences a, b, c, d, e, f, g, and h (cf. Figure 5), with a smoother transition of $X$ between the two zones. Their evolutionary tracks, however, have the same general characteristics as the tracks of the models consisting of two homogeneous zones. For instance, it is found that the sequences 5 and c are very similar. It is also seen that the only difference between the models is that in sequence c the hydrogen is somewhat less concentrated to the outer layers than in sequence 5.

The tracks h1, 1, 2, 3, f, and g pass through the region occupied by M15 in Figure 3. It should be emphasized that this circumstance by no means proves that the stars in M15 have precisely the properties of the corresponding models, even if the interpretation of the horizontal branch as a hydrogen-burning sequence is realistic. First, due to the uncertainty in the absolute magnitude of the RR Lyrae stars, the location of M15 in the

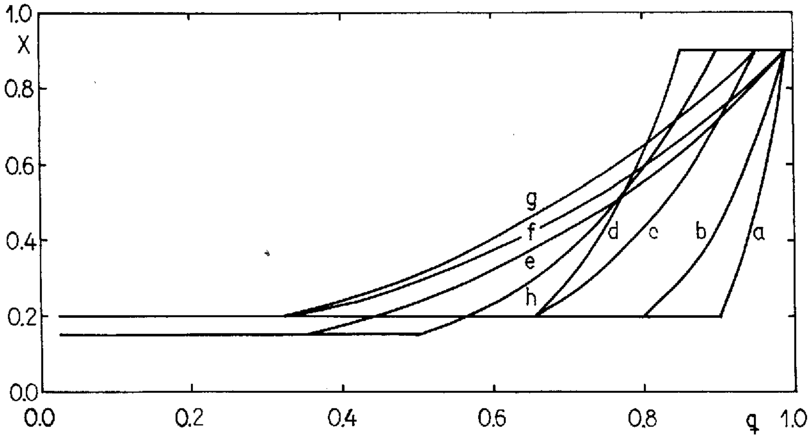


Figure 5. – $X(q)$ for the first model of the series $a$–$h$.

diagram is open to discussion, and secondly, it is possible that models with different masses and different distributions of the hydrogen can fall in the same region of the colour-magnitude diagram.

## 5. Discussion

Figure 3 shows that the computed model sequences fall within a region of the colour-magnitude diagram occupied by the observed horizontal branches of globular clusters. Therefore, our models might be realistic models for horizontal branch stars. But whether such sequences can provide a detailed description of the observed diagrams, or not, is a more difficult question.

From the series 1–3 we should expect a width of at least six tenths of a magnitude, corresponding to an increase in luminosity for such objects of the same amount during the first slow phase of evolution with hydrogen conversion in a convective core. This luminosity increase is clearly a characteristic feature of models containing 30 % of hydrogen in an inner homogeneous zone. The subsequent models show that the growth of luminosity in the slow phase decreases with diminishing hydrogen content. For $X = 0.25$ we find $\Delta M_V \simeq 0.4$ and for $X = 0.20$ $\Delta M_V \simeq 0.3$. To describe horizontal branches as narrow as observed in M 15, therefore, a rather small $X$-value must be assumed. The width $\Delta M_V \simeq 0.3$ for $X = 0.20$ is clearly a minimum width, which will only be present in a cluster if all the horizontal branch stars in the cluster have very nearly the same mass. A broader sequence does not necessarily imply a higher hydrogen content; variations in mass and chemical composition may produce the same effect.

All the partially mixed sequences investigated here have the same mass 1.2 $M_\odot$. If a larger mass is assumed, the sequences will become more luminous and of a somewhat higher temperature. Also the content of heavy elements $Z$ was the same for all the sequences, 0.005. As is well-known, lower $Z$-values give higher temperatures and somewhat higher luminosities. However, there seems to be no reason to investigate in detail how changes in mass and chemical composition will affect this case, as the sequences already computed cover the region where horizontal branch stars are found, and as differences in mixing have a much stronger effect than small differences in mass and chemical composition.

The time scale for the horizontal branch phase according to these evolutionary sequences is about $150 \times 10^6$ years for series 1–3, while, for the more luminous series which contain only 20 % hydrogen in the central region, about $70 \times 10^6$ years were found. This is less than the time scale $230 \times 10^6$ years determined by Sandage for M 3, but considerably more than the time scale for helium-burning sequences.

As described in Section 2, differences in concentration of blue stars, RR Lyrae and yellow stars in the horizontal branches suggest a mass loss by the helium flash or at the RR Lyrae stage, and an evolution from the left to the right along the horizontal branch. Generally, the computed tracks run from left to right, but they also extend upwards in the diagrams, and no evolution from the blue end to the red end of the branch is predicted. It therefore seems that the observed differences in distribution of different groups of stars cannot directly be explained by our theory in the same way as done by Woolf (1964).

In Section 2 we mentioned (Table 1, the right-hand columns) the observed differences in stellar content among globular clusters of varying abundances of heavy elements. We can now describe these differences by means of the model sequences. In the clusters with very low content of metal, $[M/H] = -2.3$, nearly all stars are found to the left of the RR Lyrae gap. According to our interpretation, the models which fall in this region are either homogeneous or partially mixed with less than about 15 % of the mass in the outer zone with the original chemical composition. The typical clusters of Group 2 with $[M/H] \simeq -1.5$ have a considerable fraction of their horizontal branch stars in the yellow part of the branch, i.e. they contain fewer completely mixed stars than Group 1 clusters, but many stars with an outer zone containing 15 % of the mass or more. Finally, in the clusters with strong metal lines, only yellow horizontal branch stars are found, all having a considerable fraction of the mass in the outer zone.

The question why these differences occur, is outside the scope of the present investigations. But provided that the interpretation of horizontal branches as hydrogen-burning sequences is correct, we can understand the observed horizontal branches of globular clusters by the simple assumption that the mixing by the helium flash is most violent for extreme population II stars. The degree of mixing must depend on the content of heavy elements in such a way that stars with negligible content of heavy elements are mostly completely mixed, while intermediate population II stars in most cases only become partially mixed, and the stars of nearly normal population I composition always contain a considerable fraction of the mass in an outer unmixed zone.

Without doubt the mixing will also depend on the mass of the star. But as it seems likely that the mass of horizontal branch stars in all globular clusters are nearly the same, this dependence might be very little conspicuous. However, in this connection it is worth mentioning that the rather young cluster M 13 contains very few yellow horizontal branch stars even if it belongs to Group 2 ($[M/H] = -1.4$). This might be due to slightly larger masses in M 13 than in the other clusters.

Finally, we shall mention a few possible observational tests of the theoretical picture developed.

By an investigation of the pulsations of the models placed in the RR Lyrae gap it is possible to determine the dependence of the pulsation periods on the observed $(B-V)$ values, and a comparison with the detailed observations for M 3 (Roberts and Sandage 1955) can be carried out. By means of determinations of detailed element abundances it is in principle possible to distinguish between the following three possible schemes for the horizontal branch evolution:

(1) Helium burning without mixing
(2) The scheme proposed by Woolf (1964)
(3) The scheme discussed here.

According to (1) the abundances determined for a horizontal branch star should be the same as those determined for other stars in the same cluster. According to (2), as described by Woolf, helium and carbon enrichment will be observed in the yellow part of the horizontal branches, while according to (3), helium and carbon enrichment should be found for stars in the blue part of the horizontal branches, but not in the yellow part.

## Appendix: Computational Technique

The computer programme, written in GIER-ALGOL language, is essentially the same as the one described by REIZ and PETERSEN (1964), in particular the fitting procedure and the handling of the opacity tables are not altered. However, a more complicated technique must be used in dealing with the atmosphere and the outer zone, as some of the models have effective temperatures so low that the outer convective zone cannot be neglected.

The functions $\ln P$, $\ln T$, $q = M(r)/M$, $f = L(r)/L$ are obtained by means of integration of the four basic differential equations from the surface of the star to a pre-assigned fitting-point and from the centre to the fitting-point. The method of integration and the fitting-procedure are exactly as described by REIZ and PETERSEN.

For the rate of energy generation by hydrogen-burning nuclear reactions, the formulae given by REEVES (1965) are used directly, and the energy release $\varepsilon_g$ due to changes of the stellar structure with time is computed according to the formula (cf. SCHWARZSCHILD 1958):

$$\varepsilon_g = -\frac{3}{2}\varrho^{2/3}\frac{\partial}{\partial t}(p_\varrho^{-5/3}).$$

However, it turns out that this term in the expression for the energy-generation rate has very little influence on the evolution of the models considered here, even in the fast evolutionary stage following the hydrogen exhaustion at the centre.

The opacity tables for the mixtures with $Z = 0.005$ and $X = 0.00$, $0.30$, $0.60$, and $0.90$, respectively, were computed at the Institute for Space Studies, New York, and made available to the Copenhagen Observatory by B. STRÖMGREN, while the tables for the mixtures $Z = 0.001$, $X = 0.00$ and $0.90$, $Z = 0.044$, $X = 0.00$ and $0.602$, published (after the present computations were performed) by COX, STEWART, and EILERS (1965) were made available by R. KIPPENHAHN. Both sets of tables have been computed according to the method used by COX (1965). For intermediate $X$-values linear interpolation in the sets of tables is used to compute the opacities. The small uncertainties in the opacity values have very little influence in investigations in which, as in the present case, only moderate accuracy is attempted.

In order to reduce computing time, the integrations from the outside of the star to the fitting-point are not started at the atmospheres. By means of tables for the outer zones, computed by means of a separate programme, it is possible to use one of the three values for the relative mass $q = 0.99$, 0.95, or 0.85 as starting value for the inward integrations. At these $q$-values, the values of the corresponding relative radius $x$, $\ln T$, and $\ln \varrho$ are calculated by interpolation in the tables stored in the computer. By this method the very time-consuming integrations through the outer layers are replaced by fast interpolation in previously computed tables, and thus it is possible to start the inward integrations deep inside the star.

The tables for the outer layers have to be computed in a grid in the $(\log R/R_{\odot}, \log L/L_{\odot})$ plane, sufficiently narrow to allow accurate interpolation. It was found that second-order interpolation in a grid with intervals equal to 0.10 in the parameters $\log R/R_{\odot}$ and $\log L/L_{\odot}$ was entirely satisfactory for the sequences computed here; in fact the variations in the tables were often very nearly linear.

The integrations through the atmosphere and the outer layers, where ionization of hydrogen and helium must be taken into account in detail and non-adiabatic convection zones may occur, are performed by a procedure very similar to the method used by Baker and Kippenhahn (1962) and Baker (1963). In the present case, however, the influence of the radiation on the ionization is very slight and has therefore been disregarded.

If we use $\ln P$ as independent variable in the outer zone, the differential equations for $x = r/R$, $q = M(r)/M$ and $\ln T$ become:

$$\left.\begin{aligned} \frac{dx}{d\ln P} &= -\frac{Rx^2P}{GMq\varrho} \\ \frac{dq}{d\ln P} &= \frac{4\pi R^3}{M}x^2\varrho\frac{dx}{d\ln P} \\ \frac{d\ln T}{d\ln P} &= \nabla \end{aligned}\right\} \quad (1)$$

In non-adiabatic convection zones the gradient $\nabla$ is computed from the mixing-length theory of Böhm-Vitense (1958) by the method described by Baker (1963). The equation for $\nabla$

$$(\nabla - \nabla_{\text{ad}}) - (\nabla_R - \nabla_{\text{ad}}) + \frac{9}{8U}\left(\sqrt{\nabla - \nabla_{\text{ad}} + U^2} - U\right)^3 = 0 \tag{2}$$

with

$$U = \frac{12\sigma T^3}{C_p \varrho^2 \varkappa l}\left(\frac{8\mu}{RTQ}\right)^{1/2} \tag{3}$$

is solved by the following method described by Reiz (1964). Defining $\xi$ by

$$\nabla - \nabla_{\text{ad}} = \xi^2 + 2U\xi, \tag{4}$$

which implies

$$\nabla - \nabla_R = -\frac{9}{8U}\xi^3,$$

we get by means of (2)

$$\xi^3 + \frac{8U}{9}\xi^2 + \frac{16U^2}{9}\xi - (\nabla_R - \nabla_{\text{ad}})\frac{8U}{9} = 0. \tag{5}$$

Equation (5) has one positive root $\xi = \xi_0$, which is determined by a simple Newton-iteration. For

$$f(\xi) = \xi^3 + \frac{8U}{9}\xi^2 + \frac{16U^2}{9}\xi - (\nabla_R - \nabla_{\text{ad}})\frac{8U}{9}$$

we have

$$\frac{df}{d\xi} = 3\xi^2 + \frac{16U}{9}\xi + \frac{16U^2}{9}.$$

A good approximation to $\xi_0$ is always available. Near the boundaries of a convective zone $\xi_0 \simeq 0$, and inside a zone the value of $\xi_0$ from the previous integration step can be used as an approximate solution. Applying this method, it was found that 2–3 iterations were sufficient for securing a relative accuracy of $10^{-4}$ in $\xi_0$.

The integrations through the outer zones are performed by Merson's modified Runge-Kutta method (Merson 1958), which allows a prescribed accuracy in the integrated functions to be achieved by a minimum number of integration steps. The first part of the calculation of the derivatives consists in solving the Saha-equations for the ionizations of hydrogen and helium. This is done by a simple iterative procedure for $E$, the average number of free electrons per atom (cf. Baker and Kippenhahn (1962), Appendix A). However, at higher temperatures ($T > 10^6\ {}^0K$) the usual approximation: to replace the partition functions by their first term, is not permissible; effectively complete ionization occurs. In the computations we have simply assumed complete ionization for $T > 10^6\,{}^0K$, and the errors introduced in the model atmospheres by this procedure are negligible for the models considered here. After the ionization degrees have been determined, the density $\varrho$ can be obtained from the equation of state, and $\nabla_R$ and $\nabla_{\mathrm{ad}}$ can be found. If convection occurs, the root $\xi_0$ of (5) is determined as described, and $\nabla$ is found by means of (4). Then the derivatives can be obtained from the formulae (1).

The computation of one outer zone, including output of $x$, $\log T$ and $\log\varrho$ for the relative mass-values $q$ equal to 0.99, 0.95, and 0.85, requires about 5 minutes on the GIER-computer. It may be mentioned here that all the evolutionary sequences of partially mixed models described in Section 4, were computed by means of one table for outer zones containing results from 63 envelopes. The total computing time for this table was about 5 hours, which, however, is less than 10 per cent. of the time used for the computation of the evolutionary sequences.

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