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ON THE ANALYTIC PROPERTIES OF THE 4-POINT FUNCTION IN PERTURBATION THEORY

BY

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Synopsis

The analytic properties of the 4-point function as a function of 6 complex invariants are studied in simplest perturbation theory examples. This is a generalization of the work by Källén and Wightman on the vertex function. The singularity manifolds are: one 4-point singularity manifold, 4 sets of the 3-point manifolds of the type discussed by KW, and 6 cuts. These are determined in three different ways, including an explicit evaluation of the 4-fold Feynman parameter integral which results in a sum of 192 Spence functions. It is shown from the existence of the non-trivial geometric envelopes that the regularity domain D_4^{pert} is in general not entirely bounded by the analytic hypersurfaces. The boundary of the domain is illustrated with the aid of the 1-mass surfaces in some typical configurations of the 6 complex variables, showing that the 4-point boundary will in general carve out bubble singularities from the 3-point boundary. It is hoped that the results here may give some insight into the problem of finding the envelope of holomorphy of the 4-point domain determined by the axioms of the local field theory alone.

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I. Introduction*

In the study of the general structure of the local field theory on the basis of a few generally accepted postulates[1] (viz., field operators transforming according to the representations of the proper Lorentz group; positivity of energy of physical states; local commutativity for space-like separations; etc.), one is led to the investigation of the analytic properties of the vacuum expectation values of a product of field operators[2] and of related quantities such as the retarded commutators[3]. Several significant physical applications in this field have been made in recent years, e. g., the proofs of the dispersion relations[4], the CPT-theorem[5], and the connection between spin and statistics[6].

The significance of the vacuum expectation value of products of two fields (in short, the 2-point function) has been known for quite some time[7]. The complete 3-point analyticity domain $E(D_3)$ has been determined by Källén and Wightman[8] as a consequence of the above axioms without mass spectrum, and more recently the integral representations of the Bergman-

* Preliminary results of Sec. IV were reported by J. S. Toll at the Naples Conference (April, 1959) (see, ref. 13). I would like to thank Professor Toll for this.

[1] See, e. g., A. S. Wightman, Phys. Rev. **101**, 860 (1956). See also, Wightman, in *Les Problèmes Mathématiques de la Théorie Quantique des Champs*, Lille (1957).

[2] For a comprehensive survey of the properties of such Wightman functions, see, e. g., R. Jost's Lecture Notes in the *International Spring School of Physics*, Naples (1959); and also Jost's article in „*Theoretical Physics in the Twentieth Century*", ed. Fierz and Weisskopf, *Interscience Publishers*, New York (1960).

[3] See, e. g., H. Lehmann, K. Symanzik, and W. Zimmermann, Nuovo Cimento **1**, 205 (1955); and *ibid.* **6**, 319 (1957); V. Glaser, H. Lehmann, and W. Zimmermann, Nuovo Cimento **6**, 1122 (1957); O. Steinmann, Helv. Phys. Acta **33**, 257 (1960); and *ibid.* **33**, 347 (1960).

[4] See, e. g., N. N. Bogoliubov, B. V. Medvedev, and M. K. Polivanov, Lecture Notes (translated at Institute for Advanced Study, Princeton, 1957), and Fizmatgiz, Moscow (1958); H. J. Bremermann, R. Oehme, and J. G. Taylor, Phys. Rev. **109**, 2178 (1958); H. Lehmann, Nuovo Cimento **10**, 579 (1958).

[5] R. Jost, Helv. Phys. Acta **30**, 409 (1957).

[6] N. Burgoyne, Nuovo Cimento 8, 607 (1958); cf. also G. Lüders and B. Zumino, Phys. Rev. **110**, 1450 (1958).

[7] In a 1951 paper by H. Umezawa and S. Kamefuchi, Prog. Theor. Phys. **6**, 543 (1951), one finds, e. g., the assumption about the positive definite energy of all physical states clearly stated. Furthermore, this paper also contains an explicit example of a reduction formula, viz., for the problem of vacuum polarization. See, further, G. Källén, Helv. Phys. Acta. **25**, 417 (1952); H. Lehmann, Nuovo Cimento **11**, 342 (1954).

[8] G. Källén and A. S. Wightman, Mat. Fys. Skr. Dan. Vid. Selsk. **1**, No. 6 (1958). This paper will be referred to as KW.

Weil type have been given[9] as a most general representation for a function analytic in $E(D_3)$ and with arbitrary singularities outside.

The present investigation consists of a generalization to the 4-point case of a very special feature which was treated by KW in their discussion of the 3-point domain[10]. To make things perfectly clear as to how this might fit into the general framework in the 4-point case, it will perhaps be helpful to sketch briefly the necessary steps needed in the systematic exploitation of the analyticity domains of the n-point functions.

For an n-point function, one starts in the space of $(n-1)$ real 4-vectors ξ_i. The axiom of positivity of energy immediately allows an analytic continuation to the (complex) tube domain R_{n-1} with $\zeta_i = \xi_i - i\eta_i$ and all η_i lying inside the forward light-cone. Now there are three subsequent steps:

a) The Hall-Wightman theorem[11] maps this tube R_{n-1} into a domain $\boldsymbol{M}_{n-1}$ in the inner-product space of the $1/2\, n(n-1)$ complex variables[12]. The first problem is then to determine this primitive domain $\boldsymbol{M}_{n-1}$ (i. e., to characterize the boundary $\partial \boldsymbol{M}_{n-1}$). $\boldsymbol{M}_{n-1}$ is a natural domain of holomorphy[13].

b) By permuting the original vectors, one gets a permuted n-point function and thus a permuted domain $\mathfrak{P}\boldsymbol{M}_{n-1}$. Now by the axiom of strong locality, these permuted functions coincide on a certain space-like region $\boldsymbol{S}$. If $\boldsymbol{S} \cap \{\mathfrak{P}\boldsymbol{M}_{n-1}\} \neq 0$, then one gets a function analytic in the domain $D_n = \cup \{\mathfrak{P}\boldsymbol{M}_{n-1}\}$.

c) The domain D_n (because of the above union) is not a natural domain of holomorphy[14]. The final step is to find the envelope of holomorphy $E(D_n)$ of D_n[15].

We now briefly discuss separately the cases for $n \leq 4$.

Case 1) *2-point domain*: $\boldsymbol{M}_1$ is trivial; it is just the cut-plane (as is obvious from squaring a single (difference) vector ζ). The cut is along the positive real-axis. Steps (b) and (c) are unnecessary. $\boldsymbol{M}_1 = D_2 = E(D_2)$.

[9] G. Källén and J. S. Toll, in Pauli Memorial Volume, Helv. Phys. Acta. **33**, 753, (1960).

[10] See KW Appendix III and Section VII.

[11] D. Hall and A. S. Wightman, Mat. Fys. Medd. Dan. Vid. Selsk. **31**, No. 5 (1957).

[12] For $n \geq 5$, the number of independent inner products is reduced to $2(2n-5)$ by linear dependence of more than 4 vectors in 4-dimensional space-time.

[13] For $n \leq 3$, this is clear, since $\boldsymbol{M}_{1,2}$ are both bounded by analytic hypersurfaces, and one knows that one can go no further. For $n = 4$, one gets non-analytic hypersurfaces, however, this is still proved by Källén and Toll (private communication; and Toll's Lecture Notes in International Spring School of Physics, Naples (1959)).

[14] Cf., for example, D. Ruelle, Helv. Phys. Acta **32**, 135 (1959) and thesis (1959), Bruxelles.

[15] For basic notions of the theory of functions of several complex variables, see, e. g., H. Behnke and P. Thullen, *Theorie der Funktionen mehrerer komplexer Veränderlichen*, Ergebn. Math. **3** Nr. 3, Berlin (1934). For a physicist's summary, cf., e. g., KW Sec. VI ff.

Case 2) 3-*point domain*:

a) Part of $\boldsymbol{M}_2$ was first treated by D. Hall[16]; it was simplified and exhausted by KW who show that $\boldsymbol{M}_2$ is bounded by the following pieces of analytic hypersurfaces:

$$F_{12}:\ z_3 = z_1 + z_2 + r + z_1 z_2/r,\ 0 < r < \infty, \qquad (\text{for } \boldsymbol{Im}\, z_1 \cdot \boldsymbol{Im}\, z_2 > 0);$$
$$S:\ z_3 = z_1(1-k) + z_2(1-1/k),\ 0 < k < \infty,\ (\text{for } \boldsymbol{Im}\, z_1 \cdot \boldsymbol{Im}\, z_2 < 0),$$

and the cuts in z_1 and z_2.

b) Permutation is straightforward.

c) $E(D_3)$ turns out to be bounded also by analytic hypersurfaces:

$$\text{Cuts: } z_k = \varrho \geqslant 0,\ k = 1, 2, 3. \quad (0 < \varrho < \infty).$$
$$F'_{ij}:\ z_k = z_i + z_j - \varrho - z_i z_j/\varrho, \quad (\text{for } \boldsymbol{Im}\, z_k \cdot \boldsymbol{Im}\, z_i < 0,\ \boldsymbol{Im}\, z_k \cdot \boldsymbol{Im}\, z_j < 0);$$
$$\mathfrak{F}:\ z_1 z_2 + z_2 z_3 + z_3 z_1 - \varrho(z_1 + z_2 + z_3) + \varrho^2 = 0$$
$$(\text{for } \boldsymbol{Im}\, z_1 \cdot \boldsymbol{Im}\, z_2 > 0,\ \boldsymbol{Im}\, z_1 \cdot \boldsymbol{Im}\, z_3 > 0).$$

Case 3) 4-*point domain*:

a) Part of the boundary of the primitive domain $\boldsymbol{M}_3$ has been very elegantly characterized by Jost[17] with a set of 3×3 matrices M = DANAD, where $M = \|(\zeta_i \cdot \zeta_i)\|$, D is diagonal with positive diagonal elements, A is symmetric real except for diagonal elements which have positive imaginary parts, and N has zero diagonal elements and 1 everywhere else. That $\mathbf{M}_3$ is indeed a natural domain of holomorphy has been shown by Källén and Toll[18], who have also shown that $\boldsymbol{M}_3$ is *not* everywhere bounded by analytic hypersurfaces.

b) The permuted domain remains to be determined. This can be accomplished by the present technique if sufficient and careful work is carried through.

c) The real difficulty lies in the problem of finding the envelope of holomorphy $E(D_4)$, which is at the present moment completely unknown. It is therefore entirely an open question as to whether or not $E(D_4)$ will be bounded by analytic hypersurfaces.

At this point, we want to discuss the role of the domain D_n^{pert}, which one gets from simple yet non-trivial examples in perturbation theory. Let us recall the following facts:

[16] D. Hall, Ph. D. thesis, Princeton (1956).
[17] See Ref. cited in footnote 2.
[18] See Ref. cited in footnote 13.

Case 1a) $n = 2$: $D_2^{\text{pert}} = E(D_2)$.

Case 2a) $n = 3$: D_3^{pert} gives about three-fourths of the answer to $E(D_3)$, i. e., D_3^{pert} is bounded by cuts and F'_{kl} surfaces. *The only thing D_3^{pert} fails to tell is the $\mathfrak{F}$-surface* (which corresponds to the case when all ***Im*** z_k have the same sign). (In fact, it should perhaps be pointed out that it would be extremely difficult to discover the exact shape of $E(D_3)$ if one didn't know beforehand D_3^{pert}; a knowledge of which then enabled KW to actually prove the final results.)

It is in this spirit that the present study of the D_4^{pert} is undertaken. Namely, it is hoped that perhaps D_4^{pert} might again give some insight into the envelope of holomorphy $E(D_4)$ in the axiomatic approach.

The work divides itself into two parts. The first part (Sections II–V) is devoted to the explicit location of the singularities of the 4-point function in perturbation theory and their relevance criteria. The second part (Section VI) is to determine what constitutes the boundary of the domain; the study of this boundary is our primary interest.

The main result of this study is that D_4^{pert} is also *not* entirely bounded by analytic hypersurfaces. A lengthy analysis of the problem of the geometric envelopes for the 4-point singularity manifold is made (Section VI). The 4-mass envelopes and the 3-mass envelopes, although they can also exist, are shown to be trivial and cannot contribute to the boundary of the domain. On the other hand, the two-mass envelopes are quite non-trivial and have most natural relations with the 3-point boundary F'_{kl} surfaces. In principle, with the aid of an electronic computer, the boundary of D_4^{pert} can be explicitly plotted. However, we only give here the equations and illustrate instead the one-mass curves (which are analytic) for some typical configurations in the space of six complex variables to show the presence of the 2-mass envelopes.

It is evident that the fact that D_4^{pert} is not bounded by analytic hypersurfaces will make the problem for D_4^{pert} to provide some answer to $E(D_4)$ much less transparent than the previous 3-point case. Of course, it is trivial that $E(D_4) \subset D_4^{\text{pert}}$. However, as already mentioned above, it is still an open question whether or not $E(D_4)$ is bounded by analytic hypersurfaces. *If* D_4^{pert} does have anything to do with $E(D_4)$, then the present investigation gives a negative answer.

II. Simple Examples of the 4-Point Function

II. 1 The Vacuum Expectation Value of Products of Four Fields in Ward Theory (x-space)

We consider in perturbation theory an interaction via a Lagrangian $g\Phi_1\Phi_2\Phi_3\Phi_4$, where the Φ_i's are neutral scalar fields with field quanta m_i. Expanding in powers of g, we have

$$\Phi_j(x) = \Phi_j^{(0)}(x) + g\int dx'\, \Delta_R(x-x';\ m_j)\, \Phi_k^{(0)}(x')\,\Phi_l^{(0)}(x')\,\Phi_m^{(0)}(x') + \dots\dots \qquad (1)$$

where $(jklm)$ is a permutation of (1234).

To the first non-trivial order, the vacuum expectation value of the four fields reads:

$$\left.\begin{aligned}
&<0\,|\,\Phi_1(x_1)\,\Phi_2(x_2)\,\Phi_2(x_3)\,\Phi_4(x_4)\,|\,0> \\
&= \frac{-g}{(2\pi)^9}\iiint dq_1\,dq_2\,dq_3\cdot\exp\left[i\,(q_1x_{14}+q_2x_{24}+q_3x_{34})\right] \\
&\times\Bigg[\Theta(-q_2)\,\Theta(-q_3)\,\Theta(q_1+q_2+q_3)\,\frac{\delta(q_2^2+m_2^2)\,\delta(q_3^2+m_3^2)\,\delta((q_1+q_2+q_3)^2+m_4^2)}{(q_1^2+m_1^2)_R} \\
&+\Theta(q_1)\,\Theta(-q_3)\,\Theta(q_1+q_2+q_3)\,\frac{\delta(q_1^2+m_1^2)\,\delta(q_3^2+m_3^2)\,\delta((q_1+q_2+q_3)^2+m_4^2)}{(q_2^2+m_2^2)_R} \\
&+\Theta(q_1)\,\Theta(q_2)\,\Theta(q_1+q_2+q_3)\,\frac{\delta(q_1^2+m_1^2)\,\delta(q_2^2+m_2^2)\,\delta((q_1+q_2+q_3)^2+m_4^2)}{(q_3^2+m_3^2)_R} \\
&+\Theta(q_1)\,\Theta(q_2)\,\Theta(q_3)\,\frac{\delta(q_1^2+m_1^2)\,\delta(q_2^2+m_2^2)\,\delta(q_3^2+m_3^2)}{((q_1+q_2+q_3)^2+m_4^2)_R}\Bigg],
\end{aligned}\right\} \qquad (2)$$

where $x_{ij} = x_i - x_j$. The Θ's are the usual step functions

$$\Theta(x_0) = \begin{cases} 0, & \text{for } x_0 < 0 \\ 1, & \text{for } x_0 > 0. \end{cases}$$

The scalar products for 4-vectors are here defined with the metric $(+++-)$.

For the choice of the 4-point function in x-space, a more convenient expression results if we multiply (2) with a suitable weight function $\mathfrak{G}(m_1^2, m_2^2, m_3^2, m_4^2)$ and integrate over all masses $m_k^2 \geqslant 0$. In particular, following KW, we choose

$$\mathfrak{G}(m_1^2, m_2^2, m_3^2, m_4^2) = \prod_{k=1}^{4} \bar{\Delta}(-m_k^2;\, a_k), \quad a_k > 0, \qquad (3)$$

where

$$\bar{\Delta}(\lambda, \sigma) = \frac{1}{(2\pi)^4}\,\mathfrak{P}.\int dp\,\frac{e^{ip\alpha}}{p^2+\sigma}\Bigg|_{\alpha^2=\lambda} \tag{4}$$

in which $\mathfrak{P}.$ denotes the usual Cauchy principal part.

Using

$$\frac{2}{\pi}\,\mathfrak{P}.\int_0^\infty \frac{d\lambda\,\bar{\Delta}(-\lambda;\sigma)}{p^2+\lambda} = \Delta^{(1)}(p^2;\sigma) \equiv \frac{1}{(2\pi)^3}\int d\xi\,e^{ip\xi}\,\delta(\xi^2+\sigma) \tag{5}$$

and

$$\Delta_R(x) = 2\,\Theta(x)\,\bar{\Delta}(x), \tag{6}$$

we have the expression

$$\left.\begin{aligned}
I &\equiv \iiiint_0^\infty dm_1^2\,dm_2^2\,dm_3^2\,dm_4^2\,\mathfrak{G}(m_1^2, m_2^2, m_3^2, m_4^2) < 0\,|\,\Phi_1(x_1)\,\Phi_2(x_2)\,\Phi_3(x_3)\,\Phi_4(x_4)\,|\,0 > \\
&= \frac{-g}{32\,(2\pi)^8}\iiint dq_1\,dq_2\,dq_3\cdot\exp\left[i(q_1x_{14}+q_2x_{24}+q_3x_{34})\right] \\
&\qquad\times[\Delta^{(1)}(q_1^2;a_1)\;\Delta_A(q_2^2;a_2)\;\Delta_A(q_3^2;a_3)\;\Delta_R((q_1+q_2+q_3)^2;a_4) \\
&\qquad+\Delta_R(q_1^2;a_1)\,\Delta^{(1)}(q_2^2;a_2)\;\Delta_A(q_3^2;a_3)\;\Delta_R((q_1+q_2+q_3)^2;a_4) \\
&\qquad+\Delta_R(q_1^2;a_1)\;\Delta_R(q_2^2;a_2)\,\Delta^{(1)}(q_3^2;a_3)\;\Delta_R((q_1+q_2+q_3)^2;a_4) \\
&\qquad+\Delta_R(q_1^2;a_1)\;\Delta_R(q_2^2;a_2)\;\Delta_R(q_3^2;a_3)\,\Delta^{(1)}((q_1+q_2+q_3)^2;a_4)] \\
&= \frac{-g}{32\,(2\pi)^{11}}\iiiint d\xi_1\,d\xi_2\,d\xi_3\,d\xi_4\,\delta(\xi_1+\xi_4+x_{14})\,\delta(\xi_2+\xi_4+x_{24})\,\delta(\xi_3+\xi_4+x_{34}) \\
&\qquad\times\left\{\frac{\delta(A_1)}{A_2A_3A_4}+\frac{\delta(A_2)}{A_1A_3A_4}+\frac{\delta(A_3)}{A_1A_2A_4}+\frac{\delta(A_4)}{A_1A_2A_3}\right\},
\end{aligned}\right\} \tag{7}$$

where

$$A_k \equiv \xi_k^2 + a_k.$$

With the aid of the well-known identities

$$\sum_{\text{cyclic}}\frac{\delta(A_1)}{A_2A_3A_4} = -\iiiint_0^1 d\alpha_1\,d\alpha_2\,d\alpha_3\,d\alpha_4\,\delta(1-\Sigma\alpha_k)\,\delta^{(3)}(\Sigma\alpha_k\,a_k) \tag{8}$$

and

$$\frac{1}{\pi}\int d^4\xi\,\delta^{(3)}(\xi^2+A) = -\frac{\partial}{\partial A}\left(\mathfrak{P}.\frac{1}{A}\right) \tag{9}$$

the integrals over all ξ_k can be easily carried out. The result is:

$$\left.\begin{aligned}
I &= \frac{g}{64\,(2\pi)^{10}}\iiiint_0^1\frac{d\alpha_1\,d\alpha_2\,d\alpha_3\,d\alpha_4\,\delta(1-\Sigma\alpha_k)}{(\alpha_1\alpha_2z_1+\alpha_1\alpha_3z_2+\alpha_1\alpha_4z_3+\alpha_2\alpha_3z_4+\alpha_3\alpha_4z_5+\alpha_4\alpha_2z_6-\Sigma\alpha_k\,a_k)^2} \\
&\qquad a_k > 0,
\end{aligned}\right\} \tag{10}$$

where the six z's are defined as follows:

$$\left.\begin{array}{ll} z_1 = -x_{14}^2, & z_4 = -x_{12}^2 \\ z_2 = -x_{24}^2, & z_5 = -x_{23}^2 \\ z_3 = -x_{34}^2, & z_6 = -x_{13}^2. \end{array}\right\} \quad (11)$$

Equation (10) is the expression we shall take for the 4-point function $I(z;\, a)$ as a function of the six complex z's and four real a's.

II. 2 The Time-Ordered Product of Four Currents (p-space)

For completeness, we mention that the Fourier transform of the time-ordered product of four currents in perturbation theory gives rise in p-space to exactly the same integral expression (10). The expression for the square-loop Feynman graph is too well-known to warrant a derivation here[19]. Since, as we shall see later, the singularity manifold has a natural geometrical interpretation in terms of such graphs, we shall briefly sketch the necessary notations.

Consider also four scalar fields $\varphi_k^{(0)}(x)$, with characteristic masses m_k, $k = 1, \ldots, 4$. Write $a_k = m_k^2$, and

$$\begin{aligned} j_1(x) &= \varphi_4^{(0)}(x)\, \varphi_1^{(0)}(x) \\ j_2(x) &= \varphi_1^{(0)}(x)\, \varphi_2^{(0)}(x) \\ j_3(x) &= \varphi_2^{(0)}(x)\, \varphi_3^{(0)}(x) \\ j_4(x) &= \varphi_3^{(0)}(x)\, \varphi_4^{(0)}(x). \end{aligned}$$

Then

$$\left.\begin{aligned} F(z) &= \langle 0 \mid T\{j_1(x_1)\, j_2(x_2)\, j_3(x_3)\, j_4(x_4)\} \mid 0 \rangle \\ &= \frac{1}{16}\, \Delta_F(x_{12};\, a_1)\, \Delta_F(x_{23};\, a_2)\, \Delta_F(x_{34};\, a_3)\, \Delta_F(x_{41};\, a_4) \\ &= \frac{i\pi^2}{(2\pi)^{16}} \iiint dp_{12}\, dp_{23}\, dp_{34} \cdot \exp.\, [i(p_{12}x_{12} + p_{23}x_{13} + p_{34}x_{14})] \\ &\qquad \times H(p_{12},\, p_{23},\, p_{34}) \end{aligned}\right\} \quad (12)$$

where double indices denote the differences $x_{ij} = x_i - x_j$.

[19] For general expressions of Feynman amplitudes, cf., e. g., J. S. R. Chisholm, Proc. Camb. Soc. 48, 300 (1952); Y. Nambu, Nuovo Cimento 6, No. 5, 1064 (1957).

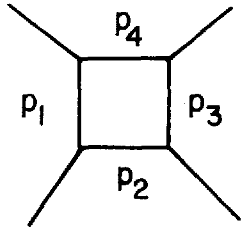

Figure 1. Square-loop Feynman graph

A standard computation then yields:

$$\left.\begin{aligned}&H(p_{12}, p_{23}, p_{34})\\&=\iiiint_0^1\frac{d\alpha_1\,d\alpha_2\,d\alpha_3\,d\alpha_4\,\delta(1-\Sigma\alpha_k)}{(\alpha_1\alpha_2\zeta_1+\alpha_1\alpha_3\zeta_2+\alpha_1\alpha_4\zeta_3+\alpha_2\alpha_3\zeta_4+\alpha_3\alpha_4\zeta_5+\alpha_4\alpha_2\zeta_6-\Sigma\alpha_k a_k)^2};\ a_k>0\end{aligned}\right\}\quad(13)$$

where the six ζ's are defined by

$$\left.\begin{aligned}\zeta_1&=-p_{12}^2, & \zeta_4&=-p_{23}^2\\ \zeta_2&=-p_{13}^2, & \zeta_5&=-p_{34}^2\\ \zeta_3&=-p_{14}^2, & \zeta_6&=-p_{42}^2.\end{aligned}\right\}\quad(14)$$

We see that expressions (13) and (10) are identical and the definitions for the z's and the ζ's are merely the same six invariants derived from a set of three independent four-vectors.[20]

III. Function of Six Complex Variables Represented by a 4-Fold Feynman Parameter Integral

III. 1 Definition of the Ψ-Manifold

Both the examples treated in Sec. II have led to the same integral expression, namely

$$I(z; a)=\iiiint_0^1\frac{d\alpha_1\,d\alpha_2\,d\alpha_3\,d\alpha_4\,\delta(1-\Sigma\alpha_k)}{D^2},\tag{15}$$

[20] Special cases of this square Feynman graph example have been treated independently for all six real variables by R. Karplus, C. M. Sommerfield, and E. H. Wichmann, Phys. Rev. **114**, 376 (1959). This was later extended to the case of two complex variables by J. Tarski, Jour. Math. Phys. **1**, 154 (1960). In both works, all the 3-point boundaries are restricted to the real domain, and all the masses (internal and external) are held fixed together with stability conditions. Subsequently, there appeared a number of papers on the methods of locating the singularities of the general Feynman amplitudes without the explicit completion of integrations. See, e. g., L. D. Landau, Proceedings of the International Conference on High Energy Nuclear

where

$$D = \alpha_1 \alpha_2 z_1 + \alpha_1 \alpha_3 z_2 + \alpha_1 \alpha_4 z_3 + \alpha_2 \alpha_3 z_4 + \alpha_3 \alpha_4 z_5 + \alpha_4 \alpha_2 z_6 - \Sigma \alpha_k a_k. \tag{16}$$

This denominator D can be written in various manners for different purposes. For instance, using the identity $\Sigma \alpha_k = 1$ under the integral, we can write

$$D = \frac{1}{2} \sum_{i,j} \Psi_{ij} \alpha_i \alpha_j \tag{17}$$

where the 4×4 symmetric matrix (Ψ_{ij}) is defined as

$$(\Psi_{ij}) = \begin{pmatrix} -2a_1 & z_1 - a_2 - a_1 & z_2 - a_3 - a_1 & z_3 - a_4 - a_1 \\ z_1 - a_1 - a_2 & -2a_2 & z_4 - a_3 - a_2 & z_6 - a_4 - a_2 \\ z_2 - a_1 - a_3 & z_4 - a_2 - a_3 & -2a_3 & z_5 - a_4 - a_3 \\ z_3 - a_1 - a_4 & z_6 - a_2 - a_4 & z_5 - a_3 - a_4 & -2a_4 \end{pmatrix}. \tag{18}$$

The determinant $|\Psi_{ij}|$ will be simply denoted by Ψ throughout this paper, and the manifold $\Psi(z; a) = 0$ will be referred to as the Ψ-manifold. It will be shown that the 4-point type singularity of our function $I(z; a)$ comes just when this linear transformation (Ψ_{ij}) becomes a singular one (Section IV). The significance and the structure of this Ψ-manifold are given in Sec. III.4.

III. 2 Symmetry of the 4-Point Function

The symmetry of the problem is contained in that of Ψ. Equivalently, we shall define a 3×3 determinant $\Lambda(z)$ (a quantity which will repeatedly appear in our later discussion), as follows:

$$\Lambda(z) = \frac{1}{2} \begin{vmatrix} -2z_1 & z_4 - z_2 - z_1 & z_6 - z_3 - z_1 \\ z_4 - z_1 - z_2 & -2z_2 & z_5 - z_3 - z_2 \\ z_6 - z_1 - z_3 & z_5 - z_2 - z_3 & -2z_3 \end{vmatrix}. \tag{19}$$

$\Lambda(z)$ has the following interpretation[21]: Let $\zeta_1, \zeta_2, \zeta_3$, be a set of three independent 4-vectors, and let the z's and ζ's be related as

Physics, Kiev (1959); J. C. Polkinghorne and G. R. Screaton, Nuovo Cimento **15**, No. 2, 289 (1960); and ibid. **15**, No. 6, 925 (1960). An inherent disadvantage of such approaches is the lack of explicit knowledge of when and only when the cancellation of singularities will not occur.

[21] For real vectors in the Euclidean space, $\Lambda(x)$ has the significance of being proportional to the square of the volume of a tetrahedron. The principal minors of $\Lambda(x)$, which are exactly the type of function $\lambda(x)$ of KW, have the meaning of being proportional to the squares of areas of triangles (cf. remark following Eq. (82)).

$$\left.\begin{array}{c} z_1 = -\zeta_1^2, \quad z_2 = -\zeta_2^2, \quad z_3 = -\zeta_3^2 \\ z_4 = -(\zeta_1 - \zeta_2)^2, \quad z_5 = -(\zeta_2 - \zeta_3)^2 \\ z_6 = -(\zeta_3 - \zeta_1)^2. \end{array}\right\} \tag{20}$$

Then

$$\Delta(z) = 4 \times \text{Gram Determinant of } (\zeta_1, \zeta_2, \zeta_3) = 4\,|(\zeta_i \cdot \zeta_j)| \tag{21}$$

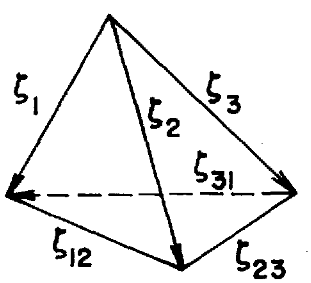

Figure 2. Tetrahedron representation for vectors

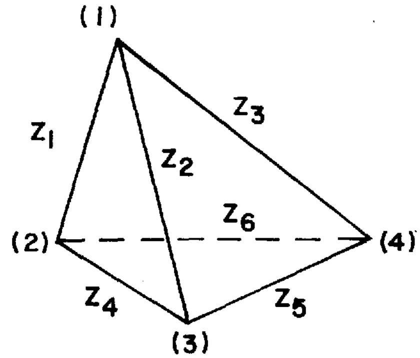

Figure 3. Tetrahedron representation for $\Delta(z)$

The situation for ζ_i is depicted in Fig. 2 together with their difference vectors. The (Lorentz) squares of the vectors are just the z's given in (20). In Fig. 3, we labelled the six edges of the "tetrahedron" T by these z's. Each of the four faces of T picks out a triplet of z's at a time. Intuitively one would expect each triplet to obey the restriction of the three-point type of KW, and this is indeed the case, as will be shown explicitly later (Sec. IV).

It is clear then that our problem has the symmetry endowed in this tetrahedron, in particular, the permutation symmetry which leaves the set of four faces of T invariant. Let us first divide the edges of T into two classes: Two edges which meet at a vertex of T will be called *adjacent* edges, otherwise *conjugate* edges. Obviously for a tetrahedron, for each edge there are four adjacent edges and only *one* conjugate edge. Thus the six z's break into three pairs of conjugate indices[22]. In our present notation, they are: (1,5), (2,6), and (3,4). For convenience, the four faces of T will be denoted by F_k, $k = 1, \ldots, 4$ and labelled in the following order: (456), (235), (136),

[22] When properly identified, energy and momentum transfer variables are conjugate to each other in this sense. It is important to note that conjugate indices, *ipso facto*, do not appear simultaneously in any one of the 3-point quantities, e. g., $\Phi(z)$ or $\lambda(z)$.

and (124). Note that this is equivalent to labelling the 4 vertices of Fig. 3 in the counter-clockwise order.

Then the operations which transform the set of all F_k into themselves are obviously the permutations among any *two pairs* of the conjugate indices. For example, $(1,5) \longleftrightarrow (2,6)$; by this we mean the following:

$$\text{either (i)} \begin{pmatrix}1\\5\end{pmatrix} \begin{matrix}\longleftrightarrow\\ \longleftrightarrow\end{matrix} \begin{pmatrix}2\\6\end{pmatrix}: \text{ thus } F_1,\, F_4 \text{ invariant; } F_2 \longleftrightarrow F_3$$

$$\text{or (ii)} \begin{pmatrix}1\\5\end{pmatrix} \times \begin{pmatrix}2\\6\end{pmatrix}: \text{ thus } F_2,\, F_3 \text{ invariant; } F_1 \longleftrightarrow F_4$$

$$\text{or (iii)} \begin{pmatrix}1\\ \updownarrow\\5\end{pmatrix} + \begin{pmatrix}2\\ \updownarrow\\6\end{pmatrix}: \text{ thus } F_1 \longleftrightarrow F_4;\ \text{and } F_2 \longleftrightarrow F_3.$$

In other words, a permutation between adjacent edges is to be accompanied by the permutation between their respective conjugate indices (e. g. cases (i) and (ii) above); and a permutation within one pair of conjugate indices is to be accompanied by the permutation within *another* pair of conjugate indices (e. g. case (iii) above). This exhausts the symmetry of the problem.

We might remark that the above symmetry property, which is purely geometrical, is not confined to the perturbation theory. The quantity $\Lambda(z)$ (or the Gram determinant of three 4-vectors) will undoubtedly play an important role in the case of the axiomatic approach. In the perturbation example Eq. (15), this symmetry is of course trivially implied by the permutation symmetry between any two $\alpha_i \longleftrightarrow \alpha_j$ in the integrand, the net result there being the proper interchange of four z's and two a's, which (apart from the associated permutation among the mass parameters) agrees exactly with our above general prescription of the permutation among two pairs of conjugate indices.

III.3 The Structure of the 3-Point Φ_k-Manifolds and The 2-Point R_μ-Manifolds

The 3-point Φ-manifold of KW has precisely the same structure as that of $\Lambda(z)$ discussed above, except that a set of three z's emerging from one vertex in Fig. 3 is now replaced by a set of three mass parameters. Thus the Φ-determinant is (apart from a trivial factor of 4) just the Gram determinant of three 4-vectors ζ_i with the diagonal elements put on some mass-shells.

In the present 4-point problem, we have in all *four* sets of such Φ, one for each face of the tetrahedron T. Thus, for example, the structure of Φ_1 can be represented by the tetrahedron T_1 in Fig. 4.

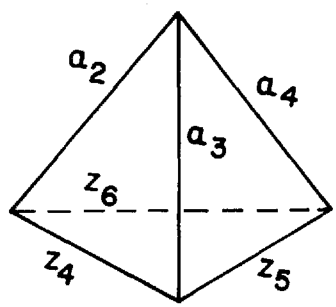

Figure 4. Tetrahedron representation for Φ_1

$$\Phi_1 = \frac{1}{2}\begin{vmatrix} -2a_2 & z_4 - a_3 - a_2 & z_6 - a_4 - a_2 \\ z_4 - a_2 - a_3 & -2a_3 & z_5 - a_4 - a_3 \\ z_6 - a_2 - a_4 & z_5 - a_3 - a_4 & -2a_4 \end{vmatrix}. \tag{22}$$

To every Φ_k-determinant, there are associated four 2×2 subdeterminants. One of them involves pure z's, i. e. the $\lambda(z)$ defined by KW, e. g.:

$$\lambda_1(456) = -\begin{vmatrix} -2z_4 & z_6 - z_4 - z_5 \\ z_6 - z_4 - z_5 & -2z_5 \end{vmatrix} \tag{23}$$

which is associated with the face with all z's in Fig. 4. To see how $\lambda(z)$ is related to $\Phi(z)$, we note that (22) can be written as

$$\Phi_1 = \frac{1}{2}\begin{vmatrix} -2a_3 & z_4 + a_3 - a_2 & z_5 + a_3 - a_4 \\ z_4 + a_3 - a_2 & -2z_4 & z_6 - z_4 - z_5 \\ z_5 + a_3 - a_4 & z_6 - z_4 - z_5 & -2z_5 \end{vmatrix} \tag{22a}$$

in which $-\lambda_1$ appeas as the first principal minor of Φ_1 when written in the form (22a). This feature will also appear in the 4-point case (cf. Sec. VI.2).

The other three quantities are the R_k-manifolds defined by KW, e. g.,

$$R_6 = -\begin{vmatrix} -2a_2 & z_6 - a_2 - a_4 \\ z_6 - a_2 - a_4 & -2a_4 \end{vmatrix} \tag{24}$$

which are associated with the faces of one z and 2 a's in Fig. 4.

It is well known that the manifold $R_k(z_k) = 0$ yields the cut in each variable z_k in the 3-point case. This feature is also carried over to the 4-point case where we have six such R-manifolds, giving rise to a cut along the positive real axis in each of the six complex variables. Note that each cut is actually an 11-dimensional manifold.

III. 4 The Structure of the 4-Point Ψ-Manifold

The generalization from the 2-point R-manifold to the 3-point Φ-manifold is strongly suggestive as to how the 4-point Ψ-manifold might be built up, and indeed the analogy turns out to be a valid one. As one can build up a tetrahedron T_k for Φ_k by adding three a's to the k-th face taken out from the tetrahedron T for $\Lambda(z)$, one may now build up a "pentahedron"[23] for Ψ by adding four legs of a's to the entire tetrahedron T as the base (Fig. 5). The remaining four hypersurfaces of this pentahedron, being tetrahedrons T_k with 3 a's and 3 z's, represent just the set of 4 Φ_k-manifolds in our problem[24].

The Ψ-determinant has the simple interpretation in the p-space as 16 times the Gram determinant $|(p_i \cdot p_j)|$ of four 4-vectors p_k such that the

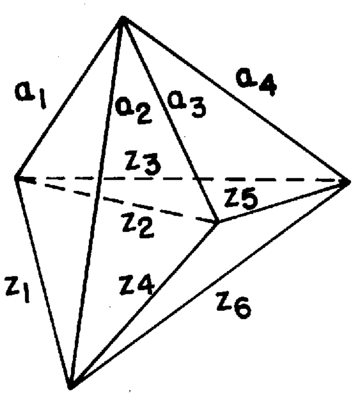

Figure 5. Pentahedron representation for $\Psi(z; a)$

diagonal elements are put on some mass-shells: $-p_k^2 = m_k^2 = a_k$, and the off-diagonal elements are re-expressed through the difference vectors, e. g.,

[23] The above intuitive terms such as "tetrahedron" and "pentahedron" should perhaps be properly changed into "n-simplex," $n = 4,5$ respectively.

[24] By our previous labelling of the four faces of T (Sec. III.2), the index k of Φ_k is such that a_k does *not* appear in Φ_k.

$$2\,p_i p_j = -(p_i - p_j)^2 - p_i^2 - p_j^2 = z_m - a_i - a_j$$

for some m. These four p_k's may just be identified with the four internal momenta of the square-loop Feynman graph (Fig. 1). On account of the momentum conservation at each vertex, these difference vectors $p_{i\,i+1} = p_i - p_{i+1}$ are just the four external momenta and the six z's are then the six invariants built up from these $p_{i\,i+1}$ (cf. Eq. (20)). From this, it is clear that the 4-point singularity manifold $\Psi = 0$ can be interpreted to arise just when the four p_k's are not linearly independent[25].

IV. Sources of Singularities of the 4-Point Function

IV. 1 General Discussion

From the integral representation (15) it is clear that, with a given set of parameters a_k, the singularities of $I(z; a)$ come from a certain manifold of z such that the denominator D vanishes somewhere within the range of integration. Of course, not all such points need be singular points of $I(z)$, as we can easily convince ourselves that the integration may very well smoothe out some of the singularities of the integrand. In fact, from the 3-point example treated in KW, we see that there are some delicate cancellations which made

a) only part of the Φ-manifold as relevant 3-point type singularities; and

b) the relevant portion of the cut (2-point singularity), in the case of non-vanishing masses, actually starts from $z_k = (\sqrt{a_m} + \sqrt{a_n})^2$, but not $(\sqrt{a_m} - \sqrt{a_n})^2$.

It will be shown in Sec. V that an inherent cancellation of this nature will again occur in the 4-point case.

In this section, we shall mainly locate the sources of all possible singularities of $I(z)$. It will be shown explicitly that these singularities arise only when the quadratic roots of D in α_i become double roots. The conditions for such double roots at each stage then yield the singularity manifolds for the 2-point, 3-point, and 4-point type, respectively.

We now briefly compare the methods we shall adopt in the 4-point case

[25] This was independently noted by LANDAU *loc. cit.*, and implicitly implied by KARPLUS, *et. al.*, *loc. cit.*, for the real case.

versus those available for the 3-point case. In the case treated by KW, the corresponding original expression is

$$H(z_1, z_2, z_3) = \int\int\int_0^1 \frac{d\alpha_1\, d\alpha_2\, d\alpha_3\, \delta(1-\Sigma\alpha_j)}{D_1}, \tag{25}$$

where

$$D_1 = \alpha_1 \alpha_2 z_3 + \alpha_2 \alpha_3 z_1 + \alpha_3 \alpha_1 z_2 - \Sigma \alpha_j a_j. \tag{26}$$

It is obvious that, when one of the 4 α's in the 4-point case becomes zero, D of (16) (apart from a trivial relabelling of the indices) goes over to D_1 of (26).

The 3-fold integration in (25) can be carried out in a straightforward manner, but the result contains a sum of 16 Spence functions[26] which are somewhat inconvenient. Instead, KW applies the differentiation $\Sigma\partial/\partial a_k$, which, on account of the identity $\Sigma\alpha_k = 1$, has the net effect of raising the power of D_1 by one for every such operation. Thus[27]

$$\sum_{k=1}^{3} \frac{\partial}{\partial a_k} H(z;\, a) = \int\int\int_0^1 \frac{d\alpha_1\, d\alpha_2\, d\alpha_3\, \delta(1-\Sigma a_j)}{D_1^2}. \tag{27}$$

Now (27) when integrated out contains only logarithms (KW (A. (46)):

$$\sum_{k=1}^{3} \frac{\partial}{\partial a_k} H(z;\, a) = -\frac{1}{2}\frac{1}{\Phi} \sum_k \frac{P_k}{\sqrt{R_k}} \log \frac{z_k - a_m - a_n + \sqrt{R_k}}{z_k - a_m - a_n - \sqrt{R_k}}, \tag{28}$$

where Φ is of the structure of (22), R_k of (24) and $P_k = \partial\Phi/\partial a_k$.

The 4-fold integration (15) can of course be carried out by force, but at first sight one is rather inclined to feel uneasy about a sum of 192 Spence functions. In this respect, it resembles (25). Unfortunately, however, the above differentiation technique will no longer save the situation, and the Spence function terms always persist in any explicit expression for $I_\nu(z)$, where ν refers to the power of D in (15). Since the case $\nu = 2$ is the simplest of all, and there is no merit in going to higher ν, we shall just stay with (15).

At this point, it is instructive to learn the lesson from the 3-point case. A study of the 3-point function $H(z)$ of (25) *in the undifferentiated form*

[26] For a comprehensive treatment of Spence functions, see, e. g., L. Lewin, *Dilogarithms and Associated Functions*, London (1958).

[27] One might note that, in a 2-dimensional (1-space, 1-time) space, the 3-point function without differentiation actually has the form (27). (This is a remark by Profs. Källén and Toll.)

led us to rederive the same singularity manifolds as those obtained from the differentiated form (28). There are two ways for this, which are essentially equivalent:

(a) The first method is to discuss the singularities under one remaining integral sign. We have, after the completion of a 2-fold integration, the following expression:

$$H(z;\, a) = \int_0^1 \frac{d\alpha}{\sqrt{N(\alpha)}} \log \chi(\alpha), \tag{29}$$

where

$$\left.\begin{aligned} N(\alpha) &= \lambda(z)\,\alpha^2 - 2\,P_3\,\alpha + R_3 = \lambda(z)\,(\alpha-\varrho_1)\,(\alpha-\varrho_2) \\ \varrho_{1,\,2} &= \frac{1}{\lambda(z)}\,[P_3 \pm 2\sqrt{z_3\,\Phi(z)}] \end{aligned}\right\}$$

and

$$\chi(\alpha) = \frac{\frac{1}{2}\left(\frac{\partial\lambda}{\partial z_1}\alpha - \frac{\partial R_3}{\partial a_2}\right) - \sqrt{N(\alpha)}}{\frac{1}{2}\left(\frac{\partial\lambda}{\partial z_1}\alpha - \frac{\partial R_3}{\partial a_2}\right) + \sqrt{N(\alpha)}} \cdot \frac{\frac{1}{2}\left(\frac{\partial\lambda}{\partial z_2}\alpha - \frac{\partial R_3}{\partial a_1}\right) - \sqrt{N(\alpha)}}{\frac{1}{2}\left(\frac{\partial\lambda}{\partial z_2}\alpha - \frac{\partial R_3}{\partial a_1}\right) + \sqrt{N(\alpha)}}. \tag{30}$$

Thus, as far as the integrand of (29) is concerned, when the 3-point roots $\varrho_{1,\,2}$ fall into the range (0,1) in the α-plane, $N(\alpha) = 0$ gives an apparent singularity. However, at this point $\log \chi(\alpha)$ becomes $\log 1 = n.\, 2\pi i$ (n = integer), in which lies the inherent cancellation. As long as the two roots remain distinct, $H(z)$ can still be defined by analytic continuation into another sheet of the Riemann surface even when one (or both) ϱ_i has (have) actually passed through the open interval (0,1), since in this case one may very well deform the path of integration to avoid meeting with the roots. The upper end $\alpha = 1$ is perfectly harmless. At the lower end $\alpha = 0$, however, one gets the R_3-manifold (which gives the cut in the z_3-plane). On the other hand, when the roots tend to coincide after they have crossed over the range (0,1) an odd number of times, then the above deformation of the integration path is no longer possible, and $H(z)$ will have a singularity. The condition for such double roots gives precisely the manifold $\Phi(z) = 0$ (apart from the trivial alternative $z_3 = 0$ which we disregard). The only other singularities $H(z)$ can have is at the coincident zeros or poles of $\chi(\alpha)$ which can be easily seen to lead to the R_1 and R_2 manifolds (cf. (46) and the remark thereto).

In this way, one is able to relocate the singularities of the 3-point function $H(z)$ in the undifferentiated form, which agrees exactly with what one gets from the explicit differentiated form (28).

(b) The second method is to carry out the last integration of (29). As already mentioned before, one gets Spence function terms besides logarithms here. However, a careful examination of these terms shows that, with proper manipulation, they are still manageable. One first learns which combination of the variables go into each of the Spence functions by explicitly differentiating them with $\Sigma \partial / \partial a_k$. From this, one sees how the Spence functions unfold and all the inherent cancellations thereof. Once this is done, one can, taking into account the symmetry of the problem, again recover the singularities of the 3-point function $H(z)$ in the Spence function form. We did this only as an exercise to get an insight into properly handling the corresponding (and more complicated) Spence function terms in the 4-point case.

In the following, these two approaches are generalized to the 4-point case.

IV.2 The One-Fold Integral Representation

We now proceed to discuss the singularities of $I(z)$ after a straightforward completion of integrations over three of the four α's. We have, before a final integration, the following expression[28]:

$$I(z;\, a) = -\frac{1}{2}\int_0^1 \frac{d\alpha}{\Lambda(z)\alpha^2 + \frac{1}{2}Q_1\alpha + \Phi_1} \sum_{j=1}^{3} \frac{M_j(\alpha)}{\sqrt{N_j(\alpha)}} \log \chi_j(\alpha). \tag{31}$$

Here the denominator in front of the summation sign has singled out, in the language of Sec. III.4, the tetrahedron T_1, viz., the set of variables $(z_4, z_5, z_6;\, a_2, a_3, a_4)$. The summation is thus extended over the remaining three T_{j+1}, $j = 1, 2, 3$, of the pentahedron of Fig. 5. (Recall that T_k was defined by deleting a_k from the pentahedron).

Now the symbols in (31) stand for the following:

$$Q_k = \frac{\partial \Psi}{\partial a_k}, \quad k = 1, \ldots, 4 \tag{32}$$

$$\Phi_k = \frac{1}{2}\Psi^{kk} \tag{33}$$

[28] We have performed the integrations over $\alpha_4, \alpha_3, \alpha_2$, The remaining integration is over α_1, where we drop the subscript. This singles out the triplet (*456*). Of course, by symmetry, the order of integration is entirely immaterial. Had one left the last integration over α_k undone for any k, the net effect would be a trivial permutation of $T_1 \longleftrightarrow T_k$ from Eq. (31).

where Ψ^{kk} denotes the k-th principal minor of the Ψ-determinant. Note that Φ_1 is explicitly given in (22). $\Lambda(z)$ is given by (19). We have also defined λ_1 in (23). The three remaining similar expressions (one for each of the remaining three triplets) can be simply defined as

$$\lambda_{j+1} = (2\Lambda)^{jj}, \quad j = 1, 2, 3. \tag{34}$$

Furthermore, we have in (31)

$$M_j(\alpha) = \frac{\partial \Lambda(z)}{\partial z_j}\alpha - \frac{\partial \Phi_1}{\partial a_{j+1}}. \tag{35}$$

The quantities $N_j(\alpha)$ and $\chi_j(\alpha)$ are precisely of the same structure as those appearing in the undifferentiated form of the 3-point function $H(z)$ in (29). Here

$$N_j(\alpha) = \lambda_{j+1}\alpha^2 - 2P_{j'}\alpha + R_{j'}, \tag{36}$$

where

$$P_{j'} = \frac{\partial \Phi_{j+1}}{\partial a_1} \tag{37}$$

and the *primed* index j' denotes the conjugate of j in the sense of Sec. III.2, viz., $j = (1, 2, 3)$; $j' = (5, 6, 4)$, respectively.

The quantity R_6 is given explicitly in (24), and the remaining five $R_\mu(z_\mu)$ are obvious from symmetry, as they can readily be read off from the principal 2×2 minors of the Ψ-determinant.

Finally we have:

$$\chi_j(\alpha) = \frac{\dfrac{\partial M_j(\alpha)}{\partial z_{k'}} - \sqrt{N_j(\alpha)}}{\dfrac{\partial M_j(\alpha)}{\partial z_{k'}} + \sqrt{N_j(\alpha)}} \cdot \frac{\dfrac{\partial M_j(\alpha)}{\partial z_{l'}} - \sqrt{N_j(\alpha)}}{\dfrac{\partial M_j(\alpha)}{\partial z_{l'}} + \sqrt{N_j(\alpha)}}, \quad j = 1, 2, 3 \tag{38}$$

in which the indices $(j'kl)$ form a triplet. The identification of indices k and l is unique for each j.

We note in passing that the integrand in (31) evaluated at $\alpha = 0$ is precisely the final expression (28) for the 3-point function in the differentiated form, now for the variables (z_4, z_5, z_6). (This is certainly to be expected, and serves as a check for (31)).

Having thus identified all the quantities that appear in (31), we proceed to note a number of identities which will be important for our subsequent discussions. We have, for $j = 1, 2, 3$,

$$M_j^2(\alpha) = \lambda_1 N_j(\alpha) + 4 z_{j'}\left[\Lambda(z)\alpha^2 + \frac{1}{2}Q_1\alpha + \Phi_1\right]. \tag{39}$$

$$\left.\begin{aligned}\left(\frac{\partial M_j(\alpha)}{\partial z_{k'}}\right)^2 &= \frac{1}{4}\left(\frac{\partial \lambda_{j+1}}{\partial z_l}\alpha - \frac{\partial R_{j'}}{\partial a_{l+1}}\right)^2 \\ &= N_j(\alpha) + 4\, z_{j'}\left[z_k\,\alpha^2 + \frac{1}{2}\frac{\partial R_k}{\partial a_1}\alpha + a_{k+1}\right].\end{aligned}\right\} \quad (40)$$

$$\left(\frac{\partial \Phi_{j+1}}{\partial a_{k+1}}\alpha + \frac{\partial \Phi_{j+1}}{\partial z_l}\right)^2 = R_k N_j(\alpha) + 4\,\Phi_{j+1}\left[z_k\,\alpha^2 + \frac{1}{2}\frac{\partial R_k}{\partial a_1}\alpha + a_{k+1}\right], \quad (41)$$

where $(j'kl)$ forms a triplet in (40) and (41).

Furthermore, we have

$$\sum_{j=1}^{3} M_j(\alpha) = -\lambda_1 \cdot (1-\alpha) \quad (42)$$

$$Q_k^2 = 16\,\Lambda(z)\,\Phi_k + 4\,\lambda_k\,\Psi(z;\,a), \qquad k = 1, \ldots, 4. \quad (43)$$

Note that, for $\alpha = 0$, (39) reads

$$\left(\frac{\partial \Phi_1}{\partial a_{j+1}}\right)^2 = \lambda_1 R_{j'} + 4\, z_{j'}\,\Phi_1 \quad (44)$$

which is just the 3-point relation (KW (A. 46 d)), now for the variable *(456)*. On the other hand, (41) reads for $\alpha = 0$

$$\left(\frac{\partial \Phi_{j+1}}{\partial z_l}\right)^2 = R_k R_{j'} + 4\, a_{k+1}\Phi_{j+1} \quad (45)$$

which is a variant of (44) in that the role of the corresponding a's and z's is now interchanged. The 4-point analogue of this will be noted in Eq. (110). Equations (43) which are the proper generalization of (44) to the 4-point case will also play a dominant role in our later discussion of the boundary (Sec. VI). It might be of some interest to point out that identities of the types (43) and (44) have a rather natural interpretation in terms of the determinant expansion by means of a theorem due to Jacobi[29]. An illustration of this is given in Appendix D.

For completeness, we might mention that the quadratic expression $(z_k\,\alpha^2 + 1/2\,\partial R_k/\partial a_m\,\alpha + a_n)$ appearing in (40) and (41) is the 2-point analogue of the 3-point quantity $N_j(\alpha)$ defined in (36), or (30). In fact, this is the expression used by KW to discuss the singularity on the cut, viz. (cf. KW (A.47)):

[29] See, Appendix D.

$$\frac{1}{\sqrt{R_k}} \log \frac{z_k - a_m - a_n + \sqrt{R_k}}{z_k - a_m - a_n - \sqrt{R_k}} = - \int_0^1 \frac{d\alpha}{z_k \alpha^2 + \frac{1}{2} \frac{\partial R_k}{\partial a_m} \alpha + a_n}. \tag{46}$$

Note that this 2-point denominator is what one gets by multiplying the numerator and the denominator of the individual factor in (38) (cf. (40)). Therefore we see that the zeros or poles of $\chi_j(\alpha)$, which give apparent singularities to the logarithms in the integrand of (31), are really confined to the individual cuts in the z's.

We see from (46), (29), and (31) that in the passage from the 2-point to the 3-point and to the 4-point functions, there is a perfect pattern of generalization, especially in the respective denominators of the integrands before the final stages of integration, viz.:

$$\left.\begin{array}{llll} & \textit{Quadratic Form:} & & \textit{Discriminant:} \\ \text{2-Point:} & z_k \alpha^2 + \frac{1}{2} \frac{\partial R_k}{\partial a_m} \alpha + a_n; & R_k\text{:} & 2 \times 2 \text{ Determinant} \\ \text{3-Point:} & \lambda(z) \alpha^2 - 2 \frac{\partial \Phi}{\partial a_j} \alpha + R_j; & \Phi\text{:} & 3 \times 3 \text{ Determinant} \\ \text{4-Point:} & \Lambda(z) \alpha^2 + \frac{1}{2} \frac{\partial \Psi}{\partial a_k} \alpha + \Phi_k; & \Psi\text{:} & 4 \times 4 \text{ Determinant} \end{array}\right\} \tag{47}$$

A word about the definition of the branches of log $\chi_j(\alpha)$ in (31) is now in order. From the original integral representation (15), we note that, where all z's are negative real, $I(z)$ is not only analytic but also positive. Hence we may define the log $\chi_j(\alpha)$ to lie on its principal sheet for such z's and the rest is done by analytic continuation from there. With this definition, for instance, we will always have on the physical sheet $\log \chi_j(1) = \log 1 = 0$ at the upper limit of integration. Note that $\chi_j(1) \equiv 1$, independent of the z's (cf. (52) below). It should perhaps also be pointed out that for $\alpha \varepsilon (0,1)$, $N_j(\alpha)$ are all positive for all negative real z's. For general z's, the sign of the square root $\sqrt{N_j}$ is rather unimportant since log χ_j will just compensate for any change of sign in front of $\sqrt{N_j}$.

IV.3 The 4-Point Roots

The 4-point roots $r_{1,2}$ are now defined as the zeros of the 4-point quadratic expression in (47), (which is the denominator in (31)). By virtue of (43), we have

$$r_{1,2}(z) = \frac{-Q \pm \sqrt{Q_1^2 - 16\Lambda\Phi_1}}{4\Lambda(z)} = \frac{-Q \pm 2\sqrt{\lambda_1 \Psi}}{4\Lambda(z)}. \tag{48}$$

Here we see explicitly that the condition for r to be a double root corresponds to the Ψ-manifold. (The other alternative $\lambda_1(z_4, z_5, z_6) = 0$ is trivial).

Thus, from (39), it follows that

$$\frac{M_j(r_i)}{\sqrt{N_j(r_i)}} = \pm\sqrt{\lambda_1}; \quad \begin{array}{l} i = 1, 2 \\ j = 1, 2, 3 \end{array} \tag{49}$$

and, together with (42), we have in particular

$$\left.\sum_{j=1}^{3}{}' \sqrt{N_j(r_i)}\right|_{r_i = 1} = 0. \tag{50}$$

Furthermore, it can be shown that

$$\prod_{j=1}^{3}{}' \chi_j(r_i) = 1, \tag{51}$$

where the summation sign Σ' and the product sign Π' are meant to take care of the sign condition of (49).

Note that

$$\chi_j(1) = 1, \quad j = 1, 2, 3 \tag{52}$$

holds automatically from (37), regardless of the manifold

$$\Lambda(z)\, r_i^2 + \frac{1}{2} Q_1 r_i + \Phi_1 = 0.$$

Finally the special case

$$\prod_{j=1}^{3}{}' \chi_j(0) = 1 \tag{53}$$

now holds on the Φ_1-manifold. This last identity was first established in KW and played an important role in their discussion of the 3-point function in the differentiated form[30].

The identities (49) and (51) are crucial for the 4-point case. Equation (49) says that at the vanishing of the 4-point denominator in (31), all the coefficients of the logarithms become identical, which allows the three log terms to be summed. Eq. (51) guarantees that they add up to log 1.

[30] Cf. KW (A. 50). There the factor $\frac{1}{\sqrt{\lambda(z)}}$ should read $\sqrt{\lambda(z)}$.

Therefore, just as in the 3-point case, the change of the branches of this final logarithm will determine the relevance of the 4-point singularity. We shall leave this problem to Sec. V and Appendix A.

With the above preliminary, the integral (31) can now be written as

$$\left.\begin{aligned} I(z;\mathrm{a}) &= \frac{-1}{2A(z)\cdot(r_1-r_2)}\int_0^1 d\alpha \sum_{j=1}^{3}\left(\frac{M_j(r_1)}{\alpha-r_1}-\frac{M_j(r_2)}{\alpha-r_2}\right)\frac{1}{\sqrt{N_j(\alpha)}}\log\chi_j(\alpha) \\ &= -\frac{1}{2}\frac{1}{A(z)\cdot(r_1-r_2)}\sum_{j=1}^{3}[F_j(r_1)-F_j(r_2)]; \end{aligned}\right\} \tag{54}$$

where

$$F_j(r_i) = M_j(r_i)\cdot\int_0^1\frac{d\alpha}{\alpha-r_i}\cdot\frac{1}{\sqrt{N_j(\alpha)}}\log\chi_j(\alpha). \quad \begin{matrix} i=1,2 \\ j=1,2,3 \end{matrix} \tag{55}$$

The situation in the α-plane is quite clear. Namely, one has only to watch out for the three sets of roots (i. e. the 2-point, 3-point, and 4-point) of the expressions (47) versus the path of integration (0,1). Equation (54) explicitly shows that singularities of the 4-point type occur when the 4-point roots r_i become a double root *and* when there is no cancellation among the F's. We now discuss separately the two cases $r_1 \neq r_2$ and $r_1 = r_2$.

IV. 4 The 2-Point and 3-Point Singularities in the 4-Point Function

We first discuss the case when the 4-point roots are distinct: $r_1 \neq r_2$ in (54). Obviously any singularity must then come from each $F_j(r_i)$ and furthermore these singularities may still be subject to cancellation when the summation over j is carried out. The functions $F_j(r_i)$ defined in (55) are evidently multi-valued. When explicitly evaluated, they involve logarithms and a sum of 32 Spence functions for each $i = 1,2$ and $j = 1,2,3$. It is clear that the 4-point complication for each $F_j(r_i)$, as compared with the 3-point function $H(z)$ in the undifferentiated form (29), arises from the presence of the extra factor $(\alpha - r_i)^{-1}$ in (55), which at first sight may cause an apparent singularity for the integrand when r_i passes through the range (0,1). However, this is actually not a relevant source of singularity as long as $r_1 \neq r_2$, and $r_i \neq 0$, or 1, since in this case the path of integration can be easily deformed. Stated otherwise, on account of the identities (49) and (51), $\sum_{j=1}^{3} F_j(r_i)$ can still be defined by analytic continuation to a different sheet of the Riemann surface whenever a single root r_i crosses over the open interval (0,1). As already remarked above (following (47) and (52)), the upper limit of integration is entirely harmless. On the other hand, $r_i = 0$ implies the Φ_1-manifold, which is exactly the 3-point singularity corresponding to the tetrahedron T_1. The other three manifolds are $\Phi_{j+1} = 0$ which arise from the set of 3-point denominators $N_j(\alpha)$ in (55). This is evidently clear from our discussion of the 3-point function in the undifferentiated form (29). The remaining singularities in $F_j(r_i)$ in (55) then come from

a) when the 3-point roots take on the lower limit 0: giving the manifolds $R_{j'} = 0$ for each j. This results in one cut each for (z_5, z_4, z_6); and

b) when the 2-point roots (i. e. the zeros and poles of $\chi_j(\alpha)$) become double roots within the open interval (0,1). These 2-point roots result in the manifolds $R_m(z_m) = 0$ for $m = 1,2,3$ and can take on the value 0 only when the appropriate masses are zero.

We thus conclude that, for the case $r_1 \neq r_2$, the singularities of our 4-point function $I(z)$ of (54) are the degenerate ones of the 3-point and the 2-point types.

The above statement can also be explicitly verified by completing the last integration of (55) and then discussing the resulting expression. This is done in Appendix A.

We might mention that, for the case $r_1 \neq r_2$, there exists yet another way of looking at the singularities of $\sum_j F_j(r_i)$. Consider now the expression

$$\left.\begin{aligned} J(z;\,a) &= \sum_{k=1}^{4}{}' \frac{\partial}{\partial a_k}\left[-2A\cdot(r_1-r_2)\cdot I(z;\,a)\right] \\ &= \sum_{k=1}^{4}{}' \frac{\partial}{\partial a_k}\sum_j{}' \left[F_j(r_1)-F_j(r_2)\right]. \end{aligned}\right\} \quad (56)$$

As far as the singularities in z's are concerned, $J(z)$ will for all practical purposes yield as much information as $I(z)$, as long as we are away from the Ψ-manifold. Now the right-hand side of (56) is free from Spence functions; and one can readily see, after a straightforward computation, that one gets singularities of the 3-point and the 2-point type.[31]

[31] These details are contained in the Appendix B of the author's University of Maryland, Department of Physics Technical Report No. 186 (unpublished).

IV. 5 The 4-Point Singularity

Now we come to the case when the 4-point roots become coincident: $r_1 = r_2$, or we are on the Ψ-manifold. From (54), it is clear that one gets a 4-point singularity on the Ψ-manifold *unless* there is a cancellation among the $\sum_j F_j(r_i)$. For this we may divide the Ψ-manifold into Ψ^R and Ψ^{IR}, where the superscripts R and IR denote respectively the *relevant* (no cancellation) and the *irrelevant* (no jump) portions of the Ψ-manifold. It is easy to convince oneself that Ψ^{IR} is actually non-empty. Obvious examples are the cases when all $\boldsymbol{Im}\, z_\mu$ have the same sign, or when all z_μ are negative real, since in both cases we know from the original integral representation (15) that $I(z)$ is analytic there.

The relevance criteria for the Ψ-manifold are treated in Sec. V.

IV. 6 Summary of the Singularity Manifolds

In this section, we see that the 4-point function $I(z)$ admits the following types of singularities:

(a) 4-Point Singularity: on the manifold $\Psi(z; a) = 0$;

(b) 3-Point Singularity: on the manifolds $\Phi_k = 0$, $k = 1, \ldots, 4$;

(c) 2-Point Singularity: on the manifolds $R_\mu = 0$, $\mu = 1, \ldots, 6$.

In terms of the determinants, the Φ_k's and the R_μ's are just the appropriate principal minors of the Ψ-determinant (cf. Sec. III).

V. The Relevance Criteria for the 4-Point Singularity Manifold

We have seen in Sec. IV that the 4-point singularity arises when the roots r_i defined by (48) become coincident. Now we want to examine the behavior[32] of these merging roots more closely in connection with the question of distinguishing Ψ^R from Ψ^{IR}.

[32] In fact, the following technique was first applied to the 3-point case in the undifferentiated form (29) where one is able to re-derive the criteria for the change of relevance of the Φ-manifold. An explicit illustration of this is contained in the Appendix C of the reference cited in footnote 31.

To be specific, let us consider, for the sake of convenience, $z_1, \ldots, z_5$ as being fixed, the roots $r_{1,2}$ as functions of z_6 alone. Suppose we make an arbitrary path $\widehat{ab}$ in the z_6-plane, which connects a point a in the known analyticity region (such a point can always be chosen; e. g., at $-\infty$) to a point b lying on the Ψ-manifold (Fig. 6). Under the mappings $z_6 \to r_i(z_6)$, $i = 1, 2$, this path $\widehat{ab}$ is now mapped into, say $\widehat{A_1B}$ and $\widehat{A_2B}$, respectively, in the α-plane. Then there are the following possibilities:

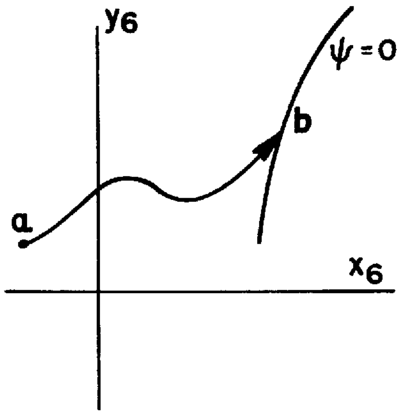

Figure 6. Path of continuation to Ψ-manifold.

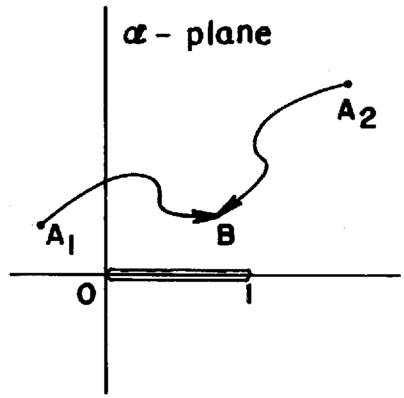

Figure 7. Behavior of the 4-point roots: Irrelevant merging.

(i) Neither of the paths $\widehat{A_iB}$ crosses the interval (0,1), e. g., Fig. 7;

(ii) One of the paths crosses over the interval (0,1) once, e. g., Fig. 8; or if more than one crossing is made, then either

(i′) the net crossing is even and without encircling the endpoints; or

(ii′) the net crossing is odd, or with encircling of the end points.

Situation (i) or (i′) is obviously harmless. For such cases, (the path of integration can be easily deformed for the case (i′)), the function $\sum_j' F_j(r)$ has no jump, hence there will be a cancellation in (54); the singularity at $r_1(b) = r_2(b)$ is thus removed, and one says that the portion of the Ψ-manifold, to which the point b belongs, must lie in Ψ^{IR}. On the other hand, for the situation (ii) or (ii′), the function does have a jump, and hence no cancellation. One gets then an actual singularity at the point b, and the portion of the Ψ-manifold to which b belongs will lie in Ψ^R.

The technique thus described, of plotting the explicit behaviors of the merging roots r_i in the α-plane versus the path of continuation in the

z-space from the known analyticity region to the part of the Ψ-manifold whose relevance is to be determined, although most primitive and tedious, is a rather useful and practical procedure to really pin down the relevance question. Except in some very special cases it is not ncessary to plot these merging roots, as one can instead rely on more general criteria. Since we do not expect the whole Ψ-manifold to be relevant, the relevance of this must change when it intersects with some other manifolds. In the following,

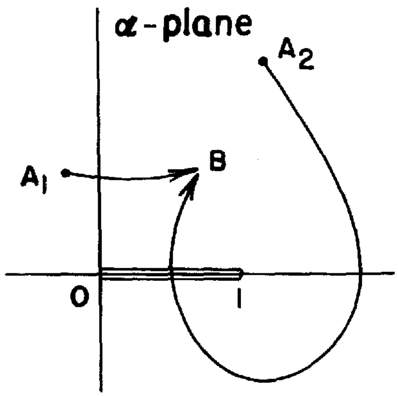

Figure 8. Behavior of the 4-point roots: Relevant merging.

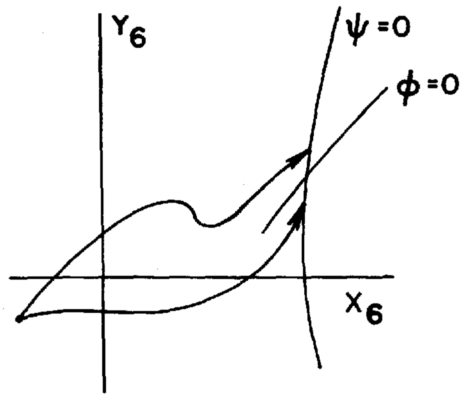

Figure 9. Path of continuation in the neighborhood of $(\Psi = 0) \cap (\Phi = 0)$.

we shall show that these other surfaces are just the relevant portions of the Φ_k-manifolds of the 3-point type.

We shall first state the relevance criteria for the 3-point singularity manifold $\Phi = 0$:

Lemma 1 (KW): The 3-point singularity manifold $\Phi = 0$ changes its relevance at its intersections with the relevant portions of the 2-point singularity manifolds $R_i = 0$, $i = 1, 2, 3$.

This statement is evident from the explicit form (28)[33].

We can now state in perfect analogy:

Lemma 2: The 4-point singularity manifold $\Psi = 0$ changes its relevance at its intersections with the relevant portions of the 3-point singularity manifolds $\Phi_k = 0$, $k = 1, \ldots, 4$.

[33] Actually in KW, the problem of choosing the relevant portion of the Φ-manifold is quite easy. Since one knows enough from the permuted domain D_3 where one must have analyticity, an explicit knowledge of the branches of the logarithms is not mandatory. A more transparent way of seeing this independently is by discussing the behavior of the 3-point roots. This is given in Appendix C of the reference cited in footnote 31.

It suffices to show this for $k = 1$, as the others will obviously follow from symmetry. There are two ways to see this:

(a) One observes that one of the two (4-point) roots, say r_1, goes through the end-point zero of the interval (0,1) in the α-plane when the z's cross the manifold $\Phi_1 = 0$ (cf. (48)), while the other root (r_2) does not and will essentially remain unchanged. Thus when the two roots tend to merge, in one case (i. e., corresponding to one side of the Φ_1-manifold), the paths of

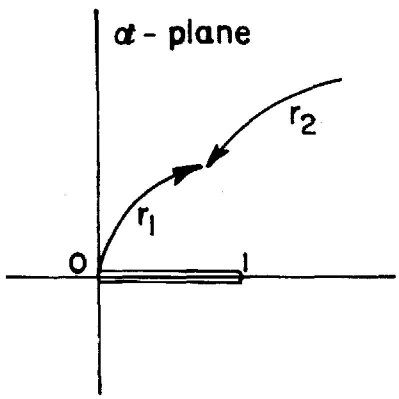

Figure 10. Merging of the 4-point roots: On one side of the Φ_1-manifold.

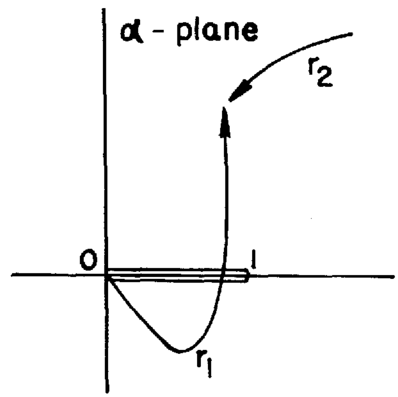

Figure 11. Merging of the 4-point roots: On the other side of the Φ_1-manifold.

the roots do not cross the cut (0,1) (Fig. 10); while for the other case (i. e. corresponding to the other side of the Φ_1-manifold), one of the roots (r_1) does cross over the cut (0,1) once (Fig. 11). Thus $\sum_j F_j(r_1)$ crosses over to a different sheet of the Riemann surface, while $\sum_j F_j(r_2)$ remains on the original sheet. Therefore, there is a cancellation on one side of the Φ_1-manifold, but not on the other side. Thus one concludes that the transition between Ψ^R and Ψ^{IR} takes place at the intersection with the Φ-manifolds.

(b) Another way to see this is by examining the explicit expression for $\sum_j F_j(r_i)$. The details are included in the Appendix A. We simply state that the results there confirm the above simple argument.

We conclude this section with a few remarks:

(1) It is clear that, since the whole Ψ-manifold cannot be all relevant, Ψ^R is non-empty only if the Ψ-manifold has an intersection with Φ_k^R (the relevant portion of the Φ_k-manifolds). Furthermore, as we shall see in Sec. VI that the singularity domain of the 4-point proper is actually compact, Ψ^R is non-trivial only if the Ψ-manifold intersects twice with the Φ_k^R. The con-

dition that such intersections occur is extremely complicated, and we shall only state later (cf. Sec. VI) some necessary conditions.

(2) The pattern of generalization from Lemma 1 to Lemma 2 strongly suggests that this feature may perhaps very well be valid for the general n-point domain in the perturbation theory. However, we do not attempt to prove (or disprove) this conjecture, since this lies outside the scope of the present investigation.

(3) Since $\partial D_3^{\text{pert}}$ is actually part of $\partial E(D_3)$, Lemma 1 is likewise valid in the axiomatic approach. In the 4-point case, from the preliminary results[34] for the $\partial D_4^{\text{prim}}$, the general spirit of Lemma 2 (i. e. deleting Ψ in leaving the question open as to whether this Ψ-manifold has any relation with the $\partial D_4^{\text{prim}}$ (cf. remark in Sec. VI.7)) seems also to be valid in the axiomatic approach.

VI. Determination of the Boundary of the 4-Point Domain

VI. 1 General Discussion

In the preceding two sections, we have shown that the 4-point singularities, subject to the relevance conditions, are confined to the manifold given by the vanishing of the following 4×4 determinant:

$$\left.\Psi(z;a)=\begin{vmatrix} -2a_1 & z_1-a_2-a_1 & z_2-a_3-a_1 & z_3-a_4-a_1 \\ z_1-a_1-a_2 & -2a_2 & z_4-a_3-a_2 & z_6-a_4-a_2 \\ z_2-a_1-a_3 & z_4-a_2-a_3 & -2a_3 & z_5-a_4-a_3 \\ z_3-a_1-a_4 & z_6-a_2-a_4 & z_5-a_3-a_4 & -2a_4 \end{vmatrix};\ a_k>0.\ \right\} \tag{57}$$

In this section, we wish to determine what constitutes the boundary surfaces for this singularity domain. As it stands, $\Psi(z_\mu;\, a_k)$ generates a 4-parameter family of surfaces in the space of six complex variables. In principle, the boundaries of such a family of surfaces could be made up from any of the following multitude of possibilities:

(1) The geometric envelope of this 4-parameter family of surfaces, which would correspond to a special path traversed by the a's in the *sedecimant* $a_k > 0$. This will be called, for convenience, the 4-*mass envelope* and will be denoted by E_{1234}.

[34] Private communication from Prof. KÄLLÉN.

(2) Subcases of (1) when *one* of the 4 a's takes on the extreme value of 0, or ∞, and the other 3 a's, taking a path in the subspace of the octant $a_j > 0$, produce a 3-*mass envelope* E_{ijk}. In principle, there could be 8 such envelopes.

(3) Still further subcases of (1) are when two of the 4 a's take on the extreme values of 0, or ∞, and the remaining two a's, taking a path in the quadrant $a_i > 0$, produce a 2-*mass envelope* E_{ik}. There could be 24 such envelopes.

(4) Finally, we have the simplest of all cases when 3 of the 4 a's take on the extreme values of 0, or ∞, leaving the remaining one single a_s to vary along the semi-axis $a_s > 0$. In all, there could be 32 such 1-*mass surfaces* E_k.

Out of all these 65 possible candidates for the boundaries to the 4-point domain D_4^{pert}, our present task is to eliminate the ineligible ones. Fortunately, we can eliminate all cases in (2)–(4) which involve any a_k to be ∞. We recall that the a's have the physical meaning of the squares of the masses associated with the internal lines in the Feynman diagrams. Now if any a_k is arbitrary large, then the thresholds for virtual production processes which correspond to the onsets of the associated cut-planes will be proportionally high. Since the 3-point boundary F'_{kl} curves will not be relevant unless they have crosses over the cut beyond the threshold (Lemma 1 of Sec. V), and furthermore, since the relevance of the 4-point boundaries depends on whether or not they have intersected the relevant 3-point curves (Lemma 2), it is clear that the ∞-portion of any a_k would not give rise to any relevant singularity. This statement is also valid in the 3-point case, if we note that all the relevant portions of the F'_{kl} curves are actually confined to the lower ends of the a_m-ranges (from $a_m = 0$ up to a finite value).

This criterion has the further consequence that the singularity domain of the 4-point proper is actually compact. Unlike the 3-point case when the F'_{12} curve extends to ∞ in the z_3-plane at the $a_3 = 0$ end, the all $a_k = 0$ end is always finite in the 4-point case (apart from the trivial case when one of the z's stays zero) (cf. Eq. (120) below).

Thus we shall from now on consider in cases (2)–(4) those extreme values of a's to be zero only. In this way, the list of candidates for boundary is now radically trimmed from 65 down to 15, viz.,

(1) 1 E_{1234}
(2) 4 E_{ijk}
(3) 6 E_{ik}
(4) 4 E_k.

In the following, we shall first examine the questions of the various envelopes listed above. *A priori*, the question is two-fold:

(a) whether such envelopes can exist at all in the allowed all-positive ranges of the a's and

(b) if they do exist under certain circumstances, then it still remains to be seen whether they are really part of the boundary of our domain.

It would perhaps be helpful to recall the corresponding situations for the Φ-manifold in the 3-point case. KW have shown that the boundaries there are made of only the 1-mass curves (analogue of case (4) above). The envelope problem for the Φ-manifold is a much simpler one than we shall encounter below. We give a concise treatment for this in Appendix B. The result there can be simply summarized as follows: *Envelopes for the 3-point Φ-manifold can exist, but they do not lie off the R-manifolds*[35] (i. e. on the cut for each z). One concludes then that the boundaries are made up by the F'_{kl}, which are simple analytic hypersurfaces.

Our results in the following subsections will show that, unlike the 3-point case, the envelopes in the 4-point case are non-trivial[36] and in general the boundary of our domain will be made of pieces of (2-mass) envelopes. Thus we have here a fundamental difference between the 4-point domain and the 3-point domain, namely, the regularity domain of the 4-point function in perturbation theory, D_4^{pert}, is in general *not* everywhere bounded by analytic hypersurfaces.

Before we go into the details for each of the above cases, we shall formulate the envelope condition as follows:

The existence of the envelopes is purely a property that is related to the algebraic structure of the manifold. Consider in general the expression for an m-parameter family of surfaces, $f(z_i; a_k) = 0$, $i = 1, \ldots, n$; $k = 1, \ldots, m$, where the a's are the parameters under consideration, which are allowed to vary over a *real* domain $\boldsymbol{A}_m$.

Definition: A point on f is said to lie on the m-envelope of f if, together with $f = 0$, the set of $(m-1)$ independent equations

$$\boldsymbol{Im}\left(\frac{\dfrac{\partial f}{\partial a_j}}{\dfrac{\partial f}{\partial a_k}}\right) = 0, \quad j, k = 1, \ldots, m \tag{58}$$

admits a set of solutions $\{a_k^*\}$ such that $\{a_k^*\} \varepsilon \boldsymbol{A}_m$.

[35] In this connection, it is very tempting to conjecture that the envelopes for the Ψ-manifold would not lie off the Φ-manifolds, but this conjecture turns out to be not true.

[36] In the sense that in general they do not lie on the Φ-manifolds. However the 4-mass and the 3-mass envelopes do not contribute to the boundary (cf. Sec. VI.2 and Sec. VI.3).

Stated in another way, the $(m-1)$ independent equations (58) can be regarded as the $(m-1)$ constraints on the m-parameters, so that in principle one can always express all the other $(m-1)$ parameters a_s, $s>1$, as functions of the remaining parameter, say a_1. Let

$$\tilde{\boldsymbol{A}}_m = \{a\colon a_1 \varepsilon \boldsymbol{A}_m;\ a_s = a_s(a_1)\}$$

which shall be referred to as the "path" for the m-envelope. Note that in general $\tilde{\boldsymbol{A}}_m$ will not be completely contained in $\boldsymbol{A}_m$. If, regardless of the configuration of the z's,

$$\tilde{\boldsymbol{A}}_m \cap \boldsymbol{A}_m = 0,$$

then it is clear that the m-envelope in question does not exist at all. Otherwise, for $\tilde{\boldsymbol{A}}_m \cap \boldsymbol{A}_m \neq 0$, we will be able to find in the a-space (i. e. $\boldsymbol{A}_m$) an allowed path $\tilde{\boldsymbol{A}}_m \cap \boldsymbol{A}_m$ such that the image of this under the mapping

$$z_n = g(z_1, \ldots, z_{n-1};\ a_k)\colon\ f = 0$$

gives the desired m-envelope. Since the a's mix the real and imaginary parts of the z's, the envelopes will evidently in general *not* be analytic hypersurfaces. Equations (58) will be referred to as the envelope conditions.

VI. 2 The 4-Mass Envelope

We now proceed to apply the general equations (58) to our specific manifold $\det|\Psi_{ij}| \equiv \Psi(z;a) = 0$ of (57). Before we do this, we shall derive a number of identities which will be crucial for the subsequent discussion of the envelopes. First, we find it useful to rewrite the Ψ-determinant such that the a's shall appear only in one column and one row (cf. (22a)). For instance, we have from (57):

$$\Psi(z;a) = \begin{vmatrix} -2a_1 & z_1+a_1-a_2 & z_2+a_1-a_3 & z_3+a_1-a_4 \\ z_1+a_1-a_2 & -2z_1 & z_4-z_2-z_1 & z_6-z_3-z_1 \\ z_2+a_1-a_3 & z_4-z_1-z_2 & -2z_2 & z_5-z_3-z_2 \\ z_3+a_1-a_4 & z_6-z_1-z_3 & z_5-z_2-z_3 & -2z_3 \end{vmatrix} \equiv |\tilde{\Psi}_{ij}|. \qquad (57\,\mathrm{a})$$

Here a_1 is singled out. Evidently there are 3 other such forms obtained by suitable permutations. For convenience, let us denote by $\tilde{\Psi}_{ij}$ the ij-th element and by $\tilde{\Psi}^{ij}$ its minor in (57a), while the corresponding uncurled quantities shall refer to those in the original from (57). Note that

$$\left.\begin{aligned}\tilde{\Psi}_{1k} &= \Psi_{1k} + 2\,a_1; \quad k \neq 1 \\ \tilde{\Psi}_{11} &= \Psi_{11}.\end{aligned}\right\} \tag{59}$$

We have already had occasion in Sec. IV to define such quantities as $Q_k = \dfrac{\partial \Psi}{\partial a_k}$ of (32). Now these have the most natural interpretation in terms of (57a), namely:
For $k = 2,3,4$

$$\tilde{\Psi}^{1k} = (-1)^k \cdot \frac{1}{2}\frac{\partial \Psi}{\partial a_k} \equiv (-1)^k \cdot \frac{1}{2} Q_k; \quad k \neq 1, \tag{60}$$

$$\tilde{\Psi}^{kk} = 2\Phi_k \equiv \Psi^{kk}; \quad k \neq 1, \tag{61}$$

$$\tilde{\Psi}^{11} = 2\,\Lambda(z). \tag{62}$$

Furthermore, from (57a), one immediately sees that

$$\sum_{i=1}^{4} Q_i \equiv \sum_{i=1}^{4} \frac{\partial \Psi}{\partial a_i} = \left(\sum_{i=1}^{4} \frac{\partial}{\partial a_i} \tilde{\Psi}_{11}\right) \cdot \tilde{\Psi}^{11}$$

since

$$\sum_{i=1}^{4} \frac{\partial}{\partial a_i} \tilde{\Psi}^{1k} = 0, \quad \text{for} \quad k \neq 1.$$

Therefore

$$\sum_{i=1}^{4} Q_i = -4\,\Lambda(z), \tag{63}$$

in which the right-hand side is independent of the a's. Next, with the aid of (60), we have

$$\begin{aligned}\Psi &= \sum_{i=1}^{4} (-1)^{i+1} \tilde{\Psi}_{1i} \tilde{\Psi}^{1i} = \tilde{\Psi}_{11} \tilde{\Psi}^{11} + \sum_{k \neq 1} (-1)^{k+1} \tilde{\Psi}_{1k} \tilde{\Psi}^{1k} \\ &= -\frac{1}{2} \Psi_{11} \sum_{i=1}^{4} Q_i + \frac{1}{2} \sum_{k \neq 1} (-1)^{2k+1} \tilde{\Psi}_{1k} Q_k.\end{aligned}$$

Using (59), we get

$$\Psi = -\frac{1}{2} \sum_{j=1}^{4} \Psi_{ij} \cdot \frac{\partial \Psi}{\partial a_j} \tag{64}$$

for all $i = 1, \ldots, 4$.
Identities (63) and (64) will be of great importance to us in the following discussions. Another set of identities which we will need here already ap-

peared in (43)[37]. We are now ready to write down the envelope conditions for E_{1234} according to (58):

$$\boldsymbol{Im}\left(\frac{\dfrac{\partial \Psi}{\partial a_j}}{\dfrac{\partial \Psi}{\partial a_k}}\right) = \boldsymbol{Im}\left(\frac{Q_j}{Q_k}\right) = 0; \qquad \begin{array}{l} j, k = 1, \ldots, 4 \\ j \neq k \text{ (otherwise trivial).} \end{array} \tag{65}$$

In view of the identity (63), we can now define a set of four *real* numbers γ_k such that on E_{1234}

$$Q_k = -4\gamma_k \cdot \Lambda(z) \tag{66}$$

with

$$\sum_k \gamma_k = 1; \quad \boldsymbol{Im}\, \gamma_k = 0. \tag{67}$$

From (43) we have on the manifold $\Psi = 0$:

$$Q_k = \pm 4\sqrt{\Lambda \Phi_k}. \tag{68}$$

Therefore we have on E_{1234}

$$\frac{\Phi_j}{\Phi_k} = \frac{\gamma_j^2}{\gamma_k^2} > 0; \quad j, k = 1, \ldots, 4. \tag{69}$$

In principle, the system of equations (69) together with $\Psi = 0$ contain all the information there is about the 4-mass envelope. (In fact, as we shall see later in Sec. VI.4 for a 2-mass envelope, one has only one such equation which actually exhausts the envelope condition). However, a frontal attack on (69) for both the 4-mass and the 3-mass envelopes could lead to tremendous algebraic complications. We find it much more convenient to go back to the system of equations (64). We have for $\Psi = 0$:

$$\sum_{k=1}^{4} \Psi_{ik} Q_k = 0, \quad i = 1, \ldots, 4. \tag{70}$$

Now with (66), we get ($\Lambda(z) \neq 0$)

$$\sum_{k=1}^{4} \Psi_{ik} \gamma_k = 0, \quad \sum_k \gamma_k = 1. \tag{71}$$

We emphasize that the γ_k's are real, so that the system of equations (71) is equivalent to the following set of 9 real linear algebraic equations[38]

[37] For a proof of such identities, see Appendix D.
[38] The fact that $\Psi = 0$ is automatically satisfied is obvious from (72).

$$\sum_{k=1}^{4} (\boldsymbol{Im}\,\Psi_{ik})\,\gamma_k = 0\,, \qquad i = 1, \ldots, 4, \tag{72a}$$

$$\sum_{k=1}^{4} (\boldsymbol{Re}\,\Psi_{ik})\,\gamma_k = 0\,, \qquad i = 1, \ldots, 4, \tag{72b}$$

$$\sum_{k=1}^{4} \gamma_k = 1. \tag{72c}$$

Since, according to (72c), the solution with all γ's being equal to zero is unacceptable, it follows that the determinants of the coefficients of any four equations taken at a time (out of the eight in (72a-b)) should vanish. We shall first discuss the consequences of (72a) which contain the most powerful restrictions on E_{1234}:

$$\det|\boldsymbol{Im}\,\Psi_{ij}| \equiv \begin{vmatrix} 0 & y_1 & y_2 & y_3 \\ y_1 & 0 & y_4 & y_6 \\ y_2 & y_4 & 0 & y_5 \\ y_3 & y_6 & y_5 & 0 \end{vmatrix} = 0\,. \tag{73}$$

Note that this determinant is equivalent to the λ-function of products of conjugate variables, viz:

$$\lambda(y_k y_{k'}) \equiv -\begin{vmatrix} -2y_1y_5 & y_2y_6 - y_1y_5 - y_3y_4 \\ y_2y_6 - y_1y_5 - y_3y_4 & -2y_3y_4 \end{vmatrix} = 0\,. \tag{73a}$$

From now on, we shall be more specific by keeping the other 5 z's fixed, and project everything into the z_6-plane. We see that the 4-mass envelope E_{1234} can only be satisfied on the two horizontal straight lines obtained by solving (73), viz:

$$y_6 = \frac{y_1y_5 + y_3y_4 \pm 2\sqrt{y_1y_3y_4y_5}}{y_2} \tag{74}$$

or

$$\pm\sqrt{y_2y_6} = \sqrt{y_1y_5} \pm \sqrt{y_3y_4}\,. \tag{74a}$$

From (74), it follows immediately that *there exists no 4-mass envelope whenever*

$$y_1y_3y_4y_5 < 0\,. \tag{75}$$

More generally, in view of (74a), we can state that *the necessary condition for the existence of the 4-mass envelope E_{1234} is that the three products of* $y_k y_{k'}$, $k = 1, 2, 3$ (k' = conjugate of k, cf. Sec. III.2) *must have the same sign*, or

$$\frac{y_1 y_5}{|y_1 y_5|} = \frac{y_2 y_6}{|y_2 y_6|} = \frac{y_3 y_4}{|y_3 y_4|} \tag{76}$$

which we shall refer to as the *sign convention* for the existence of the 4-mass envelope $E^{\pm}_{1234}$.

An obvious example which satisfies this sign condition but where $E^{\pm}_{1234}$ is entirely irrelevant is furnished by the configuration whenever 5 z's lie in the same half-plane. Then the $E^{\pm}_{1234}$ in the 6-th variable must also lie in this same half-plane. As we have already mentioned in Sec. V, the original function (15) has no singularity for all $6z$'s having the same sign in the $\boldsymbol{Im}\, z$'s. Here we have a situation where the entire lines are irrelevant. For the other configurations[39], however, the situations are much more complicated, as we shall see below.

So far, we have only explored the existence condition of E_{1234} based on the consequence of the imaginary part equations (72a). A brief examination of the real part equations (72b) will convince oneself that there is no algebraic contradiction among the two sets of equations, so that in principle E_{1234}, satisfying (76), can exist provided that all the parameters a_k could be found to be positive at least for some configurations of the x's. This we now proceed to show.

To be specific, let the y's be given, satisfying (76); one can explicitly compute the γ_k's from (72a) and (72c)[40] (cf. Appendix C) in terms of the y's. Equations (72b) may now be regarded as those governing the a_k's. The solutions may be written as follows:

$$a_k = -\frac{1}{2}\sum_{i,j} X_{ij}\gamma_i\gamma_j + \sum_j X_{kj}\gamma_j; \quad k, i, j = 1, \ldots, 4 \tag{77}$$

with

$$\sum_j \gamma_j = 1,$$

where the matrix X is given by

$$X = \begin{pmatrix} 0 & x_1 & x_2 & x_3 \\ x_1 & 0 & x_4 & x_6 \\ x_2 & x_4 & 0 & x_5 \\ x_3 & x_6 & x_5 & 0 \end{pmatrix}. \tag{78}$$

[39] The distinct configurations for which (76) is satisfied are given in Appendix C.
[40] Note that the following ratios hold on E_{1234}:
$\gamma_1^2 : \gamma_2^2 : \gamma_3^2 : \gamma_4^2 := y_4 y_5 y_6 : y_2 y_3 y_5 : y_1 y_3 y_6 : y_1 y_2 y_4$, the right-hand side can be regarded as Φ_k evaluated at all $x = 0$ and all $a = 0$ (cf. Appendix C).

Note that

$$X_{ij} = \boldsymbol{Re}\, \Psi_{ij}\big|_{a_k = 0}.$$

It is clear then that the configurations of the x's must be such that $\bigcap_k \{a_k > 0\} \neq 0$, or

$$-\frac{1}{2}\sum_{i,j} X_{ij}\gamma_i\gamma_j + \sum_j X_{kj}\gamma_j > 0, \quad \text{for} \quad k = 1, \ldots, 4. \tag{79}$$

The set of equations (79) which is linear and homogeneous in the six x's defines a region of the x's in the six-dimensional space R^6, which can be visualized as the intersection of the "positive sides" of the four linear manifolds defined by setting the left-hand side of (79) equal to zero for each k. Let Ω_x denote this intersection. The fact that Ω_x is non-empty is trivial (since the dimensionality of the variables (x's) exceeds the number of constraints by two). It may be of some interest to note the subset of Ω_x for which the a_k's are positive definite (i. e., regardless of the γ_k's). For this we may rewrite (77) in the following matrix notation:

$$a_k = -\frac{1}{2}\gamma^{(k)T} L^{(k)} \gamma^{(k)}, \quad k = 1, \ldots, 4 \tag{80}$$

in which $\gamma^{(k)}$ denotes a 3×1 column matrix of the γ_j's with the deletion of the γ_k, e.g., $\gamma^{(1)} = \begin{pmatrix}\gamma_2\\ \gamma_3\\ \gamma_4\end{pmatrix}$, etc.. $L^{(k)}$ is a set of 3×3 symmetric matrices in the x's:

$$L^{(1)} = \begin{pmatrix} -2x_1 & x_4 - x_2 - x_1 & x_6 - x_3 - x_1 \\ x_4 - x_1 - x_2 & -2x_2 & x_5 - x_3 - x_2 \\ x_6 - x_1 - x_3 & x_5 - x_2 - x_3 & -2x_3 \end{pmatrix},$$

$$L^{(2)} = \begin{pmatrix} -2x_1 & x_2 - x_4 - x_1 & x_3 - x_6 - x_1 \\ x_2 - x_1 - x_4 & -2x_4 & x_5 - x_6 - x_4 \\ x_3 - x_1 - x_6 & x_5 - x_4 - x_6 & -2x_6 \end{pmatrix},$$

$$L^{(3)} = \begin{pmatrix} -2x_2 & x_1 - x_4 - x_2 & x_3 - x_5 - x_2 \\ x_1 - x_2 - x_4 & -2x_4 & x_6 - x_5 - x_4 \\ x_3 - x_2 - x_5 & x_6 - x_4 - x_5 & -2x_5 \end{pmatrix},$$

$$L^{(4)} = \begin{pmatrix} -2x_3 & x_1 - x_6 - x_3 & x_2 - x_5 - x_3 \\ x_1 - x_3 - x_6 & -2x_6 & x_4 - x_5 - x_6 \\ x_2 - x_3 - x_5 & x_4 - x_6 - x_5 & -2x_5 \end{pmatrix}.$$

Note that[41]

$$\det L^{(k)} = 2\Lambda(x), \quad \text{for} \quad k = 1, \ldots, 4. \tag{81}$$

Now with the a_k's regarded as the quadratic forms in the γ_k's in (80), a standard procedure of diagonalization immediately shows that the subset ω_x of Ω_x for which the a_k's are positive definite is given by

$$\omega_x = \left\{ x: x_\mu > 0, \lambda_k(x) < 0, \Lambda(x) < 0, \quad \begin{matrix} k = 1, \ldots, 4 \\ \mu = 1, \ldots, 6 \end{matrix} \right\}. \tag{82}$$

It is trivial to check that ω_x is non-empty. Thus $0 \neq \omega_x \subset \Omega_x$. Geometrically, $-\lambda(z_i, x_j, x_k) = 16$ times the squares of the area of the triangle with the

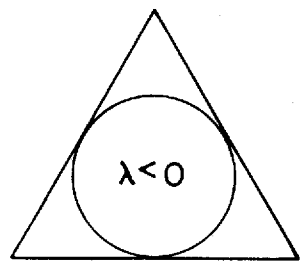

Figure 12. Projection of the $\lambda(x)$-cone.

sides $\sqrt{x_i}$, $\sqrt{x_j}$, $\sqrt{x_k}$; and $-\Lambda(x) = 144$ times the square of the volume of the tetrahedron formed by the six edges with lengths $\sqrt{x_\mu}$. In a 3-dimensional space, the region $\lambda(x) < 0$ is the interior of a cone[42] (tangent to all coordinate planes) within the octant $x_i, x_j, x_k > 0$ (cf. Fig. 12 above as projection). Now in the 6-dimensional space, one first goes to the *sexaginta-quadrant* $x_\mu > 0$, then takes the intersection of 4 sets of the λ-cones in the sub-3-spaces, and finally inscribes the surface of $\Lambda(x) = 0$ (which will be tangent to all four λ_k-cones). (No attempt is made to draw such a picture here, not even the projection).

This establishes that with suitably given z's (***Im*** z's satisfying (76), ***Re*** z's satisfying (79), and in particular (82)), the four parameters a_k can indeed be found simultaneously positive on the 4-mass envelope E_{1234}, and with this we conclude the existence of the 4-mass envelope.

41 We note in passing that the structures of the $L^{(k)}$-matrices can be easily understood with the aid of the tetrahedron T of Sec. III.2. The diagonal elements in $L^{(k)}$ correspond to those edges emerging from the k-th vertex of Fig. 3, and the off-diagonal elements to the edges conjugate to this vertex (i. e. the k-th face).

42 A beautiful picture of such λ-cone appeared in a recent paper of A. S. Wightman and H. Epstein, Annals of Phys. **11**, 201 (1960), in an entirely different context.

We now proceed to discuss the relevance of E_{1234}. To be specific, consider $y_1, \ldots, y_5$ given according to (76), compute the y_6 from (74) (i. e. we get two horizontal lines $E^{\pm}_{1234}$ in the z_6-plane). From these 6 y's, compute the γ_k's from (72a). Now given more or less arbitrary $x_1, \ldots, x_5$, the linearity of (77) implies that a_k has one zero only on each of the $E^{\pm}_{1234}$. A

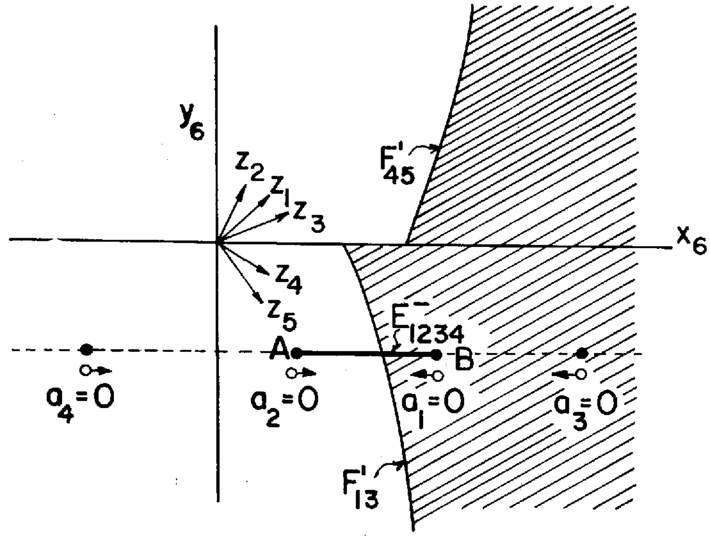

Figure 13. Straight-line segment in z_6-plane as the 4-mass envelope.

typical case is illustrated in Fig. 13. We use the symbol $0 \rightarrow$ to show the direction in which that particular a_j is positive. Hereafter, E_{1234} shall properly denote the allowed region of existence of the 4-mass envelope on which the intersection of all $a_k > 0$ has been taken (e. g., the segment between $a_2 = 0$ and $a_1 = 0$ in Fig. 13). By definition, E_{1234} is contained in Ω_x of (79); however, $E_{1234} \cap \omega_x$ may be empty. The case when $E_{1234} \cap \omega_x \neq 0$ has some pertinent features which we leave to the Appendix C. In general there are the following possibilities:

(a) E_{1234} is either empty for a particular configuration of the z's or is entirely contained inside the 3-point singularity domain: in such cases, the 4-mass envelopes are entirely irrelevant.

(b) E_{1234} is unbounded at one end which lies outside the 3-point singularity domain: In this case (imagine all $\leftarrow 0$ pointing to the left in Fig. 13), it is also easy to dispose of by observing that the extreme far end of E_{1234} (which corresponds to all $a_k \to \infty$) is never relevant. Since the relevance of E_{1234} does not change unless it has an intersection; otherwise, the case is reduced to (c) below.

(c) E_{1234} is finite and partly lies outside the 3-point singularity domain (Figs. 13, 24, 25). This is the only outstanding situation of the 4-mass envelope which needs further discussion. It is clear that the path in the a-space corresponding to such a finite E_{1234} is a straight-line segment bounded by two 3-dimensional sub-spaces. As will be shown in Sec. VI.3, at the end $a_m = 0$ of E_{1234} comes the 3-mass envelope $E_{jkl}\,(j \neq k \neq l \neq m)$. Stated otherwise, the fact that E_{1234} suddenly comes to a stop must mean that there is another curve which would also pass through that point. For this, we must defer the remaining discussion of the role of the 4-mass envelope until we have treated the 3-mass envelopes in the next sub-section.

VI. 3 The 3-Mass Envelope

We have seen that the restrictions of the 4-mass envelope are so strong that one gets only rather trivial situations where E_{1234} is confined to a straight-line segment in the z-plane. The envelope condition (58) or (66) is relaxed when one goes from an m-envelope to an $(m-1)$-envelope; since, by definition, one of the parameters a_i now takes on the fixed extreme value 0, the corresponding restriction of $\partial \Psi / \partial a_i$ is then to be removed. We shall now sketch the necessary modification for the treatment of the 3-mass envelopes E_{jkl}. To be specific, let us consider E_{123}, the 3-mass envelope formed by a special path in the (a_1, a_2, a_3) 3-space. For this, we set once for all, $a_4 = 0$, in the expression for $\Psi(z; a)$. Strictly speaking, Q_4 which was defined as $\partial \Psi / \partial a_4$ is now meaningless, however, as a shorthand notation, we shall still use it as

$$Q_4 = \left(\frac{\partial \Psi}{\partial a_k}\right)\bigg|_{a_4 = 0}$$

which, as stated above, is no longer restricted by the reality condition of (66). However, the identities (70) still hold with $a_4 = 0$. We may now define on E_{123} a set of 5 real γ_s, $s = 1, \ldots, 5$ such that

$$\left.\begin{aligned} Q_j &= \gamma_j\, G(z;\,a), \quad j = 1, \ldots, 3 \\ G(z;\,a) &= \sum_{j=1}^{3} Q_j = -4\,\Delta(z) - Q_4\,; \\ &\sum_{j=1}^{3} \gamma_j = 1 \end{aligned}\right\} \tag{83}$$

and

$$Q_4 = (\gamma_4 + i\,\gamma_5)\cdot G(z;\,a).$$

Substituting (83) into (70) and dividing by $G(z;\,a)$, we get, after taking the imaginary and the real parts:

$$\sum_{k=1}^{4} (\boldsymbol{Im}\,\Psi_{ik})\,\gamma_k + (\boldsymbol{Re}\;\Psi_{i4})\,\gamma_5 = 0\,; \tag{84a}$$

$$\sum_{k=1}^{4} (\boldsymbol{Re}\;\Psi_{ik})\,\gamma_k - (\boldsymbol{Im}\,\Psi_{i4})\,\gamma_5 = 0\,; \quad \text{for} \quad i = 1, \ldots, 4. \tag{84b}$$

Now taking the fourth equation of (84b) together with (84a), we have

$$\sum_{t=1}^{5} U_{st}\,\gamma_t = 0\,, \quad s = 1, \ldots, 5 \tag{85}$$

where

$$U = \begin{pmatrix} 0 & y_1 & y_2 & y_3 & \xi_1^{(4)} \\ y_1 & 0 & y_4 & y_6 & \xi_2^{(4)} \\ y_2 & y_4 & 0 & y_5 & \xi_3^{(4)} \\ y_3 & y_6 & y_5 & 0 & 0 \\ \xi_1^{(4)} & \xi_2^{(4)} & \xi_3^{(4)} & 0 & 0 \end{pmatrix} \tag{86}$$

with

$$(\xi_j^{(4)}) \equiv \begin{pmatrix} x_3 - a_1 \\ x_6 - a_2 \\ x_5 - a_3 \end{pmatrix} \tag{87}$$

where the superscript (4) is a reminder of $a_4 = 0$. Note that this column corresponds to the edges emerging from the 4-th vertex of the tetrahedron T of Fig. 3.

Since the det$|U|$ must vanish for non-trivial solutions of the γ_s's, we have

$$\det|U| = -\,\xi^{(4)^T}\,V\,\xi^{(4)} = 0, \tag{88}$$

where V is a symmetric 3×3 *singular* matrix involving the y's alone:

$$V = \begin{pmatrix} 2\, y_4 y_5 y_6 & y_5 (y_1 y_5 - y_3 y_4 - y_2 y_6) & y_6 (y_2 y_6 - y_1 y_5 - y_3 y_4) \\ y_5 (y_1 y_5 - y_3 y_4 - y_2 y_6) & 2\, y_2 y_3 y_5 & y_3 (y_3 y_4 - y_1 y_5 - y_2 y_6) \\ y_6 (y_2 y_6 - y_3 y_4 - y_1 y_5) & y_3 (y_3 y_4 - y_1 y_5 - y_2 y_6) & 2\, y_1 y_3 y_6 \end{pmatrix}. \tag{89}$$

Eq. (88) can be easily solved. The result is

$$\sum_j W_{ij}\, \xi_j^{(4)} = 0\,, \quad \text{for} \quad i = 1\,,\, 2\,,\, 3\,, \tag{90}$$

where W is also a 3×3 singular matrix (but in general unsymmetric):

$$W = \begin{pmatrix} V_{11} & V_{12} \mp \sqrt{\lambda (y_k y_{k'})} & V_{13} \pm \sqrt{\lambda (y_k y_{k'})} \\ V_{21} \pm \sqrt{\lambda (y_k y_{k'})} & V_{22} & V_{23} \mp \sqrt{\lambda (y_k y_{k'})} \\ V_{31} \mp \sqrt{\lambda (y_k y_{k'})} & V_{32} \pm \sqrt{\lambda (y_k y_{k'})} & V_{33} \end{pmatrix} \tag{91}$$

in which $\lambda (y_k y_{k'})$ is the determinant (73a) which vanishes on the 4-mass envelope.

From (91), it immediately follows that *the 3-mass envelopes cannot exist if*

$$\lambda (y_k y_{k'}) < 0\,. \tag{92}$$

This implies that

(a) If the y's satisfy the sign convention (76), then *the 3-mass envelopes can only lie outside the region bounded by the two lines of* (74). In particular, (92) implies that E_{jkl} *can never cross over* E_{1234} (cf. Fig. 15).
(b) On the other hand, if the y's do not obey the sign convention (76), then $\lambda (y_k y_{k'}) > 0$ always, and E_{jkl} may exist while E_{1234} cannot.

For case (a), i. e. when E_{1234} exists, we assert that E_{jkl} intersects with E_{1234} at the point which corresponds to the remaining parameter $a_m = 0$ on E_{1234}. This is intuitively clear since, at the point $(a_j^*,\, a_k^*,\, a_l^*;\; a_m = 0)$ on E_{1234}, we are in the 3-space of $(a_j,\, a_k,\, a_l)$ in which lies a path for E_{jkl}; now this point must actually lie on the path for E_{jkl}, since the condition for E_{1234} is sufficient for that of E_{jkl}. This statement can be explicitly verified by elementary computation. Considering the case $a_4 = 0$, we note that the following ratios hold for the γ_s's on E_{123}:

$$\left.\begin{aligned}\gamma_1:\gamma_2:\gamma_3:\gamma_4:\gamma_5: &= y_5\left[(y_1y_5-y_2y_6-y_3y_4)\pm\sqrt{\lambda(y_ky_{k'})}\right]\\ &:2y_2y_3y_5:y_3\left[(y_3y_4-y_1y_5-y_2y_6)\mp\sqrt{\lambda(y_ky_{k'})}\right]:\\ &:y_2\left[(y_2y_6-y_1y_5-y_3y_4)\pm\frac{\xi_3^{(4)}y_3+\xi_1^{(4)}y_5}{\xi_3^{(4)}y_3-\xi_1^{(4)}y_5}\cdot\sqrt{\lambda(y_ky_{k'})}\right]\\ &:\frac{\mp 2y_2y_3y_5}{\xi_3^{(4)}y_3-\xi_1^{(4)}y_5}\cdot\sqrt{\lambda(y_ky_{k'})}.\end{aligned}\right\}\quad(93)$$

It is clear then that, at $E_{1234}\cap E_{123}$, we have $\gamma_5=0$. Then the remaining four γ's will have exactly the same ratio as those in the case of the 4-mass envelope (cf. footnote 40, and Appendix C), and the solution to the 3-mass envelope will coincide with the solution to the 4-mass envelope E_{1234} at $a_4=0$ on the latter. This establishes our above statement that $E_{ijk}\cap E_{1234}\neq 0$.

We now return to the discussion of the situation (c) of E_{1234} in the last sub-section, in which E_{1234} has a finite strip lying outside the relevant 3-point

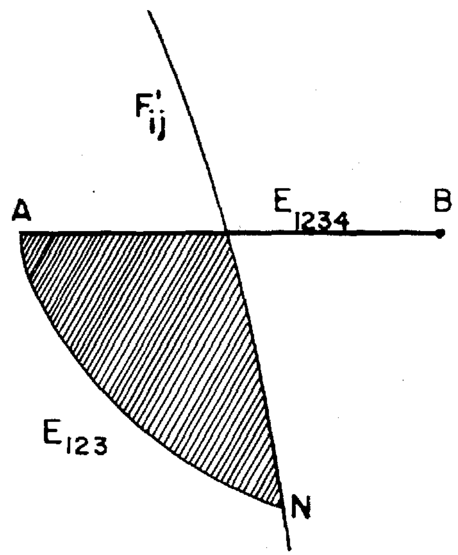

Figure 14. Inadmissible corners formed by the intersection between the 3-mass and the 4-mass envelopes.

singularity domain (cf. Fig. 13). The end-point A of E_{1234} corresponds to one particular $a_m=0$ on E_{1234}, say $m=4$. As we have just seen that E_{123} can only lie on one side of E_{1234} (e.g., below the segment AB in the z_6-plane, cf. Fig. 14 above) and that E_{123} actually touches this end-point A. Let us imagine that E_{123} is depicted by some curve $\widehat{AN}$ in the z_6-plane (Fig. 14). The exact shape of E_{123} will not be important to us (cf. remark in connection with Fig. 15 below). Our discussion up to this point does not exclude the possibility that the shaded region in Fig. 14 might contain the 4-point singularity. But this we now proceed to show as inadmissible.

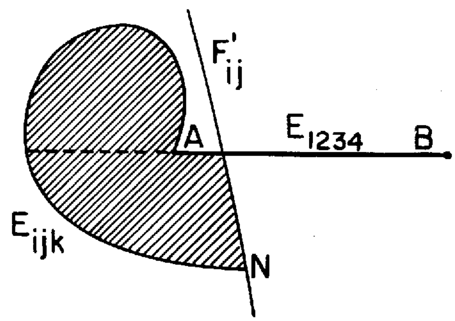

Figure 15. Admissible (but non-occurring) corners.

If this were really the case, the intersections of E_{ijk} with E_{1234} would be of such a kind that we had a corner in our domain. Since, as is characteristic of the theory of several complex variables, such corners are vulnerable to further analytic continuation[43], they cannot be part of the actual boundary of a natural domain of holomorphy. Note that if it were possible for E_{ijk} to cross over the 4-mass envelope like in the situation shown in Fig. 15, then this would in principle be admissible (since, in this case, the regularity domain would be the intersection of the two rather than the union as in Fig. 14). But our discussion of the 3-mass envelopes definitely excludes the possibility of such double intersections between E_{ijk} and E_{1234}. This leaves the only alternative of the corner as shown in Fig. 14, which one can reject as unacceptable for the boundary of our domain. Thus one concludes that the 4-mass envelope and the 3-mass envelopes do not contribute to

[43] A standard theorem is the well-known "Kantensatz". See, e. g., Behnke-Thullen, *loc. cit.*, p. 52; KW's Sec. VI; and H. Kneser, Math. Ann. **106**, 656 (1932). Although, strictly speaking, this theorem has only been proved for corners formed by analytic surfaces, while in our present case we are presumably dealing with the non-analytic surfaces, one can in the neighborhood of such corners construct tangential analytic surfaces so that the shaving of the corner received from the "Kantensatz" on the enveloping analytic surfaces will automatically affect our present corner proper. I would like to thank both Professors Jost and Källén for comments on this point.

the boundary[44]. In cases when the shaded region of Fig. 14 contains actual singularities, there must be another surface passing through and cover up this corner of Fig. 14. For this, we must go over to the treatment of the 2-mass envelopes.

VI. 4 The 2-Mass Envelope

As already mentioned in Sec. VI.3, the farther we go down to the envelopes of lower hierarchy, the less restrictions there are on the Q_k's. We shall first establish the intersection of the 2-mass envelope E_{ik} with the 3-mass envelope E_{ijk}. The method is quite analogous to the previous treatment of the 3-mass envelope.

We introduce a set of 6 real parameters γ_μ. For specificity, let us set $a_4 = a_2 = 0$ and consider E_{13} (i. e. the 2-mass envelope formed by a path in the quadrant $a_1 > 0$, $a_3 > 0$). As before, the quantities Q_4, Q_2 shall now be understood to stand for

$$Q_4 = \left(\frac{\partial \Psi}{\partial a_4}\right)\Bigg|_{(a_4 = a_2 = 0)}$$

$$Q_2 = \left(\frac{\partial \Psi}{\partial a_2}\right)\Bigg|_{(a_4 = a_2 = 0)}.$$

Since the envelope condition for E_{13} requires that

$$\boldsymbol{Im}\left(\frac{Q_1}{Q_3}\right) = 0,$$

we may set

$$\left.\begin{aligned} &Q_i = \gamma_i\, h(z;\, a), \quad i = 1, 3\\ &\gamma_1 + \gamma_3 = 1\\ &h(z;\, a) \equiv -4\Lambda(z) - Q_2 - Q_4\\ &Q_2 = (\gamma_2 + i\gamma_6)\cdot h(z;\, a)\\ &Q_4 = (\gamma_4 + i\gamma_5)\cdot h(z;\, a). \end{aligned}\right\} \quad (95)$$

Substituting (95) into (70) and dividing by $h(z;\, a)$, we get, after taking the imaginary and the real parts:

[44] The role of the 3-mass envelopes in the case when the 4-mass envelope does not exist will not be discussed here. In view of the above feature for $m = 4$ that the $(m-1)$-envelope can only lie on one side of the m-envelope (i. e. meet at most tangentially), which will be seen later (Lemma 3) to be also valid for $m = 3$, one feels more confident that the 2-mass envelopes are actually more important even in this case.

$$\sum_{k=1}^{4} (\boldsymbol{Im}\,\Psi_{ik})\,\gamma_k + (\boldsymbol{Re}\;\Psi_{i2})\,\gamma_6 + (\boldsymbol{Re}\;\Psi_{i4})\,\gamma_5 = 0; \tag{96a}$$

$$\sum_{k=1}^{4} (\boldsymbol{Re}\;\Psi_{ik})\,\gamma_k - (\boldsymbol{Im}\,\Psi_{i2})\,\gamma_6 - (\boldsymbol{Im}\,\Psi_{i4})\,\gamma_5 = 0. \tag{96b}$$

Combining the second and the fourth equations of (96b) with (96a), we have:

$$\sum_{\nu=1}^{6} T_{\mu\nu}\gamma_\nu = 0, \quad \mu = 1, \ldots, 6 \tag{97}$$

where $(T_{\mu\nu})$ is a 6×6 symmetric matrix:

$$(T_{\mu\nu}) = \begin{pmatrix} 0 & y_1 & y_2 & y_3 & \xi_1 & \xi_3 \\ y_1 & 0 & y_4 & y_6 & 0 & \xi_6 \\ y_2 & y_4 & 0 & y_5 & \xi_4 & \xi_5 \\ y_3 & y_6 & y_5 & 0 & \xi_6 & \\ \xi_1 & 0 & \xi_4 & \xi_6 & 0 & -y_6 \\ \xi_3 & \xi_6 & \xi_5 & 0 & -y_6 & 0 \end{pmatrix} \tag{98}$$

where

$$\left.\begin{aligned} \xi_i &= x_i - a_1, \quad i = 1, 3 \\ \xi_j &= x_j - a_3, \quad j = 4, 5 \\ \xi_6 &= x_6. \end{aligned}\right\} \tag{99}$$

Let $T \equiv \det |\, T_{\mu\nu} \,|$. Now making use of the Jacobi theorem[45] on the expansion of the determinant in terms of the minors, we have

$$T = \frac{T^{55}\,T^{66} - (T^{56})^2}{\lambda(y_k\, y_{k'})}, \tag{100}$$

where $T^{\mu\nu}$ = minor of $T_{\mu\nu}$, being 5×5 determinants.

One immediately recognizes that T^{55} and T^{66} are precisely the determinants of the type (whose matrix is defined in (86)) for the 3-mass envelopes E_{123} (at $a_2 = 0$) and E_{134} (at $a_4 = 0$) respectively. From this, the intersection of the 2-mass envelope with the 3-mass envelope is quite obvious. Consider, e. g., $E_{13} \cap E_{123}$. Since T must vanish on E_{13}, and T^{55} vanishes on E_{123}, and consequently on $E_{13} \cap E_{123}$, we have

[45] See Appendix D.

$$\gamma_6 = 0$$

and

$$T^{55} = 0 . \tag{101}$$

A straightforward computation with the aid of (90) will reveal that (101) reduces to the second equation of (96b) with $\gamma_6 = 0$ and $\gamma_1, \ldots, \gamma_5$ expressed

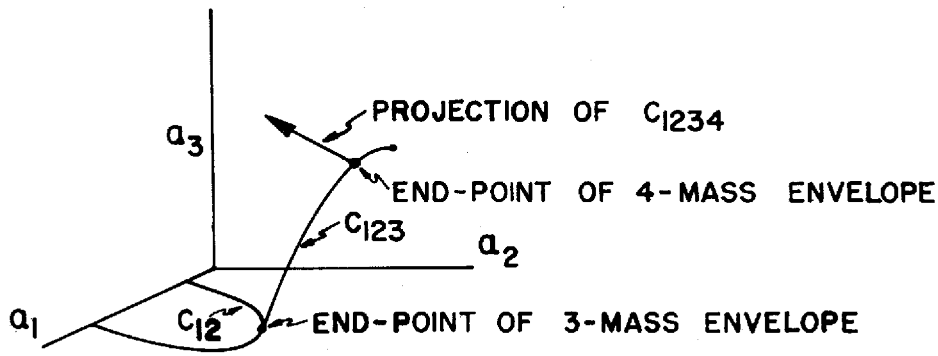

Figure 16. Typical paths for the various envelopes in the a-space.

by those on the 3-mass envelope E_{123} (cf. (93)). This means that (101) is automatically satisfied on $E_{13} \cap E_{123}$; hence there is no internal inconsistency. This shows that in general $E_{ik} \cap E_{ijk} \neq 0$.

This is also intuitively clear since the path C_{ijk} in the octant of all positive a's corresponding to the relevant portion of E_{ijk} is in general bounded by the coordinate 2-planes $a_m = 0$, $m = i, j$, or k. Since the envelope condition for E_{ijk} is sufficient for E_{ik}, the end-points of C_{ijk} (in the finite case) must then necessarily lie on the path, say, C_{ik} for E_{ik}. This situation is depicted in Fig. 16, showing that the path of one of the $(m-1)$-envelopes passes through one of the end-points of the path for the m-envelope, $m = 2, 3, 4$. One further consequence for $E_{ik} \cap E_{ijk}$ is the following:

From (100), we have, since $T = 0$ on E_{13},

$$(T^{56})^2 = T^{55}\, T^{66}. \tag{102}$$

(102) can only be satisfied when T^{55} and T^{66} have the same sign. In the case when E_{134} and E_{123} are distinct, we have in the neighborhood of $E_{13} \cap E_{123}$, $T^{55} \approx 0$; while T^{66} (i. e. the determinant corresponding to E_{134}) will essentially remain unchanged in sign. Thus (102) immediately implies that T^{55} cannot change its sign in the neighborhood of $E_{13} \cap E_{123}$, i. e. E_{13} cannot cross over E_{123}. The same statement holds for E_{134}.

Collecting with this our previous result for $E_{ijk} \cap E_{1234}$, we have established the cases $m = 3, 4$ of the following:

Lemma 3: The intersection between the envelopes $E_f^{(m-1)}$ and $E_f^{(m)}$, for $m = 2, 3, 4$,

(1) is non-empty,
(2) occurs at the ends of $E_f^{(m)}$, and
(3) is "tangential".

Remark: (a) The subscript f is used to denote the case when the path $C^{(m)}$ for the m-envelope $E^{(m)}$ is finite (i. e. $C^{(m)}$ is bounded by the sub-$(m-1)$-spaces). Otherwise, in the case when $C^{(m)}$ is unbounded, one can always show that the corresponding envelopes are irrelevant.

(b) The term "tangential" is understood as saying that $E_f^{(m-1)}$ can only lie on one side of $E_f^{(m)}$ (i. e. cannot cross over $E_f^{(m)}$ at the intersection[46], in the z-space).

(c) Lemma 3 says *nothing* about the relationship between an $E^{(m-2)}$ and an $E^{(m)}$. Thus, for instance, a 2-mass envelope can cross over the 4-mass envelope to swallow the corner of Fig. 15 (cf. Fig. 18 below).

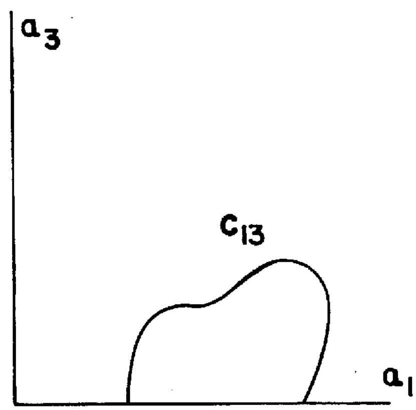

Figure 17. Path for the 2-mass envelope E_{13}.

(d) Whether $E_f^{(m)}$ will always contain the actual singularity is not fully settled here. This is true, however, in the 3-point case: while the $E_f^{(3)}$ and $E_f^{(2)}$, although not contributing to the boundary, do lie inside the sin-

[46] This feature seems to be also valid for the envelopes in the primitive domain of the 4-point function in the axiomatic approach. (Private communication from Professor G. KÄLLÉN).

gularity domain (on the cut)[47]. However, in the present case, we have one explicit example (cf. Fig. 24) where the corner formed by $E_f^{(4)}$ with $E_f^{(3)}$ is actually singular. (Of course, the case when $E_f^{(m)}$ is entirely contained inside the 3-point singularity domain is trivial).

It remains to say a few words about the case $m = 2$ in Lemma 3 which involves the 1-mass envelopes (strictly speaking, they are not envelopes).

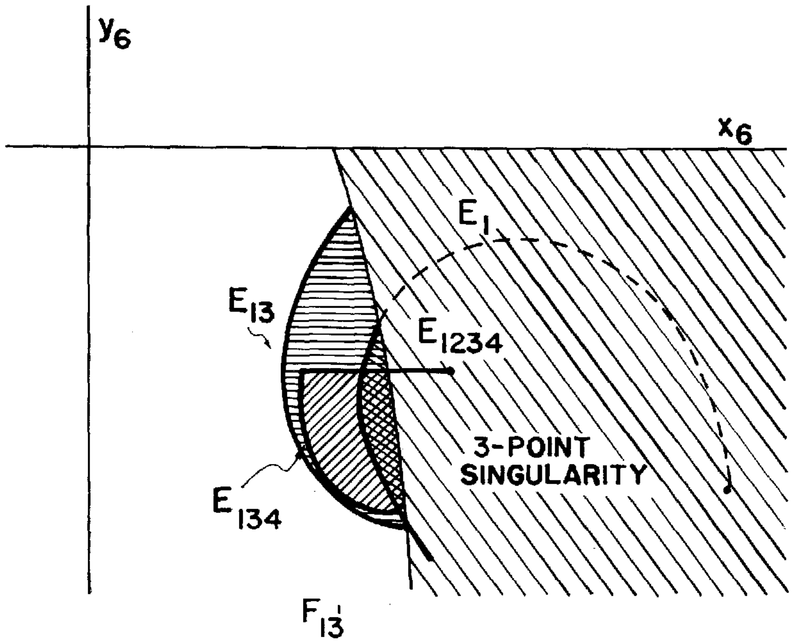

Figure 18. Envelopes in the z_6-plane.

One can, of course, explicitly show their intersections with the 2-mass envelopes in a perfectly analogous manner as was done above for $E_f^{(2)} \cap E_f^{(3)}$; we shall, however, omit this elementary computation here. Intuitively, it is clear in the quadrant $a_i > 0$, $a_k > 0$, since a finite path C_{ik} for E_{ik} must necessarily terminate on the semi-axes. A typical situation is shown in Fig. 17 in which C_{13} is bounded by the same axis. The image in the z_6-plane is shown in Fig. 18 where the 2-mass envelope E_{13} rides on top of the one-mass surface E_1, and the singularity domain is the union of the regions bounded by these two.

For completeness, we mention that, in the 3-point case, there occurs

[47] See Appendix B.

a peculiar situation where $E^{(1)} \cap E^{(2)} \cap E^{(3)} \neq 0$. This does not happen in general for the 4-point case. The only exception for $E^{(m-2)} \cap E^{(m-1)} \cap E^{(m)} \neq 0$ to occur would be when there are coincident zeros of the a's on $E^{(m)}$, e. g. Fig. 19. However, the situation in the 3-point case is actually of a slightly different nature than that of Fig. 19. There, the image of the 2-mass envelope in the z-plane happens to be a constant, so that E_{12} (the analogue of which in the 4-point case are the 3-mass envelopes) actually shrinks to a point which serves as the junction between the 1-mass F'-curves and the 3-mass envelope there[48].

The rest of this sub-section is devoted to the discussion of the connection

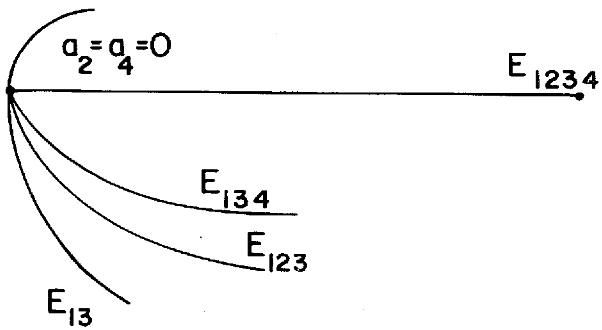

Figure 19. Multiple intersections among the envelopes in the 4-point case (Non-occurrence of).

of the 2-mass envelopes with the boundaries of the 3-point singularity domain, F'_{kl}- curves, and the equations for the former.

The conditions for the 2-mass envelopes are all contained in equations of type (96). However, for the 2-mass envelope, it is actually more convenient to take (94) together with the identities (68), (i. e. (69)). Thus, we have, for instance, on E_{13},

$$\left(\frac{\Phi_1}{\Phi_3}\right)_{a_2 = a_4 = 0} = \sigma^2, \qquad \sigma \text{ real.} \tag{103}$$

Since, for $z_6 \neq 0$, we may write

$$\left(\frac{\Phi_1}{\Phi_3}\right)_{a_2 = a_4 = 0} = \frac{a_3\,(z_6 - z_6^{(3)})}{a_1\,(z_6 - z_6^{(1)})} = \sigma^2 > 0, \tag{104}$$

where

[48] See Appendix B.

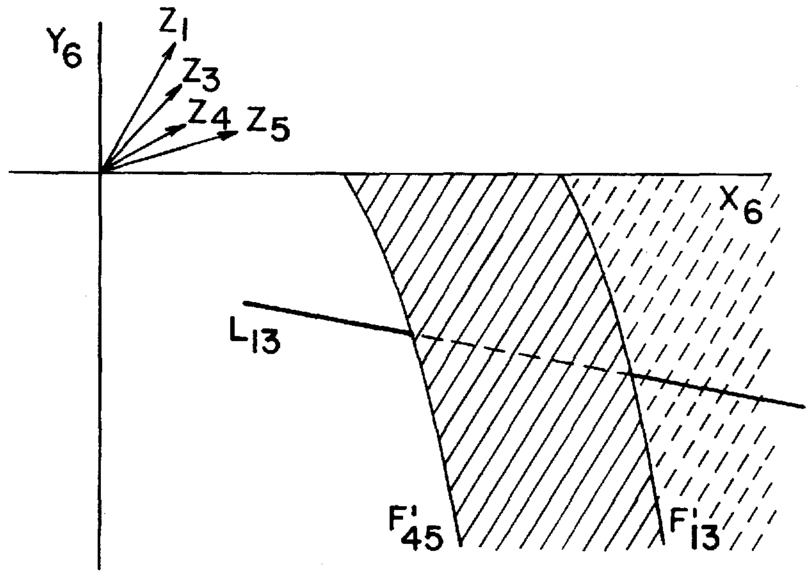

Figure 20. Allowed region for the 2-mass envelope: (outside the solid-line shaded region) when the two sets of F'-curves are in the same half-plane.

$$z_6^{(3)} = z_4 + z_5 - a_3 - \frac{z_4 z_5}{a_3} \qquad (105\,\mathrm{a})$$

$$z_6^{(1)} = z_1 + z_3 - a_1 - \frac{z_1 z_3}{a_1} \qquad (105\,\mathrm{b})$$

are the points on the F'_{45} and F'_{13}, respectively. Equation (104) allows a simple visualization of the location of the 2-mass envelope. Consider a point (a_1^*, a_3^*) on C_{13}, then in the z_6-plane one can locate two points $z_6(a_i^*)$, $i = 1, 3$, on F'_{13} and F'_{45}, respectively, according to (105). One sees then that the condition (104) for E_{13} at (a_i^*) can only be satisfied on the line L_{13} passing through $z_6(a_1^*)$, $z_6(a_3^*)$, excluding the segment between them. In other words, the 2-mass envelopes cannot exist in the region bounded by the two F'-curves, such as the shaded regions in Figs. 20 and 21. The exact image of the point (a_1^*, a_3^*) in the z_6-plane is given by the intersection of this line L_{13} with the Ψ-manifold, which now reads for $a_2 = a_4 = 0$:

$$z_6 = \frac{(z_1 - a_1)(z_5 - a_3) + (z_3 - a_1)(z_4 - a_3) \pm 2\sqrt{a_1 a_3 (z_6 - z_6^{(1)})(z_6 - z_6^{(3)})}}{(z_2 - a_1 - a_3)}. \qquad (106)$$

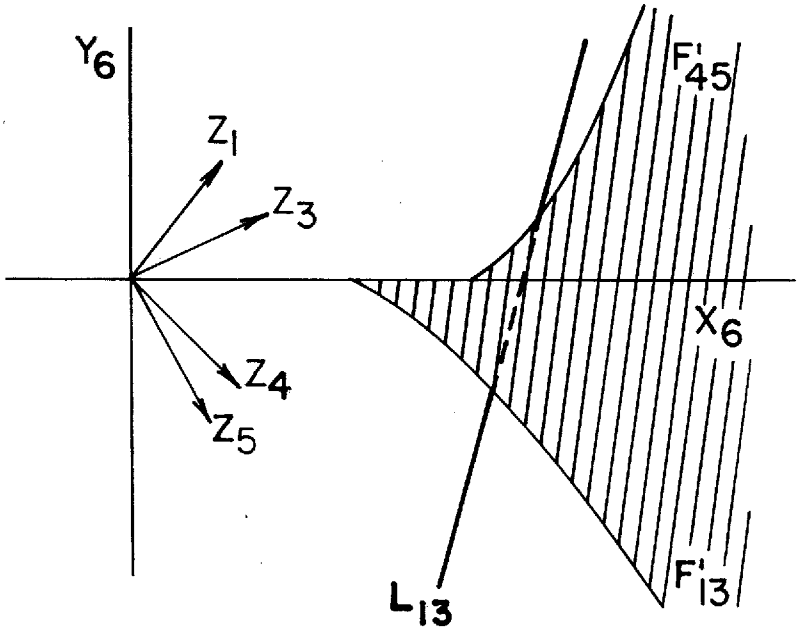

Figure 21. Allowed region for the 2-mass envelope: (outside the shaded region) where the two sets of F'-curves are in the opposite half-plane.

The elimination of z_6 from (104) and (106) is straightforward, and the resulting equation reads:

$$\left.\begin{aligned} 0 = \sigma^2 & \left\{a_1^3 - a_1^2 a_3 - a_1^2 (z_1 + z_3 - z_4 - z_5 + z_2)\right. \\ & \left. - a_1 [z_1 z_5 + z_3 z_4 - z_1 z_3 - z_2 (z_1 + z_3)] + a_3 z_1 z_3 - z_2 z_1 z_3\right\} \\ + 2\sigma & \left\{a_1^2 a_3 - a_1 a_3^2 - a_1 a_3 (z_1 + z_3 - z_4 - z_5) - a_1 z_4 z_5 + a_3 z_1 z_3\right\} \\ - & \left\{a_3^3 - a_3^2 a_1 - a_3^2 (z_4 + z_5 - z_1 - z_3 + z_2)\right. \\ & \left. - a_3 [z_1 z_5 + z_3 z_4 - z_4 z_5 - z_2 (z_4 + z_5)] + a_1 z_4 z_5 - z_2 z_4 z_5\right\}. \end{aligned}\right\} \quad (107)$$

Note that this equation is symmetric under the simultaneous permutation of

$$\left\{ a_1 \longleftrightarrow a_3; \begin{pmatrix} z_1 \longleftrightarrow z_4 \\ z_3 \longleftrightarrow z_5 \end{pmatrix}; \quad \sigma \longleftrightarrow \frac{1}{\sigma} \right\}$$

and the variation thereof (cf. Sec. III.2). The real and imaginary parts of (107) give two equations for the 3 parameters a_1, a_3, and σ. In principle, from these one is able to express two of the parameters in terms of the remaining one, say σ; thus, for fixed $z_1, \ldots, z_5$, one gets:

$$\begin{matrix} a_1 = a_1(\sigma) \\ a_3 = a_3(\sigma) \end{matrix} ; \quad \sigma^2 > 0. \tag{108}$$

(108) then defines the path C_{13} when taken in the positive quadrant $a_1 > 0$, $a_3 > 0$. With this substituted into the equation resulting by solving for z_6 from (106), one gets the final equation for E_{13} in the z_6-lane, which for all other z's being fixed, reads

$$z_6 = z_6(\sigma). \tag{109}$$

In actual computation, however, the solutions of (108) from (107) involve great computational labor[49]. The other five E_{jk} envelopes are, fortunately, slightly less complicated. But we shall not go into all this.

Since, in the solutions (108) for the a's, the ***Re*** z's and the ***Im*** z's are well mixed, it is clear that (109) no longer gives an equation for an analytic hypersurface. Since, as we shall show in Sec. VI.5, the 1-mass surfaces (which are analytic) do not in general constitute the whole boundary to the 4-point domain, and since we have shown that in general the higher envelopes lead to the pathological situations shown in Fig. 14, the process of successive elimination forces the 2-mass envelopes to be the only remaining eligible candidates for our boundary. And indeed for one explicit configuration (cf. Fig. 24 in Sec. VI.5) we have shown that the 2-mass envelope does come in.

With this we conclude that non-analytic hypersurfaces do serve as part of the boundary to the 4-point domain in perturbation theory. In the final sub-section, we shall study those 1-mass surfaces[50] E_k and shall illustrate in some typical configurations the explicit behavior of E_k which indicates the presence of the envelopes.

VI. 5 The 1-Mass Surfaces

The 1-mass surfaces, as compared with the various envelopes we have discussed above, are much simpler objects, as they are simply the images of the four coordinate semi-axes in the a-space. Applying the technique of the determinant expansion of Appendix D, we have the following identity:

[49] With σ as a running parameter, one gets usually a 6th degree algebraic equation involving one final a_i.

[50] Chronologically, these 1-mass curves were investigated first. From these, we can easily convince ourselves that they do not give the whole boundary. One is then forced to undertake a lengthy treatment of the envelope problem which is summarized above.

$$\Psi = \frac{4\,\Phi_2\Phi_4 - (\Psi^{24})^2}{-R_2}. \tag{110}$$

Therefore, on the Ψ-manifold, we get

$$\Psi^{24} = \pm 2\sqrt{\Phi_2\Phi_4}. \tag{111}$$

Note that

$$\Psi^{24} = \frac{1}{2}\frac{\partial\Psi}{\partial z_6}. \tag{112}$$

Identity (110) is the proper generalization of (45) which holds in the 3-point case. In terms of z_6, (111) is equivalent to

$$z_6 = \frac{(z_4 - a_2 - a_3)\dfrac{\partial\Phi_2}{\partial z_5} + (z_1 - a_1 - a_2)\dfrac{\partial\Phi_2}{\partial z_3} \pm 2\sqrt{\Phi_2\Phi_4}}{R_2}. \tag{113}$$

For completeness, we mention that the analogue of KW (A. 48c) reads in the 4-point case as follows: On the Ψ-manifold

$$\pm\sqrt{\Phi_1} = \frac{\dfrac{\partial\Phi_4}{\partial z_1}\sqrt{\Phi_2} \pm \dfrac{\partial\Phi_2}{\partial z_3}\sqrt{\Phi_4}}{R_2} \tag{114a}$$

$$\pm\sqrt{\Phi_3} = \frac{\dfrac{\partial\Phi_4}{\partial z_4}\sqrt{\Phi_2} \pm \dfrac{\partial\Phi_2}{\partial z_5}\sqrt{\Phi_4}}{R_2} \tag{114b}$$

and the permutation thereof. (114) follows directly from (113), or equivalently also from (70) with the aid of (68).

The expressions for the 1-mass surfaces E_k (i. e. $a_k \neq 0$, for one k, all other a's being zero), which immediately follow from (113) by setting to zero 3 a's at a time, are summarized as follows:

For E_1: $a_1 > 0$:

$$z_6 = \frac{a_1}{(z_2 - a_1)^2}\left(\sqrt{z_4 w_4} \pm \sqrt{z_5 w_5}\right)^2, \tag{115}$$

For E_3: $a_3 > 0$:

$$z_6 = \frac{a_3}{(z_2 - a_3)^2}\cdot\left(\sqrt{z_1 w_1} \pm \sqrt{z_3 w_3}\right)^2, \tag{116}$$

where the w's are defined as

$$w_1 = z_1 - z_2 - z_4 + a_3 + \frac{z_2 z_4}{a_3} \tag{117a}$$

$$w_3 = z_3 - z_2 - z_5 + a_3 + \frac{z_2 z_5}{a_3} \tag{117b}$$

$$w_4 = z_4 - z_1 - z_2 + a_1 + \frac{z_1 z_2}{a_1} \tag{117c}$$

$$w_5 = z_5 - z_2 - z_3 + a_1 + \frac{z_2 z_3}{a_1} \tag{117d}$$

which *vanish* on the appropriate F'_{kl}-curves.

For E_2: $a_2 > 0$:

$$z_6 = \frac{1}{z_2}\left[z_1 z_5 + z_3 z_4 + a_2(z_2 - z_3 - z_5) \pm 2\sqrt{z_3 z_5 [a_2^2 + a_2(z_2 - z_1 - z_4) + z_1 z_4]}\right]. \tag{118}$$

For E_4: $a_4 > 0$:

$$z_6 = \frac{1}{z_2}\left[z_1 z_5 + z_3 z_4 + a_4(z_2 - z_1 - z_4) \pm 2\sqrt{z_1 z_4 [a_4^2 + a_4(z_2 - z_3 - z_5) + z_3 z_5]}\right]. \tag{119}$$

With $z_1, \ldots, z_5$ fixed, the above 4 curves E_k in the z_6-plane start from a common point G which corresponds to all $a_i = 0$ (for a given choice of the sign in front of the square root, cf. remark following (124) below)

$$z_6^{(0)} = \frac{\zeta}{z_2} \tag{120}$$

with

$$\zeta = z_1 z_5 + z_3 z_4 \pm 2\sqrt{z_1 z_3 z_4 z_5}.$$

The 4 curves E_k start from G with the following slopes:

$$\left(\frac{\partial z_6}{\partial a_k}\right)_G = \pm \frac{\sqrt{\Lambda(z)}}{z_2 \sqrt{z_1 z_3 z_4 z_5}} \sqrt{\Phi_k^{(0)}}, \tag{121}$$

where $\Phi_k^{(0)}$ is Φ_k evaluated at all a's being zero, viz:

$$(\Phi_k^{(0)}) = \begin{pmatrix} z_4 z_5 z_6 \\ z_2 z_3 z_5 \\ z_1 z_3 z_6 \\ z_1 z_2 z_4 \end{pmatrix}. \tag{122}$$

On account of the identities (63) and (68), we have

$$\sum_{k=1}^{4}\left(\frac{\partial z_6}{\partial a_k}\right)_G = \mp \frac{\Delta(z)}{z_2\sqrt{z_1 z_3 z_4 z_5}}. \tag{123}$$

One may note the analogy between the ratios among these slopes and those among the γ's on the 4-mass envelope (cf. footnote 40) if one replaces all the z's by ***Im*** z's.

Next we come to the asymptotic behavior of the E_k. For E_1 and E_3 in the z_6-plane, we have respectively (for other a's being zero or finite)

$$\lim_{a_1\to\infty} z_6 = \left(\sqrt{z_4}\pm\sqrt{z_5}\right)^2 \tag{124a}$$

$$\lim_{a_3\to\infty} z_6 = \left(\sqrt{z_1}\pm\sqrt{z_3}\right)^2. \tag{124b}$$

In other words, E_1 and E_3 terminate at finite points in the z_6-plane corresponding to $\lambda_1(z_4, z_5, z_6) = 0$ and $\lambda_3(z_1, z_3, z_6) = 0$, respectively. On the other hand, E_2 and E_4 extend to infinity in the z_6-plane as $a_2 \to \infty$ and $a_4 \to \infty$, with the following slopes:

$$z_6(a_2 \neq 0) \xrightarrow[a_2\to\infty]{} \frac{1}{z_2}\left[z_2 - z_3 - z_5 \pm 2\sqrt{z_3 z_5}\right] a_2 \tag{124c}$$

$$z_6(a_4 \neq 0) \xrightarrow[a_4\to\infty]{} \frac{1}{z_2}\left[z_2 - z_1 - z_4 \pm 2\sqrt{z_1 z_4}\right] a_4. \tag{124d}$$

We now proceed to investigate the relevance problem[51] of these 1-mass curves. First of all, the sign in front of the root in equations (115)–(119) should be chosen in such a way that one gets an enhancement rather than a cancellation among the terms. The latter is entirely irrelevant. This situation is also true for the lower order singularity manifolds. We recall that, in the 2-point case, the relevant cuts start from $z_k = (\sqrt{a_m} + \sqrt{a_n})^2$, but not from $(\sqrt{a_m} - \sqrt{a_n})^2$. In the 3-point case, the F'-curves are gotten by also choosing the sign which would add up terms (while the opposite sign gives exactly zero there). Of course, for complex quantities under the square roots, the sign is meaningful only with a suitable convention of the branches, which we shall take as the one with the positive imaginary part.

[51] To be precise, in view of the fact that part of the singular portion of E_k may be overriden by a 2-mass envelope (cf. Fig. 24), we are here seeking only the relevant portion of E_k in the following sense:

(i) it has actual singularities, *and*
(ii) it lies outside the 3-point singularity domain (but not necessarily as the actual 4-point boundary).

It is a consequence of Lemma 2 that E_k has a relevant portion if E_k intersects twice with the relevant portions of the dominating F'-curves *and* if the bubble formed by such double intersections lies outside the 3-point singularity domain. The condition for such double intersections between E_k and F' can in principle be stated algebraically as follows: Consider, for example, $E_1 \cap F'_{45}$. After rewriting (115) for E_k in the form

$$\left.\begin{aligned} &\lambda_1(z_4, z_5, z_6)\, a_1^2 - 2\, a_1 \{ z_6 [z_2 z_6 - z_1 z_5 - z_3 z_4 + 2\, z_4 z_5 - z_2 (z_4 + z_5)] \\ &\quad - (z_4 - z_5)(z_1 z_5 - z_3 z_4) \} + \lambda\, (z_k z_{k'}) = 0\,, \quad 0 < a_1 < \infty, \end{aligned}\right\} \quad (125)$$

and with the relevant portion of F'_{45} given by

$$\left.\begin{aligned} &z_6 = z_4 + z_5 - \varrho - \frac{z_4 z_5}{\varrho} \\ &0 < \varrho \leqslant \frac{\boldsymbol{Im}\, z_4 z_5}{\boldsymbol{Im} (z_4 + z_5)}, \end{aligned}\right\} \quad (126)$$

the problem is to find the condition on the configuration of the other 5 z's such that the system of equations (125), (126) admits at least two solutions for a_1 (or ϱ) in their respectively allowed ranges, as indicated above. This can be done by brute force, but the result is so complicated that we do not wish to display it here. The conditions are obviously dependent on the moduli (as well as the arguments) of the 5 z's, and we have not been able to deduce from it a concise statement about the desired configuration. (However, cf. (129)).

Instead, we shall in the following classify the configurations of the 5 z's by the location of the starting point G of E_k. There are three distinct cases:

Case (1); G lies *outside* the 3-point singularity domain;
Case (2): G lies *deep inside* the 3-point singularity domain;
Case (3): G lies *on* or *slightly inside* the 3-point singularity boundary.

From our studies of the E_k curves, we find that the first two cases do not yield anything of interest. They correspond to the situations where E_k has no intersection or non-relevant intersections with the F' curves. Therefore we shall concentrate on case (3) above, which also has an intuitively appealing feature for the desired intersections between the E_k and the dominant F'-curves.

The condition is then to require that at least one of the slopes for E_k

at G given by (121) has an intersection with the dominant F' curves. For the case when the latter are hyperbolas (i. e. $0 < \arg z_k + \arg z_l < \pi$), this implies that[52]

$$\arg z_k + \arg z_l < \arg \left(\frac{\partial z_6}{\partial a_i}\right)_G < \pi + \arg z_k + \arg z_l \quad i = 1, \ldots, 4. \tag{127}$$

Condition (127), however, like the solutions to (125) and (126), is again dependent on the lengths of the z's in addition to their arguments.

It is clear that (127) is not sufficient to guarantee a double intersection even when G is chosen to lie on or slightly inside the F'-curves. However, only in such cases will the 1-mass curves E_k provide a useful hint as to how the 2-mass envelopes would come in. We illustrate this statement with the following 4 pictures: Fig. (22a) and Fig. (22b) show situations where the E_k's have the wrong slopes, and are irrelevant. In such cases, the envelopes are also irrelevant. Fig. (22c) shows a situation when one E_j comes out of the F'-region, while one other E_k stays inside. Although neither makes double intersections with F' (hence neither is relevant *per se*), the corresponding 2-mass envelope E_{jk} may very well form a bubble with F', which will serve as the 4-point boundary. Finally, in Fig. (22d), one sees a situation where one E_i does make a bubble with F' (the bubble can be shown to be relevant). On the other hand, another E_j also comes out of F', which by itself gives no contribution to the boundary; however, their 2-mass envelope E_{ij} may enlarge the bubble formed previously by E_i alone. This last phenomenon is what we have called the "overriding" of the relevant portion of 1-mass curves by a 2-mass envelope.

We shall now study some explicit examples. Let us first fix, for the sake of convenience, two (out of three in all) pairs of the conjugate variables (in the sense of Sec. II.2), say z_1, z_3, z_4, z_5. Ideally one would like to plot simultaneously in the product planes of the remaining pair of conjugate variables (i. e. z_2 and z_6), but for simplicity and practicality, we shall only plot in the z_6-plane (i. e. a 2-dimensional slice in the space of 12 dimensions) with suitable reference to the location of its conjugate variable z_2. The restriction on z_2 is as follows:

[52] The 3-point analogy of this condition is obvious: The relevance condition of the F'_{kl} curve itself, $z_m = z_k + z_l - r - z_k z_l / r$, can also be easily discussed by investigating the slope of the curve (actually the asymptote here for the hyperbola) at $r = 0$ (i. e. the analogue of the point G). Since one knows that the whole piece of F'_{kl} changes its relevance at its intersection with the cut along the positive x_m-axis, the relevance condition of F'_{kl} is to require that the slope at $r = 0$ should at least intersect with this x_m cut, i. e., $\pi < \arg(-z_k z_l) < 2\pi$, from which follows immediately the desired condition of the configuration: $0 < \arg z_k + \arg z_l < \pi$.

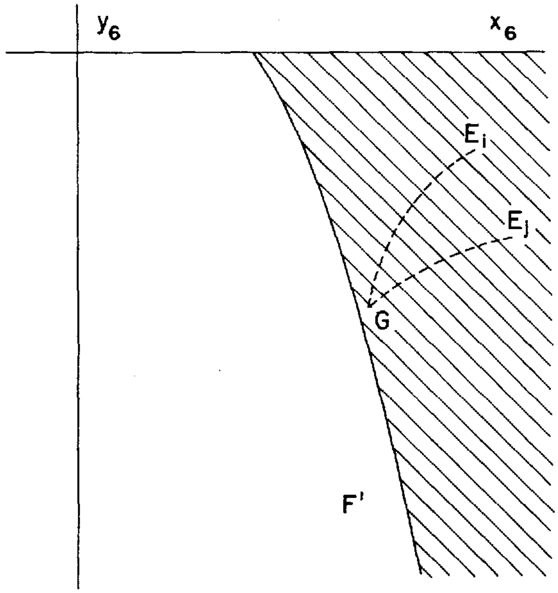

Figure 22a. Starting slopes of the 1-mass curves: (Irrelevant).

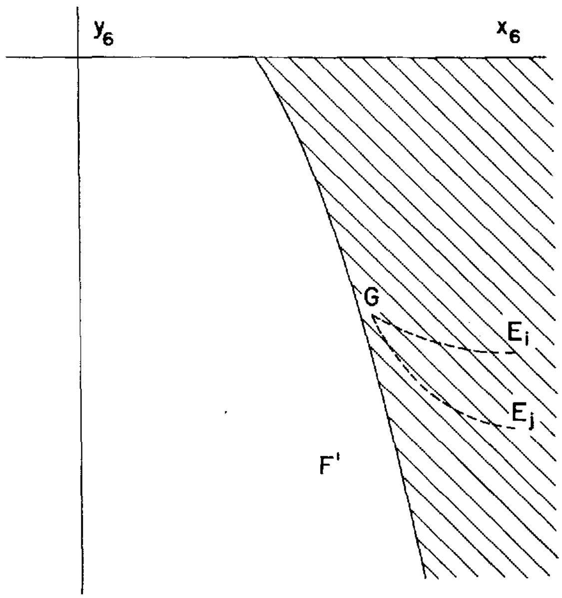

Figure 22b. Starting slopes of the 1-mass curves: (Irrelevant).

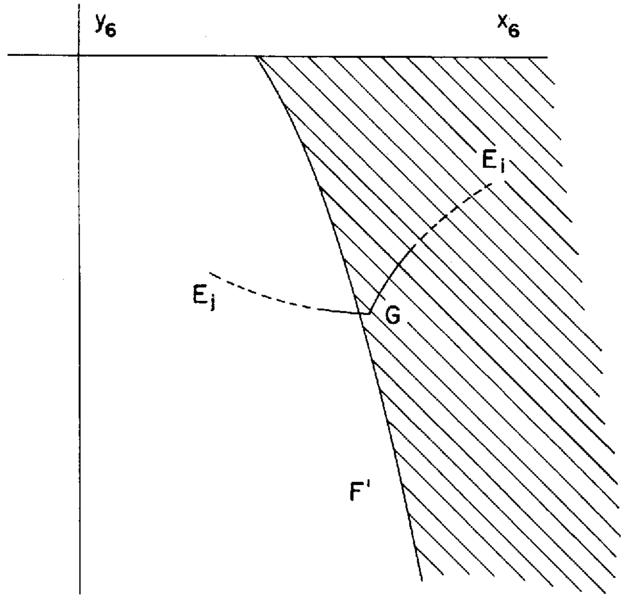

Figure 22c. Starting slopes of the 1-mass curves: (Irrelevant 1-mass curves, but relevant 2-mass envelopes).

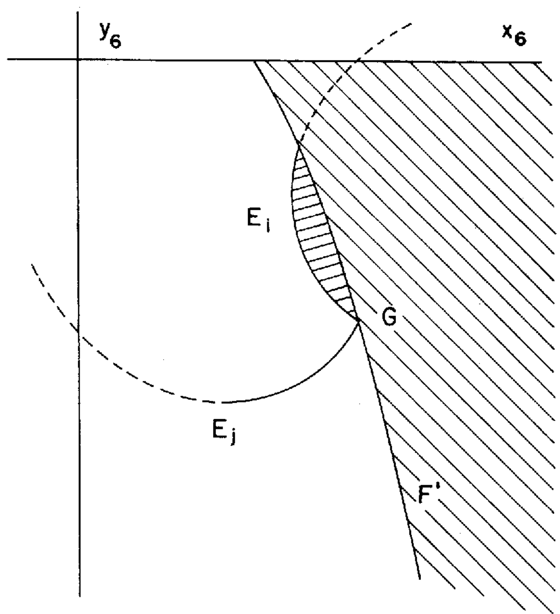

Figure 22d. Starting slopes of the 1-mass curves: (Relevant 1-mass curve and further 2-mass envelopes).

$$\left.\begin{array}{l}\text{(1) } z_2 \text{ shall not lie inside the relevant portions of the} \\ \text{3-point singularity manifolds } \Phi_2 = 0 \text{ and } \Phi_4 = 0, \text{ and} \\ \text{(2) } z_2 \text{ shall be such that } G \text{ of (120) lies on or slightly inside} \\ \text{the dominating boundaries to the manifolds } \Phi_1 = 0 \text{ and } \Phi_3 = 0 \\ \text{in the } z_6\text{-plane.}\end{array}\right\} \cdots\cdots (128)$$

Clearly there exists a limiting case of (120) when $\arg z_6^{(0)}$ and $\arg z_2$ approach respectively those of the asymptotes of the two dominating F' for z_6 and z_2. This implies, after a simple computation, the following necessary condition for the relevance of E_k for the case when $\boldsymbol{Im}\, z_i$, $i = 1, 3, 4, 5$, have the same sign[53]

$$2 \operatorname*{Max}_i \left\{\arg z_i\right\} < \sum_i \arg z_i, \quad i = 1, 3, 4, 5. \tag{129}$$

In the following, we shall confine ourselves to the consideration of those configurations for which the four sets of the 3-point Φ_k-manifolds are simultaneously relevant. (A few remarks are, however, made near the end of the text, regarding the degenerate cases, cf. Lemma 4 of Sec. VI.6). This means that, if one is looking at the triplet (ijk) in the z_k-plane, one requires that the following 3-point conditions are to be satisfied:

$$\text{(a)} \quad 0 < \arg z_i + \arg z_j < \pi, \qquad \text{if } y_i y_j > 0, \; y_i > 0. \tag{130a}$$

$$\text{(b)} \quad 3\pi < \arg z_i + \arg z_j < 4\pi, \quad \text{if } y_i y_j > 0, \; y_i < 0. \tag{130b}$$

$$\text{(c)} \quad \arg z_j > \pi + \arg z_i, \qquad \text{if } y_i y_j < 0, \; y_i > 0. \tag{130c}$$

One recalls that the configurations (a) and (b) yield hyperbolas and the configuration (c) gives a bubble in the z_k-plane.

There are *five* distinct configurations in the distribution of the 4 z_i's, $i = 1, \ldots, 4$. The first *four* cases correspond to $y_1 y_3 y_4 y_5 > 0$ (which imply the existence of the 4-mass envelope) and the remaining case is for $y_1 y_3 y_4 y_5 < 0$ (where E_{1234} does not exist).

(A) All 4 Up: (Two sets of hyperbolas each for z_6 and z_2).

In this case, we have:

$$y_1 y_3 y_4 y_5 > 0: \; y_i > 0. \tag{131}$$

[53] For other configurations with mixed signs of $Im\, z_i$, condition (129) can be easily modified by replacing some appropriate $\arg z_m$ by $2\pi - \arg z_m$ (Cf., e. g., Eq. (136)).

The 3-point boundaries are $F'_{13} \cup F'_{45}$ in the z_6-plane; and $F'_{14} \cup F'_{35}$ in the z_2-plane. (When there is no intersection among the two F' curves, one of the Φ-manifolds will be imbedded in the other, and the dominating F' curve is the one which corresponds to the *smaller* sum of the arguments. Otherwise, one has to take both of them into account.) The 3-point conditions are (130a) taken four times, or

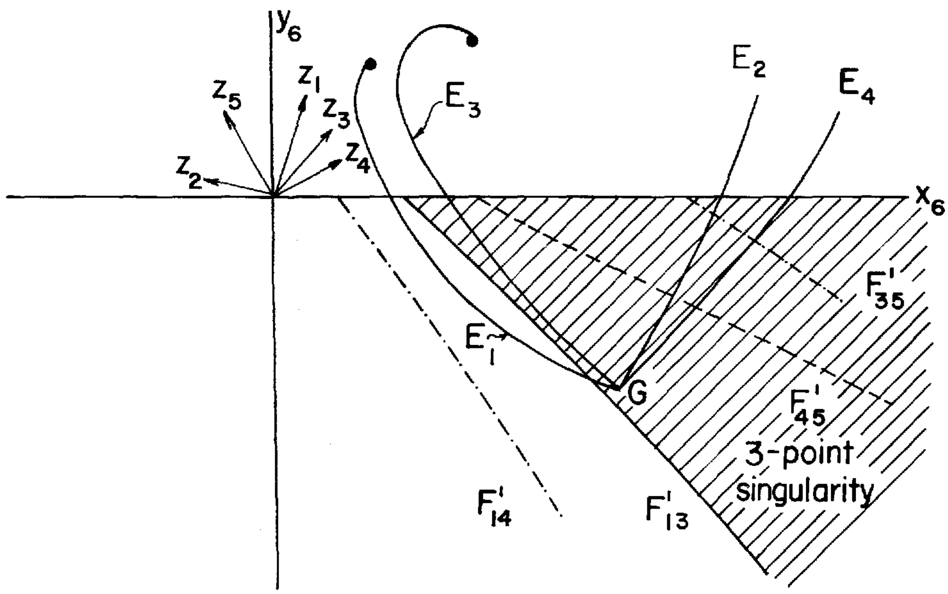

Figure 23. 1-mass curves in z_6-plane for the configuration (A): All 4 up (Two sets of hyperbolas each for z_6 and z_2).

$$\arg z_i + \arg z_j < \pi, \quad \text{for} \quad i, j = 1,3,4,5,: \quad \left.\begin{array}{l} i \neq j \text{ and} \\ i \neq \text{conjugate of } j. \end{array}\right\} \tag{132}$$

The 4-point condition (129) reads:

$$2 \operatorname*{Max}_i \{\arg z_i\} < \sum_i \arg z_i < 2\pi, \quad i = 1, 3, 4, 5. \tag{133}$$

In plotting in the z_6-plane, z_2 is to be chosen according to (128). A typical situation for this case is shown in Fig. 23.

(*B*) *Two Up and Two Down*: (*hyperbolas for* z_6: *bubbles for* z_2).

In this and the immediate next configurations, conjugate variables lie in the opposite half-planes. Here,

$$y_1 y_3 y_4 y_5 > 0: \; y_1 y_3 > 0; \; y_1 y_4 < 0. \tag{134}$$

The 3-point boundaries are $F'_{13} \cup F'_{45}$ in the z_6-plane; and $F'_{23} \cup F'_{25} \cup F'_{12} \cup F'_{24}$ in the z_2-plane. The 3-point conditions are explicitly (consider the case $y_1 > 0$)

$$\left.\begin{aligned} 0 < \arg z_1 + \arg z_3 < \pi \\ 3\pi < \arg z_4 + \arg z_5 < 4\pi \\ \arg z_4 > \pi + \arg z_1 \\ \arg z_5 > \pi + \arg z_3 . \end{aligned}\right\} \quad (135)$$

The 4-point condition (129) now takes the modified form

$$\text{(i)} \quad 2 \operatorname*{Max}_i \{\arg z_i\} < \sum_i \arg z_i < 5\pi \qquad (136\,\text{a})$$

if

$$\operatorname{Max}\{\arg z_1, \arg z_3\} < 2\pi - \operatorname{Min}\{\arg z_4, \arg z_5\}$$

or

$$\text{(ii)} \quad 3\pi < \sum_i \arg z_i < 4\pi + 2 \operatorname{Min}\{\arg z_i\} < 5\pi \qquad (136\,\text{b})$$

if

$$\operatorname{Max}\{\arg z_1, \arg z_3\} > 2\pi - \operatorname{Min}\{\arg z_4, \arg z_5\}.$$

A typical case is shown in Fig. 24.

Note: Figure 24 gives a very interesting example: E_1 makes a bubble with F'_{13} which can be shown to be singular. On the other hand, E_3 lies outside, and by itself is not relevant. Thus we have the situation shown in Fig. 22*d*. Now, if one takes the path $a_1 = a_3$ in the positive (a_1, a_3)-quadrant $(a_2 = a_4 = 0)$, one finds that its image in the z_6-plane makes another bubble with F'_{13}, which is also singular, but not contained by E_1. This shows definitely that

(a) *The 1-mass surfaces E_k do not in general give the whole boundary of D_4^{pert}*, and
(b) *Envelopes actually exist.*

Another curve, which corresponds to the path $a_1 = a_2 = a_3 = a_4$ in the positive sediciment, is also plotted in Fig. 24. However, it is not relevant in this case.

A plot of one of the simplest envelopes in the z_6-plane, namely E_{24}, is also made, but in this particular case, it is completely submerged inside the 3-point singularity domain.

Finally the 4-mass envelope E_{1234} is finite in this case, being bounded by $a_2 = 0$ and $a_1 = 0$. This is exactly the situation illustrated in Fig. 13.

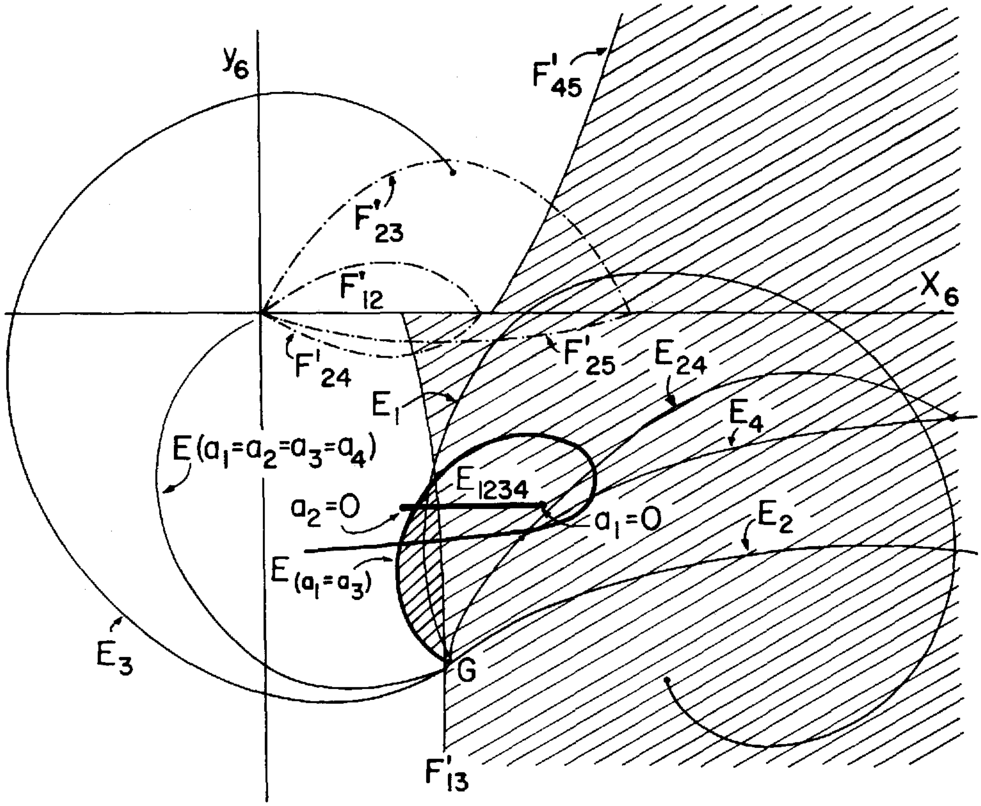

Figure 24. 1-mass curves in z_6-plane for the configuration (B): Two up and two down (hyperbolas for z_6; bubbles for z_2).

The end-point $a_2 = 0$ lies outside the F'_{13} and E_1 as well as $E_{a_1=a_3}$. The 3-mass envelope E_{134} can only come from below the E_{1234}. One will then have essentially a final situation similar to that shown in Fig. 18.

(*C*) *Two Up and Two Down*: (*bubbles for* z_6; *hyperbolas for* z_2).

This one gets from (*B*) by simply permuting within one pair of conjugate indices. The net result (cf. Sec. III. 2) is the interchange of the role of z_6 and z_2.

Thus, e. g., if one permutes z_3 and z_4 from (*B*),:

$$y_1 y_3 y_4 y_5 > 0: \; y_1 y_4 > 0, \; y_1 y_3 < 0. \tag{137}$$

The 3-point boundaries are $F'_{16} \cup F'_{36} \cup F'_{46} \cup F'_{56}$ in the z_6-plane, and $F'_{35} \cup F'_{14}$ in the z_2 plane. The 3-point and 4-point conditions are literally the same as (135) and (136) if one permutes z_3 and z_4. The E_k's are shown in Fig. 25. This suggests a 2-mass envelope.

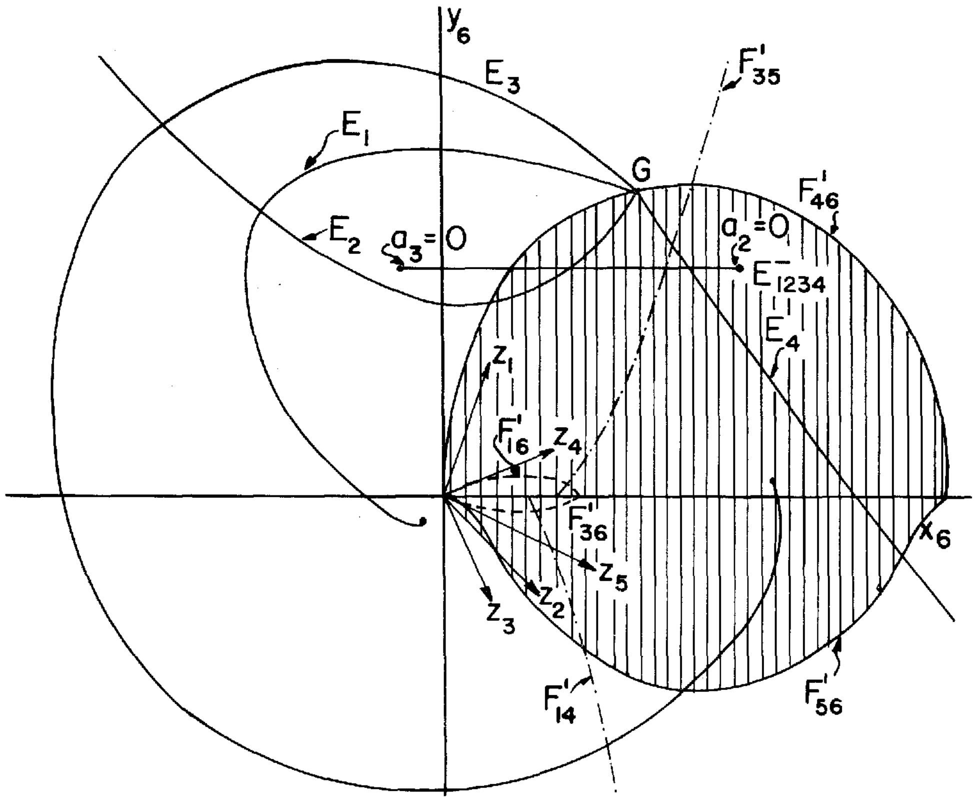

Figure 25. 1-mass curves in z_6-plane for the configuration (C): Two up and two down (bubbles for z_6; hyperbolas for z_2).

Note: In Fig. 25, one sees again the situation of Fig. 14. Here the 4-mass envelope E^-_{1234} is terminated at $a_3 = 0$. Now the 3-mass envelope E_{124} will intersect this point in the z_6-plane from above the line E^-_{1234} (since the other line E^+_{1234} in this case lies below E^-_{1234}, and from our analysis of (92), the 3-mass envelope must lie outside the region bounded by these two lines.) One gets again a corner in the intersection $E_{1234} \cap E_{124}$. A 2-mass envelope, say, E_{14}, is then expected to cover this corner. The situation is depicted in Fig. 25a.

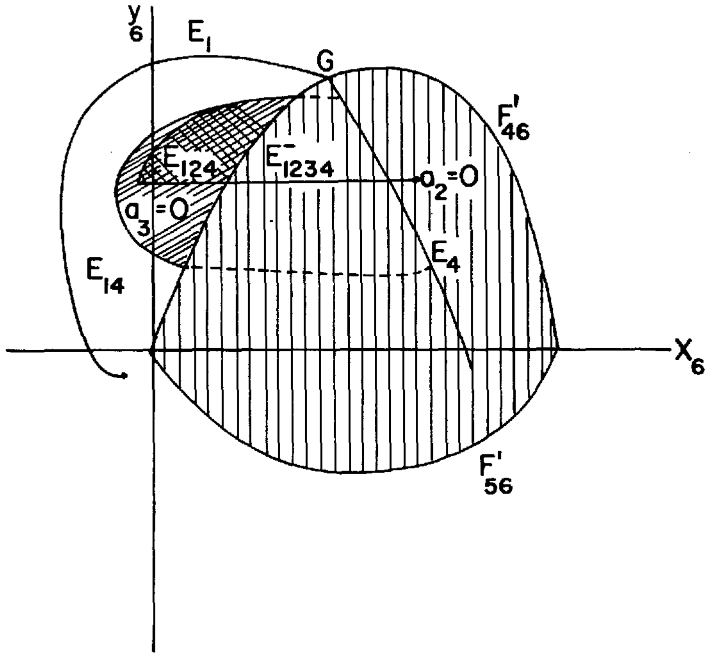

Figure 25a. Envelope situation for Fig. 25.

(D) Two Up and Two Down: (bubbles for z_6; bubbles for z_2).

One obtains this configuration from (A) when one shifts one pair of conjugate indices *(1,5)* or *(3,4)* to the opposite half-plane. Here we have, e. g.:

$$y_1 y_3 y_4 y_5 > 0: \ y_1 y_5 > 0; \ y_1 y_3 < 0. \tag{138}$$

The 3-point boundaries are $F'_{16} \cup F'_{36} \cup F'_{46} \cup F'_{56}$ in the z_6-plane, and $F'_{23} \cup F'_{25} \cup F'_{12} \cup F'_{24}$ in the z_2-plane. The 3-point condition is in this case (with *(3,4)* down)

$$\mathrm{Min}\left\{\arg z_3, \arg z_4\right\} > \pi + \mathrm{Max}\left\{\arg z_1, \arg z_5\right\} \tag{139}$$

and the 4-point condition reads:

$$\text{(i)} \quad \sum_i \arg z_i < 4\pi + 2\,\mathrm{Min}\left\{\arg z_i\right\}, \quad i = 1, \ldots, 4 \tag{140a}$$

if

$$\mathrm{Max}\left\{\arg z_1, \arg z_5\right\} > 2\pi - \mathrm{Min}\left\{\arg z_3, \arg z_4\right\}$$

or

$$\text{(ii)} \quad \sum_i \arg z_i > 2\,\mathrm{Max}\{\arg z_i\}, \quad i = 1, \ldots, 4 \tag{140b}$$

if

$$\mathrm{Max}\{\arg z_1, \arg z_5\} < 2\pi - \mathrm{Min}\{\arg z_3, \arg z_4\}.$$

(E) Three Up and One Down: (1 hyperbola and 1 bubble each for z_6 and z_2). Here

$$y_1 y_3 y_4 y_5 < 0. \text{ Consider, for example:}$$

$$y_1, y_3, y_4 > 0, \text{ and } y_5 < 0. \tag{141}$$

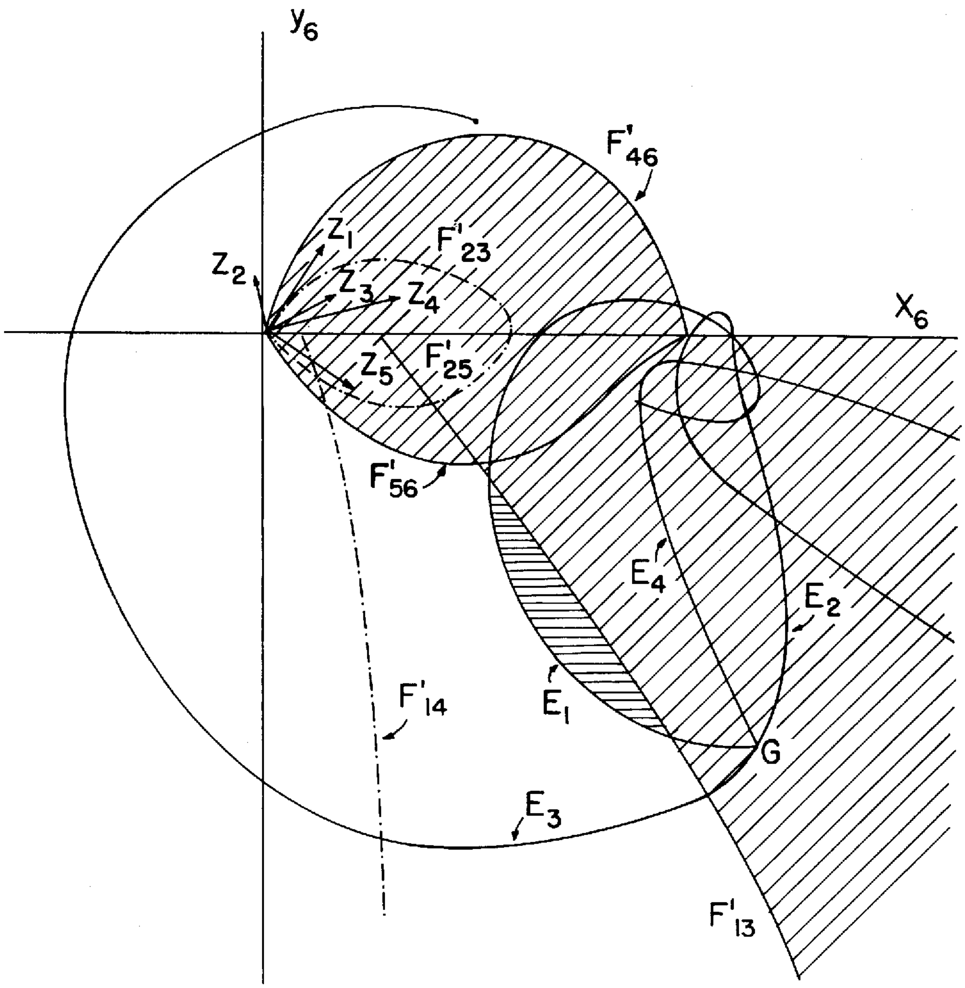

Figure 26. 1-mass curves in z_6-plane for the configuration (E): Three up and one down (one hyperbola and one bubble each for z_6 and z_2).

The 3-point boundaries are then $F'_{13} \cup F'_{46} \cup F'_{56}$ in the z_6-plane, and $F'_{14} \cup F'_{23} \cup F'_{25}$ in the z_2-plane. The 3-point conditions are:

$$\left.\begin{aligned} &\arg z_1 + \operatorname{Max}\{\arg z_3, \arg z_4\} < \pi \\ &\arg z_5 > \pi + \operatorname{Max}\{\arg z_3, \arg z_4\} \end{aligned}\right\} \quad (142)$$

and the 4-point condition reads for this case:

$$2 \arg z_5 + 2 \operatorname{Max}\{\arg z_1, \arg z_3, \arg z_4\} - 2\pi < \sum_i \arg z_i < 2 \arg z_5. \quad (143)$$

A typical case is shown in Fig. 26 (which suggests a 2-mass envelope).

VI. 6 Brief Remarks on the Degenerate Cases.

In our above description of the 1-mass curves, we have only considered the configurations where all four sets of the Φ_k-manifolds are simultaneously relevant. It would be of interest to see how the 4-point boundary changes its character when one or more Φ_k-manifolds become irrelevant. While we shall not attempt to enter into the discussion for this in detail, we offer two remarks on such degenerate cases:

(1) **Lemma 4:** Non-relevance of 2 sets of Φ-manifolds must imply the non-relevance of at least one more set.

Proof:

It suffices to show this for one particular configuration, say, in the case when 4 of the 6 z's are all in the upper half-plane, $0 < \arg z_i < \pi$, $i = 1, 3, 4, 5$ (cf. configuration (A) of Sec. VI.5), (since the proof for the other configurations can be easily carried through with only trivial modifications).

Suppose Φ_2 and Φ_4 manifolds are both irrelevant in the z_2-plane, then

$$\begin{aligned} &\arg z_1 + \arg z_4 > \pi \\ &\arg z_3 + \arg z_5 > \pi. \end{aligned}$$

Assume Φ_3-manifold to be relevant in the z_6-plane (otherwise, nothing is to be proved), so

$$\arg z_1 + \arg z_3 < \pi.$$

Then Φ_1-manifold must be irrelevant, since

$$\arg z_4 + \arg z_5 > 2\pi - \arg z_1 - \arg z_3 > \pi.$$

(1a) An immediate consequence of Lemma 4 is the following. When Φ_2 and Φ_4 manifolds are both irrelevant (thus, e. g., one has the cut-plane in z_2), then in the z_6-plane, one has at most one set of relevant F' curves for the 3-point boundary. In this case, the 2-mass envelopes will not be expected to play a role, and the 4-point boundary will then be at most made up of the 1-mass surfaces which are analytic.
(2) The case when all 4 sets of Φ_k-manifolds are simultaneously irrelevant is, of course, trivial. Absence of any relevant 3-point boundary implies no change of relevance for the 4-point boundary. Since the latter cannot be entirely relevant, it must be entirely irrelevant. Thus, in this case, one gets the cut-planes.

VI. 7 Conclusion

It should be emphasized that we have by no means exhausted the boundary of the 4-point domain in perturbation theory. In fact, we have only explored it to the extent that we have shown how the 4-point correction to the already existing 3-point singularity might look. Our studies of the domain D_4^{pert} shows that the relevant 4-point singularities will carve out some bubbles from the dominating F' curves of D_3^{pert}. The singularity domain of the 4-point proper is seen to be compact. We have demonstrated that in general the 1-mass surfaces will not constitute the whole boundary of D_4^{pert} and that the presence of the envelopes implies that D_4^{pert} is not everywhere bounded by analytic hypersurfaces. Of the various envelopes we have discussed, the 2-mass envelopes are the most important ones.

It is hoped that, if the 3-point analogy is again valid in the 4-point case, the results derived here might be of some use to the problem of finding the holomorphy envelope $E(D_4)$ based on the axioms of local field theory alone.

We conclude by posing a question. One recalls again from the 3-point case that the domain D_3^{prim} is bounded by the F-curves (say, for the case when both $\boldsymbol{Im}\, z_j$, $\boldsymbol{Im}\, z_k$ have the same sign) of KW, which differ from the holomorphy envelope F'-curves only by the exactly opposite signs of the range of the parameters a_i (which, in the p-space, has the significance of being m_i^2). Intuitively, this can be understood as follows: If one starts from the original tube domain R_{n-1} of the vectors p_i where one requires $\boldsymbol{Im}\, p_i \varepsilon V$, this automatically forces one to go off the mass-shells and in particular one finds it convenient to go to negative values of the mass-squares $a_i = m_i^2 < 0$.

(This situation is clear, for example, in the proof of dispersion relations, with the technique of BOGOLIUBOV[54].) So $\partial D_3^{\text{prim}}$ essentially involves the manifold with the parameters still in the range $a_i < 0$. The problem of finding the holomorphy envelope then furnishes the necessary analytic continuation from $a_i < 0$ to $a_i > 0$, which is by no means trivial. This is exactly the relation between the F-curve of D_3^{prim} and the F'-curve of $E(D_3)$, or the F'-curve of D_3^{pert}, as shown by KW.

Therefore it will be of interest to see whether or not this analogy is a valid one in the 4-point case, viz., whether $\partial D_4^{\text{pert}}$ can be compared with $\partial D_4^{\text{prim}}$ with only a possible difference of the signs of the parameters*. Of course, the problem is much more complicated in the 4-point case, since one is dealing with the envelopes in both $\partial D_4^{\text{prim}}$ and $\partial D_4^{\text{pert}}$. An answer in the affirmative sense would further strengthen one's hope that $\partial D_4^{\text{pert}}$ may have something to do with $\partial E(D_4)$. But this we shall leave to a separate investigation.

Acknowledgement

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[54] See, reference cited in footnote 4.

* *Note added in proof*: For the 3-point case, reversing the sign of the parameter a_i in the F_{jk}-curve is equivalent to changing the sign of $(z_i - z_j - z_k)$. The latter scheme is better suited in a generalization to the 4-point case. Preliminary investigation shows that the analogy of such a simple relationship between $\partial D_3^{\text{pert}}$ and $\partial D_3^{\text{prim}}$ may very well break down in the 4-point case.

Appendix A

Explicit Form of the 4-Point Function

Here we discuss the singularities of $I(z)$ of (54) after explicitly carrying out the final integration for $F_j(r_i)$ in (55). For the case $r_1 \neq r_2$, the singularities are found on the Φ_k-manifolds and the R_μ-manifolds (cf. Sec. IV. 4). Finally, for the case $r_1 = r_2$, the change of relevance of the Ψ-manifold is shown to occur at $(\Psi = 0) \cap (\Phi_k = 0)$ (cf. Sec. V).

By symmetry, it suffices to write down $F_j(r_i)$ only, say, for $j = 1$, $i = 1$, namely for the first half of the terms for the triplet (z_2, z_3, z_5). A straightforward computation from (55) yields:

$$\left.\begin{aligned} F_1(r_1) = \sqrt{\lambda_1}\Big\{\log\frac{1-r_1}{-r_1}\log\chi_1(r_1) + \log\frac{\eta_1(1)}{\eta_1(0)}\log\chi_1(r_1) \\ + S_1(\eta_1(r_1)) - S_1(\eta_2(r_1))\Big\} \end{aligned}\right\} \tag{A. 1}$$

in which each $S_j(\eta_k(r_i))$ is a sum of 16 Spence functions:

$$S_1(\eta_1(r_1)) = \sum_{\mu=1}^{8}\varepsilon_\mu\left[\varphi\left(\frac{\eta_1(r_1)-\eta_1(1)}{\eta_1(r_1)-\omega_\mu}\right) - \varphi\left(\frac{\eta_1(r_1)-\eta_1(0)}{\eta_1(r_1)-\omega_\mu}\right)\right], \tag{A. 2}$$

where

$$\left.\varepsilon_\mu = \begin{cases} 1, & \mu = 1, \ldots, 4 \\ -1, & \mu = 5, \ldots, 8 \end{cases}\right\} \tag{A. 3}$$

$$\varphi(\zeta) = \int_1^\zeta \frac{dt}{t}\log(1+t) \tag{A. 4}$$

and

$$\eta_{1,2}(\alpha) = \frac{1}{\lambda_2}\left[\frac{1}{2}\frac{\partial N_1(\alpha)}{\partial\alpha} \pm \sqrt{\lambda_2 N_1(\alpha)}\right] \tag{A. 5}$$

$$\left.\begin{aligned} \omega_{1,2} &= \frac{2z_5}{\lambda_2}\cdot\frac{P_2 \pm \sqrt{\lambda_2 R_2}}{\frac{1}{2}\frac{\partial\lambda_2}{\partial z_3} - \sqrt{\lambda_2}} \\ \omega_{3,4} &= \frac{2z_5}{\lambda_2}\cdot\frac{P_3 \pm \sqrt{\lambda_2 R_3}}{\frac{1}{2}\frac{\partial\lambda_2}{\partial z_2} - \sqrt{\lambda_2}} \end{aligned}\right\} \tag{A. 6}$$

where

$$\left.\begin{aligned} \omega_{5,6} &= A_3\omega_{1,2} \\ \omega_{7,8} &= A_2\omega_{3,4} \\ A_m &= \frac{\dfrac{1}{2}\dfrac{\partial\lambda_2}{\partial z_m} - \sqrt{\lambda_2}}{\dfrac{1}{2}\dfrac{\partial\lambda_2}{\partial z_m} + \sqrt{\lambda_2}}, \qquad m = 2, 3. \end{aligned}\right\} \tag{A. 6a}$$

The following identities can be easily verified:

$$\frac{4 z_5}{\lambda_2^2}\Phi_2 = \eta_1(\alpha)\,\eta_2(\alpha) = A_3\omega_1\omega_2 = A_2\omega_3\omega_4 = A_3^{-1}\omega_5\omega_6 = A_2^{-1}\omega_7\omega_8 \tag{A.7}$$

and

$$\left.\log\chi_1(\alpha) = \log A_2 A_3 \frac{\prod\limits_{\mu=1}^{4}(\eta_1(\alpha)-\omega_\mu)}{\prod\limits_{\mu=5}^{8}(\eta_1(\alpha)-\omega_\mu)} = -\log A_2 A_3 \frac{\prod\limits_{\mu=1}^{4}(\eta_2(\alpha)-\omega_\mu)}{\prod\limits_{\mu=5}^{8}(\eta_2(\alpha)-\omega_\mu)}\right\} \tag{A.8}$$

We now briefly discuss the singularities of $\sum\limits_j F_j(r_1)$, with $F_1(r_1)$ given by (A.1), for the case $r_1 \neq r_2$.

(1) The first term is $\log\dfrac{1-r_1}{-r_1}\cdot\log\chi_j(r_1)$. The point $r_1 = 0$ corresponds to the Φ_1-manifold (cf. (48)). It is clear that a cancellation of the 3-point type occurs here when the summation over j is carried out. Finally, for $r_1 \neq 0$, or 1, the zeros and poles of $\chi_j(r_1)$ can at most lead to the cuts in the z's. (cf. (51) and Sec. IV.4).

(2) The next term is $\log\dfrac{\eta_1(1)}{\eta_1(0)}\log\chi_j(r_1)$. Here the vanishing of $\eta_1(1)$ or $\eta_1(0)$ gives the Φ_{j+1}-manifold.

(3) Now we come to the Spence function terms. Each Spence function $\varphi(\zeta)$ is defined with a cut in the ζ-plane starting from its branch point at $\zeta = -1$ to infinity. Now the branch points in (A.2) occur at

$$\left.\begin{aligned} \omega_\mu - \eta_1(1) &= 0, \\ \text{and} \qquad \omega_\mu - \eta_1(0) &= 0. \end{aligned}\right\} \quad \mu = 1, \ldots, 8 \tag{A.9}$$

With the aid of (A.8), we see that this happens at the two ends of the integration interval. Again the point $\alpha = 1$ is irrelevant. But the point $\alpha = 0$

leads to the Φ_1-manifold, which is to be expected. Note that the points $r_1 = 1$ or 0, which give $\varphi(0) = -\pi^2/12$, are entirely harmless for the Spence functions.

Another source of singularity for the Spence function $\varphi(\zeta)$ is at infinity. Now this happens when

$$\eta_1(r_1) - \omega_\mu = 0, \quad \mu = 1, \ldots, 8 \tag{A.10}$$

or, according to (A.8), this implies zeros or poles of $\chi_j(r_1)$. But these can at most correspond to the individual cut in each of the z's.

Thus we conclude that from the explicit expression (A.1) and its permuted form for the case $r_1 \neq r_2$, the singularities of the 4-point function $I(z)$ of (54) are confined to the 4 sets of Φ_k-manifolds and the 6 cuts, one for each z along the positive real axis. This agrees with our simple argument in Sec. IV.4.

Finally, from the representation (A.1) and its permuted forms of $\sum_j F_j(r_i)$, we now briefly discuss the change of relevance of the Ψ-manifold in the case $r_1 = r_2$. Here the expression (54) gets essentially a contribution from the first term in (A.1) (summed over j) in the neighborhood of the Φ_1-manifold:

$$\left.\begin{aligned} &\frac{1}{r_1 - r_2}\left[\log r_1 \log \prod_j \chi_j(r_1) - \log r_2 \log \prod_j \chi_j(r)\right] \\ &\quad \sim \frac{n \cdot 2\pi i}{r_1 - r_2} \cdot \log \frac{r_1}{r_2}\bigg|_{\Psi = 0} \sim \frac{nm\,(2\pi i)^2}{\sqrt{\Psi}}, \end{aligned}\right\} \tag{A.11}$$

where

$$\log \prod_j \chi_j(r_i) = \log 1 = n \cdot 2\pi i, \quad \text{by virtue of (51),}$$

and

$$\log \frac{r_1}{r_2}\bigg|_{\Psi = 0} \sim \log 1 = m \cdot 2\pi i; \quad n, m, \text{ integers.}$$

On the Φ_1-manifold, one of the r_i, say r_1, becomes zero, while the other is finite. Thus on one side of Φ_1-manifold, $m = 0$ (if we are on the principal sheet to start with, e. g., for all z's being negative real), but on the other side, $m \neq 0$. This shows a change of relevance of the Ψ-manifold at its intersection with the Φ_1-manifold. In a quite similar fashion, e. g., from the second term in (A.2), there develops a change of relevance across the Φ_{j+1}-manifold. To show this, it suffices to note that at $(\Psi = 0) \cap (\Phi_{j+1} = 0)$, one gets $N_j(r_i) = 0$, whence $\log \chi_j(r_i) = \log 1$ also, for each $j = 1,2,3$.

This confirms Lemma 2 in a more explicit way.

Appendix B

Envelope Problem for the Φ-manifold

In stating that the 3-point domain is bounded by analytic hypersurfaces F'_{kl} (obtained by setting two of the three mass-parameters equal to zero from the Φ-manifold), it is understood that the envelopes of the Φ-manifold are trivial, in the sense that they do not exist off the cuts and hence never actually contribute to the boundary (apart from what one has already on the cut). The purpose of this appendix is twofold:

(a) To give a proof of the above statement[55], and
(b) Since the Φ-manifold is of a much simpler structure, the analysis here actually serves as a prototype for the treatment of the Ψ-manifold (cf. Sec. IV), despite the fact that the final situations are quite different in two cases.

The notation here for the variables in the Φ-manifold follows that of KW.

I. 3-*Mass Envelope* E_{123}:

Let

$$\Phi(z; a) = \frac{1}{2} \begin{vmatrix} -2a_1 & z_3 - a_2 - a_1 & z_2 - a_3 - a_1 \\ z_3 - a_1 - a_2 & -2a_2 & z_1 - a_3 - a_2 \\ z_2 - a_1 - a_3 & z_1 - a_2 - a_3 & -2a_3 \end{vmatrix}. \tag{B.1}$$

The analogue of (63) is

$$\sum_{k=1}^{3} P_k = \lambda(z), \qquad P_k = \frac{\partial \Phi}{\partial a_k}. \tag{B.2}$$

The analogue of (64) is

$$\Phi = -\frac{1}{2} \sum_{k=1}^{3} \Phi_{ik} \frac{\partial \Phi}{\partial a_k}, \quad \text{for} \quad i = 1, 2, 3, \tag{B.3}$$

where the Φ_{ik}'s denote the elements in the determinant (B.1) without, however, the factor 1/2.

The Analogue of (70) now reads on the Φ-manifold:

$$\sum_{k=1}^{3} \Phi_{ik} P_k = 0, \quad i = 1, 2, 3. \tag{B.4}$$

[55] This is previously known to KW, but remained unpublished. My sincere thanks are due Professor Källén for his many enlightening discussions on this, and for his kind permission to include it here.

On the envelope E_{123}, we have

$$\frac{\dfrac{\partial \Phi}{\partial a_i}}{\dfrac{\partial \Phi}{\partial a_k}} = \frac{P_i}{P_k} = \frac{\gamma_i}{\gamma_k} \tag{B.5}$$

where the γ_k's are *real*, such that

$$\left. \begin{aligned} P_k &= \gamma_k \lambda(z) \\ \sum_k \gamma_k &= 1 . \end{aligned} \right\} \tag{B.6}$$

Thus the analogues of (72) are

$$\sum_k (\boldsymbol{Im}\, \Phi_{ik})\, \gamma_k = 0 \tag{B.7}$$

$$\sum_k (\boldsymbol{Re}\, \Phi_{ik})\, \gamma_k = 0 . \tag{B.8}$$

Now from (B.7) follows immediately the analogue of (73):

$$0 = \det | \boldsymbol{Im}\, \Phi_{ik} | = 2\, y_1 y_2 y_3 . \tag{B.9}$$

In general, for given $y_1, y_2, \neq 0$, (B.9) implies that y_3 must be zero on the 3-mass envelope. Or in other words:

No 3-mass envelope for the Φ-manifold can exist off the real axis.

This is also a horizontal line in the z_3-plane (cf. E_{1234} of (74) in the 4-point case). At this point, one can immediately see that E_{123} is irrelevant: It cannot be relevant on the negative real axis. Then *at most* E_{123} can lie on the positive real axis, which is already the cut.

The following, however, is devoted to an explicit solution to the real part equations (B.8), showing that E_{123} (as well as the 2-mass envelopes discussed below) is actually non-empty, and in one particular configuration (i. e. bubble) the 3-mass and the 2-mass envelopes are rather amusing (cf. Fig. 28).

With $y_3 = 0$, it follows further from (B.7) that

$$\left. \begin{aligned} \gamma_3 &= 0 \\ \frac{\gamma_1}{\gamma_2} &= \frac{-y_1}{y_2} . \end{aligned} \right\} \tag{B.10}$$

Or, when normalized according to (B.6),

$$\gamma_1 = \frac{y_1}{y_1 - y_2}; \quad \gamma_2 = \frac{-y_2}{y_1 - y_2}. \tag{B.10a}$$

With these explicit values of the γ_k's, the real part equations (B.8) yield

$$y_1 a_1 + (-y_2) a_2 + (y_1 - y_2) a_3 = (x_2 y_1 - x_1 y_2) \tag{B.11}$$

$$y_1^2 a_1 - y_2^2 a_2 = 0 \tag{B.12}$$

$$x_3 = \left(1 - \frac{y_1}{y_2}\right) a_1 + \left(1 - \frac{y_2}{y_1}\right) a_2. \tag{B.13}$$

The path C_{123} in the a_k-space (which would give rise to E_{123}) is then the straight-line intersection of the two planes given by (B.11) and (B.12), within the octant $a_k > 0$. We now divide our discussion into two parts:

Case 1: $y_1 y_2 < 0$ (Bubble configuration).

Without loss of generality, we may take $y_1 > 0$. In this case, (B.11) is compact within the octant $a_k > 0$. Therefore its intersection with (B.12) gives

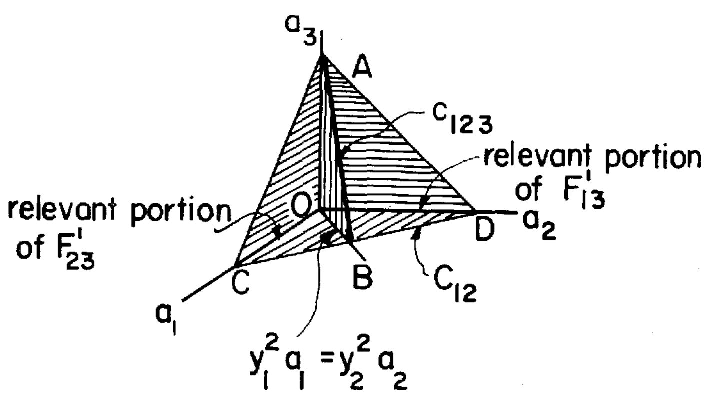

Figure 27. Paths in the a-space for the 3-mass and 2-mass envelopes and the 1-mass curves for the 3-point Φ-manifold: $y_1 y_2 < 0$.

a finite straight-line segment AB (Fig. 27). The image of AB in the z_3-plane is given by (B.13). More explicitly, we have from (B.12) and (B.13)

$$x_3 = \left(1 - \frac{y_1}{y_2}\right)^2 a_1 \tag{B.14}$$

which, together with (B.12), implies that a_1 and a_2 are positive if and only if x_3 is positive. Furthermore, one gets from (B.11)

$$a_3 = \frac{(-y_1 y_2)}{(y_1 - y_2)^2} (x_3^{(0)} - x_3), \tag{B.15}$$

where

$$x_3^{(0)} = \frac{(y_1 - y_2)}{(-y_1 y_2)} (x_2 y_1 - x_1 y_2) \tag{B.16}$$

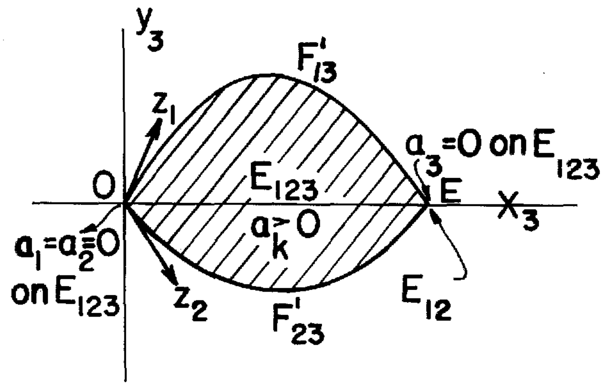

Figure 28. Φ-manifold envelopes for the bubble configuration in the z_3-plane.

is precisely the abscissa of the point E (Fig. 28) which is the common intersection of F'_{13} and F'_{23} with the x_3-cut.

Note that

$$x_3^{(0)} > 0, \quad \text{for} \quad \arg z_2 > \pi + \arg z_1,$$

which is the relevance criterion for the bubble of Fig. 28. From (B.14), (B.12), and (B.15), it is clear now that OE is the image of AB, since all $a_k > 0$ if and only if

$$0 < x_3 < x_3^{(0)}. \tag{B.17}$$

This shows that in the case *when the 3-point boundary is given by the bubble, the 3-mass envelope for the Φ-manifold is actually the segment of the cut on the real axis lying inside the bubble.* It will be shown later that the end point E (where $a_3 = 0$ on the 3-mass envelope) actually constitutes the 2-mass envelope E_{12} for the Φ-manifold in this case.

Case 2: $y_1 y_2 > 0$ (Hyperbola configuration).

In this case, the results become dependent on the ratios of the real and imaginary parts of z_1 and z_2.

(i) when $y_1 = y_2$, we must have also $x_1 = x_2$ as a consequence of (B.11) and (B.12). The allowed region in the a_k-space becomes unbounded, being the whole plane (B.12) within the octant $a_k > 0$ (i. e., $a_1 = a_2$, a_3 arbitrary). The image in the z_3-plane is a single point $x_3 = 0$, viz., E_{123} is at the origin.

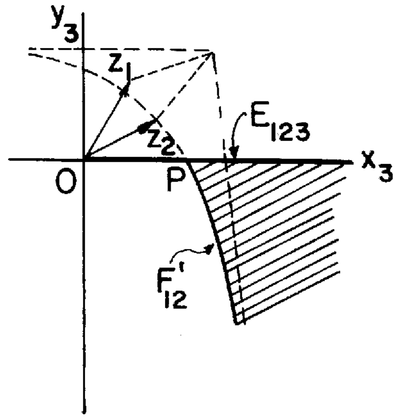

Figure 29. Φ-manifold envelopes in the z_3-plane for the hyperbola configuration: $x_3^{(0)} \leqslant 0$.

(ii) $y_1 \neq y_2$. (B.14) and (B.15) now imply that all $a_k > 0$ if and only if

$$x_3 > \mathrm{Max}\left\{0,\, x_3^{(0)}\right\}. \tag{B.18}$$

Thus

(iia) if $x_3^{(0)} \leqslant 0$, E_{123} is the whole cut $x_3 \geqslant 0$. (Fig. 29).

(iib) if $x_3^{(0)} > 0$, E_{123} starts from $x_3 = x_3^{(0)}$. However, this point has no significance for the case $y_1 y_2 > 0$, since the hyperbola F'_{12} (Fig. 30) intersects the real axis at P with

$$x_3^{(P)} = \frac{y_1 y_2 \left[(x_1 - x_2)^2 + (y_1 + y_2)^2\right]}{(y_1 + y_2)(x_1 y_2 + x_2 y_1)}. \tag{B.19}$$

In this case one has both

$$\left.\begin{aligned} x_3^{(P)} &> x_3^{(0)} \\ \text{and} \qquad x_3^{(P)} &> 0 \end{aligned}\right\} \tag{B.20}$$

for $\arg z_1 + \arg z_2 < \pi$, which is the criterion for F'_{12} to be relevant.

II. *Two-Mass Envelopes.*

It can be easily seen that the 2-mass envelopes still lie on the cut along the real axis.

We shall only treat E_{12} here with a_3 set equal to zero; for the others the analysis can be easily adapted. The envelope condition reads:

$$\frac{P_1}{P_2} = \frac{\sqrt{R_1}}{\sqrt{R_2}} = \sigma, \quad \text{a real number.} \tag{B.21}$$

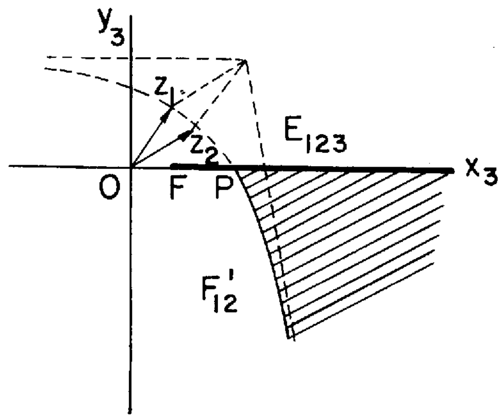

Figure 30. Φ-manifold envelopes in the z_3-plane for the hyperbola configuration: $x_3^{(0)} > 0$.

Case 1: $y_1 y_2 < 0$

(B.12) and (B.14) now no longer hold, however, (B.11) with $a_3 = 0$ is equivalent to (B.21). Furthermore, (B.13), which can be regarded as the equation for the Φ-manifold in this case, is still valid. From these, one gets rather unexpectedly that E_{12} is just a single point at $x_3 = x_3^{(0)}$, (viz., the point E of Fig. 28). Geometrically, in Fig. 27, CD is now the path for E_{12} in the positive quadrant. The entire segment CD is mapped into the point E, which is exactly the end-point $a_3 = 0$ of E_{123}.[56]

In this case, it is interesting to note that the path OC along the a_1-axis and the path OD along the a_2-axis in Fig. 27 map respectively into the rele-

[56] The fact that the path for E_{12} is simply the projection of the plane for E_{123} in 3-space onto the 2-plane must be regarded again as a peculiarity of the 3-point case. This is not true in the 4-point case (cf. Sec. VI), where we have shown that, although the path for E_{1234} is also a straight line in the 4-space, the paths for E_{ijk} and E_{ik} are both not projections, and are very far from being straight lines.

vant portion of F'_{23} and F'_{13} in Fig. 28 (with the common end-point E besides the origin). The occurrence of such multiple intersections of $E_{123} \cap E_{12} \cap E_1 \cap E_2$ must be regarded as a 3-point peculiarity (cf. Fig. 19 and the accompanying remark). In the 4-point case, we have seen, however, that in general we have only the intersection between an m-envelope and an $(m-1)$-envelope (cf. Lemma 3).

Case 2: $y_1 y_2 > 0$

Following E_{123} in this case, and the E_{12} for the above case, we see that E_{12} for this case also consists of a point at $x_3 = x_3^{(0)}$. Now

(i) If $x_3^{(0)} < 0$, E_{12} is irrelevant, and
(ii) If $x_3^{(0)} > 0$, E_{12} is the point F in Fig. 30, which is imbedded in the cut.

Appendix C

Some Algebraic Details for the 4-Mass Envelope

We give here the details for the values of the γ_k's on E_{1234}, and the dependence of their relative signs on the configuration of the y's.

Solving (72a), one gets

$$\frac{\gamma_1 \gamma_2}{\gamma_3 \gamma_4} = \frac{y_5}{y_1}; \quad \frac{\gamma_1 \gamma_3}{\gamma_2 \gamma_4} = \frac{y_6}{y_2}; \quad \frac{\gamma_1 \gamma_4}{\gamma_2 \gamma_3} = \frac{y_4}{y_3} \tag{C.1}$$

or equivalently:

$$\left.\begin{aligned} \frac{\gamma_1}{\gamma_2} &= \frac{y_3 y_4 \pm \sqrt{y_1 y_3 y_4 y_5}}{-y_2 y_3} \\ \frac{\gamma_3}{\gamma_2} &= \frac{y_1 y_5 \pm \sqrt{y_1 y_3 y_4 y_5}}{-y_2 y_5} \\ \frac{\gamma_4}{\gamma_2} &= \frac{\pm \sqrt{y_1 y_3 y_4 y_5}}{y_3 y_5}. \end{aligned}\right\} \tag{C.1a}$$

These may then be normalized according to (67). The $(\pm)$ signs correspond to the sign of $E_{1234}^{\pm}$ in (74). From these, one immediately notes that, for example,

on E_{1234}^{+},

$$\gamma_1 \gamma_3 \gtrless 0 \text{ according as } y_1 y_3 \gtrless 0,$$

and

$$\gamma_2 \gamma_4 \gtrless 0 \text{ according as } y_3 y_5 \gtrless 0.$$

The exactly opposite statements hold on E^{-}_{1234}.

We summarize the results in Table 1:

Table 1: Relative Signs of γ_k on $E^{\pm}_{1234}$ Versus Configurations

<table>
<tr><th rowspan="2">Cases</th><th colspan="2">Configuration of y</th><th colspan="2">Relative Signs of γ_k's</th></tr>
<tr><th>Up</th><th>Down</th><th>On E^{+}_{1234}</th><th>On E^{-}_{1234}</th></tr>
<tr><td rowspan="2">I</td><td rowspan="2">1, 3, 4, 5</td><td rowspan="2">2, 6</td><td rowspan="2">all $\gamma_k > 0$</td><td>(i) $y_1, y_4 \| y_2, y_3$</td></tr>
<tr><td>(ii) $y_1, y_2 \| y_3, \gamma_4$</td></tr>
<tr><td rowspan="2">II</td><td rowspan="2">1, 2, 5, 6</td><td rowspan="2">3, 4</td><td rowspan="2">$\gamma_1, \gamma_2 \| \gamma_3, \gamma_4$</td><td>(i) $\gamma_2, \gamma_4 \| \gamma_1, \gamma_3$</td></tr>
<tr><td>(ii) all $\gamma_k > 0$</td></tr>
<tr><td rowspan="2">III</td><td rowspan="2">2, 3, 5</td><td rowspan="2">1, 4, 6</td><td>(i) $\gamma_1 \| \gamma_3, \gamma_2, \gamma_4$</td><td rowspan="2">$\gamma_4 \| \gamma_1, \gamma_2, \gamma_3$</td></tr>
<tr><td>(ii) $\gamma_3 \| \gamma_1, \gamma_2, \gamma_4$</td></tr>
<tr><td rowspan="2">IV</td><td rowspan="2">1, 2, 3</td><td rowspan="2">4, 5, 6</td><td>(i) $\gamma_2 \| \gamma_4, \gamma_1, \gamma_3$</td><td rowspan="2">$\gamma_3 \| \gamma_1, \gamma_2, \gamma_4$</td></tr>
<tr><td>(ii) $\gamma_4 \| \gamma_2, \gamma_1, \gamma_3$</td></tr>
</table>

Remark: (a) These are the only four distinct configurations of the y's for which $E^{\pm}_{1234}$ exists. The remaining case with all $y_\mu > 0$ is disregarded here, since the 4-mass envelope is entirely irrelevant in this case (cf. remark following (76)). The permutation of *(3, 4)* with *(1, 5)* in case II is trivial. So is the permutation of $(2 \leftrightarrow 6)$ in cases III and IV.

(b) The subdivision into (i) and (ii) is based on

(i) $|y_1 y_5| > |y_3 y_4|$

(ii) $|y_1 y_5| < |y_3 y_4|$, respectively. Note that, when $y_1 y_5 = y_3 y_4$, one of the lines $E^{\pm}_{1234}$ coincides with the cut.

(c) All signs except in the case when all $\gamma_k > 0$ are meant only in a relative sense. Thus we use the double bars to denote that the γ's lying on the same side of the double bar have the same sign, while any two γ's lying on the opposite sides of the double bar have opposite signs.

(d) The above results can be briefly stated as follows:

(1) When the signs of the 6 y's break into 4 || 2, the signs of the 4 γ's break into 4 || 0, or 2 || 2.

(2) When the signs of the y's break into 3 || 3, then those of the γ's break into 3 || 1.

With Table 1, one can readily infer from (77) or (80) the signs of the a_k's on the 4-mass envelope at $x_6 \to \pm\infty$. For a_1 and a_3, no other information is needed; however, for a_2 and a_4, there is a further dependence on the magnitude of γ_2 and γ_4 (when the latter are positive). Table 2 illustrates the situation for $z_6 \to -\infty$. Exactly opposite statements hold for the signs of the a_k at the other end $x_6 \to +\infty$.

TABLE 2: The Signs of a_k on $E^{\pm}_{1234}$ at $x_6 \to -\infty$.

Cases*		a_1 and a_3		a_2		a_4	
		On E^{+}_{1234}	On E^{-}_{1234}	On E^{+}_{1234}	On E^{-}_{1234}	E^{+}_{1234}	E^{-}_{1234}
I	(i)	+	−	−		−	
	(ii)	+	−	−		−	
II	(i)	−	+				
	(ii)	−	+		−		−
III	(i)	+	−	$\mp(\gamma_2 \lessgtr 1)$	$\pm(\gamma_2 \lessgtr 1)$	$\mp(\gamma_4 \lessgtr 1)$	−
	(ii)	+	−	$\mp(\gamma_2 \lessgtr 1)$	$\pm(\gamma_2 \lessgtr 1)$	$\mp(\gamma_4 \lessgtr 1)$	−
IV	(i)	−	+	−	$\mp(\gamma_2 \lessgtr 1)$	$\pm(\gamma_4 \lessgtr 1)$	$\mp(\gamma_4 \lessgtr 1)$
	(ii)	−	+	$\pm(\gamma_2 \lessgtr 1)$	$\mp(\gamma_2 \lessgtr 1)$	−	$\mp(\gamma_4 \lessgtr 1)$

* For the cases III and IV in Table 2, the signs of 4 γ's break into $3 \,\|\, 1$. Table 2 assumes that 3 γ's > 0 and one $\gamma < 0$.

The remainder of this appendix is devoted to the discussion of the case when the (all a_k positive) segment E_{1234} has an intersection with the set ω_x of (82). For this, it will be convenient to divide the discussion into the following two classes of configurations:

(1) *All γ_k positive.*

In this case, we have

$$0 < \gamma_k < 1\,, \quad \Sigma\gamma_k = 1\,. \tag{C.2}$$

From Table 1, we see that this happens only for the following two configurations (Figs. 31–32). We recall from Table 1 that all $\gamma_k > 0$ hold for the configuration (Fig. 32) only for $y_1 y_5 < y_3 y_4$ (otherwise 2 of the γ's become negative). When $y_1 y_5 - y_3 y_4 \to 0$, the line E_{1234}^- collapses into the cut on the real x_6-axis. On E_{1234}, we have, in general, by virtue of (72):

$$\left.\begin{aligned} \sum_{i,j} (\boldsymbol{Re}\,\Psi_{ij})\,\gamma_i\gamma_j &= 0 \\ \sum_{i,j} (\boldsymbol{Im}\,\Psi_{ij})\,\gamma_i\gamma_j &= 0. \end{aligned}\right\} \quad (\text{C.3})$$

Now the γ's of (C.2) may just be identified as playing the same role as our original integration variables α_k's. Therefore for this case, the denominator D

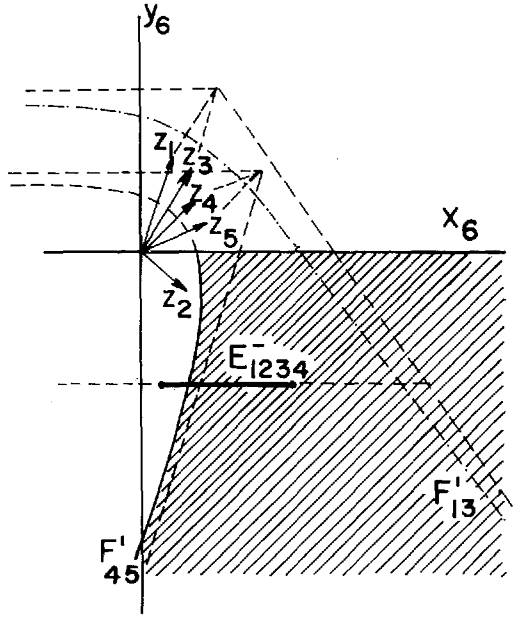

Figure 31. All $\gamma_k' > 0$ on the 4-mass envelope for the configuration (I) of Table 1.

of (15) will indeed vanish identically on E_{1234} (where all α_k's are positive). When this segment has an intersection with the Φ_k-manifolds, part of it will have actual singularities.

One observes from Table 2 that E_{1234} are finite for both of these configurations, since two of the a's (viz., a_2, a_4) are negative at $x_6 \to -\infty$, and the other two (viz. a_1, a_3) are negative at the other end ($x_6 \to +\infty$).

(2) *Not all γ_k's positive*:

In this case, identification of γ_k with α_k is not possible, thus (C.3) do not automatically imply that (16) will vanish on E_{1234}. In fact, it can be easily seen that $\boldsymbol{Re}\,D$ never vanishes for $x \varepsilon \omega_x$ of (82).

One notes from (16), after the substitution $\alpha_4 = 1 - \sum_{i=1}^{3} \alpha_i$,

$$\left.\begin{aligned} -\boldsymbol{Re}\, D &= -\frac{1}{2}\sum_{i,j} (\boldsymbol{Re}\, \Psi_{ij})\, \alpha_i \alpha_j = \\ &= \left[x_6 (\alpha_2 + \alpha_2^0)^2 - \frac{\lambda_1(x)}{4\, x_6} (\alpha_3 + \alpha_3^0)^2 + \frac{\Lambda(x)}{\lambda_1(x)} (\alpha_1 + \alpha_1^0)^2 - \frac{\Psi(x;\, a)}{4\Lambda(x)} \right] \end{aligned}\right\} \quad (C.4)$$

where

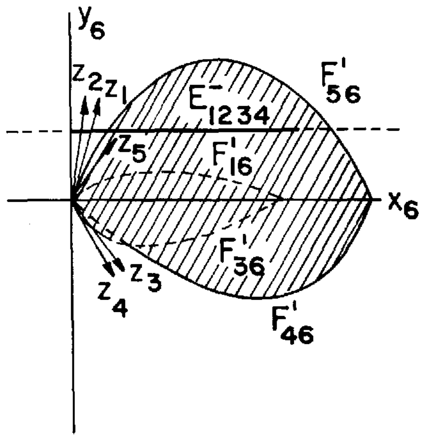

Figure 32. All γ_k's > 0 on the 4-mass envelope for the configuration (II) of Table 1.

$$\left.\begin{aligned} \alpha_2^0 &= -\frac{1}{2\, x_6} \left[\frac{\partial^2 \Lambda(x)}{\partial x_1 \partial x_2} \alpha_3 + \frac{\partial^2 \Lambda(x)}{\partial x_4 \partial x_2} \alpha_1 - \frac{\partial^2 \Phi_1(x;\, a)}{\partial x_4 \partial a_3} \right] \\ \alpha_3^0 &= \frac{1}{\lambda_1(x)} \left[\frac{\partial \Lambda(x)}{\partial x_2} \alpha_1 - \frac{\partial \Phi_1(x;\, a)}{\partial a_3} \right] \quad \text{(Cf. Eq. (35));} \\ \alpha_1^0 &= \frac{Q_1(x;\, a)}{4\Lambda(x)} \end{aligned}\right\} \quad (C.5)$$

and $\Psi(x;\, a) \equiv \det |\, \boldsymbol{Re}\, \Psi_{ji} \,|$
which vanishes identically on E_{1234}. Thus we see that (C.4) is positive definite for $x \varepsilon \omega_x$, unless simultaneously

$$\left.\begin{aligned} \alpha_i &= -\alpha_i^0, \quad \text{for} \quad i = 1, 2, 3 \\ \text{and} \qquad \alpha_i^0 &< 0\,, \ \sum_i (-\alpha_i^0) < 1. \end{aligned}\right\} \quad (C.6)$$

It is now easy to see that (C. 6) cannot happen when at least one of the γ_k's is negative. We have on E_{1234}, after treating the det $|\boldsymbol{Re}\,\Psi_{ij}|$ with exactly the same procedure which led to (64),

$$\gamma_k = -\frac{Q_k(x;\, a)}{4\,\Lambda(x)}. \tag{C.7}$$

Now, without loss of generality, we may take[57] $\gamma_1 < 0$. Then (C.5) implies that $\alpha_1^0 = -\gamma_1 > 0$, and (C.6) clearly cannot happen. Thus for all cases with γ_k not simultaneously positive, $-\boldsymbol{Re}\,D$ is positive definite on $\omega_x \cap E_{1234}$, and it follows that this portion of the 4-mass envelope can never be a relevant part of the boundary.

For completeness, we note the following identity *on the 4-mass envelope*:

$$2\,\boldsymbol{Re}\,D \equiv \sum_{i,j} \boldsymbol{Re}\,\Psi_{ij}\,\alpha_i\,\alpha_j = \sum_{i,j} X_{ij}\,(\alpha_i - \gamma_i)\,(a_j - \gamma_j) \tag{C.8}$$

which can be easily verified with the aid of (77).

Appendix D

Note on the Determinant Expansion

We here observe that a great number of identities which have played an essential role in our preceding discussion, such as (43), (44), (45), (100), and (110), have a most natural interpretation in terms of their associated determinants. Take, for example, (43), which reads:

$$\left(\frac{1}{2}\frac{\partial \Psi}{\partial a_k}\right)^2 = 4\,\Lambda\,(z)\,\Phi_k + \lambda_k\,\Psi\,(z;\, a). \tag{D.1}$$

Recalling the quantities following (57 a), we have, for $k = 2$

$$\left.\begin{aligned} \frac{1}{2}\frac{\partial \Psi}{\partial a_2} &= \tilde{\Psi}^{12} \\ 2\,\Lambda(z) &= \tilde{\Psi}^{11} \\ 2\,\Phi_2 &= \tilde{\Psi}^{22} \\ -\lambda_2 &= \tilde{\Psi}^{12,\,12} \end{aligned}\right\} \tag{D.2}$$

[57] Otherwise, a trivial permutation will bring (C.4) into the form where the last α_k^0 corresponds to the desired negative γ_k.

where $\tilde{\Psi}^{12,12}$ refers to the minor complementary to the 2×2 minor

$$\begin{vmatrix} \tilde{\Psi}_{11} & \tilde{\Psi}_{12} \\ \tilde{\Psi}_{21} & \tilde{\Psi}_{22} \end{vmatrix}$$

in the $\tilde{\Psi}$-determinant of (57a). Then (D.1) takes the form

$$\left.\Psi \equiv \hat{\Psi} = \frac{\begin{vmatrix} \tilde{\Psi}^{11} & \tilde{\Psi}^{12} \\ \tilde{\Psi}^{21} & \tilde{\Psi}^{22} \end{vmatrix}}{\begin{vmatrix} \tilde{\Psi}_{33} & \tilde{\Psi}_{34} \\ \tilde{\Psi}_{43} & \tilde{\Psi}_{44} \end{vmatrix}} \quad\right\} \qquad (D.3)$$

Now identity (D.3) can be easily verified to hold for a general 4×4 determinants Thus (D.1) is established for $k = 2$, and by symmetry the others follow. At this point, the corresponding identities for the 3-point case (KW (A46d)) are seen to be also derivable from such a determinant expansion.

Ir appears, however, that identities of the form (D.3) are actually very special cases of a general theorem, which, in various forms, has been dated back to Gauss (also for symmetric determinants) and others. We shall here quote a theorem due to Jacobi[58], which states that

Any minor of order k in A^{-1} is equal to the complementary signed minor in A' (the adjoint of A), multiplied by $|A|^{-1}$.

In other words, this technique of determinant expansion relates the block I in (D.4) with the block II in (D.5), their determinants being off by a factor of the original determinant:

$$A^{-1}: \qquad k\left\{\left(\begin{array}{c:c} \overbrace{\mathrm{I}}^{k} & \\ \hdashline & \end{array}\right)\right. \qquad (D.4)$$

[58] See, e.g., an elementary text by A. C. Aitken, *Determinants and Matrices*, 3rd ed., Edinburgh (1944).

$$A': \qquad \left(\begin{array}{c:c} & \\ \hdashline & \underbrace{\mathrm{II}}_{(n-k)} \end{array}\right)\Big\} (n-k) \tag{D.5}$$

It is then a simple matter to derive all the identities we mentioned by simply writing down the desired $k \times k$ minors in this fashion.

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