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EFFECT OF THE DIPOLE-QUADRUPOLE  
INTERACTION ON THE WIDTH AND THE  
STRUCTURE OF THE GIANT DIPOLE LINE  
IN SPHERICAL NUCLEI

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### **Synopsis**

A phenomenological study is made of the effect of a dipole-quadrupole interaction on the width and the structure of the giant dipole line in spherical nuclei. The broadening of the line is evaluated through the method of moments. The effect of the interaction on the structure of the line is also studied by constructing the interaction matrix and diagonalizing it. A classical limit is derived for the case of strong coupling. An attempt to correlate the calculated widths with those observed experimentally is presented.

## 1. Introduction

Since the dipole state does not correspond to an eigenstate of the nuclear Hamiltonian, it is distributed over a large number of levels and the giant dipole line acquires thereby a structure of considerable complexity. This situation can be pictured as arising from a coupling of the dipole vibration to other modes of motion. The difficulty of studying theoretically the structure of the dipole line stems from the fact that, in the present state of nuclear physics, we do not possess a detailed knowledge of these other modes of motion and, a fortiori, of the form of the interaction between them and the dipole vibration. This makes it necessary to study first the interaction which seems likely to be responsible for most of the broadening of the line and then to improve this first picture by considering further types of coupling. In the case of  $O^{16}$ , for instance, one may start with a description of the dipole state in terms of single-particle excitations. The particle-hole interaction splits the five  $I = 1^-$  states in such a way that most of the dipole strength is concentrated in two of the states. One can then study the broadening of each of these sharp lines which results from other types of coupling (as, for instance, the coupling to states in the continuum).

It is the purpose of the present paper to examine a mechanism which may be expected to give rise to much of the broadening of the dipole line in spherical nuclei, namely the interaction between the dipole and quadrupole vibrations. Both theoretically and experimentally it is a well-known fact that an axially symmetric static deformation of the nuclear surface splits the dipole line into two peaks. Similarly, the dynamic deformation resulting from the quadrupole vibrations of a spherical nucleus determines to a large extent the shape and the width of the line by spreading the dipole state over many levels. We will tackle this problem in a purely phenomenological way, limiting our description of the nuclear motion to that provided by the variables representing the collective degrees of freedom of the nucleus. While this approach makes the problem easily manageable, it obviously prevents us from making any estimate of the broadening of the individual levels through the admixture of configurations from the continuum.

## 2. The Hamiltonian

The dipole vibrations may be described in terms of the three collective coordinates  $\alpha_{1\mu}$  specifying the relative distance between the centers of mass of the neutrons and the protons, and the quadrupole vibrations in terms of the five coefficients  $\alpha_{2\mu}$  introduced in the familiar expansion of the nuclear surface<sup>(1)</sup>

$$R(\theta, \varphi) = R_0 \left( 1 + \sum_{\mu} \alpha_{2\mu} Y_{2\mu}^*(\theta, \varphi) \right). \quad (1)$$

In the harmonic approximation, the Hamiltonian corresponding to these two collective modes is given by

$$H_0 = \sum_{\lambda=1}^2 \left[ \frac{1}{2B_{\lambda}} \sum_{\mu} \pi_{\lambda\mu}^* \pi_{\lambda\mu} + \frac{1}{2} C_{\lambda} \sum_{\mu} \alpha_{\lambda\mu}^* \alpha_{\lambda\mu} \right], \quad (2a)$$

where the  $\pi_{\lambda\mu}$  are the momenta canonically conjugate to the  $\alpha_{\lambda\mu}$ . The inertia and restoring force parameters  $B_{\lambda}$  and  $C_{\lambda}$  are quantities determined empirically. The form of the dipole-quadrupole interaction is established in a unique way from invariance considerations, if one limits oneself to the cubic terms containing no time-derivatives:

$$H_I = \frac{K}{\sqrt{5}} \sum_{\nu} (-)^{\mu} \langle 11\nu\mu - \nu | 2\mu \rangle \alpha_{1\nu} \alpha_{1\mu - \nu} \alpha_{2-\mu}. \quad (3)$$

This is precisely the interaction which is responsible for the splitting of the dipole line in deformed nuclei. For the coupling constant  $K$  we use an estimate<sup>(4)</sup> based on the assumption that the dipole frequency is inversely proportional to the nuclear radius:

$$K = -\frac{5}{2} \sqrt{\frac{3}{2\pi}} C_1. \quad (4)$$

It is convenient to make a transformation from the  $\alpha_{\lambda\mu}$  and  $\pi_{\lambda\mu}$  to the creation and annihilation operators  $b_{\lambda\mu}^{\dagger}$  and  $b_{\lambda\mu}$ :

$$\left. \begin{aligned} \alpha_{\lambda\mu} &= \sqrt{\frac{\hbar}{2B_{\lambda}\omega_{\lambda}}} (b_{\lambda\mu}^{\dagger} + (-)^{\mu} b_{\lambda-\mu}) \\ \pi_{\lambda\mu} &= i \sqrt{\frac{\hbar B_{\lambda}\omega_{\lambda}}{2}} (-)^{\mu} (b_{\lambda-\mu}^{\dagger} - b_{\lambda\mu}) \end{aligned} \right\} \quad (5)$$

where

$$\omega_\lambda = \sqrt{C_\lambda/B_\lambda}. \quad (6)$$

In terms of the new operators the uncoupled Hamiltonian takes the very simple form

$$H_0 = \sum_{\lambda=1}^2 \sum_{\mu} (b_{\lambda,\mu}^+ b_{\lambda,\mu} + 1/2) \hbar \omega_\lambda, \quad (6b)$$

with the eigenvalues

$$E_0(N_1, N_2) = (N_1 + 3/2) \hbar \omega_1 + (N_2 + 5/2) \hbar \omega_2, \quad (7)$$

$N_1$  and  $N_2$  being the number of dipole and quadrupole phonons, respectively.

A simple glance at the coupling term  $H_I$  reveals that neither  $N_1$  nor  $N_2$  is a good quantum number. However, for the sake of simplicity, we shall always work in subspaces where the number of dipole phonons  $N_1$  is constant. In view of the very large value of  $\hbar \omega_1$ , this is an excellent approximation since the conservation of parity would allow only the admixing of uncoupled states differing by at least two dipole phonons. This approximation is equivalent to retaining only that part of  $H_I$  which commutes with  $\sum_{\mu} b_{1\mu}^+ b_{1\mu}$ :

$$H' = \frac{K\hbar}{B_1\omega_1} \sqrt{\frac{\hbar}{10B_2\omega_2}} \sum_{\nu\mu} (-)^{\nu} \langle 11\nu\mu - \nu | 2,\mu \rangle b_{1\nu}^+ b_{1\nu-\mu} (b_{2-\mu}^+ + (-)^{\mu} b_{2\mu}). \quad (8)$$

We then have for the total Hamiltonian

$$H = H_0 + H'. \quad (9)$$

As a consequence of this approximation the  $N_1 = 0$  states are unaffected by the presence of  $H'$ . In particular, the vacuum  $|0\rangle$ , defined as the state for which  $N_1 = N_2 = 0$ , is an eigenstate of both  $H_0$  and  $H_0 + H'$ .

A complete solution of the problem involves a determination of the eigenvalues and eigenvectors of  $H$ . This is a clearly impossible task from the analytic point of view, since it is tantamount to the solution of a system of coupled differential equations in eight variables. One may have the temptation to apply an adiabatic approximation of the Born-Oppenheimer type, but such a procedure is unfortunately not justifiable here. In molecular spectroscopy, the projection of the electronic angular momentum along the symmetry axis of the molecule is considered as a good quantum number. No approximation of this kind can be made here: for each instantaneous

quadrupole deformation of the nuclear surface there will be three eigenfrequencies of the dipole vibration corresponding to the three principal axes

$$E_j(\beta, \gamma) = \hbar\omega_1 \left[ 1 + \frac{2K}{\sqrt{30} C_1} \beta \cos(\gamma - j2\pi/3) \right], \quad (10)$$

where

$$\beta^2 = a_0^2 + 2a_2^2, \quad \text{tg } \gamma = \sqrt{2} a_2/a_0. \quad (11)$$

The variables  $a_0$  and  $a_2$  specify the shape of the ellipsoid in the frame of reference coinciding with its principal axes<sup>(1)</sup>. Since the three eigenfrequencies  $E_j(\beta, \gamma)$  are quasi-degenerate, the corresponding eigenfunctions will surely have to be thoroughly mixed in any reasonable approximation to the actual  $N_1 = 1$  wave functions. This automatically implies the obligation to solve a system of coupled differential equations in order to determine those parts of the wave functions which describe the quadrupole vibrations. Such a situation presents a great analogy with that encountered in the study of the vibronic spectrum<sup>(2)</sup>. We will try to obtain the eigenvectors of the coupled system  $H_0 + H'$  as a superposition of those of the uncoupled Hamiltonian  $H_0$ . This will be achieved by calculating the energy matrix with respect to the uncoupled basis and then diagonalizing it. But before we carry out this program in detail, it is interesting to see how much information can be gathered on the broadening of the dipole line through a simple calculation using the method of moments.

### 3. Calculation of the width by the method of moments

Let us introduce the  $n^{\text{th}}$  moment

$$S_n = \sum_i (E_i - E_0)^n |\langle i | b_{1\mu}^+ | 0 \rangle|^2. \quad (12)$$

The kets  $|i\rangle$  are eigenvectors of the total Hamiltonian  $H$  with energies  $E_i$ . Using the closure property we can write

$$S_n = \langle 0 | b_{1\mu} (H - E_0)^n b_{1\mu}^+ | 0 \rangle. \quad (13)$$

The cross section for the excitation of a state  $i$  through dipole absorption is proportional to  $(E_i - E_0) |\langle i | b_{1\mu}^+ | 0 \rangle|^2$  and we may define the mean energy  $\bar{E}$  of the dipole line as the position of the centre of gravity of the absorption cross section:

$$\bar{E} = S_2/S_1. \quad (14)$$

A possible measure of the width of the line is the root mean square deviation  $\Delta$ :

$$\Delta^2 = \frac{\sum_i (E_i - E_0 - \bar{E})^2 (E_i - E_0) |\langle i | b_{1\mu}^+ | 0 \rangle|^2}{\sum_i (E_i - E_0) |\langle i | b_{1\mu}^+ | 0 \rangle|^2} = \frac{S_3}{S_1} - \left( \frac{S_2}{S_1} \right)^2. \quad (15)$$

The first three moments are readily evaluated and we obtain

$$\Delta^2 = \frac{K^2 \hbar^3 \omega_1^2 \omega_2}{6C_1^2 C_2} \left( 1 + \frac{\omega_2}{\omega_1} \right) - \frac{1}{\hbar^2 \omega_1^2} \left( \frac{K^2 \hbar^3 \omega_1^2 \omega_2}{6C_1^2 C_2} \right)^2. \quad (16)$$

In general, the second term will be much smaller than the first one, so that in first approximation the width of the line will increase linearly with the coupling parameter  $K$ . This would have been strictly true if we had chosen to define  $\bar{E}$  as the mean energy  $S_1$  of the dipole state; in that case  $\Delta$  would have been given by the first term of (16) only. In tables 2 and 3 of section 6 we give the value of  $2\Delta$  for various nuclei. For most of the cases considered this quantity turns out to be many times as large as  $\hbar\omega_2$ . This is the reason why we cannot study the problem by perturbation theory.

#### 4. Construction and diagonalization of the energy matrix

The precedent calculation enabled us to evaluate the effect of the dipole-quadrupole interaction on the width of the dipole line. If, however, we want to study how it modifies its structure, we must attempt to determine explicitly the eigenvalues and eigenvectors of  $H$ . To this end we now proceed to the construction of the energy matrix and to its diagonalization.

The states over which  $H'$  spreads the dipole line have  $N_1 = 1$ ,  $I = 1$  and negative parity. They can be expressed as a linear combination of eigenvectors of the uncoupled Hamiltonian:

$$|N_1 = 1; i; I = 1, M\rangle = \sum_{N_2 \nu l_2} a_{N_2 \nu l_2}^i |N_1 = 1, l_1; N_2 \nu l_2; I = 1, M\rangle. \quad (17)$$

The angular momenta  $l_1$  and  $l_2$  of the dipole and quadrupole phonons are vector coupled to a total angular momentum  $I = 1$  with projection  $M$  on the z-axis. This restricts  $l_2$  to the values 0 and 2, since it has been shown that no  $l_2 = 1$  state can arise from the quadrupole vibrations of the nuclear surface<sup>(1)</sup>. For  $N_2 \geq 4$ , it is possible to form states with the same  $l_2$  and z-component of  $l_2$  in more than one way. The need to remove at least part

of this degeneracy led RAKAVY<sup>(3)</sup> to introduce a new quantum number, the seniority  $\nu$ , which indicates how many of the  $N_2$  phonons are not coupled pairwise to an angular momentum equal to 0. A state of  $N_2$  phonons may have seniorities  $\nu = N_2, N_2 - 2, \dots, 1$  or 0. In the conventional way of labelling the irreducible representations of  $R_5$ , those would be the  $(N_2, 0), (N_2 - 2, 0), \dots, (1, 0)$  or  $(0, 0)$  representations. The decomposition of those into irreducible representations of  $R_3$  may be effected through elementary methods (see the Appendix), and table 1 gives the multiplicities of the various values of the angular momentum found in a representation of seniority  $\nu$ , for values of  $\nu$  going up to 18. Now, we observe an interesting regularity: the states with  $l_2 = 0$  or 2 never occur more than once, or both at the same time, for a given seniority. Moreover, the lowest value of  $l_2$  is equal to 0 if  $\nu = 3n$  and 2 if  $\nu = 3n \pm 1$ ,  $n$  being an integer. We checked these rules up to  $\nu = 18$ , but could not derive them through elementary methods. Of course, these regularities simplify our calculations immensely by enabling us to characterize uniquely the unperturbed states in which we are interested by the two quantum numbers  $N_2$  and  $\nu$  only. We can therefore drop the label  $l_2$  attached to the coefficients  $a$  in (17).

It is a straightforward matter to show that the matrix elements of  $H'$  are given by

$$\langle 11; N'_2 \nu' l'_2; 1M | H' | 11; N_2 \nu l_2; 1M \rangle = -\frac{K\hbar}{B_1 \omega_1} W(1l_2 1l'_2; 12) \langle N'_2 \nu' l'_2 || \alpha_2 || N_2 \nu l_2 \rangle, \quad (18)$$

where the reduced matrix element  $\langle N'_2 \nu' l'_2 || \alpha_2 || N_2 \nu l_2 \rangle$  is defined by

$$\langle N'_2, \nu', l'_2, m'_2 | \alpha_{2\mu} | N_2, \nu, l_2, m_2 \rangle = \frac{1}{\sqrt{2l'_2 + 1}} \langle l_2 2 m_2 \mu | l'_2 m'_2 \rangle \langle N'_2 \nu' l'_2 || \alpha_2 || N_2 \nu l_2 \rangle. \quad (19)$$

The very definition of the operators  $\alpha_{2\mu}$  yields the selection rules

$$\left. \begin{aligned} N'_2 &= N_2 \pm 1 \\ \nu' &= \nu \pm 1. \end{aligned} \right\} \quad (20)$$

We proceed to the evaluation of the reduced matrix elements in two steps: first, by showing that all of them can be expressed in terms of those which are taken between states for which  $\nu = N_2$  and, second, by evaluating them for that special case. Moreover, because of the trivial relation between the matrix elements of  $b$  and  $b^\dagger$ , we have to consider only the former.

We begin by noticing that a state with  $N_2 = 2\nu + \nu$ ,  $\nu$  being a positive integer, can be written as



$$|2\nu + v, v, l_2, m_2\rangle = \frac{1}{\sqrt{C(2\nu + v, v)}} \{b_2^+ b_2^+\}_0^v |v, v, l_2, m_2\rangle, \quad (21)$$

where

$$\{b_2^+ b_2^+\}_0 = \frac{1}{\sqrt{5}} \sum_{\mu} (-)^{\mu} b_{2\mu}^+ b_{2-\mu}^+. \quad (22)$$

We stick to the convention that all the kets are normalized to unity. The normalization coefficients

$$C(2\nu + v, v) = \left(\frac{2}{5}\right)^{\nu} \nu! \prod_{\alpha=1}^{\nu} (2\alpha + 2\nu + 3) \quad (23)$$

are immediately deduced from the recursion formula

$$\left. \begin{aligned} C(2\nu + v, v) &= \langle v, v, l_2, m_2 | \{b_2 b_2\}_0^v \{b_2^+ b_2^+\}_0^v |v, v, l_2, m_2\rangle \\ &= \frac{4}{5} \sum_{\alpha=1}^{\nu} [2(\nu - \alpha) + v + 5/2] \langle v, v, l_2, m_2 | \{b_2 b_2\}_0^{\nu-1} \{b_2^+ b_2^+\}_0^{\nu-1} |v, v, l_2, m_2\rangle \\ &= \frac{2\nu}{5} (2\nu + 2v + 3) C(2\nu + v - 2, v). \end{aligned} \right\} \quad (24)$$

In deriving this formula we have used the commutation relation

$$[\{b_2 b_2\}_0, \{b_2^+ b_2^+\}_0] = \frac{4}{5} N_2 + 2 = \frac{4}{5} \sum_{\mu} b_{2\mu}^+ b_{2\mu} + 2 \quad (25)$$

and the fact that  $\{b_2 b_2\}_0$  acting on a state  $|v, v, l_2, m_2\rangle$  gives zero. With the help of eqs. (21), (23) and (25) we readily obtain

$$\left. \begin{aligned} &\langle N_2 - 1, v - 1, l'_2, m'_2 | b_{2\mu} | N_2, v, l_2, m_2 \rangle \\ &= \sqrt{\frac{N_2 + v + 3}{N_2 + v + 1}} \langle N_2 - 3, v - 1, l'_2, m'_2 | b_{2\mu} | N_2 - 2, v, l_2, m_2 \rangle \\ &= \sqrt{\frac{N_2 + v + 3}{2v + 3}} \langle v - 1, v - 1, l'_2, m'_2 | b_{2\mu} | v, v, l_2, m_2 \rangle. \end{aligned} \right\} \quad (26)$$

Similarly,

$$\left. \begin{aligned} &\langle N_2 - 1, v + 1, l'_2, m'_2 | b_{2\mu} | N_2, v, l_2, m_2 \rangle \\ &= \sqrt{\frac{N_2 - v}{2}} \langle v + 1, v + 1, l'_2, m'_2 | b_{2\mu} | v + 2, v, l_2, m_2 \rangle. \end{aligned} \right\} \quad (27)$$

Then, eqs. (21), (23), the commutator

$$[b_{2\mu}, \{b_2^+ b_2^+\}_0] = \frac{2}{\sqrt{5}} (-)^\mu b_{2-\mu}^+ \quad (28)$$

and the relation between the matrix elements of  $b$  and  $b^+$  yield

$$\left. \begin{aligned} & \langle N_2 - 1, v + 1, l'_2, m'_2 | b_{2\mu} | N_2, v, l_2, m_2 \rangle \\ & = \sqrt{\frac{N_2 - v}{2v + 5}} \langle v, v, l_2, m_2 | (-)^\mu b_{2-\mu} | v + 1, v + 1, l'_2, m'_2 \rangle. \end{aligned} \right\} \quad (29)$$

In order to evaluate the matrix elements

$$\langle v - 1, v - 1, l'_2, m'_2 | b_{2\mu} | v, v, l_2, m_2 \rangle$$

we must consider separately the three cases  $v = 3n$ ,  $3n + 1$  and  $3n - 1$ .

$$\boxed{v = 3n}$$

Since  $l_2 = 0$  when  $v = 3n$ , we have

$$(-)^\mu b_{2\mu} | v, v, 0, 0 \rangle = d(-\mu) | v - 1, v - 1, 2, -\mu \rangle. \quad (30)$$

The coefficients  $d(-\mu)$  are easily seen to be independent of  $\mu$ , so that

$$\langle v, v, 0, 0 | \sum_{\mu} b_{2\mu}^+ b_{2\mu} | v, v, 0, 0 \rangle = 5d^2 = v \quad (31)$$

and

$$d = \sqrt{v/5}. \quad (32)$$

$$\boxed{v = 3n - 1}$$

An annihilation operator acting on  $|v = 3n - 1, v, 2, M\rangle$  produces a  $v = 3n - 2$  state with  $l'_2 = 2$  or  $4$ ; moreover we see from table 1 that the  $l'_2 = 4$  state never occurs more than once, so that we can write

$$\left. \begin{aligned} \sum_{\mu} \langle 22\mu M | LM + \mu \rangle (-)^\mu b_{2-\mu} | v, v, 2, M \rangle &= e(L, M + \mu) | v - 1, v - 1, L, M + \mu \rangle, \\ &(L = 2 \text{ or } 4). \end{aligned} \right\} \quad (33)$$

The coefficients  $e$  are independent of  $M + \mu$  and, therefore, dropping that index, we obtain from (33)

$$(-)^\mu b_{2-\mu} | v, v, 2, M \rangle = \sum_L \langle 22\mu M | LM + \mu \rangle e(L) | v - 1, v - 1, L, M + \mu \rangle, \quad (34)$$

whence

$$\left. \begin{aligned} \langle v, v, 2, M | \sum_{\mu} b_{2\mu}^+ b_{2\mu} | v, v, 2, M \rangle &= v = \sum_{L, \mu} \langle 22\mu M | LM + \mu \rangle^2 e(L)^2 \\ &= e(2)^2 + \frac{9}{5} e(4)^2 \end{aligned} \right\} \quad (35)$$

and

$$\left. \begin{aligned} \langle v, v, 2, M | \sum_{\mu} \mu b_{2\mu}^+ b_{2\mu} | v, v, 2, M \rangle &= M \\ &= -e(2)^2 \sum_{\mu} \mu \langle 22\mu M | 2M + \mu \rangle^2 - e(4)^2 \sum_{\mu} \mu \langle 22\mu M | 4M + \mu \rangle^2. \end{aligned} \right\} \quad (36)$$

For  $M = 0$  this relation yields an identity, but for  $M \neq 0$  it gives a unique relation

$$e(2)^2 - \frac{12}{5} e(4)^2 = 2. \quad (37)$$

Eqs. (35) and (37) can be solved for  $e(2)$

$$e(2) = \sqrt{\frac{4v+6}{7}}, \quad (38)$$

which enables us to evaluate the matrix element

$$\langle v-1, v-1, 2, M+\mu | (-)^{\mu} b_{2-\mu} | v, v, 2, M \rangle = \langle 22\mu M | 2M + \mu \rangle e(2). \quad (39)$$

$$\boxed{v = 3n+1}$$

Relation (29) gives

$$\langle v-1, v-1, 0, 0 | b_{2\mu} | v, v, 2, \mu \rangle = \sqrt{\frac{2v+3}{2}} \langle v, v, 2, \mu | (-)^{\mu} b_{2-\mu} | v+1, v-1, 0, 0 \rangle. \quad (40)$$

Now, we see from table 1 that

$$(-)^{\mu} b_{2-\mu} | v+1, v-1, 0, 0 \rangle = f | v, v, 2, \mu \rangle + g | v, v-2, 2, \mu \rangle, \quad (41)$$

whence we deduce

$$\langle v+1, v-1, 0, 0 | \sum_{\mu} b_{2\mu}^+ b_{2\mu} | v+1, v-1, 0, 0 \rangle = v+1 = 5(f^2 + g^2). \quad (42)$$

The coefficient  $g$  may be evaluated from eqs. (26), (30) and (32):

$$g = \langle v, v-2, 2, \mu | (-)^{\mu} b_{2-\mu} | v+1, v-1, 0, 0 \rangle = \sqrt{\frac{2v+3}{2v+1} \left( \frac{v-1}{5} \right)}. \quad (43)$$

TABLE 1. Multiplicity  $d_v(l_2)$  of the representations  $l_2$  of  $R_3$  contained in a representation  $(v, 0)$  of  $R_5$ .

$v \setminus l_2$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36						
0	1																																										
1		1																																									
2			1																																								
3		1		1																																							
4			1		1																																						
5			1		1		1																																				
6		1		1		1		2		1		1		1																													
7			1		1		1		1		2		1		1		1																										
8			1		1		1		1		2		1		2		1		1		1																						
9		1		1		1		2		1		2		2		1		2		1		1		1																			
10			1		1		1		1		2		1		2		2		2		2		1		2		1		1		1		1		1								
11			1		1		1		1		2		1		2		2		2		2		2		1		2		1		1		1		1		1						
12		1		1		1		2		1		2		2		1		3		2		2		2		2		1		2		1		1		1		1					
13			1		1		1		1		2		1		2		2		2		2		2		3		2		2		2		2		2		2		2				
14			1		1		1		1		2		1		2		2		2		2		2		3		2		3		2		2		2		2		2				
15		1		1		1		2		1		2		2		1		3		2		2		3		3		2		3		2		2		2		2		2			
16			1		1		1		1		2		1		2		2		2		2		2		3		2		3		3		3		3		3		2		3		
17			1		1		1		1		2		1		2		2		2		2		2		3		2		3		3		3		3		3		3		2		
18		1		1		1		2		1		2		2		1		3		2		2		3		3		2		4		3		3		3		3		3		2	

From eqs. (42) and (43) we then obtain

$$f = \sqrt{\frac{2v+4}{5(2v+1)}} \quad (44)$$

and from eq. (40)

$$\langle v-1, v-1, 0, 0 | b_{2\mu} | v, v, 2, \mu \rangle = \sqrt{\frac{(2v+3)(v+2)}{5(2v+1)}}. \quad (45)$$

We remark that the signs of  $d$ ,  $e(L)$ ,  $f$  and  $g$  could not be determined from the precedent algebraic relations. However, we shall consider that our arbitrary choice determines the overall phase of the  $|v, v, l'_2, m'_2\rangle$  states. Such an overall phase is immaterial since we will be interested only in the squares of the coefficients.

Using eqs. (5), (18), (19), (26), (29), (30), (32), (38), (39) and (45), we can write down the matrix elements of  $H$ :

$$\begin{aligned} \langle N'_1 = 1, N_2 - 1, v' | H | N_1 = 1; N_2, v \rangle = & \\ \left. \begin{aligned} k \sqrt{\frac{(N_2 + v + 3)v}{15(2v+3)}} & \quad \text{if } v' = v-1 \quad \text{and} \quad v = 3n, \\ k \sqrt{\frac{N_2 + v + 3}{30}} & \quad \text{if } v' = v-1 \quad \text{and} \quad v = 3n-1, \\ k \sqrt{\frac{(N_2 + v + 3)(v+2)}{15(2v+1)}} & \quad \text{if } v' = v-1 \quad \text{and} \quad v = 3n+1, \\ k \sqrt{\frac{(N_2 - v)(v+3)}{15(2v+5)}} & \quad \text{if } v' = v+1 \quad \text{and} \quad v = 3n, \\ k \sqrt{\frac{(N_2 - v)(v+1)}{15(2v+5)}} & \quad \text{if } v' = v+1 \quad \text{and} \quad v = 3n-1, \\ k \sqrt{\frac{N_2 - v}{30}} & \quad \text{if } v' = v+1 \quad \text{and} \quad v = 3n+1, \end{aligned} \right\} \quad (46) \end{aligned}$$

where

$$k = -\frac{K\hbar}{B_1\omega_1} \sqrt{\frac{\hbar}{2B_2\omega_2}}. \quad (47)$$

We are now in a position to construct the complete energy matrix. The rows and the columns are arranged in order of increasing  $N_2$  and, for each  $N_2$ , of decreasing  $v$ . It is convenient to express the whole matrix in units of  $\hbar\omega_2$ . If the diagonal terms corresponding to  $5/2(1 + \omega_1/\omega_2)$  times the unit matrix are omitted, the matrix becomes

$$\begin{array}{cccccccc}
 0 & \eta\sqrt{1/3} & & & & & & \\
 \eta\sqrt{1/3} & 1 & \eta\sqrt{7/30}\eta\sqrt{2/15} & & & & & \\
 & \eta\sqrt{7/30} & 2 & 0 & \eta\sqrt{1/5}\eta\sqrt{1/15} & & & \\
 & \eta\sqrt{2/15} & 0 & 2 & 0 & \eta\sqrt{7/15} & & \\
 & & \eta\sqrt{1/5} & 0 & 3 & 0 & \eta\sqrt{\frac{22}{45}}\eta\sqrt{\frac{2}{45}} & \\
 & & \eta\sqrt{1/15}\eta\sqrt{7/15} & 0 & 3 & 0 & \eta\sqrt{3/10}\eta\sqrt{4/15} & \\
 & & & \eta\sqrt{\frac{22}{45}} & 0 & 4 & 0 & 0 & \eta\sqrt{\frac{13}{30}}\eta\sqrt{\frac{4}{45}} & \\
 & & & \eta\sqrt{\frac{2}{45}}\eta\sqrt{3/10} & 0 & 4 & 0 & 0 & \eta\sqrt{\frac{11}{45}}\eta\sqrt{2/15} & \\
 & & & & \eta\sqrt{\frac{4}{45}} & 0 & 0 & 4 & 0 & 0 & \eta\sqrt{3/5} & \\
 & & & & & \eta\sqrt{\frac{13}{30}} & 0 & 0 & 5 & 0 & 0 & \\
 & & & & & \eta\sqrt{\frac{4}{45}}\eta\sqrt{\frac{11}{45}} & 0 & 0 & 5 & 0 & & \\
 & & & & & & \eta\sqrt{\frac{2}{45}}\eta\sqrt{3/5} & 0 & 0 & 5 & \dots\dots & 
 \end{array}$$

where

$$\eta = k/\hbar\omega_2. \quad (48)$$

This matrix was diagonalized on the GIER computer of the Astronomical Observatory of Copenhagen. As  $\eta$  was increased, larger and larger matrices had to be considered in order to decrease the error due to their finite size. The eigenvalues  $E_i$  of the total Hamiltonian

$$H|N_1 = 1; i; I = 1, M\rangle = E_i|N_1 = 1; i; I = 1, M\rangle \quad (49)$$

and the corresponding coefficients  $a_{N_1, \nu}^i$  of the eigenvectors were evaluated for some values of  $\eta$  ranging from  $\sqrt{3}$  to  $6\sqrt{3}$ . Figs. 1-4 represent the distribution of the dipole state amongst the various levels. The length of the lines is proportional to the square of the quantity  $a_{00}^i$

$$a_{00}^i = \langle i|b_{1\mu}^+|0\rangle \quad (50)$$

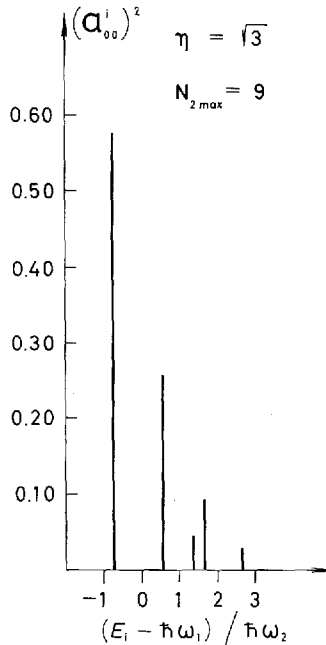


Fig. 1.

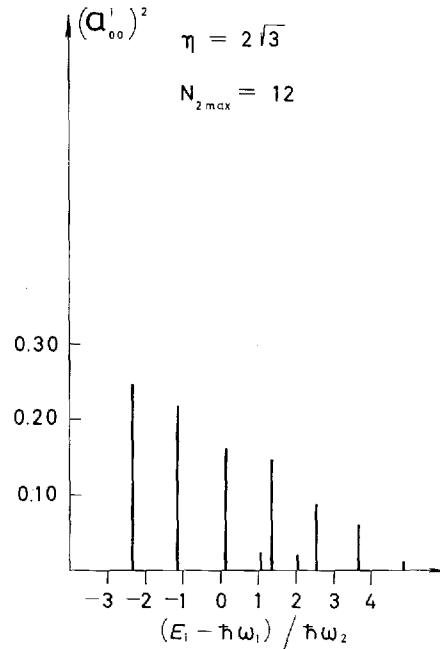


Fig. 2.

$N_{2\max}$  is the largest number of phonons considered.

involved in the calculation of the cross sections. The coupling between the dipole and quadrupole modes appears to give rise to quite a complicated structure of the dipole line for sufficiently large values of  $\eta$ . It is difficult to predict how much of this structure will persist once the intrinsic widths of the levels have been added to the picture. The whole situation is made even more complicated by the fact that the dipole mode may be coupled to other modes beside the quadrupole vibrations.

### 5. A classical approximation

In the limit  $\Delta \gg \hbar\omega_2$  it is possible to derive a classical expression for the probability  $P(E)$  of exciting an eigenstate at an energy  $E$  through dipole absorption:

$$\begin{aligned}
 P(E) &= \sum_i |\langle i | b_{1\mu}^+ | 0 \rangle|^2 \delta(E - E_i + E_0) \\
 &= \langle 0 | b_{1\mu} \delta(E - H + E_0) b_{1\mu}^+ | 0 \rangle.
 \end{aligned}
 \quad \left. \vphantom{\sum_i} \right\} (51)$$

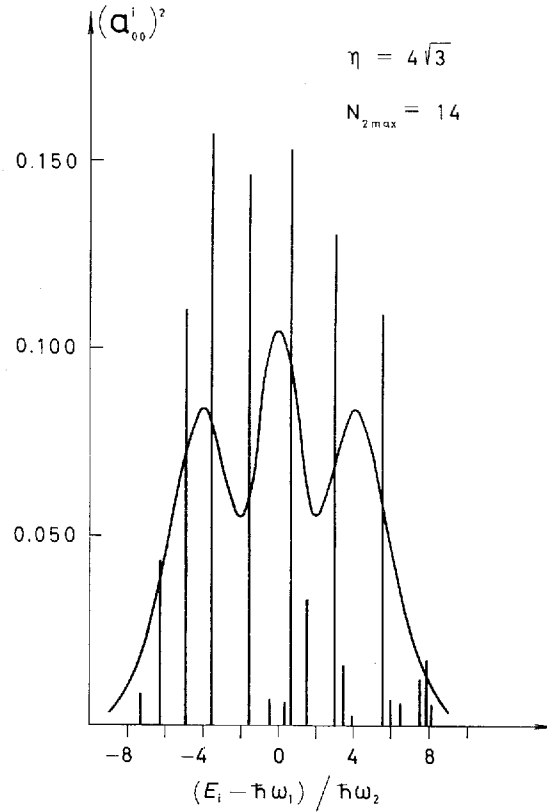


Fig. 3.

$N_{2\max}$  is the largest number of phonons considered. The curve shows the classical approximation.

We can then drop the part of the Hamiltonian which describes the quadrupole vibrations and the coupling between the normal dipole modes through the angular velocity of the intrinsic axes, keeping only the terms

$$H \cong \sum_i (b'_{1j}{}^+ b'_{1j} + 1/2) \left[ 1 + \frac{2K}{\sqrt{30} C_1} \beta \cos(\gamma - 2\pi j/3) \right] \hbar\omega_1, \quad (52)$$

where  $b'_{1j}{}^+$  and  $b'_{1j}$  are the creation and destruction operators for a dipole phonon along the axis  $j$ . Substituting (52) into (51), expressing  $b_{1\mu}^+$  and  $b_{1\mu}$  in terms of the primed operators and integrating over the Euler angles, we obtain

$$P(E) = \frac{1}{3} \sum_j \int |\Psi_0(\beta, \gamma)|^2 \delta \left[ E - \hbar\omega_1 \left( 1 + \frac{2K}{\sqrt{30} C_1} \beta \cos(\gamma - 2\pi j/3) \right) \right] \beta^4 d\beta |\sin^3 \gamma| d\gamma \quad (53)$$



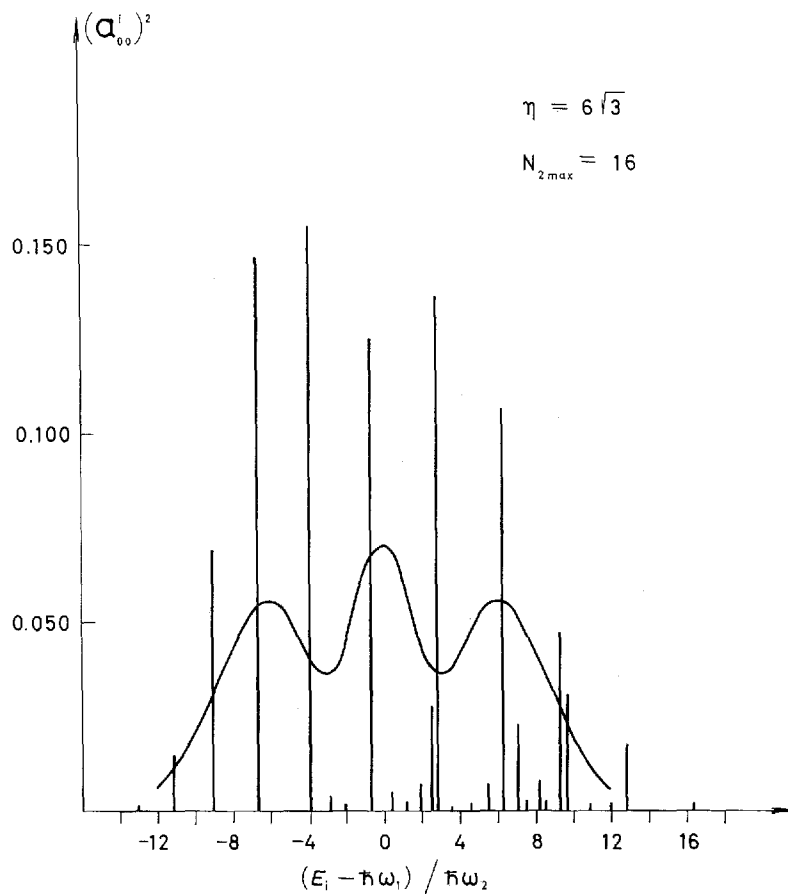


Fig. 4.

$N_{2max}$  is the largest number of phonons considered. The curve shows the classical approximation.

This expression has an immediate physical meaning: since the absorption of the dipole radiation takes place in a time which is very short in comparison with the period of the quadrupole vibrations<sup>1</sup>, a nucleus possessing a deformation  $(\beta, \gamma)$  will absorb only photons with the energy required to excite one of the three normal dipole modes; the probability that this happens is proportional to  $\Psi_0(\beta, \gamma)$ , the ground state wave function for quadrupole oscillations. We average over the three intrinsic axes, because unpolarized nuclei are considered.

<sup>1</sup> It is possible to form a wave packet with an energy spread large compared with the quadrupole energy but small compared with the width of the dipole line.

The integral in (53) is evaluated without difficulty and yields

$$P(E) = \frac{2}{3} \sqrt{\frac{\alpha}{\pi}} \left\{ e^{-\alpha(E - \hbar\omega_1)^2} [3\alpha(E - \hbar\omega_1)^2 - 1] + 2e^{-4\alpha(E - \hbar\omega_1)^2} \right\}, \quad (54)$$

where

$$\alpha = \frac{15B_1^2\omega_1^2B_2\omega_2}{2K^2\hbar^3}. \quad (55)$$

The classical result is shown in figs. 3 and 4. When comparing it with the exact solution, one should not forget that  $P(E)$  is a probability density, so that it should be compared with  $\sum (a_{00}^i)^2 / \Delta E$ , the summation being extended to all levels in the interval  $\Delta E$ . For the values of  $\eta$  considered, however, the level density is still too low to make the construction of a histogram meaningful. The exact solution tends slowly towards the classical limit and it is doubtful whether the latter is of any validity for the cases of physical interest. This may seem surprising at first sight, since it is possible to study an analogous problem where the exact solution tends rather rapidly towards the classical limit, namely the motion of two coupled harmonic oscillators in one dimension with frequencies  $\omega_1$  and  $\omega_2$  such that  $\omega_1 \gg \omega_2$ . The situation, however, is more complex here: the classical limit has a structure and it will not be a good approximation to the exact solution until there are many lines under each of its three peaks.

## 6. Comparison with the experimental results

We will now attempt to compare the calculated widths with those observed experimentally. For many reasons this is not an easy task. The quantity  $\Delta$  which we calculated is the root mean square deviation of the dipole absorption cross section, if one neglects the intrinsic width of each line. Of course no such quantity is known experimentally and the best we can do is to compare  $2\Delta$  with some parameter characterizing the width of the observed line. For the sake of simplicity, we choose  $I$ , the full width at half maximum cross section, as this parameter. Since the two compared quantities are not exactly the same, we cannot expect to do more than reproduce the general trends of the variation of the width as a function of the parameters involved. The comparison is made yet more difficult by the fact that, for many of the nuclei considered, the cross section  $\sigma(\gamma, n)$  is used without any correction for the neutron multiplicity, so that we tend to have an overestimate of the actual width. Since we consider nuclei with  $A \geq 50$ ,

the neglect of the  $(\gamma, p)$  process should not introduce a great error. Another difficulty lies in the choice of  $\hbar\omega_1$ . We take for this quantity the value of the energy at which the cross section  $\sigma(\gamma n)$  has its maximum.

Strictly speaking, our theory is applicable to spherical even-even nuclei only. Unfortunately there is a scarcity of experimental data on the dipole line of those nuclei. For this reason, we also consider a certain number of odd- $A$  nuclei, assuming that the odd nucleon does not have a great influence on the shape of the dipole line and, in particular, that its interaction with the nuclear surface does not perturb the interaction between the dipole and quadrupole modes. The parameters  $\hbar\omega_2$  and  $C_2$  are taken to be those of the adjacent even-even nuclei. When the experimental spectrum of the odd- $A$  nucleus, with a certain amount of optimism, can be interpreted as resulting from the coupling of the odd nucleon to the surface vibrations of one of the adjacent nuclei, the vibrational parameters of the latter are used. In all other cases, at least in so far as the available data enable us to do so, we use an average of the parameters of both neighbours. When, however, the two sets of parameters are widely different (near a closed shell), we give separately the widths predicted for each of them.

$\Delta$  may be calculated from (16) or, alternatively, the eigenvalues and eigenvectors obtained through the diagonalization of the matrix may be used in conjunction with eqs. (15) and (50):

$$\Delta^2 = \frac{\sum_i (E_i - E_0 - \bar{E})^2 (E_i - E_0) (a_{00}^i)^2}{\sum_i (E_i - E_0) (a_{00}^i)^2}. \quad (56)$$

The first method is an exact one, whereas the second one contains a certain inaccuracy stemming from the finite size of the matrix and from its numerical diagonalization. A comparison between the results obtained in both ways is useful in that it provides an indication of the order of magnitude of this inaccuracy. The discrepancy turns out to be of a few percent only.

The experimental widths, the exact theoretical quantities  $2\Delta$  and all the relevant parameters are presented in tables 2 and 3 for even-even and odd- $A$  nuclei, respectively. The values of the width due to the dipole-quadrupole interaction are seen to vary between one third and one half of the experimental ones. This is hardly surprising, partly because of the uncertainties in the comparison which were indicated above, partly because of the very nature of the present calculations. We have assumed that the quadrupole phonons do not interfere with each other although the inaccuracy introduced

TABLE 2.

Nucleus	$\hbar\omega_2$ (MeV)	$C_2$ (MeV)	Ref.	$\hbar\omega_1$ (MeV)	$\Gamma$ (MeV)	Ref.	$2\Delta$ (MeV)
${}_{30}\overline{\text{Zn}}$	1.03	49	5	16.3	6.3	11	3.4
${}_{32}\text{Ge}^{70}$	0.88	53	5	17.5	8.0	12	3.2
${}_{38}\text{Sr}^{88}$	1.85	258	6	16.3	4.0	13	2.1
${}_{40}\text{Zr}^{92}$	0.92	151	7	16.9	5.5	13	1.9
${}_{42}\text{Mo}^{92}$	1.52	173	8	16.0	2.7 <sup>1</sup>	14	2.2
${}_{48}\text{Cd}^{112}$	0.610	42	9	16.0	5.1 <sup>2</sup>	11	2.8
${}_{50}\overline{\text{Sn}}$	1.19	235	9	15.8	5.0 <sup>2</sup>	15	1.7
${}_{50}\text{Sn}^{112}$	1.26	280	10	16.0	5.0	16	1.6
${}_{50}\text{Sn}^{124}$	1.13	240	9	15.5	5.0	16	1.6
${}_{82}\text{Pb}^{208}$	—	—	—	13.8	3.5 <sup>2</sup>	31	$\sim 0$

<sup>1</sup> Measured by the method of residual activity.

<sup>2</sup> Corrected for neutron multiplicity.

when dropping the anharmonic and higher order terms of the quadrupole Hamiltonian is probably not a negligible one; moreover, we have considered only the dipole-quadrupole interaction and neglected completely the coupling to other modes of motion. The present results indicate that these other effects are of the same order of magnitude as those due to the dipole-quadrupole interaction. We therefore conclude that the latter plays an appreciable part in the broadening of the giant dipole line in spherical nuclei, although it is clearly insufficient to explain the whole width of the line.

The author wishes to express his gratitude for the hospitality extended to him at the Institute for Theoretical Physics, University of Copenhagen. He is deeply indebted to Professor BEN MOTTELSON for suggesting this problem as well as for many stimulating discussions and invaluable suggestions. Special thanks are due Professor ÅGE BOHR for his interest in this work and for many fruitful conversations, as well as Dr. J. H. GUNN for diagonalizing the matrices. It is a pleasure to acknowledge discussions with many members of the Institute, in particular with Dr. K. T. HECHT on the classification of the states of surface vibrations and with Dr. D. BRINK on the classical limit.

The award of a CIBA Fellowship is gratefully acknowledged.

TABLE 3.

Nucleus	Adjacent nuclei	$\hbar\omega_2$ (MeV)	$C_2$ (MeV)	Ref.	$\hbar\omega_1$ (MeV)	$\Gamma$ (MeV)	Ref.	$2\Delta$ (MeV)
$^{23}\text{V}^{51}$	$^{24}\text{Cr}^{52}$	1.43	79	23	19.0 <sup>2</sup>	7.5 <sup>1</sup>	17	3.7
$^{25}\text{Mn}^{55}$	$^{24}\text{Cr}^{54}$ , $^{26}\text{Fe}^{56}$	0.85	58	24	18.2 <sup>2</sup>	6 <sup>1</sup>	18	3.2
$^{27}\text{Co}^{59}$	$^{26}\text{Fe}^{58}$ , $^{28}\text{Ni}^{60}$	1.07	50	24	17.9 <sup>2</sup>	7 <sup>1</sup>	17	3.8
$^{29}\text{Cu}^{63}$	$^{28}\text{Ni}^{62}$	1.17	79	25	16.3	8.5	19	2.9
$^{29}\text{Cu}^{65}$	$^{28}\text{Ni}^{64}$	1.34	122	25	17.0	8.1 <sup>3</sup>	19	2.6
$^{38}\text{As}^{76}$	$^{32}\text{Ge}^{74}$ , $^{34}\text{Se}^{76}$	0.58	15.5	5	17.3	9.0	20	4.7
$^{38}\text{Sr}^{87}$	$^{38}\text{Sr}^{88}$	1.85	258	7	15.8	5.3	13	2.0
$^{39}\text{Y}^{89}$	$^{38}\text{Sr}^{88}$	1.85	258	7	16.3	3.8	13	2.1
$^{40}\text{Zr}^{91}$	$^{40}\text{Zr}^{92}$	0.92	151	7	16.5	5.0	13	1.9
$^{41}\text{Nb}^{93}$	$^{40}\text{Zr}^{92}$	0.92	151	7	17.0	6.8	20	1.9
	$^{42}\text{Mo}^{94}$	0.874	72	9				2.7
	$^{40}\text{Zr}^{92}$ , $^{42}\text{Mo}^{94}$	0.90	112	7, 9				2.2
$^{45}\text{Rh}^{103}$	$^{44}\text{Ru}^{102}$ , $^{46}\text{Pd}^{104}$	0.515	24	9	16.4 <sup>2</sup>	6 <sup>1</sup>	21	3.4
$^{47}\text{Ag}^{107}$	$^{46}\text{Pd}^{106}$	0.513	25	9	16.0	6 <sup>1</sup>	21	3.2
$^{49}\text{In}^{115}$	$^{48}\text{Cd}^{114}$	0.555	36	9	15.4 <sup>2</sup>	5.5 <sup>1</sup>	21	2.7
	$^{50}\text{Sn}^{116}$	1.268	260	9				1.6
	$^{48}\text{Cd}^{114}$ , $^{50}\text{Sn}^{116}$	0.912	148	9				1.8
$^{53}\text{I}^{127}$	$^{52}\text{Te}^{126}$	0.673	64	9	15.5	4.9 <sup>1</sup>	11	2.3
$^{57}\text{La}^{139}$	$^{56}\text{Ba}^{138}$ , $^{58}\text{Ce}^{140}$	1.513	346	26, 27	15.5	5.2 <sup>1</sup>	15	1.5
$^{58}\text{Ce}^{141}$	$^{58}\text{Ce}^{140}$	1.596	379	27	16.0	4.5 <sup>1</sup>	15	1.6
	$^{58}\text{Ce}^{142}$	0.630	78	27				2.1
	$^{58}\text{Ce}^{140}$ , $^{142}$	1.113	229	27				1.6
$^{59}\text{Pr}^{141}$	$^{58}\text{Ce}^{140}$ , $^{60}\text{Nd}^{142}$	1.583	372	27	15.0	3.0 <sup>1</sup>	22	1.5
$^{79}\text{Au}^{197}$	$^{78}\text{Pt}^{196}$ , $^{80}\text{Hg}^{198}$	0.385	69	28, 29	14.2	4.7 <sup>1</sup>	11	1.5
$^{81}\text{Tl}$	$^{80}\text{Hg}^{202}$ , $^{204}$	0.433	330	29	14.6	4.6 <sup>1</sup>	11	0.8
	$^{82}\text{Pb}^{204}$ , $^{206}$	0.832	1337	30				0.6
	$^{80}\text{Hg}^{202}$ , $^{204}$ and $^{82}\text{Pb}^{204}$ , $^{206}$	0.633	834	29, 30				0.6
$^{82}\text{Pb}^{207}$		—	—		13.8	3.5 <sup>1</sup>	31	~ 0
$^{83}\text{Bi}^{209}$		—	—		13.8	3.5 <sup>1</sup>	31	~ 0

<sup>1</sup> Corrected for neutron multiplicity.

<sup>2</sup> Nucleus for which a splitting of the giant dipole line was observed. The two peaks at  $E_a$  and  $E_b$  have an integrated cross section  $\sigma_a$  and  $\sigma_b$ , respectively. We take  $\hbar\omega_1 = \frac{\sigma_a E_a + \sigma_b E_b}{\sigma_a + \sigma_b}$ .

<sup>3</sup> Total formation cross section, including  $\sigma(\gamma, p)$ .

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## Appendix

RAKAVY<sup>(3)</sup> indicated an elementary method to determine the representations  $l_2$  of  $R_3$  contained in a representation  $(v, 0)$  of  $R_5$ . He showed that  $d_v(l_2)$ , the number of times  $l_2$  that appears in  $(v, 0)$ , is given by the simple formula

$$d_v(l_2) = b_v(l_2) - b_{v-2}(l_2) - [b_v(l_2 + 1) - b_{v-2}(l_2 + 1)], \quad (\text{A1})$$

where  $b_N(M)$  is the number of ways in which  $N$  quadrupole phonons may have a projection  $M$  on the  $z$ -axis. For more than a few phonons, however, the evaluation of the quantities  $b_N(M)$  becomes extremely tedious if one undertakes to determine explicitly all the possible ways of projecting  $N$  phonons on the  $z$ -axis, and we should like to indicate a graphical method which permits to obtain  $b_N(M) - b_{N-2}(M)$  very rapidly.

If  $n_\mu$  phonons have a projection  $\mu$  on the  $z$ -axis, we have

$$N = N_2 + N_1 + N_0 \quad (\text{A2})$$

$$M = 2\nu_2 + \nu_1, \quad (\text{A3})$$

where

$$N_\mu = n_\mu + n_{-\mu}$$

and

$$\nu_\mu = n_\mu - n_{-\mu}.$$

From (A2) it is obvious that

$$0 \leq N_2 \leq N \quad (\text{A4})$$

and

$$0 \leq N_1 \leq N - N_2. \quad (\text{A5})$$

For a given value of  $N_\mu$ ,  $\nu_\mu$  can be equal to  $N_\mu, N_\mu - 2, N_\mu - 4, \dots, -N_\mu$ . One can then write down the array shown in table A1. It indicates all the possible ways of forming the pair  $(\nu_1, \nu_2)$  with  $N$  phonons and, with the help of (A3), may be used to evaluate the quantities  $b_N(M)$ . If one is

TABLE A1.

$N_2$	Possible values of $\nu_2$	$N_1$	Possible values of $\nu_1$
$N$	$N, N-2, N-4 \dots -N$	0	0
$N-1$	$N-1, N-3 \dots -N+1$	1	1 -1
		0	0
$N-2$	$N-2, N-4 \dots -N+2$	2	2 0 -2
		1	1 -1
		0	0
.....			
0	0	$N$	$N, N-2, N-4 \dots -N$
		$N-1$	$N-1, N-3, N-5 \dots -N+1$
		$N-2$	$N-2, N-4 \dots -N+2$
.....			
		1	1 -1
		0	0

TABLE A2.

$N_2$	Possible values of $\nu_2$	$N_1$	Possible values of $\nu_1$
$N$	$N, -N$	0	0
$N-1$	$N-1, -N+1$	1	1 -1
		0	0
$N-2$	$N-2, -N+2$	2	2 0 -2
		1	1 -1
		0	0
.....			
0	0	$N$	$N, N-2, N-4 \dots -N$
		$N-1$	$N-1, N-3, N-5 \dots -N+1$
		$N-2$	$N-2, N-4 \dots -N+2$
.....			
		1	1 -1
		0	0

interested in the differences  $b_N(M) - b_{N-2}(M)$  only, it is possible to introduce a considerable simplification by constructing a similar array for  $N-2$  phonons and "subtracting" it from that corresponding to  $N$  phonons, with the result presented in table A2. The simple symmetry properties of this array suggest to construct a tableau in the  $\nu_1 - \nu_2$  plane such that at each point corresponding to the pair of integers  $(\nu_1, \nu_2)$  we write down the num-

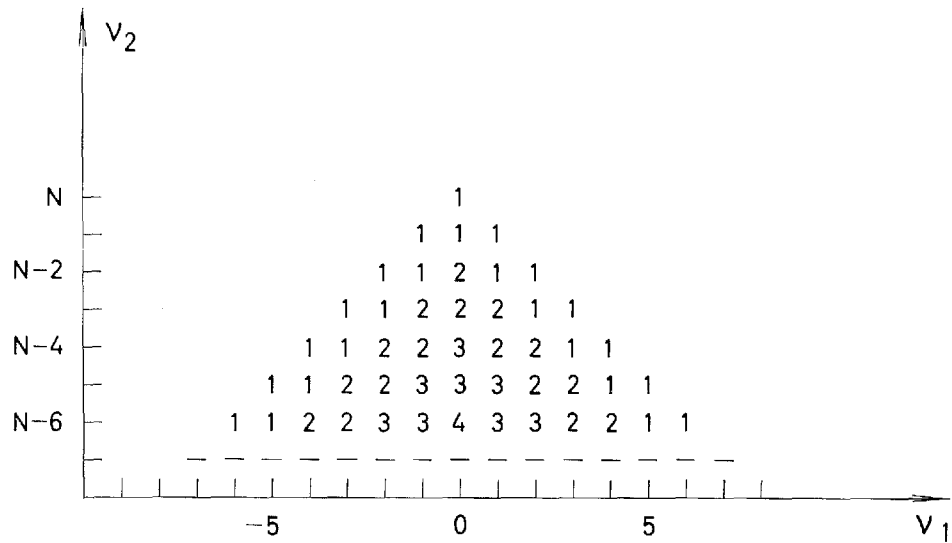


Fig. A1. Structure of the tableau permitting to determine  $b_N(M) - b_{N-2}(M)$ .

ber of times this pair occurs in table A2. It is immediately obvious from the latter that the tableau will have the structure shown in fig. A1 and possess reflectional symmetry with respect to the  $v_1$  axis. If we then draw the straight line  $M = 2v_2 + v_1$ , and add up all the numbers of the tableau that fall on it, we obtain the quantity  $b_N(M) - b_{N-2}(M)$ . Once we know how to build the tableau, we can write it down immediately for any value of  $N$ . An example is given in fig. A2 for  $N = 8$ .



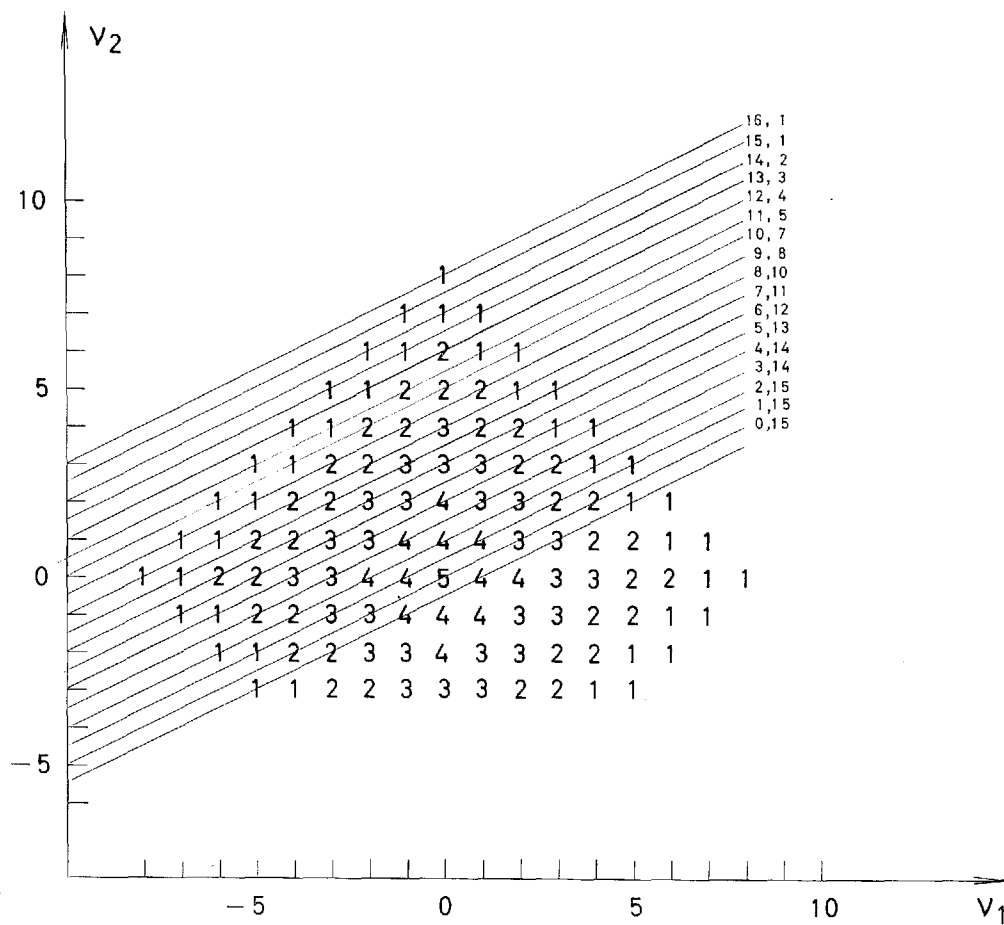


Fig. A.2. The two numbers at the end of each line are  $M$  and  $b_N(M) - b_{N-2}(M)$ , respectively.

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