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ON THE FORMULATION OF THE DYNAMICAL LAWS IN THE QUANTUM THEORY OF FIELDS

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CONTENTS

Synopsis

A new formulation of the dynamical laws for a system of elementary particles is proposed. In addition to simple assumptions of a kinematical nature, the formulation rests on the principle that a classical field and classical sources which, according to the classical theory, describe the same physical situation, also do so when the classical system is coupled to a quantum field. For the simple example of the Hurst-Thirring field, it is shown that this principle may be formulated in finite mathematical terms and may serve as a substitute for the formal field equation of the renormalization theory. To the third order in the coupling constant—and presumably to all orders—the perturbation expansion gives the same result as the usual theory.

No infinities or similar mathematical ambiguities appear in the theory.

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1. Introduction

The study of the fundamental assumptions of the relativistic quantum field theory has to a large extent been concerned with axioms of a general nature, such as f. inst. the axiom of microscopic causality, the asymptotic condition, and the requirement of unitarity. Basic assumptions of this type, valid in general, could be referred to as kinematical assumptions. It is well known how to express the kinematical assumptions either directly in terms of the field operators or in terms of various mathematical quantities closely connected with the field operators. In particular certain distributions, such as the τ-functions and the r-functions, have been studied. In terms of such distributions, one may express the kinematical assumptions in closed form[1a, 1b, 2, 3a, 3b].

Various suggestions for the incorporation into such formulations of the dynamical laws valid for a specific system of interacting elementary particles have been discussed. Thus LEHMANN, SYMANZIK and ZIMMERMANN[1a] have pointed out that the forces between elementary particles may be characterized by means of boundary conditions superimposed on the system of equations for the τ-functions*. Recently, NISHIJIMA[3b] and MURASKIN and NISHIJIMA[4] have proposed to use a postulated dispersion relation in terms of which the boundary conditions may be formulated in a simple manner. It might be true that the forces between elementary particles most conveniently are expressed in terms of boundary conditions imposed on equations of a purely kinematical character. Still, we hardly know the best way of characterizing the forces. It might therefore be of interest to investigate also formulations in which the basic assumptions are directly concerned with the dynamical properties of the system, and in which boundary conditions are used to exclude solutions of irregular behaviour only. So far, no direct formulation of the dynamical laws, such as, for example, an explicit construction in terms of the field operator of the source term in the field equation, could be given. More indirect approaches might therefore be acceptable.

* See also ref. 9.

The formulation investigated in the present work does not utilize field equations but is based, instead, on an assumption which concerns the behaviour of the quantum system when in interaction with a classical system of the same type as the quantum system. The assumption has a direct physical interpretation for the case of quantum electrodynamics. Consider the situation in which photons and electrons interact with an external electromagnetic system. The physical state of the classical electromagnetic system may be described in terms of a classical distribution of current and charge $j_\mu(x)$. In this case, the interaction between the quantum system and the classical system enters into the theory by addition of the classical source $j_\mu(x)$ to the operator source of the photon field. However, this is not the only possibility. According to the classical theory, we might also describe the physical situation of the classical electromagnetic system by the electromagnetic field which, according to the Maxwell equations of the classical theory, is produced by the classical distribution of four current $j_\mu(x)$. If this possibility is chosen, the interaction between the classical system and the quantum system is expressed by an additional term $ie\gamma_\mu A_\mu \boldsymbol{\psi}$ in the field equation for the electron field, $\boldsymbol{\psi}$ being the electron field operator. In quantum electrodynamics, it has always been assumed that these two possibilities give the same physical result.*

We shall consider an assumption of this type as a basic principle of quantum physics. Admittedly, a direct physical interpretation of such a principle is possible only for the case of quantum electrodynamics. However, the principle may be generalized to other cases and formulated as a definite mathematical relation. To make the principle and some of its implications clear we study, in the present work, the simple example of the Hurst-Thirring field, i. e. the quantized version of the classical real field which satisfies the classical field equation

$$(-\Box + m^2)\, A(x) = gA^2(x). \tag{1.1}$$

Let us for the moment apply the formal version of the quantized form of (1.1). If the system interacts with an external field $A(x)$ and an external source $j(x)$, the real quantum field $\boldsymbol{A}(x)$ satisfies the field equation

$$(-\Box + m^2)\, \boldsymbol{A}(x) = \boldsymbol{j}(x) + 2g\, A(x)\, \boldsymbol{A}(x) + j(x), \tag{1.2}$$

where the formal expression for the source operator is

$$\boldsymbol{j}(x) = g\, \boldsymbol{A}^2(x). \tag{1.3}$$

* A proof was given by J. Schwinger, Phys. Rev. **76**, 790 (1949).

To emphasize that the field operator depends on both $A(x)$ and $j(x)$ we use the notation $\boldsymbol{A}[A, j; x]$.

Consider first the situation where $j = o$. If retarded boundary conditions are used we have the integral equation

$$\boldsymbol{A}(x) = \boldsymbol{A}_{in}(x) + \int \Delta_R(x-x')\{\boldsymbol{j}(x') + 2\,gA(x')\boldsymbol{A}(x')\}\,d^4x'. \tag{1.4}$$

The notations used will be found in Appendix A. To give a definite meaning to the dependence of $\boldsymbol{A}$ on the physical state of the external system, we choose a representation of the free field operator $\boldsymbol{A}_{in}$ in which this operator is independent of the external field and source. If the formal expression (1.3) for the source operator is used, one finds that the operator

$$\hat{\boldsymbol{A}}(x) = A(x) + \boldsymbol{A}[A, o; x]$$

satisfies the integral equation

$$\hat{\boldsymbol{A}}(x) = \boldsymbol{A}_{in}(x) + \int \Delta_R(x-x')\,g\hat{\boldsymbol{A}}^2(x')\,d^4x' + A(x) - \int \Delta_R(x-x')\,gA^2(x')\,d^4x'.$$

Thus, if we define A_{in} and j by the classical field equation

$$A(x) = A_{in}(x) + \int \Delta_R(x-x')\{gA^2(x') + j(x')\}\,d^4x', \tag{1.5}$$

where

$$(-\Box + m^2)A_{in}(x) = o,$$

we have

$$\hat{\boldsymbol{A}}(x) = \boldsymbol{A}_{in}(x) + A_{in}(x) + \int \Delta_R(x-x')\{g\hat{\boldsymbol{A}}^2(x') + j(x')\}\,d^4x'.$$

In order to remove the classical radiation field A_{in}, we apply the time-independent unitary transformation*

$$\boldsymbol{U}[A_{in}] = exp(-i\int \boldsymbol{A}_{in}\frac{\overleftrightarrow{\partial}}{\partial y_o}A_{in}(y)\,d^3\vec{y}). \tag{1.6}$$

If one observes that

$$\boldsymbol{U}^\dagger[A_{in}]\,\boldsymbol{A}_{in}(x)\,\boldsymbol{U}[A_{in}] = \boldsymbol{A}_{in}(x) - A_{in}(x),$$

it is easily seen from the equation for $\hat{\boldsymbol{A}}$ that $\boldsymbol{U}^\dagger[A_{in}]\hat{\mathbf{A}}(x)\,\boldsymbol{U}[A_{in}]$ satisfies the integral equation which determines $\boldsymbol{A}[o, j; x]$. Thus, we have the relation

$$\boldsymbol{A}[o, j; x] = A(x) + \boldsymbol{U}^\dagger[A_{in}]\,\boldsymbol{A}[A, o; x]\,\boldsymbol{U}[A_{in}], \tag{1.7}$$

where the sources j and A_{in} are connected with the field A by the classical field equation (1.5) for the Hurst-Thirring field.

* We employ the notation: $f(x)\frac{\overleftrightarrow{\partial}}{\partial x_o}g(x) = f(x)\frac{\partial g(x)}{\partial x_o} - \frac{\partial f(x)}{\partial x_o}g(x)$.

The equation (1.7) is the mathematical expression for the assumption that sources j and A_{in} and a field A, which describe the same physical situation, according to the classical theory also do so when the classical system is coupled to a quantum system.

In a similar manner, one derives the more useful equation

$$\boldsymbol{A}[A, j+\delta j; x] = \delta A(x) + \boldsymbol{U}^{\dagger}[\delta A_{in}]\,\boldsymbol{A}[A+\delta A, j; x]\,\boldsymbol{U}[\delta A_{in}], \tag{1.8}$$

where the connection between the source variations δj, δA_{in} and the variation of the field is given by the varied form of the classical field equation, i. e.,

$$\left.\begin{aligned} \delta A(x) = \delta A_{in}(x) + \int \Delta_R(x-x')\left\{ g(\delta A(x'))^2 \right. \\ \left. + 2gA(x')\,\delta A(x') + \delta j(x') \right\} d^4x'. \end{aligned}\right\} \tag{1.9}$$

The proof given of (1.8), (1.9) is completely formal of course. However, these relations are in themselves meaningful mathematical expressions and we may assume that eqs. (1.8) and (1.9) are valid quite apart from the proof given. The formal derivation makes it plausible that, by such an assumption, essential characteristics of the dynamics are introduced in the theory. In fact, similar formal calculations with another expression for the operator source lead to a completely different result. This is also indicated by the fact that the characteristic non-linearity of the Hurst-Thirring field appears explicitly in equation (1.9).

We shall take (1.8), (1.9) as a basic assumption of the theory. It will be shown that such a postulate may be utilized in very much the same way as the formal field equation of the renormalization theory. The advantage gained is of course that we may maintain the attitude of ordinary mathematics that divergent quantities are allowed neither in the fundamental equations nor in any intermediate step of the calculations.

In paragraph 2, a list is given of the assumptions on which we propose to build a consistent formulation of the quantum theory of the Hurst-Thirring field. It will appear that (1.8) and (1.9), which we shall refer to as the variational equations for the field operator, are not totally of a dynamical nature. In fact, as a special result, we obtain from the variational equation for the field operator the reduction formulae of Lehmann, Symanzik and Zimmermann[1b] and of Nishijima[3a]. Thus, the asymptotic conditions are superfluous. The boundary conditions which are necessary to avoid solutions of irregular behaviour are discussed in paragraph 3. It is found that these boundary conditions may be formulated as a principle of maximum regularity of certain distributions, called π_R-functions, related to the source operator. The perturbation expansion is studied in paragraph 4, where it is shown that the theory gives unambiguous answers to the third order (and presumably to all orders) in the coupling constant. The results found for the source operator agree with those of the renormalization theory.

The present investigation is of a rather preliminary nature. Several important problems have not been solved. Presumably, the validity of the variational equation for the field operator may be "proved" in the framework of the renormalization theory. From the mathematical point of view such a proof would be as formal as that given here. Still, a proof should be given in order to ensure that the correct results of the renormalization theory are reproduced to all orders in the coupling constant in the present formulation. This question has not yet been considered. The assumption of unitarity is not needed for the unique characterization of the theory. Ultimately we shall therefore be faced with the problem to prove the existence of a scattering matrix. This question has not been considered either. Only the Hurst-Thirring field has been studied and it is well known that this theory is not quite typical in several respects.

The mathematical techniques used are presented in the usual language of mathematical physics. Thus, the technical language of modern distribution theory is avoided, although a certain not too low standard of mathematical rigour should be maintained as regards questions of distribution theory. In other respects we benefit from the advantages of a purely formal approach, in particular with regard to topological questions in the underlying Hilbert space. For the purpose of the present investigation this is not dangerous. In fact it is easily seen that the situation may be remedied by a strict adherence to the weak topology, i. e. all definitions and calculations may be interpreted as relations between definite matrix elements in the Hilbert space. However, whether the weak topology is the appropriate one for a more thorough study of the theory is an open question.

2. The basic equations of the theory

To give a precise formulation of the variational equation for the field operator we need the connection between the quantum field $\boldsymbol{A}(x)$ and the incoming field $\boldsymbol{A}_{in}(x)$. We assume that $\boldsymbol{A}(x)$ is so regular that

$$\int \Delta_R(x-x')(-\Box'+m^2)\boldsymbol{A}(x')\,d^4x' \tag{2.1}$$

exists as a convolution integral, and that the operator

$$\boldsymbol{A}_{in}(x) = \boldsymbol{A}(x) - \int \Delta_R(x-x')(-\Box'+m^2)\boldsymbol{A}(x')\,d^4x' \tag{2.2}$$

is independent of the external fields and sources, and is quantized in the usual way.* These assumptions involve two kinematical postulates, one concerning the distribution character of $\boldsymbol{A}(x)$, which serves to guarantee the existence of $\boldsymbol{A}_{in}(x)$, and a quantum rule which characterizes this operator.

For the special case of infinitesimal variations the fundamental dynamical assumption may now, in accordance with (1.8) and (1.9), be given in the following form:

$$\left.\begin{aligned}\int \frac{\delta \boldsymbol{A}(x)}{\delta j(y)} \delta j(y)\, d^4 y = \delta A(x) + \int \frac{\delta \boldsymbol{A}(x)}{\delta A(y)} \delta A(y)\, d^4 y \\ - i \int [\boldsymbol{A}(x), \boldsymbol{A}_{in}(y)] \frac{\overleftrightarrow{\partial}}{\partial y_0} \delta A_{in}(y)\, d^3 \vec{y},\end{aligned}\right\} \tag{2.3}$$

where

$$\delta j(x) = (-\Box + m^2 - 2\, g\, A(x))\, \delta A(x), \tag{2.4}$$

and

$$\delta A_{in}(x) = \delta A(x) - \int \Delta_R(x-x')(-\Box' + m^2)\, \delta A(x')\, d^4 x'. \tag{2.5}$$

A possible definition of the Volterra derivative is given in the Appendix B. As already mentioned in the Introduction, this postulate allows a derivation of the reduction formula.

As a final kinematical assumption we take the variational equation†, ††

$$\frac{\delta \boldsymbol{A}(x)}{\delta j(y)} = i\, \vartheta(x-y)\, [\boldsymbol{A}(x), \boldsymbol{A}(y)], \tag{2.6}$$

first proved by PEIERLS[5]. As is well known, this equation holds in the renormalized theory.

It will be seen in the next paragraph that the variational equations (2.3) and (2.6) have more than one solution. These equations should therefore be supplemented with subsidiary conditions, which excludes solutions of too irregular a behaviour. One such condition is the requirement of relativistic invariance. In the following it should be understood that only relativistically invariant solutions are admitted. The non-trivial question of the necessary boundary conditions will be discussed in the next paragraph.

Before taking up the discussion of the boundary conditions we derive

* Cf. Appendix A.

† In the present work only the vacuum expectation value of Peierls' variational equation is used.

†† For the Hurst-Thirring model it may be assumed that the commutator of two field operators is so regular that the right-hand side of (2.6) exists as a limit of $i\,\vartheta_\tau(x-y)\,[\boldsymbol{A}(x), \boldsymbol{A}(y)]$ where $\vartheta_\tau(x)$ is a sequence of testing functions which for $\tau \to 0$ converges (in the topology of the space of distributions) to the distribution $\vartheta(x)$. In practice this means that the retarded commutator may be treated as an ordinary product. Such a regularity assumption is not possible for the commutator of two source operators.

some direct consequences of the basic equations. First, we show that the reduction formula follows directly from the variational equation for the field operator.

Derivation of the reduction formula.

To derive the reduction formula we use twice the variational equation for the field operator. In the case of $\delta A_{in}(x) = o$, we find from (2.3)*

$$(K_y - 2gA(y))\frac{\delta \boldsymbol{A}(x)}{\delta j(y)} = \frac{\delta \boldsymbol{A}(x)}{\delta A(y)} + \delta(x-y), \tag{2.7}$$

while for $\delta j(x) = o$ the result is

$$o = \delta A(x) + \int \frac{\delta \boldsymbol{A}(x)}{\delta A(y)} \delta A(y)\, d^4 y - i \int [\boldsymbol{A}(x), \boldsymbol{A}_{in}(y)] \frac{\overleftrightarrow{\partial}}{\partial y_0} \delta A_{in}(y)\, d^3 \vec{y}, \tag{2.8}$$

where

$$\left.\begin{aligned} &\delta A(x) = \delta A_{in}(x) + \Delta_R\, 2gA(x)\, \delta A(x), \\ &K_x\, \delta A_{in}(x) = o. \end{aligned}\right\} \tag{2.9}$$

The conditions (2.9) should not be ignored in the derivation of the reduction formula, as these conditions severely limit the domain of the variations $\delta A(x)$. Thus, in the formula which results from (2.7) and (2.8),

$$\left.\begin{aligned} &\int \delta A(y)(K_y - 2gA(y)) \frac{\delta \boldsymbol{A}(x)}{\delta j(y)}\, d^4 y = \\ &+ i\int [\boldsymbol{A}(x), \boldsymbol{A}_{in}(y)] \frac{\overleftrightarrow{\partial}}{\partial y_o} \delta A_{in}(y)\, d^3 \vec{y}, \end{aligned}\right\} \tag{2.10}$$

an integration by parts is not permitted. Instead, we use

$$K_y\, \delta A(y) = 2gA(y)\, \delta A(y), \tag{2.11}$$

and find

$$\left.\begin{aligned} &\int \left\{ \delta A(y) K_y \frac{\delta \boldsymbol{A}(x)}{\delta j(y)} - \frac{\delta \boldsymbol{A}(x)}{\delta j(y)} K_y\, \delta A(y) \right\} d^4 y = \\ &+ i\int [\boldsymbol{A}(x), \boldsymbol{A}_{in}(y)] \frac{\overleftrightarrow{\partial}}{\partial y_o} \delta A_{in}(y)\, d^3 \vec{y}. \end{aligned}\right\} \tag{2.12}$$

* In the following we use for convenience the notations

$$K_x f(x) = (-\Box + m^2) f(x), \text{ and } \Delta_R f(x) = \int \Delta_R(x-x') f(x')\, d^4 x'.$$

Due to the retarded character of $\delta \boldsymbol{A}(x) / \delta j(y)$ we have

$$\int \Delta_R (y-z) K_y \frac{\delta \boldsymbol{A}(x)}{\delta j(y)} d^4 y = \frac{\delta \boldsymbol{A}(x)}{\delta j(z)}. \tag{2.13}$$

In fact, the difference between these two expressions vanishes for $x_0 < z_0$ and satisfies as a function of z the homogeneous wave equation. If, further, the relation

$$\Delta_R K_x \delta A(x) = \delta A(x) - \delta A_{in}(x) \tag{2.14}$$

is taken into account, we find from (2.12)

$$\int \delta A_{in}(y) K_y \frac{\delta \boldsymbol{A}(x)}{\delta j(y)} d^4 y = +i \int [\boldsymbol{A}(x), \ \boldsymbol{A}_{in}(y)] \frac{\overleftrightarrow{\partial}}{\partial y_0} \delta A_{in}(y) d^3 \vec{y}. \tag{2.15}$$

With the aid of the well-known solution of the initial value problem of the wave equation

$$\delta A_{in}(y) = -\int \Delta (y-z) \frac{\overleftrightarrow{\partial}}{\partial z_0} \delta A_{in}(z) d^3 \vec{z}, \tag{2.16}$$

we find from (2.15) the reduction formula for the field operator

$$[\boldsymbol{A}(x), \ \boldsymbol{A}_{in}(y)] = -i \int \Delta (y-z) K_z \frac{\delta \boldsymbol{A}(x)}{\delta j(z)} d^4 z, \tag{2.17}$$

which alternatively, due to (2.6), may be written in the usual form

$$[\boldsymbol{A}(x), \ \boldsymbol{A}_{in}(y)] = \int \Delta (y-z) K_z \vartheta (x-z) [\boldsymbol{A}(x), \ \boldsymbol{A}(z)] d^4 z. \tag{2.18}$$

Thus, the reduction formula gets a heuristic motivation in the formulation studied here. Further, it might be remarked that in the derivation we have not made use of asymptotic formulae, which in fact do not form a part of the basic assumptions of the theory.

For the discussion of the contents of the formulation proposed here, we found it convenient to work with the source operator instead of the field operator itself. For the Hurst-Thirring field, in the presence of an external field and external sources, the source operator $\boldsymbol{j}(x)$ is most conveniently defined by the equation

$$(-\Box + m^2) \boldsymbol{A}(x) = \boldsymbol{j}(x) + 2 g A(x) \boldsymbol{A}(x) + j(x), \tag{2.19}$$

whence by (2.2)

$$\boldsymbol{A}(x) = \boldsymbol{A}_{in}(x) + \int \Delta_R (x-x') \left\{ \boldsymbol{j}(x') + 2 g A(x') \boldsymbol{A}(x') + j(x') \right\} d^4 x'. \tag{2.20}$$

We shall refer to the equation (2.20) as the field equation.

In terms of the source operator, the variational equation (2.7) takes the form of the variational equation for the source operator, viz.

$$[K_y - 2gA(y)]\frac{\delta \boldsymbol{j}(x)}{\delta j(y)} = \frac{\delta \boldsymbol{j}(x)}{\delta A(y)} + 2g\,\delta(x-y)\,\boldsymbol{A}(x), \tag{2.21}$$

while the reduction formula (2.17) yields the reduction formula for the source operator

$$[\boldsymbol{j}(x),\ \boldsymbol{A}_{in}(y)] = -i\int \Delta(y-z)\,K_z\frac{\delta \boldsymbol{j}(x)}{\delta j(z)}\,d^4 z. \tag{2.22}$$

Explicit expressions in terms of the field operator for the variational derivatives of $\boldsymbol{j}(x)$ and for $\delta \boldsymbol{A}(x)/\delta A(y)$ may be found from (2.6), (2.7) and the field equation (2.19). As an example, we quote the formula

$$\left.\begin{aligned}\frac{\delta \boldsymbol{j}(x)}{\delta A(y)} &= [K_x - 2gA(x)]\,[K_y - 2gA(y)]\,i\vartheta(x-y)\,[\boldsymbol{A}(x),\ \boldsymbol{A}(y)]\\ &\quad - [K_x - 2gA(x)]\,\delta(x-y) - 2g\,\delta(x-y)\,\boldsymbol{A}(x).\end{aligned}\right\} \tag{2.23}$$

The coupling constant appears explicitly here, where it plays a role in the characterization of the singularity at $x = y$. Outside the singularity the expression simplifies to

$$\frac{\delta \boldsymbol{j}(x)}{\delta A(y)} = i\vartheta(x-y)\,[\boldsymbol{j}(x),\ \boldsymbol{j}(y)],\quad \text{for } x_o \neq y_o. \tag{2.24}$$

This expression is well known from the formal canonical theory[6a, b)], where the expression is assumed to cover the singularity for $x = y$ as well. It may easily be seen that the extrapolation of (2.24) and the corresponding expressions for $\delta\boldsymbol{j}/\delta j$ and $\delta\boldsymbol{A}/\delta A$ to all values of $x-y$ give the correct result if the commutation relations between $\boldsymbol{A}$, $\dot{\boldsymbol{A}}$ and $\boldsymbol{j}$ of the formal canonical theory are valid.

We see from (2.22) that, if the operator

$$\boldsymbol{j}(x;y) = K_y\frac{\delta \boldsymbol{j}(x)}{\delta j(y)} \tag{2.25}$$

is expanded in the series*

$$\left.\begin{aligned}\boldsymbol{j}(x;y) &= f(x;y) + \int f(x;y,\ 1)\,\boldsymbol{A}_{in}(1)\ d(1)\\ &\quad + \frac{1}{2!}\int f(x;y,\ 1,\ 2) : \boldsymbol{A}_{in}(1)\,\boldsymbol{A}_{in}(2) : d(12) + \ldots,\end{aligned}\right\} \tag{2.26}$$

* Haag[7)]. In the absence of bound states, the expansion functions f are c-numbers.

where :...: denotes the Wick product (se Appendix A), we have for the source operator

$$\left.\begin{aligned} \boldsymbol{j}(x) = f(x) + \int f(x;1)\,\boldsymbol{A}_{in}(1)\,d(1) \\ + \frac{1}{2!}\int f(x;1,\ 2):\boldsymbol{A}_{in}(1)\,\boldsymbol{A}_{in}(2):d(12) \\ + \frac{1}{3!}\int f(x;1,\ 2,\ 3):\boldsymbol{A}_{in}(1)\,\boldsymbol{A}_{in}(2)\,\boldsymbol{A}_{in}(3):d(123)+\ldots \end{aligned}\right\} \tag{2.27}$$

Here, $f(x) = \langle 0\,|\,\boldsymbol{j}(x)\,|\,0\rangle$ is not yet determined, but may be assumed to be subject to the boundary condition

$$\langle 0\,|\,\boldsymbol{j}(x)\,|\,0\rangle = f(x) = o, \quad \text{for } A = j = o. \tag{2.28}$$

We close this paragraph by a few comments on the formal theory which is obtained if (2.24) is extrapolated in a naive fashion to all values of x and y. In this formal theory the system of basic equations is easily seen to be complete. We take $j = A = o$ and find, by (2.21) and the extrapolated form of (2.24),

$$\boldsymbol{j}(x;y) = i\vartheta(x-y)\,[\boldsymbol{j}(x),\ \boldsymbol{j}(y)] + 2g\,\delta(x-y)\,\boldsymbol{A}(x), \quad \text{(wrong)}$$

where

$$\boldsymbol{A}(x) = \boldsymbol{A}_{in}(x) + \Delta_R\,\boldsymbol{j}(x).$$

These two equations determine the source operator anyhow if perturbation theory applies. To the lowest order in the coupling constant we find from (wrong)

$$\boldsymbol{j}(x;y) = 2g\,\delta(x-y)\,\boldsymbol{A}_{in}(x),$$

whence by (2.27) and (2.28) we find for the lowest order term in the source operator

$$\boldsymbol{j}(x) = g:\boldsymbol{A}_{in}^2(x):.$$

When this expression is inserted into the right-hand side of the equation (wrong) we find to the second order in g

$$\begin{aligned} \boldsymbol{j}(x;y) = i\vartheta(x-y)\,[g:\boldsymbol{A}_{in}^2(x):,\ g:\boldsymbol{A}_{in}^2(y):] \\ + 2g\,\delta(x-y)\,\boldsymbol{A}_{in}(x) + 2g\,\delta(x-y)\,\Delta_R\,g:\boldsymbol{A}_{in}^2(x):. \end{aligned}$$

Proceeding in this manner we obtain the perturbation theory. However, already the second order expression demonstrates that the naive extrapolation of (2.24) is indeed not possible. The vacuum expectation value of the first term on the right-hand side of the above expression leads to the well-known divergent expression for the self-mass of the meson and to a wave function renormalization, this being finite in the present model.

Still, the formal approach is not without interest. It shows that the axioms we have chosen are as complete as the usual axiomatic foundation of the formal canonical theory. It also demonstrates that, in order to have a complete dynamical theory, we must find the correct solution to (2.24), regarded as an equation for $\delta \boldsymbol{j}(x) / \delta A(y)$.

A direct approach to the multiplication problem (2.24) has not been found. The complete mathematical solution to the equation (2.24) involves an arbitrary distribution wich vanishes outside the subspace $x = y$, and what in particular complicates matters is, that this distribution is operator valued. We have instead chosen a more indirect method of investigation. In this method the arbitrary distribution is c-number valued and the correct solution may easily be characterized. The drawback of the method is that the external field has to be kept finite until the end of the calculations. This complicates somewhat the algebraic part of the calculations.

3. Discussion of the boundary conditions

In the following we take the external source equal to zero. Due to

$$\frac{\delta \boldsymbol{j}(x)}{\delta j(y)} = \int \Delta_R(z-y)\, K_z \frac{\delta \boldsymbol{j}(x)}{\delta j(z)}\, d^4 z, \tag{3.1}$$

we have, by (2.21) and (2.25),

$$[K_y - 2B(y)] \int \boldsymbol{j}(x;1)\, \Delta_R(1-y)\, d(1) = g\frac{\delta \boldsymbol{j}(x)}{\delta B(y)} + 2g\,\delta(x-y)\, \boldsymbol{A}(x), \tag{3.2}$$

where we have employed the notation

$$B(x) = g A(x).$$

Obviously only this combination is relevant for the problem. The discussion in the present paragraph as well as the explicit calculations in the next paragraph will be based on the equation (3.2), the reduction formula, which will be used in the form of the connection between the Haag series for $\boldsymbol{j}(x;y)$ and $\boldsymbol{j}(x)$ given by (2.26) and (2.27), and the equations

$$\left.\begin{aligned} g\frac{\delta \boldsymbol{j}(x)}{\delta B(y)} - g\frac{\delta \boldsymbol{j}(y)}{\delta B(x)} &= i\,[\boldsymbol{j}(x),\ \boldsymbol{j}(y)], \\ g\frac{\delta \boldsymbol{j}(x)}{\delta B(y)} &= o, \quad \text{for } y_o > x_o, \end{aligned}\right\} \tag{3.3}$$

which are a direct consequence of the explicit expression (2.23). Actually, as will become clear, we need (3.3) only in the vacuum subspace of the Hilbert space.

For $j(x) = o$ the field equation (2.20) reads

$$\boldsymbol{A}(x) = \boldsymbol{A}_{in}(x) + \Delta_R(\boldsymbol{j}(x) + 2B(x)\boldsymbol{A}(x)). \tag{3.4}$$

Consider for the moment the perturbation solution. Here we regard $\boldsymbol{j}$ as of at least the order g. For $g = o$ we have by the field equation

$$\boldsymbol{A}^{(o)}(x) = \boldsymbol{A}_{in}(x) + \Delta_R 2B(x)\boldsymbol{A}^{(o)}(x).$$

By the solution of this equation we find $2g\delta(x-y)\boldsymbol{A}^{(o)}(x)$, i. e. the right-hand side of (3.2), to the first order in g. Hence we may calculate $\boldsymbol{j}(x;y)$ to the first order in g. To discuss in general terms how to proceed, assume that $\boldsymbol{j}(x)$ is known to the order g^n as a functional of B. The right-hand side of (3.2) and thus $\boldsymbol{j}(x;y)$ may then be calculated to the order g^{n+1}. Hence by the reduction formula, or more directly by (2.26), (2.27), we find $\boldsymbol{j}(x) - \langle 0|\boldsymbol{j}(x)|0\rangle$ to the order g^{n+1}. To be able to proceed in the iteration procedure we need $\langle 0|\boldsymbol{j}(x)|0\rangle$ to the order g^{n+1} or alternatively $g\delta\langle 0|\boldsymbol{j}(x)|0\rangle/\delta B(y)$ to the order g^{n+2}. This is the point where the vacuum expectation value of the system (3.3) comes into play. By the knowledge of $\boldsymbol{j}(x) - \langle 0|\boldsymbol{j}(x)|0\rangle$ to the order g^{n+1} we can calculate

$$C(x;y) = -C(y;x) = i\langle 0|[\boldsymbol{j}(x), \boldsymbol{j}(y)]|0\rangle \tag{3.5}$$

to the order g^{n+2}. Thus, what we need is to solve the system of equations

$$\left.\begin{aligned} g\frac{\delta\langle 0|\boldsymbol{j}(x)|0\rangle}{\delta B(y)} - g\frac{\delta\langle 0|\boldsymbol{j}(y)|0\rangle}{\delta B(x)} &= C(x;y), \\ g\frac{\delta\langle 0|\boldsymbol{j}(x)|0\rangle}{\delta B(y)} &= o, \quad \text{for } y_o > x_o, \end{aligned}\right\} \tag{3.6}$$

where the functional $C(x;y)$ is known. Observe that C is real valued.

We shall take the perturbation argument as an indication of the fact that, if we can characterize that solution of the system (3.6) which should be used in physics, we have a well-defined formalism for the Hurst-Thirring field. We therefore proceed to discuss the system (3.6) and from now on drop the assumption of the perturbation expansion. Thus we have converted the problem of the $i\vartheta(x-y)[\boldsymbol{j}(x), \boldsymbol{j}(y)]$ multiplication into a similar, but simpler, problem where only c-number valued distributions are involved.

The complete solution of the system (3.6) consists of a particular solution added to the complete solution of the corresponding homogeneous system, viz.

$$\left.\begin{aligned} &\frac{\delta l(x)}{\delta B(y)} - \frac{\delta l(y)}{\delta B(x)} = o, \\ &\frac{\delta l(x)}{\delta B(y)} = o, \ \text{for} \ x \neq y. \end{aligned}\right\} \tag{3.7}$$

Due to the requirement of relativistic invariance we need only discuss the relativistically invariant solutions of the homogeneous equations. Hence we have replaced the condition $x_0 > y_0$, which for the homogeneous system is extended to $x_0 \neq y_0$, by the condition $x \neq y$.

A functional of a function $B(x)$ determines an infinite set of distributions which we take as the expansion coefficients of the formal Volterra series. For $l(x)$ we denote these distributions by

$$l_{4n}(x-y_1, x-y_2, \ldots, x-y_n) = \delta^n l(x)/\delta B(y_1)\,\delta B(y_2)\ldots\delta B(y_n)\big|_{B=o}, \tag{3.8}$$

where we have used a notation which reflects the invariance of l under displacements in space-time. For the discussion of the equations (3.7) we found it necessary to restrict the domain of solutions to functionals analytical in the sense that they are determined uniquely by the set of expansion coefficients of the formal Volterra series. It need not be assumed that the formal Volterra series is convergent. To indicate the one-to-one correspondence between the set of distributions l_{4n} and the functional l, we write*

$$l(x) \approx \sum_{n=1}^{\infty} \frac{1}{n!} \int l_{4n}(x-y_1, \ldots, x-y_n)\, B(y_1)\ldots B(y_n)\, d^4 y_1 \ldots d^4 y_n. \tag{3.9}$$

The distributions $l_{4n}(z_1, z_2, \ldots z_n)$ are of course symmetric in $z_1, z_2, \ldots, z_n$ and are invariant under the homogeneous Lorentz group. By the second equation (3.7), $l_{4n}(x-y, x-y_2, \ldots, x-y_n)$ vanishes outside the subspace $x = y$, and hence, by the symmetry, $l_{4n}(z_1, z_2, \ldots, z_n)$ vanishes outside the intersection of the subspaces $z_\nu = o$, $\nu = 1, 2, \ldots n$, i. e. except at the single point $z_1 = z_2 = \ldots = z_n = o$. Thus, by a well-known theorem in the theory of distributions, $l_{4n}(z_1, z_2, \ldots, z_n)$ is a finite linear combination of $\delta(z_1)\delta(z_2)\ldots\delta(z_n)$ and its derivatives, viz.

$$l_{4n}(z_1, z_2, \ldots, z_n) = P\left(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \ldots, \frac{\partial}{\partial z_n}\right) \delta(z_1)\,\delta(z_2)\ldots\delta(z_n). \tag{3.10}$$

Here, P is a symmetric relativistically invariant polynomial.

* The summation starts at $n=1$ due to $\langle 0 | j(x) | 0 \rangle |_{B=o} = o$.

We have not yet taken the first of the equations (3.7) into account. This equation restics the distribution l_{4n} by the condition that $l_{4n}(x_1-x_2,\ x_1-x_3,\ \ldots,\ x_1-x_{n+1})$ is symmetric in $x_1, x_2, \ldots,\ x_{n+1}$ as well. Hence, $l(x)$ may be represented in the form

$$l(x) = \frac{\delta\,\Theta\,[B]}{\delta\,B\,(x)}, \tag{3.11}$$

where Θ is associated with the formal Volterra series

$$\left.\begin{aligned}\Theta \approx \sum_{n=2}^{\infty} \frac{1}{n!} \int l_{4(n-1)}\,(x_1-x_2, x_1-x_3, \ldots, x_1-x_n)\,B(x_1)\,B(x_2)\ldots B(x_n)\\ \cdot\, d^4 x_1\, d^4 x_2 \ldots d^4 x_n.\end{aligned}\right\} \tag{3.12}$$

Thus, due to (3.10), we have

$$\Theta\,[B] = \int \mathfrak{L}\,(B(x),\ \partial_\mu B(x),\ \partial_\mu \partial_\nu B(x), \ldots)\, d^4 x, \tag{3.13}$$

where the density function $\mathfrak{L}$ is an ordinary function of $B(x)$ and the derivatives of $B(x)$*. The function $\mathfrak{L}$ may involve derivatives of $B(x)$ of arbitrarily high order, but should be in accordance with (3.10) and (3.12).

The results may conveniently be expressed in terms of the Fourier transform of the distributions. We define

$$\left.\begin{aligned} l_{4n}\,(z_1, z_2, \ldots, z_n) = \frac{1}{(2\pi)^{4n}} \int \mathfrak{P}\,(q_1, q_1, \ldots, q_n)\\ \times \exp\,(-i\,q_1\,z_1 - i\,q_2\,z_2 - \ldots - i\,q_n\,z_n)\, d^4 q_1\, d^4 q_2 \ldots d^4 q_n,\end{aligned}\right\} \tag{3.14}$$

and have the result: $\mathfrak{P}(q_1, q_2, \ldots, q_n)$ is a symmetric and Lorentz invariant polynomium, i. e.

* An interpretation of $\Theta[B]$ may be given in the following manner. It may be shown that the first variational derivative of a scattering operator for the system is given by

$$-i\,\frac{\delta\,\boldsymbol{S}}{\delta\,A\,(x)} = \boldsymbol{S}\,\boldsymbol{j}\,(x).$$

Here the operator $\boldsymbol{S}$ is defined up to a phase factor by the equation

$$\boldsymbol{A}_{out}\,(x) = \boldsymbol{S}^{\dagger}\,\boldsymbol{A}_{in}\,(x)\,\boldsymbol{S},$$

and the conditions of unitarity and causality. It is easily seen that the condition of causality restricts the arbitrary phase factor to the form $\exp\, i\Theta$, with Θ given by an expression of the form (3.13). If the source operator is known to the order g^n in perturbation theory, we find $\boldsymbol{S}$, apart from such a phase factor, to the order g^n, and hence $\boldsymbol{j}$ to the order g^{n+1} is given by the formula

$$\boldsymbol{j}\,(x) = -i\,g\,\boldsymbol{S}^{\dagger}\,\frac{\delta\,\boldsymbol{S}}{\delta\,B\,(x)} + g\,\frac{\delta\,\Theta}{\delta\,B\,(x)},$$

where Θ is of the type described but otherwise unknown.

Thus the discussion in the following would be made superfluous if a characterization of the c-number phase factor of this scattering operator could be found by other means.

$$\mathfrak{P}(q_1, q_2, \ldots, q_n) = \mathfrak{P}(-q_1^2, -q_2^2, \ldots, -q_n^2, q_1 q_2, q_1 q_3, \ldots, q_{n-1} q_n),$$

and is symmetric in the indices. This is the result which follows from the second of the equations (3.7). The restrictions on $\mathfrak{P}$ which follow from the first of these equations will not be discussed further here.

The boundary conditions.

We now return to the system (3.6). Let $\langle 0 | \boldsymbol{j}(x) | 0 \rangle$ be represented by the formal Volterra series*

$$\left.\begin{aligned} \langle 0 | \boldsymbol{j}(x) | 0 \rangle \approx \sum_{n=1}^{\infty} \frac{1}{n! g^n} \int \pi_R(x-y_1, \ldots, x-y_n) B(y_1) \ldots B(y_n) \\ \cdot d^4 y_1 \ldots d^4 y_n . \end{aligned}\right\} \qquad (3.15)$$

It follows from the discussion that, if distributions $\overset{\circ}{\pi}_R(x-y_1, \ldots, x-y_n)$ define a particular solution to the system (3.6), then

$$\pi_R(z_1, \ldots z_n) = \overset{\circ}{\pi}_R(z_1, \ldots, z_n) + P\left(\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_n}\right) \delta(z_1) \ldots \delta(z_n), \qquad (3.16)$$

where P is a polynomium of the type described.

Thus the boundary conditions we need are those which may serve to characterize coefficients of the polynomial $P(\partial/\partial z)$.

A simple boundary condition suggests itself at this place. It is tempting to require that the solutions should be regular at the origin, in other words, that there should be no δ-like singularity. Such a requirement gives the same (meaningsless) result as the naive approach to the $i\vartheta \cdot [\boldsymbol{j}, \boldsymbol{j}]$-multiplication problem. The point is that no such solution exists, and an enforcement of such an inconsistent requirement leads to mathematical inconsistencies which manifest themselves in terms of the well-known divergent integrals.

This is illustrated by a consideration of the simple example of the function $\pi_R(x-y)$. For this function we simply have the expression $i \langle 0 | [\boldsymbol{j}(x), \boldsymbol{j}(y)] | 0 \rangle |_{B=o}$, for the right-hand side of the first of the equations (3.6). It is easily seen that this expression may be represented in the form†

$$i \langle 0 | [\boldsymbol{j}(x), \boldsymbol{j}(o)] | 0 \rangle |_{B=o} = -\int \varepsilon(t) \delta(x^2 + \tau) f(\tau) d\tau, \qquad (3.17)$$

* In the formal theory the interpretation of the functions π_R is easily seen to be

$$\pi_R(x-y_1, \ldots, x-y_n) = \langle 0 | R(\boldsymbol{j}(x); \boldsymbol{j}(y_1) \ldots \boldsymbol{j}(y_n)) | 0 \rangle,$$

where R denotes the retarded product. This formal expression is ambiguous, and not of much use.

† See f. inst. the work of Gårding and Roos, reported in the lecture notes of Gårding and Lions [8]).

where $f(\tau)$ is a distribution with support on the positive part of the real τ-axes, $o \leqslant \tau < \infty$. If we split $\pi_R(x)$ in the symmetric part $\bar{\pi}(x) = \frac{1}{2}(\pi_R(x) + \pi_R(-x))$ and the antisymmetric part $\pi(x) = -\pi_R(x) + \pi_R(-x)$, we have according to (3.6) and (3.18)

$$\pi_R(x) = \bar{\pi}(x) - \frac{1}{2}\pi(x), \tag{3.18}$$

where

$$\pi(x) = \int \varepsilon(t)\,\delta(x^2+\tau)\,f(\tau)\,d\tau. \tag{3.19}$$

For $\bar{\pi}$ we now have the equation

$$\bar{\pi}(x) = -\frac{1}{2}\varepsilon(t)\,\pi(x), \quad \text{for } t \neq o. \tag{3.20}$$

If $f(\tau)$ is assumed to be sufficiently regular at the origin, we have the solution

$$\bar{\pi}(x) = -\frac{1}{2}\int \delta(x^2+\tau)\,f(\tau)\,d\tau. \tag{3.21}$$

However, for simple approximations, f. inst. in second order perturbation theory, we find that $f(\tau)$ is of the general type

$$f(\tau) \approx \delta'(\tau), \tag{3.22}$$

apart from regular terms, where δ' denotes the derivative of Dirac's δ-function. Assume, for the sake of the argument, that (3.22) is correct*. In this case, the expression (3.21) as it stands is without any mathematical meaning. To illustrate this point, we proceed in the calculations with complete disregard of the validity of the formal operations, and find the formal result

$$\bar{\pi}(x) = \frac{1}{2}\delta'(x^2) = \frac{1}{8rt}(\delta'(t-r) + \delta'(t+r)).$$

To see whether this expression makes sense we apply it to a testing function $\varphi(r, t)$, and find

$$\int \bar{\pi}(x)\,\varphi(r,t)\,d^4x = -\frac{\pi}{2}\int_0^\infty (\dot{\varphi}(r,r) - \dot{\varphi}(r,-r))\,dr + \frac{\pi}{2}\int_0^\infty \frac{\varphi(r,r)+\varphi(r,-r)}{r}\,dr.$$

* It may be seen that the assumption (3.22) essentially is equivalent to the assumption that the well-known spectral function $\Pi(\varkappa^2)$ in the Källén-Lehmann representation for the vacuum expectation value of the commutator of two source operators behaves like a constant for large values of $\varkappa^2$.

The last term diverges logarithmically at the origin. This of course is a manifistation of the well-known divergent self-energy. Thus the symbol $\delta'(x^2)$ does not represent a distribution.*

In the renormalization theory the situation is saved by an additional term in the solution of the type $\delta m^2 \delta(x)$, where δm^2 is a conveniently chosen logarithmically divergent constant. The renormalization theory gives of course the right result, but the detour over the mathematically undefined (divergent) expressions should be, and in fact is, superfluous.

As the above example shows, in our formulation the inconsistencies originate from mathematically inconsistent regularity assumptions. The mathematical form of (3.16) suggests a formulation of consistent regularity conditions in terms of the notion of the order of a distribution at a point (here the origin). Such a formulation may easily be given and might indeed be the most adequate one for the investigation of the fundamental problems of the theory, such as, for instance, the problem of the existence of rigorous solutions.

However, the concept of the order of a distribution at a point is difficult to work with in practical calculations, where concepts pertaining to the momentum space formulation are much more convenient. To avoid mathematical complications we assume that the π_R-functions are tempered distributions such that they possess a Fourier transform, and study the asymptotic behaviour of the Fourier transforms of the distributions instead of the singularity at the origin of the distributions themselves.

We here adopt a simple characterization of the asymptotic behaviour in momentum space of a tempered distribution given f. inst. by MEDVEDEV[9]. Let $T(z) = T(z_1, \ldots, z_n)$ be a tempered distribution with the Fourier transform $\tilde{T}(q_1, \ldots, q_n)$. We define the *rate of growth* at infinity in momentum space of $T(z)$ as the smallest integer $N = N(T)$ such that $\xi^{N+\alpha}$ increases faster than $\tilde{T}(\xi q_1, \ldots, \xi q_n)$ for $\xi \to \infty$ and any value of $\alpha > o$. Correspondingly, we call S more regular than T if $N(S) < N(T)$. If $N(T)$ is negative we simply say that T is regular. This ordering of distributions with respect to regularity is quite rough, as is illustrated by the remark that T and $T + P(\partial/\partial z)\delta(z)$ are equally regular whenever the degree of the polynomium P does not exceed $N(T)$. Still, the above characterization is sufficient for the discussion of the perturbation theory in the next paragraph. A more refined ordering of distributions with respect to regularity is proposed in the Appendix C.

* In contrast hereto, the distribution symbolized by $\varepsilon(t)\ \delta'(x^2)$ is perfectly well defined, as one sees by a similar calculation. A parallel, but simpler situation would arise in a two-dimensional theory (one x and one t), where $\delta(x^2-t^2)$ is divergent, but $\varepsilon(t)\ \delta(x^2-t^2)$ is convergent, if the usual way of treating δ-functions in mathematical physics is adopted.

Consider now the π_R-functions. If the system (3.6) does not admit regular solutions, we may instead look for the most regular solution. The rate of growth for the various π_R-functions of such a solution might be looked upon as constants characteristic of the singularity required by the interaction. Additional terms of the type $P(\partial/\partial z)\delta(z)$, when they make the π_R-functions less regular, introduce singularities of a complexity not required by the interaction. It is natural to assume that such singularities do not belong to the theory. This motivates the following formulation of the boundary conditions for the equations for the source operator: *The functions* $\pi_R(z_1, \ldots, z_n)$ *are as regular as compatible with the basic assumptions.* This boundary condition will be referred to as the principle of maximum regularity.

The principle of maximum regularity is not new, but has always been adhered to in the usual formulation of the theory. An explicit formulation of the principle may be found in the book by Bogoliubov and Shirkov[10]. In the renormalization theory one simply introduces such renormalization constants, only, as are required to remove divergences according to the experience from the perturbation theory.

If the rate of growth at infinity in momentum space of a π_R-function is N, an additional singularity of the type $P(\partial/\partial z)\delta(z)$, where the degree of P does not exceed N, is left arbitrary. The coefficients of the various terms in the polynomial are thus arbitrary constants in the theory so far formulated. If this situation should occur, these constants should be determined by further conditions of the character of normalization conditions. Presumably no such arbitrary constants remain in the case of the Hurst-Thirring field. The situation in this respect might be different in, for instance, the π-meson theory, this theory being more singular. Alternatively such constants could be regarded as subject to physical interpretation, and thus as parameters belonging to the theory in the same way as the mass and the coupling constant.

4. Perturbation theory

As mentioned in the Introduction, we have not yet been able to prove that the theory proposed here gives the same results as the usual theory to all orders in the perturbation expansion. In the absence of a general proof we show in this paragraph that the two theories agree to the third order in the coupling constant. As the methods of calculation are somewhat different from the usual methods we present the third order calculation in some detail.

In perturbation theory we assume that the source operator is at least of the order g, and expand in powers of g. When only results for $B = o$ are desired, it suffices to expand in powers of B as well. To indicate for a quantity F the sum of all terms of order $g^{\varrho} B^{\sigma}$, with $\varrho \leqslant r$ and $\sigma \leqslant s$, we employ the symbol $F^{(r;s)}$. It is easily seen that in the n'th order calculation we must calculate all terms $\boldsymbol{j}(x;y)^{(r;s)}$, with $r+s \leqslant n$ and $s<n$.

Second order perturbation theory.

To the first order in g and for $B = o$ we find from the variational equation for the source operator (3.2)

$$\boldsymbol{j}(x;y)^{(1;0)} = 2g\,\delta(x-y)\,\boldsymbol{A}_{in}(x); \tag{4.1}$$

whence, by (2.26) and (2.27),

$$\boldsymbol{j}(x)^{(1;0)} = g:\boldsymbol{A}_{in}^2(x):. \tag{4.2}$$

For $g = o$ and to the first order in B we have, by the field equation (3.4),

$$\boldsymbol{A}(x)^{(0;1)} = \boldsymbol{A}_{in}(x) + 2\int \varDelta_R(x-1)\,B(1)\,\boldsymbol{A}_{in}(1)\,d(1). \tag{4.3}$$

These formulae allow us to calculate the operator $\boldsymbol{j}(x;y)^{(1;1)}$ from (3.2). From the resulting expression

$$\left.\begin{aligned}\boldsymbol{j}(x;y)^{(1;1)} = {} & 2g\,\delta(x-y)\,\boldsymbol{A}_{in}(x) + 4g\,\delta(x-y)\int \varDelta_R(x-1)\,B(1)\,\boldsymbol{A}_{in}(1)\,d(1)\\ & + 4g\,\boldsymbol{A}_{in}(x)\,\varDelta_R(x-y\,B(y)\end{aligned}\right\} \tag{4.4}$$

one finds, by the use of (2.26) and (2.27),

$$\left.\begin{aligned}\boldsymbol{j}(x)^{(1;1)} - \langle 0\,|\,\boldsymbol{j}(x)\,|\,0\rangle^{(1;1)} = {} & g:\boldsymbol{A}_{in}^2(x):\\ & + 4g\int \varDelta_R(x-1)\,B(1):\boldsymbol{A}_{in}(x)\,\boldsymbol{A}_{in}(1):d(1).\end{aligned}\right\} \tag{4.5}$$

As explained in the beginning of paragraph 3, we determine the unknown vacuum expectation value $\langle 0|\boldsymbol{j}(x)|0\rangle^{(1;1)}$ by means of the system (3.6). A simple calculation gives for the right-hand side of the first of the equations (3.6) the result

$$\left.\begin{aligned}C(x)^{(2;0)} &= i\langle 0\,|\,[\boldsymbol{j}(x)^{(1;0)}, \boldsymbol{j}(o)^{(1;0)}]\,|\,0\rangle\\ &= -\int_{4m^2}^{\infty} \varDelta(x;\varkappa^2)\,\Pi^{(2)}(\varkappa^2)\,d\varkappa^2,\end{aligned}\right\} \tag{4.6}$$

with

$$\Pi^{(2)}(\varkappa^2) = \frac{g^2}{8\pi^2}\sqrt{\frac{\varkappa^2-4m^2}{\varkappa^2}}. \tag{4.7}$$

Thus, to this order, the system (3.6) becomes

$$\left.\begin{aligned} &\pi_R(x)^{(2;0)} - \pi_R(-x)^{(2;0)} = -\int_{4m^2}^{\infty} \varDelta(x;\varkappa^2)\, \Pi^{(2)}(\varkappa^2)\, d\varkappa^2, \\ &\pi_R(x)^{(2;0)} = o, \quad \text{for } x_o < o. \end{aligned}\right\} \tag{4.8}$$

We have here used

$$\frac{g\,\delta \langle 0 \,|\, \boldsymbol{j}(x) \,|\, 0 \rangle^{(1;1)}}{\delta B(y)} = \pi_R(x-y)^{(2;0)}. \tag{4.9}$$

It is easily seen that, for $\pi_R(x)^{(2;0)}$, the naive ϑ-multiplication leads to a meaningless result. However, a particular solution to the system (4.8) is easily obtained by means of the identity $(-\Box + a^2)\varDelta(x;\varkappa^2) = (a^2 - \varkappa^2)\varDelta(x;\varkappa^2)$. For the sake of convenience, we choose $a^2 < 4m^2$ and have the solution

$$\mathring{\pi}_R(x)^{(2;0)} = (-\Box + a^2)\int_{4m^2}^{\infty} \frac{\varDelta_R(x;\varkappa^2)}{a^2 - \varkappa^2}\, \Pi^{(2)}(\varkappa^2)\, d\varkappa^2. \tag{4.10}$$

Hence, the complete solution to (4.8) is given by

$$\pi_R(x)^{(2;0)} = \mathring{\pi}_R(x)^{(2;0)} + g^2 c_o\, \delta(x) + \ldots + g^2 c_N (-\Box)^N \delta(x), \tag{4.11}$$

where $c_o, c_1, \ldots, c_N$ are constants and N is an arbitrary positive integer. By the use of (4.5) and the above expression for the right-hand side of (4.9) we find, from (3.2),

$$\left.\begin{aligned} \boldsymbol{j}(x;y)^{(2;0)} &= \pi_R(x-y)^{(2;0)} + 2g\,\delta(x-y)\,\boldsymbol{A}_{in}(x) \\ &+ 2g^2 \int \varDelta_R(x-1) : \boldsymbol{A}_{in}^2(1) : d(1)\,\delta(x-y) \\ &+ 4g^2\, \varDelta_R(x-y) : \boldsymbol{A}_{in}(x)\,\boldsymbol{A}_{in}(y) : . \end{aligned}\right\} \tag{4.12}$$

Following the general pattern we next obtain $\boldsymbol{j}(x)$, to the same order, with the aid of (2.26) and (2.27). Due to $-\Box \boldsymbol{A}_{in}(x) = -m^2 \boldsymbol{A}_{in}(x)$ and the general formula

$$\int \mathring{\pi}_R(x-y)\,\boldsymbol{A}_{in}(y)\, d^4 y = c'\,\boldsymbol{A}_{in}(x), \tag{4.13}$$

where c' is a constant, we find

$$\left.\begin{aligned} \boldsymbol{j}(x)^{(2;0)} &= c''\boldsymbol{A}_{in}(x) + g : \boldsymbol{A}_{in}^2(x) : \\ &+ 2g^2 \int \varDelta_R(x-1) : \boldsymbol{A}_{in}(x)\,\boldsymbol{A}_{in}^2(1) : d(1). \end{aligned}\right\} \tag{4.14}$$

Here the constant c'' is given by $c'' = c' + g^2 c_o + \ldots + g^2(-m^2)^N c_N$. To fulfill the regularity condition that (2.1) exists as a convolution integral, $c'' = o$

is required in order to avoid a $\delta(p^2+m^2)^2$—catastrophe in momentum space. Hence, by the field equation (3.4), we find the complete second-order expression for the field operator, viz.

$$\left.\begin{aligned} \boldsymbol{A}(x)^{(2;0)} &= \boldsymbol{A}_{in}(x) + \int \Delta_R(x-1)\, g : \boldsymbol{A}^2_{in}(1) : d(1) \\ &+ 2g^2 \int \Delta_R(x-1)\, \Delta_R(1-2) : \boldsymbol{A}_{in}(1)\, \boldsymbol{A}^2_{in}(2) : d(12). \end{aligned}\right\} \quad (4.15)$$

The function $\pi_R(x)^{(2;0)}$ may now be calculated with the aid of the explicit expression (2.23). A simple calculation gives the result

$$\left.\begin{aligned} \pi_R(x-y)^{(2;0)} &= g\,\frac{\delta\langle 0 \,|\, \boldsymbol{j}(x) \,|\, 0\rangle^{(1;1)}}{\delta B(y)} \\ &= K_x^2 \int_{4m^2}^{\infty} \frac{\Delta_R(x-y;\varkappa^2)}{(m^2-\varkappa^2)^2}\, \Pi^{(2)}(\varkappa^2)\, d\varkappa^2. \end{aligned}\right\} \quad (4.16)$$

From (4.16) we find $c_2 = \ldots = c_N = o$, and explicit expressions for c_0 and c_1 could be found. The results (4.16) and (4.15) are of course the well-known results of the renormalization theory.

Thus we see that, in the second order approximation, the function $\pi_R(x-y)$ is uniquely determined without the use of the principle of maximum regularity. Indeed, it may be shown by similar considerations as that above that this result is exactly true on the assumption that the spectral function $\Pi(\varkappa^2)$ in the Källén-Lehmann representation for $i\langle 0 \,|\, [\boldsymbol{j}(x), \boldsymbol{j}(o)] \,|\, 0\rangle\,|_{B=o}$ is bounded for large values of $\varkappa^2$. If this is true one finds, as above, that the first of the π_R-functions is given by

$$\pi_R(x) = K_x^2 \int_{4m^2}^{\infty} \frac{\Delta_R(x;\varkappa^2)}{(m^2-\varkappa^2)^2}\, \Pi(\varkappa^2)\, d\varkappa^2. \quad (4.17)$$

This expression gives the well-known result for he polarization of the vacuum by a weak external field.

Third order perturbation theory.

For the higher-order calculations, the principle of maximum regularity is needed to determine the π_R-functions depending on two or more variables $z_1, z_2, \ldots$ It will be convenient to have a notation for the terms in a quantity $F^{(r;s)}$ which are proportional to $g^r B^s$. We shall denote these terms by the symbol $F^{(\mathbf{r};\mathbf{s})}$. Thus $\boldsymbol{j}(x)^{(3;0)} = \boldsymbol{j}(x)^{(\mathbf{3};\mathbf{0})} + \boldsymbol{j}(x)^{(2;0)}$. Hence, in the third order calculation, as we already know $\boldsymbol{j}(x)^{(2;0)}$, we need only calculate $\boldsymbol{j}(x)^{(\mathbf{3};\mathbf{0})}$.

To calculate this operator, we start from $\boldsymbol{j}(x;y)^{(1;2)}$ and follow the method outlined in the beginning of paragraph 3. By the variational equation for the source operator (3.2) we find

$$\boldsymbol{j}(x;y)^{(1;2)} = 2g\,\delta(x-y)\,\boldsymbol{A}(x)^{(0;2)} + \int \boldsymbol{j}(x;z)^{(1;1)}\,\Delta_R(z-y)\,d^4z\,2B(y), \tag{4.18}$$

where due to (4.4)

$$\left.\begin{aligned} \boldsymbol{j}(x;z)^{(1;1)} &= 4g\,\delta(x-z)\int \Delta_R(x-1)\,B(1)\,\boldsymbol{A}_{in}(1)\,d(1) \\ &\quad + 4g\,\boldsymbol{A}_{in}(x)\,\Delta_R(x-z)\,B(z), \end{aligned}\right\} \tag{4.19}$$

while $\boldsymbol{A}(x)^{(0;2)}$ is given by the field equation (3.4), viz.

$$\boldsymbol{A}(x)^{(0;2)} = \int \Delta_R(x-1)\,2B(1)\,\Delta_R(1-2)\,2B(2)\,\boldsymbol{A}_{in}(2)\,d(12). \tag{4.20}$$

Application of (2.26) and (2.27) — i. e. of the reduction formula for the source operator — yields the result

$$\left.\begin{aligned} &\boldsymbol{j}(x)^{(1;2)} - \langle 0\,|\,\boldsymbol{j}(x)^{(1;2)}\,|\,0\rangle \\ &\quad = 8g\int \Delta_R(x-1)\,B(1)\,\Delta_R(1-2)\,B(2) : \boldsymbol{A}_{in}(x)\,\boldsymbol{A}_{in}(2) : d(12) \\ &\quad + 4g\int \Delta_R(x-1)\,\Delta_R(x-2)\,B(1)\,B(2) : \boldsymbol{A}_{in}(1)\,\boldsymbol{A}_{in}(2) : d(12). \end{aligned}\right\} \tag{4.21}$$

According to the methods described in paragraph 3, the next step in the calculation consists in the evaluation of $g\langle 0\,|\,\delta\boldsymbol{j}(x)^{(1;2)}/\delta B(y)\,|\,0\rangle$ by the aid of the system (3.6). By (3.5)

$$C(x;y)^{(2;1)} = i\langle 0\,|\,[\boldsymbol{j}(x)^{(1;1)}, \boldsymbol{j}(y)^{(1;0)}]\,|\,0\rangle - (x \longleftrightarrow y), \tag{4.22}$$

where the relevant source operators are given by (4.5). One finds

$$C(x;y)^{(2;1)} = \int K(x,y;z)\,B(z)\,d^4z, \tag{4.23}$$

where the kernel $K(x,y;z)$ is given by

$$\left.\begin{aligned} K(x,y;z) &= 4g^2\,\Delta_R(x-z)\left\{\Delta^{(1)}(z-y)\,\Delta(y-x) - \Delta(z-y)\,\Delta^{(1)}(y-x)\right\} \\ &\quad - 4g^2\,\Delta_R(y-z)\left\{\Delta^{(1)}(z-x)\,\Delta(x-y) - \Delta(z-x)\,\Delta^{(1)}(x-y)\right\}. \end{aligned}\right\} \tag{4.24}$$

In this case, the formal solution of the system (3.6), i. e.

$$g\,\frac{\delta\langle 0\,|\,\boldsymbol{j}(x)^{(1;2)}\,|\,0\rangle}{\delta B(y)} = \frac{1}{g}\int \pi_R(x-y, x-z)^{(3;0)}\,B(z)\,d^4z, \tag{4.25}$$

with

$$\left.\begin{aligned} \pi_R(x-y, x-z)^{(3;0)} &= 4g^3\,\Delta_R(x-y)\left\{\Delta_R(x-z)\,\Delta^{(1)}(z-y)\right. \\ &\quad \left. + \Delta_R(y-z)\,\Delta^{(1)}(z-x)\right\} \\ &\quad + 4g^3\,\Delta_R(x-z)\,\Delta_R(z-y)\,\Delta^{(1)}(x-y) \end{aligned}\right\} \tag{4.26}$$

has a meaning, and defines a distribution. Simple considerations show that this distribution is regular in the sense defined in paragraph 3. Further, it is easily verified that all of the basic equations are fulfilled by the corresponding expression for $\boldsymbol{j}(x)^{(1;2)}$. The solution (4.26) is thus the one required by the principle of maximum regularity.

We may now calculate $\boldsymbol{j}(x;y)^{(2;1)}$. By (3.2)

$$\left.\begin{aligned} \boldsymbol{j}(x;y)^{(2;1)} &= 2g\,\delta(x-y)\,\boldsymbol{A}(x)^{(1;1)} + g\,\frac{\delta \boldsymbol{j}(x)^{(1;2)}}{\delta B(y)} \\ &+ \int \boldsymbol{j}(x;z)^{(2;0)}\,\Delta_R(z-y)\,d^4 z\,2B(y). \end{aligned}\right\} \tag{4.27}$$

Here $\boldsymbol{A}(x)^{(1;1)}$ may be obtained from the field equation (3.4) and the formula

$$\left.\begin{aligned} \boldsymbol{j}(x)^{(1;1)} &= \frac{1}{g}\int \pi_R(x-1)^{(2;0)}\,B(1)\,d(1) \\ &+ 4g\int \Delta_R(x-1)\,B(1) : \boldsymbol{A}_{in}(x)\,\boldsymbol{A}_{in}(1) : d(1), \end{aligned}\right\} \tag{4.28}$$

which is a consequence of (4.5) and (4.9). Thus,

$$\left.\begin{aligned} \boldsymbol{A}(x)^{(1;1)} &= \frac{1}{g}\int \Delta_R(x-1)\,\pi_R(1-2)^{(2;0)}\,B(2)\,d(12) \\ &+ 4g\int \Delta_R(x-1)\,\Delta_R(1-2)\,B(2) : \boldsymbol{A}_{in}(1)\,\boldsymbol{A}_{in}(2) : d(12) \\ &+ 2g\int \Delta_R(x-1)\,B(1)\,\Delta_R(1-2) : \boldsymbol{A}_{in}^2(2) : d(12). \end{aligned}\right\} \tag{4.29}$$

An expression for $g\,\delta \boldsymbol{j}(x)^{(1;2)}/\delta B(y)$ may be found from (4.21) and (4.25). Finally, $\boldsymbol{j}(x;y)^{(2;0)}$ is obtained from the expression (4.12). In this way $\boldsymbol{j}(x;y)^{(2;1)}$ may be calculated. The result is

$$\boldsymbol{j}(x;y)^{(2;1)} = \langle 0 \,|\, \boldsymbol{j}(x;y)^{(2;1)} \,|\, 0 \rangle + \boldsymbol{j}(x;y)_{\mathrm{II}}^{(2;1)}, \tag{4.30}$$

where $\boldsymbol{j}(x;y)_{\mathrm{II}}^{(2;1)}$ denotes a two-particle term. The vacuum expectation value is found to be

$$\left.\begin{aligned} \langle 0 \,|\, \boldsymbol{j}(x;y)^{(2;1)} \,|\, 0 \rangle &= 2\delta(x-y)\int \Delta_R(x-1)\,\pi_R(1-2)^{(2;0)}\,B(2)\,d(12) \\ &+ 2\int \pi_R(x-1)^{(2;0)}\,\Delta_R(1-y)\,d(1)\,B(y) \\ &+ (1/g)\int \pi_R(x-y,\,x-1)^{(3;0)}\,B(1)\,d(1), \end{aligned}\right\} \tag{4.31}$$

while the two-particle part of the operator $\boldsymbol{j}(x;y)^{(2;1)}$ becomes identical with the (meaningful) expression found by the application of the formal unrenormalized canonical theory. The first two terms on the right-hand side of (4.31) originate from the first and the last term on the right-hand side of (4.27), respectively. Thus both these "dangerous" terms are brought into the third order calculation not by the second term on the right-hand side of (4.27), but by the two other terms and in the already properly normalized form obtained by the second order calculation.

By the expression for $\boldsymbol{j}(x;y)^{(2;1)}$ and by (2.26) and (2.27) we find $\boldsymbol{j}(x)^{(2;1)} - \langle 0 | \boldsymbol{j}(x)^{(2;1)} | 0 \rangle$. It is easily seen that, in fact, the vacuum expectation value is equal to zero. For by (3.6)

$$g \frac{\delta \langle 0 | \boldsymbol{j}(x)^{(2;1)} | 0 \rangle}{\delta B(y)} - (x \leftrightarrow y) = C^{(3;0)}(x;y), \tag{4.32}$$

where by (3.5)

$$C(x;y)^{(3;0)} = i \langle 0 | [\boldsymbol{j}(x)^{(2;0)}, \boldsymbol{j}(y)^{(1;0)}] | 0 \rangle - (x \leftrightarrow y), \tag{4.33}$$

i. e. by (4.14)

$$\left.\begin{aligned} C(x;y)^{(3;0)} = 2ig^3 \int \Delta_R(x-1) \langle 0 | [:\boldsymbol{A}_{in}(x)\boldsymbol{A}^2_{in}(1):, :\boldsymbol{A}^2_{in}(y):] | 0 \rangle \\ - (x \leftrightarrow y). \end{aligned}\right\} \tag{4.34}$$

Hence

$$C(x;y)^{(3;0)} = o, \tag{4.35}$$

due to the fact that the vacuum expectation value is required for an odd number of incoming fields. It is now easily seen that the principle of maximum regularity requires

$$g \frac{\delta \langle 0 | \boldsymbol{j}(x)^{(2;1)} | 0 \rangle}{\delta B(y)} = o. \tag{4.36}$$

The equation (3.2) gives the expression

$$\boldsymbol{j}(x;y)^{(3;0)} = 2g\delta(x-y)\boldsymbol{A}(x)^{(2;0)} + g \frac{\delta \boldsymbol{j}(x)^{(2;1)}}{\delta B(y)}. \tag{4.37}$$

Here $\boldsymbol{A}(x)^{(2;0)}$ is given by (4.15) and $\boldsymbol{j}(x)^{(2;1)}$ is known already. From the resulting expression for $\boldsymbol{j}(x;y)^{(3;0)}$ and with the aid of equations (2.26) and (2.27) one finally finds the result, well known from the renormalization theory, that

$$\boldsymbol{j}(x)^{(3;0)} = \boldsymbol{j}(x)^{(3;0)}_{\mathrm{II}} + \boldsymbol{j}(x)^{(3;0)}_{\mathrm{IV}}, \tag{4.38}$$

where the two-particle part is given by

$$\left.\begin{aligned} \boldsymbol{j}(x)^{(3;0)}_{\mathrm{II}} = g \int \pi_R(x-1)^{(2;0)} \Delta_R(1-2) :\boldsymbol{A}^2_{in}(2): d(12) \\ + 2g^3 \int \Delta_R(x-1) \Delta_R(x-2) \Delta^{(1)}(1-2) :\boldsymbol{A}_{in}(1)\boldsymbol{A}_{in}(2): d(12) \\ + 4g^3 \int \Delta_R(x-1) \Delta^{(1)}(x-2) \Delta_R(1-2) :\boldsymbol{A}_{in}(1)\boldsymbol{A}_{in}(2): d(12), \end{aligned}\right\} \tag{4.39}$$

while the four-particle part* $\boldsymbol{j}(x)^{(3;0)}_{\mathrm{IV}}$ is identical with the four-particle part found by use of the formal theory.

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APPENDIX A

The notations used for the theory of free mesons

We use the pseudo-Euclidean metric, $x = (x_1, x_2, x_3, x_4)$ where $x_4 = i x_0$. Further, $x^2 = x_\mu x_\mu = \vec{x}^2 - x_0^2$, and $-\Box = -\Delta + \partial^2/\partial x_0^2$. The free field operators $\boldsymbol{A}_{in}(x)$ are self-adjoint operator valued distributions which satisfy the commutation relations

$$[\boldsymbol{A}_{in}(x), \boldsymbol{A}_{in}(y)] = i\Delta(x-y),$$

where

$$\Delta(x-y) = -i(2\pi)^{-3}\int d^4p\, \varepsilon(p_0)\, \delta(p^2+m^2) \exp ipx.$$

Here, $\varepsilon(p_0) = p_0/|p_0|$. The retarded Greens function is $\Delta_R(x) = -\vartheta(x)\Delta(x)$, where $\vartheta(x)$ is the Heaviside function

$$\vartheta(x) = \begin{cases} 1, \text{ for } x_0 > o \\ o, \text{ for } x_0 < o. \end{cases}$$

The value of $\vartheta(x)$ for $x_0 = o$ is not important.

The Wick product is denoted by $:\ldots:$, and designates that the operators inside the double dots are ordered such that any positive frequency part stands to the right of any negative frequency part. Useful rules for the calculation with Wick products are found for instance in the book by BOGOLIUBOV and SHIRKOV[10].

* I. e. an operator of the form

$$\frac{1}{4!}\int f(x;1,2,3,4){:}\boldsymbol{A}_{in}(1)\,\boldsymbol{A}_{in}(2)\,\boldsymbol{A}_{in}(3)\,\boldsymbol{A}_{in}(4){:}\,d(1234).$$

APPENDIX B

Volterra derivatives

Volterra derivatives seem to be an indispensable mathematical tool for the theory proposed here. It might therefore be useful to give a brief introduction to the theory of variational derivatives.

Consider a functional $\Phi[j]$, which maps from a certain space of functions j into the complex plane. Useful definitions of differentiability of such functionals are all of the following general type: The functional Φ is called differentiable of j if the variation of Φ is of the form

$$\Phi[j+\delta j]-\Phi[j]=\int \Psi[j;x]\,\delta j(x)\,d^4x+o[j,\delta j], \tag{B. 1}$$

where $\int \Psi[j;x]\,\delta j(x)\,d^4x$ is a linear functional of δj and $o[j,\delta j]$ has certain properties. Roughly speaking, it is required that for j fixed and $\delta j\to o$, $o[j,\delta j$ tends to zero "faster than" δj. Hence, to give a precise definition of differentiability, one has to specify

(i) the meaning of $\delta j\to o$, and
(ii) the meaning of the term "faster than".

For any such specification, we call $\Psi[j;x]$ the Volterra derivative of Φ, and use the symbol

$$\Psi[j;x]=\frac{\delta\Phi[j]}{\delta j(x)}. \tag{B.2}$$

A simple possibility is the following:

(i) $\delta j\to o$ means $\delta j=\xi j^{(1)}$, where $j^{(1)}$ is fixed and $\xi\to o$,
(ii) $o[j,\xi j^{(1)}]$ is required to be $o(\xi)$ for j and $j^{(1)}$ fixed, i. e.

$$\lim_{\xi\to o}\frac{o[j,\xi j^{(1)}]}{\xi}=o.$$

This immediately leads to the relation

$$\int \Psi[j;x]\,j^{(1)}(x)\,d^4x=\left.\frac{d\Phi[j+\xi j^{(1)}]}{d\xi}\right|_{\xi=o}. \tag{B.3}$$

We stress here that, to our knowledge, no argument is known which indicates that this particular definition is the most adequate one for use in the quantum field theory. However, other reasonable definitions seem to be more restrictive. Thus, in general, the existence of the right-hand side of (B. 3) will be a necessary condition for differentiability.

As mentioned in the Introduction, when operator valued functionals are considered, the weak topology in the Hilbert space may be used to give the relations in the text a well defined meaning. If further the definition (B. 3) is adopted, and if the field operator is regarded as an operator valued distribution, a symbol like $\delta \boldsymbol{A}(x)/\delta j(y)$ becomes endowed with the interpretation

$$\langle A \,|\int \frac{\delta \boldsymbol{A}(x)}{\delta j(y)}\, \psi(x)\, d^4x \,|\, B\rangle = \frac{\delta}{\delta j(y)} \langle A \,|\, \boldsymbol{A}(x)\, \psi(x)\, d^4x \,|\, B\rangle, \tag{B. 4}$$

where $\psi(x)$ is a testing function for the operator valued distribution $\boldsymbol{A}(x)$, and the variational derivative on the right-hand side is the one defined above.

Finally, a remark about certain interchanges of limiting processes, frequently performed in the text, may be in its place. As an example consider the relation

$$(-\Box_x + m^2) \frac{\delta \boldsymbol{A}(x)}{\delta j(y)} = \frac{\delta}{\delta j(y)} (-\Box_x + m^2)\, \boldsymbol{A}(x). \tag{B. 5}$$

If the possible interpretation mentioned above is employed, this equation means

$$\left.\begin{aligned} &\frac{d}{d\xi}(-\Box_x + m^2) \langle A \,|\, \boldsymbol{A}[j + \xi j^{(1)}; x] \,|\, B\rangle \\ &\quad = (-\Box_x + m^2) \frac{d}{d\xi} \langle A \,|\, \boldsymbol{A}[j + \xi j^{(1)}; x] \,|\, B\rangle \end{aligned}\right\} \tag{B. 6}$$

for $\xi = o$. This only requires an interchange of two ordinary differential operators. The validity of such relations is assumed in the text.

APPENDIX C

On the formulation of the principle of maximum regularity

In paragraph 3 the concept of the rate of growth at infinity in momentum space was used to formulate the principle of maximum regularity. Such a formulation is satisfactory due to its simplicity, but might not always work. It presupposes that the π_R-functions are tempered distributions for which the Fourier transforms are functions for large values of the momenta. Both these properties might be difficult to prove without recourse to an approximation method. In this appendix we propose an ordering of distributions with respect to regularity which avoids these problems. Only standard notions in the theory of distributions are used, and for these we refer to the book of L. SCHWARTZ[11].

We consider distributions $T(\xi)$ defined on a f-dimensional Euclidean space, $\xi = (\xi_1, \xi_2, \ldots, \xi_f)$. Differential operators are denoted by $D^\alpha = \partial^{\alpha_1+\alpha_2+\ldots+\alpha_f}/\partial\xi_1^{\alpha_1}\, \partial\xi_2^{\alpha_2} \ldots \partial\xi_f^{\alpha_f}$, and $|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_f$ is the degree of the differential operator. As only local properties are considered, we need not specify the type of the distribution. Let $\mathfrak{O}$ be an open *bounded* set and $D(\mathfrak{O})$ the totality of testing functions which vanish outside $\mathfrak{O}$. To define the concept of the order of a distribution in $\mathfrak{O}$, we need the seminorms

$$p_k(\psi) = \max_{|\alpha|=k} \max_{\xi \varepsilon \mathfrak{O}} |D^\alpha \psi(\xi)|. \tag{C. 1}$$

The notion of the order of a distribution, as given in the book of Schwartz, is easily seen to be equivalent to the following definition (valid for bounded sets $\mathfrak{O}$ only): The order of the distribution T in $\mathfrak{O}$ is the smallest integer $m_T(\mathfrak{O})$ for which there exists a constant C, such that

$$\left| \int T(\xi)\, \psi(\xi)\, d\xi \right| \leqslant C\; p_{m_T(\mathfrak{O})}(\psi), \tag{C. 2}$$

for all $\psi \varepsilon D(\mathfrak{O})$. We define the amplitude $A_T(\mathfrak{O})$ of T in $\mathfrak{O}$ as the infimum of the possible constants C in (C. 2). The order and the amplitude of the distribution T at a point, say the origin o of ξ-space, may now be defined as follows: *The order of T at o is the smallest integer m_T for which there exists a neighbourhood $\mathfrak{O}$ of o in which the order of T is m_T. The amplitude of T at o, A_T, is the infimum of the amplitude over all neighbourhoods of o, i. e. $A_T = \inf_{\mathfrak{O} \ni o} A_T(\mathfrak{O})$.*

The two numbers, m_T and A_T, may now serve to order distributions with respect to their behaviour at $\xi = o$. We say that S is more regular than T at the origin if $m_S < m_T$ or if $m_S = m_T$ but $A_S < A_T$. In this manner all distributions may be compared to each other, and in particular to derivatives of δ. However, the comparison is quite rough. In general, if $m_S < m_T$, the distributions $T + S$ and T are equally regular at the origin.

The formulation of the principle of maximum regularity may now be taken over from paragraph 3.

It is obvious that the ϑ-multiplication thus defined (although not always uniquely defined) constitutes a generalization of the ordinary product of two functions. Indeed, for sufficiently regular distributions, i. e. for distributions of the order zero and of the amplitude zero at the origin, the requirement of maximum regularity gives the same result as the "naive" ϑ-multiplication which succeeds in this case. It is easily seen that such distributions locally are measures continuous at the origin.

The concepts of the order and the amplitude of a distribution at a point are much more powerful than the concept of the rate of growth at infinity in momentum space to analyse the dominating singularity of the distribution. This fact is revealed by the following theorem, which we give without proof: *If m is the order at the origin in $(x_1, x_2, \ldots, x_n)$-space of the distribution $\pi_R(x_1, x_2, \ldots, x_n)$ there exists a unique relativistically invariant and symmetric polynomial $P_m(\partial/\partial x_1, \partial/\partial x_2, \ldots, \partial/\partial x_n)$, homogeneous of the m'th degree, such that $\pi_R - P_m \delta$ has the amplitude zero at the origin.*

The main point is that the polynomial exists and is *unique*. The relativistic invariance and the symmetry of the polynomial are then a trivial consequence of the fact that the statement that the amplitude at the origin is zero involves symmetric and invariant concepts only. In general one cannot reduce the order by a regularization process f this type.

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