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A STUDY OF NUCLEAR POTENTIAL ENERGY SURFACES AND GAMMA VIBRATIONS

BY

DANIEL R. BÈS

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i kommission hos Ejnar Munksgaard

CONTENTS

Synopsis

The theory of the collective properties of the nuclear shell model has progressed recently due to the introduction of the simple pairing force to simulate the residual nucleonic interaction. Working within the framework of the adiabatic approximation, the present paper studies the consequences of this model for the γ-dependent terms of the nuclear potential energy surface. The simplified case of nucleons in a harmonic oscillator potential is considered first. Then, the energies and transition probabilities are calculated for γ-vibrations of deformed nuclei of axial symmetric shape. In addition, numerical calculations, based on realistic wave functions for nucleons in deformed nuclei, have been performed in a few cases and are compared with empirical data.

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I. Introduction

In recent years, promising progress has been made in deriving the nuclear collective properties, starting from a description in terms of independent-particle motion.

On the one hand, it has been recognized that the shell model binding field may be associated with the effect of the long range part of the nucleonic interaction. In particular, the deformations of ellipsoidal shape can be shown to be a consequence of the quadrupole component of the effective two-body force[1][2]. On the other hand, it was recognized that there are important effects of this force which cannot be incorporated into a smoothly varying binding field, such as the inertial properties of the collective motion[3] or the potential energy of the nuclear deformation.

To represent this "residual" force, an interaction of especially simple properties has been suggested[4]. This is the so-called "pairing force" which is analogous to that used in the recent theory of superconductivity[5], and which is a generalization of the force in terms of which seniority is defined[6]. Preliminary investigations have shown that such a nuclear model contains many of the qualitative features of the observed nuclear spectra[7][8][9][10]. A more quantitative test of this model has been performed for nuclei in the regions near closed shells[11].

The aim of the present investigation is to study in greater detail some of the features of the nuclear potential energy surface which follow from this model. In particular, we consider the dependence on the parameter γ, which describes the departure from axial symmetry of an ellipsoidal nuclear deformation. We also investigate the properties of vibrations in the γ-coordinate, which are expected for nuclei of spheroidal shape.

For a quantitative analysis of the collective nuclear properties it is necessary to start from a nuclear shell model with the appropriate single-particle level spacings and wave functions. However, in order to explore some of the qualitative features, we first consider the simplified case of a harmonic oscillator well. Subsequently, we present some calculations based on a realistic single-particle spectrum, and compare the results with experimental data.

II. Formulation of the model

The basic assumption of many studies of nuclear structure is that it is sufficient to consider the degrees of freedom associated with the particles outside closed shells, the particles within closed shells manifesting themselves only through the Pauli principle and through a renormalization of the effective interparticle force.

In this chapter, we give (a) a brief description of the solution of the problem of identical particles moving in a certain shell coupled by the pairing force; (b) a discussion of the deformed part of the single-particle field; and (c) the calculation of the potential energy surface and mass parameter.

a) *The independent quasi-particle approximation.*

We assume that the matrix of the single-particle Hamiltonian has been diagonalized and that ε_ν is the eigenvalue corresponding to the degenerate single-particle states labelled by $\nu+$ and $\nu-$. These states are related by the operation of time reversal. Using the formalism of second quantization, the total single-particle Hamiltonian can be written

$$H_{\mathrm{sp}} = \sum_{\nu} \varepsilon_\nu \, (c^\dagger_{\nu+} c_{\nu+} + c^\dagger_{\nu-} c_{\nu-}). \tag{1}$$

Here, $c^\dagger_\nu$ and c_ν are, respectively, the creation and annihilation operators for the single-particle state ν. They obey the usual anti-commutation relations.

In this formalism, the pairing force is given by

$$H_{\mathrm{pair}} = -G \sum_{\nu,\,\omega} c^\dagger_{\nu+} c^\dagger_{\nu-} c_{\omega-} c_{\omega+}. \tag{2}$$

The lowest eigenvalue of the total Hamiltonian $H = H_{\mathrm{sp}} + H_{\mathrm{pair}}$ can be approximated by means of a variational procedure. One uses a trial function[5]

$$|\,0\,\rangle = \prod_{\nu} [U_\nu + V_\nu c^\dagger_{\nu+} c^\dagger_{\nu-}]\,|\,\text{vacuum}\,\rangle. \tag{3}$$

The condition $U^2_\nu + V^2_\nu = 1$ ensures that the wave function (3) is normalized. From (3) it is seen that V^2_ν is the probability that the states $\nu+$ and $\nu-$ are occupied. The V_ν are variational parameters to be determined by the condition that they minimize $\langle\, 0 \,|\, H \,|\, 0 \,\rangle$. This leads to the equation

$$2/G = \sum_{\nu} (\varepsilon^2_\nu + \Delta^2)^{-1/2}, \tag{4}$$

where $\Delta = G\sum_\nu U_\nu V_\nu$. The excitation spectrum has energies $E_\nu + E_\omega$, where

$$E_\nu = (\varepsilon_\nu^2 + \Delta^2)^{1/2}. \tag{5}$$

In consequence, the energy of the first excited state is always greater than 2Δ.

An elegant formulation, equivalent to the above procedure, has been developed by BOGOLUBOV[12] and by VALATIN[13]. It has been applied to nuclei by BELYAEV[7]. We now summarize some of his results. One starts by introducing two new operators ($\alpha_\nu^\dagger$, $\beta_\nu^\dagger$) defined by the following canonical transformation

$$\left.\begin{aligned} \alpha_\nu^\dagger &= U_\nu c_{\nu+}^\dagger - V_\nu c_{\nu-} \\ \beta_\nu^\dagger &= U_\nu c_{\nu-}^\dagger + V_\nu c_{\nu+}. \end{aligned}\right\} \tag{6}$$

Because this is a canonical transformation, the new operators obey the same anti-commutation relations as the old. Thus, they can be regarded as creation operators for "quasi-particles" obeying Fermi statistics.

By means of the transformation inverse to (6), we can express H in terms of $\alpha_\nu^\dagger$, $\beta_\nu^\dagger$, β_ν, and α_ν. Using the anti-commutation relations, H can be put into normal form, i. e., with the $\alpha_\nu^\dagger$, $\beta_\nu^\dagger$ to the left of the β_ν, α_ν. H has then the following structure:

$$H = U + H_{11} + H_{20} + H_{\text{int}}. \tag{7}$$

The term U is a constant. H_{11} contains terms proportional to $(\alpha_\nu^\dagger \alpha_\nu + \beta_\nu^\dagger \beta_\nu)$, H_{20} terms proportional to $(\alpha_\nu^\dagger \beta_\nu^\dagger + \beta_\nu \alpha_\nu)$. H_{int}, the remainder, is supposed to have a small influence on the properties of at least the lowest states. The requirement that the coefficient of $(\alpha_\nu^\dagger \beta_\nu^\dagger + \beta_\nu \alpha_\nu)$ vanishes leads to (4).

If H_{int} is neglected, the remaining

$$U + H_{11} = \sum_\nu \varepsilon_\nu 2V_\nu^2 - \Delta^2/G + \sum_\nu E_\nu(\alpha_\nu^\dagger \alpha_\nu + \beta_\nu^\dagger \beta_\nu) \tag{8}$$

describes a system of non-interacting quasi-particles. The single quasi-particle energies are given by (5). The wave functions can be characterized by the number of quasi-particles present. In particular, the ground state has no quasi-particles. Expressed in terms of the original particle-creation operators, it is just the state (3), so that

$$\alpha_\nu \mid 0 \rangle = \beta_\nu \mid 0 \rangle = 0. \tag{9}$$

The excited states all have even numbers of quasi-particles. Those with two quasi-particles are denoted by

$$\mid \nu\omega \rangle = \alpha_\nu^\dagger \beta_\omega^\dagger \mid 0 \rangle. \tag{10}$$

Unfortunately, the solutions (3) and (10) are not eigenstates of the operator representing the number of particles

$$n_{\mathrm{op}} = \sum_{\nu} (c^{\dagger}_{\nu+} c_{\nu+} + c^{\dagger}_{\nu-} c_{\nu-}). \tag{11}$$

However, we can at least ensure that the average particle number in the ground state has a prescribed value n, by using a Lagrange multiplier. That is, we replace H by $H - \lambda n_{\mathrm{op}}$. The Lagrange multiplier λ is to be determined by the condition

$$n = \langle 0 \mid n_{\mathrm{op}} \mid 0 \rangle = 2 \sum_{\nu} V_{\nu}^2. \tag{12}$$

Since the formal effect of the subtraction of λn_{op} is the replacement of the ε_{ν} by $\varepsilon_{\nu} - \lambda$, we see that λ can be interpreted as an effective Fermi energy.

b) *The deformation-dependent terms of the Hamiltonian.*

We have been using a representation in which the single-particle Hamiltonian is diagonal. However, the spherical part of this Hamiltonian is not necessarily diagonal. We denote its matrix elements by $\varepsilon^0_{\nu\omega}$. The non-spherical part, associated with ellipsoidal deformations, is represented by the scalar product of the single-particle and the total nuclear quadrupole moments. This lifts the degeneracies characteristic of the central field, and has been successfully used[14][15] in the explanation of many properties of deformed nuclei. Thus, the total single-particle matrix element is

$$\varepsilon_{\nu\omega} = \varepsilon^0_{\nu\omega} - \varkappa \sum_{\mu} Q^T_{\mu} (q_{\mu})_{\nu\omega} = \varepsilon_{\nu} \delta_{\nu\omega}, \tag{13}$$

where $\varkappa$ is a coupling constant ultimately determined by the quadrupole force, and

$$(q_{\mu})_{\nu\omega} = 4 \sqrt{\frac{\pi}{5}} \langle \nu \mid r^2 Y_{2\mu}(\theta) \mid \omega \rangle \tag{14a}$$

$$Q^T_{\mu} = 4 \sqrt{\frac{\pi}{5}} \sum_{k} \int \Psi^* \Psi r_k^2 Y^*_{2\mu}(\theta_k)\, dr_1 \ldots dr_k \ldots dr_n, \tag{14b}$$

Ψ being the total nuclear wave function. Therefore, (13) and (14b) imply the self-consistency condition that the quadrupole tensor of the field is the sum of the quadrupole tensors of the orbits determined by that field and the pairing force.

If we were to take $\langle \Psi \mid H \mid \Psi \rangle$ as the total energy, the contribution of

the quadrupole force would be $-\varkappa\sum_{\mu}(Q_{\mu}^{T})^{2}$. This is a factor of 2 too large, since it effectively counts each particle pair twice. Thus, the expectation value $\langle\Psi\left|\left(H+\frac{\varkappa}{2}\sum_{\mu}(Q_{\mu}^{T})^{2}\right)\right|\Psi\rangle$, regarded as a function of the Q_{μ}^{T}, gives the potential energy surface for quadrupole deformations. This picture is reasonable, if the frequencies associated with changes in the Q_{μ}^{T} are small compared to the single-particle frequencies (adiabatic hypothesis).

Since we prefer to treat only the degrees of freedom associated with particles outside closed shells, we should like to replace (13) by an expression involving the quadrupole tensor, Q_{μ}, of these particles alone. The ratio Q_{μ}^{T}/Q_{μ} has been studied[9][17][18] in several single-particle models for the equilibrium values of Q_{μ}^{T}. We make the additional assumption that this ratio is independent of Q_{μ}. Consequently, Q_{μ}^{T} in (13) can be replaced by Q_{μ}, and $\varkappa$ renormalized.

It is useful to perform a principal axis transformation so that the five degrees of freedom (14b) are replaced by three Eulerian angles specifying the orientation of an intrinsic system of axes, and two parameters describing the shape of the ellipsoid. In this intrinsic system $Q_1 = Q_{-1} = 0$ and $Q_2 = Q_{-2}$. Following[16], we use the shape parameters β and γ defined by*

$$\left.\begin{aligned} Q_0 &\equiv Q \equiv \beta\cos\gamma \\ Q_2 = Q_{-2} &\equiv S/\sqrt{2} \equiv \beta\sin\gamma/\sqrt{2}. \end{aligned}\right\} \tag{15}$$

This definition of β differs from that given in[16] by a factor of dimension (length)^2. Consequently, the single-particle matrix elements can be written in the form

$$\varepsilon_{\nu\omega} = \varepsilon^{0}_{\nu\omega} - \varkappa\beta\left[\cos\gamma\,(q_0)_{\nu\omega} + \sin\gamma\, s_{\nu\omega}\right] = \varepsilon_{\nu}\delta_{\nu\omega}, \tag{16}$$

where

$$s_{\nu\omega} = 1/\sqrt{2}\,[(q_2)_{\nu\omega} + (q_{-2})_{\nu\omega}]. \tag{17}$$

The self-consistency conditions can now be written

$$\left.\begin{aligned} Q &= \sum_{\nu}(q_0)_{\nu\nu}\,2V_{\nu}^{2} \\ S &= \sum_{\nu} s_{\nu\nu}\,2V_{\nu}^{2}. \end{aligned}\right\} \tag{18}$$

These can be taken into account by means of two additional Lagrange multipliers $\hat{\mu}$ and $\hat{\sigma}$. One must thus replace the ε_{ν} in (1) by

* In the following, all the directed quantities refer to the intrinsic axes, unless otherwise specified.

$$\varepsilon_{\nu\omega} = \varepsilon^0_{\nu\omega} - \lambda\delta_{\nu\omega} - \mu\,(q_0)_{\nu\omega} - \sigma s_{\nu\omega} = \varepsilon_\nu \delta_{\nu\omega}, \tag{13^1}$$

where

$$\mu = \varkappa Q + \hat{\mu} \qquad \sigma = \varkappa S + \hat{\sigma}. \tag{19}$$

In order to clarify the role of these Lagrange multipliers it is convenient to return to the original description of the quasi-particle approximation in terms of a variational procedure. The expectation value of $\langle 0 \mid H \mid 0 \rangle$ has to be minimized with respect to the V_ν. The result is a set of V_ν which depend on λ, μ, and σ, which are in turn determined from (12) and (18). Then, if we allow small variations from this set of V_ν, but keep λ, μ, and σ fixed, we have

$$0 = \delta\langle 0 \mid H \mid 0\rangle = \delta\langle 0 \mid H_{\text{spher}} \mid 0\rangle - \mu\delta Q - \sigma\delta S. \tag{20}$$

Thus,

$$\frac{\delta\langle 0 \mid H_{\text{spher}} \mid 0\rangle}{\delta Q} = \mu \qquad \frac{\delta\langle 0 \mid H_{\text{spher}} \mid 0\rangle}{\delta S} = \sigma. \tag{21}$$

The energy for the optimum V_ν, which we take to define the potential energy surface, is

$$E = \langle 0 \mid H_{\text{spher}} \mid 0\rangle + \lambda n - 1/2\,\varkappa Q^2 - 1/2\,\varkappa S^2 \tag{22}$$

(cf. p. 7), and so

$$\frac{\delta E}{\delta Q} = \mu - \varkappa Q = \hat{\mu}; \qquad \frac{\delta E}{\delta S} = \sigma - \varkappa S = \hat{\sigma}. \tag{23}$$

At equilibrium, the Lagrange multipliers $\hat{\mu}$ and $\hat{\sigma}$ therefore vanish, and the Hamiltonian used to generate the wave functions has the same deformation as the one used to calculate the energy.

c) *Calculation of the potential energy surface and mass parameters.*

We seek an expansion of the potential energy as a power series in Q and S. According to (23), it is sufficient to calculate the partial derivatives $\frac{\delta^m \mu}{\delta Q^{m-n}\delta S^n}$ and $\frac{\delta^m \sigma}{\delta Q^{m-n}\delta S^n}$. If we wish to calculate the restoring force we require second derivatives, and in most cases these are conveniently obtained as follows*. For simplicity, we treat only one independent variable which we call R. We write

$$H = H_{\text{spher}} - \varrho R_{\text{op}} \tag{24}$$

$$H_{\text{spher}} = \sum_\nu (\varepsilon^0_\nu - \lambda)\,(c^\dagger_{\nu+} c_{\nu+} + c^\dagger_{\nu-} c_{\nu-}) + H_{\text{pair}}. \tag{25}$$

* This method was suggested by A. Bohr (private communication).

We assume that the problem defined by (25) has been solved within the quasi-particle approximation. Thus we have values of λ, Δ, and the V_ν. R_{op} can then be written

$$\left.\begin{aligned} R_{\mathrm{op}} &= \sum_{\nu} r_{\nu\nu}\, 2V_\nu^2 \\ &+ \sum_{\nu\omega} r_{\nu\omega}\,(U_\nu U_\omega - V_\nu V_\omega)\,(\alpha_\nu^\dagger \alpha_\omega + \beta_\nu^\dagger \beta_\omega) \\ &+ \sum_{\nu\omega} r_{\nu\omega}\,(U_\nu V_\omega + U_\omega V_\nu)\,(\alpha_\nu^\dagger \beta_\omega^\dagger + \beta_\nu \alpha_\omega). \end{aligned}\right\} \tag{26}$$

Treating $-\varrho R_{\mathrm{op}}$ as a perturbation, the new ground state $|\,0'\rangle$ is given in first-order perturbation theory by

$$|\,0'\rangle = |\,0\rangle + \varrho \sum_{\nu\omega} \frac{r_{\nu\omega}(U_\nu V_\omega + U_\omega V_\nu)}{E_\nu + E_\omega}\,|\,\nu\omega\rangle. \tag{27}$$

Here again the value of ϱ is determined from the assigned expectation value of R_{op}, i. e.,

$$R = \langle 0'\,|\,R_{\mathrm{op}}\,|\,0'\rangle = 2\varrho \sum_{\nu,\,\omega} r_{\nu\omega}^2 \frac{(U_\nu V_\omega + U_\omega V_\nu)^2}{E_\nu + E_\omega}. \tag{28}$$

According to (23) and (28), the restoring force $C = \dfrac{\delta^2 E}{\delta R^2}$ is given by

$$C = \frac{\delta \varrho}{\delta R} - \varkappa = \frac{1}{2\sum\limits_{\nu,\,\omega} r_{\nu\omega}^2 \dfrac{(U_\nu V_\omega + U_\omega V_\nu)^2}{E_\nu + E_\omega}} - \varkappa. \tag{29}$$

A simple physical interpretation can be given for the terms in (29). The first one, which tends to preserve the spherical shape, equals the increase in the expectation value of H_{spher} due to the deformation. To second order in R,

$$\langle 0'\,|\,H_{\mathrm{spher}}\,|\,0'\rangle - \langle 0\,|\,H_{\mathrm{spher}}\,|\,0\rangle = \frac{R^2}{4\sum\limits_{\nu,\,\omega} r_{\nu\omega}^2 \dfrac{(U_\nu V_\omega + U_\omega V_\nu)^2}{E_\nu + E_\omega}}. \tag{30}$$

The second term corresponds to the expectation value of the interaction which produces the deformation.

However, some precautions must be taken when using $|\,0'\rangle$ given by (27), since its average particle number differs from that of $|\,0\rangle$. In fact, to first order,

$$\langle 0' \mid n_{\mathrm{op}} \mid 0' \rangle - \langle 0 \mid n_{\mathrm{op}} \mid 0 \rangle = \varrho \Delta^2 \sum_{\nu} \frac{r_{\nu\nu}}{E_\nu^3}. \tag{31}$$

The linear term in (31) can produce a spurious contribution in (30). In order to eliminate this linear term and thus this spurious effect, we must require $\sum_{\nu} \frac{r_{\nu\nu}}{E_\nu^3} = 0$. (A second order variation in the number of particles does not affect (30), since the expectation value of H_{spher} is stationary in the number of particles).

In the Appendix, it is shown that the additional condition $\sum \frac{r_{\nu\nu}\varepsilon_\nu}{E_\nu^3} = 0$ must also be satisfied, due to the requirement that matrix elements of the form $\langle 0' \mid H_{\mathrm{int}} \mid (\nu\omega)' \rangle$ should contain no terms linear in ϱ.

The conditions $\sum_{\nu} \frac{r_{\nu\nu}}{E_\nu^3} = 0$ and $\sum_{\nu} \frac{r_{\nu\nu}\varepsilon_\nu}{E_\nu^3} = 0$ are satisfied for quadrupole vibrations about spherical equilibrium shape. The methods of this section can also be used to study the vibrations about non-zero equilibrium deformations. In this case, the conditions $\sum_{\nu} \frac{s_{\nu\nu}}{E_\nu^3} = 0$ and $\sum_{\nu} \frac{r_{\nu\nu}\varepsilon_\nu}{E_\nu^3} = 0$ are satisfied for γ-vibrations about $\gamma = 0$ or $\gamma = \pi$. However, for β-vibrations they are not satisfied, because the quadrupole operator connects the ground state to the spurious 2-quasi-particle state. Hence, one may not fix λ and Δ and then do the perturbation calculation; one must rather determine first the effect of the perturbation on the single-particle energies and wave functions, and then solve (4) and (12) for λ and Δ. Although we will not need the general expressions so obtained, we give them in the Appendix for completeness.

We calculate the mass parameter, using time-dependent adiabatic pertubation theory (the "cranking" model). (See also eq. (15) of ref.[7]).

$$\left.\begin{aligned} B &= 2\hbar^2 \sum_{\nu\omega} \frac{|\langle \nu\omega \mid \frac{\delta}{\delta R} \mid 0 \rangle|^2}{E_\nu + E_\omega} \\ &= 2\hbar^2 \left(\frac{d\varrho}{dR}\right)^2 \sum_{\nu\omega} \frac{|\langle \nu\omega \mid \left(\frac{\delta}{\delta\varrho}\right) \mid 0 \rangle|^2}{E_\nu + E_\omega}. \end{aligned}\right\} \tag{32}$$

Using the relation $\frac{\delta H}{\delta \varrho} = \left[\frac{\delta}{\delta\varrho}, H\right]$, (32) can be shown to be equivalent to[19]

$$\left.\begin{aligned} B &= 2\hbar^2\left(\frac{d\varrho}{dR}\right)^2 \sum_{\nu,\omega} \frac{|\langle \nu, \omega \,|\, \frac{dH}{d\varrho} \,|\, 0\rangle|^2}{(E_\nu + E_\omega)^3} \\ &= 2\hbar^2\left(\frac{d\varrho}{dR}\right)^2 \sum_{\nu,\omega} \frac{|\langle \nu, \omega \,|\, R_{\mathrm{op}} \,|\, 0\rangle|^2}{(E\nu + E\omega)^3} \\ &= 2\hbar^2\left(\frac{d\varrho}{dR}\right)^2 \sum_{\nu,\omega} \frac{r_{\nu\omega}^2 (U_\nu V_\omega + U_\omega V_\nu)^2}{(E_\nu + E_\omega)^3}. \end{aligned}\right\} \quad (33)$$

The replacement of $\frac{\delta H}{\delta \varrho}$ by $-R_{\mathrm{op}}$ is valid only if λ and Δ have no first order terms in ϱ (cf. eq. (10A)). The formulae for the restoring force and mass parameters of the vibrations simplify if the single-particle quadrupole moment has only diagonal elements (e. g., the harmonic oscillator)

$$C = \frac{1}{\Delta^2 \sum_\nu \frac{r_{\nu\nu}^2}{E_\nu^3}} - \varkappa \quad (34\,\mathrm{a})$$

$$B = \hbar^2 \frac{\sum_\nu \frac{r_{\nu\nu}^2}{E_\nu^3}}{\left(2\Delta \sum_\nu \frac{r_{\nu\nu}^2}{E_\nu^3}\right)^2}. \quad (34\,\mathrm{b})$$

Another simple case arises when the single-particle states are degenerate

$$C = \frac{G\Omega}{2\theta_n \sum_{\nu,\omega} r_{\nu\omega}^2} - \varkappa \quad (35\,\mathrm{a})$$

$$B = \frac{\hbar^2}{2G\Omega\theta_n \sum_{\nu,\omega} r_{\nu\omega}^2} \quad (35\,\mathrm{b})$$

$$\hbar\omega = G\Omega\left(1 - \frac{2\varkappa\theta_n}{G\Omega} \sum_{\nu,\omega} r_{\nu,\omega}^2\right)^{1/2}. \quad (35\,\mathrm{c})$$

Here, $\theta_n = 1 - x_n^2$ and $x_n = \frac{n}{\Omega}$, while Ω is the total number of pairs of states available.

The above adiabatic treatment of the quadrupole vibrations requires the energy of the first vibrational excitation $\hbar\omega$ to be small compared to twice the quasi-particle energy. A different approach to this problem has been given by B. MOTTELSON[10]. He considers particles moving in degenerate states, and coupled by pairing and quadrupole forces. The quadrupole force affects only one of the $I = 2$ two quasi-particle states, whose energy is given

by this model to be exactly the same as (35 c). Here the conditions of validity are complementary to ours, since they effectively imply* a small depression of the vibrational state compared to the 2-quasi-particle energy. Since (35 c) holds at both limits, we may expect it to be a reasonable approximation in between.

III. Expansion of the energy about the spherical equilibrium shape

BELYAEV[7] has already studied the dependence of the nuclear surface energy on an axially symmetric deformation, using a single-particle Hamiltonian with diagonal intrinsic quadrupole moments $(q_0)_{\nu\omega} = (q_0)_{\nu\nu}\,\delta_{\nu,\omega}$, and an assumed density of states. In the following, the simplified case of nucleons moving in a harmonic oscillator shell will be treated**, but the restriction to axial symmetry will be omitted.

In order to calculate the partial derivatives $\frac{\delta^n \mu}{\delta Q^{n-m} \delta S^m}$ and $\frac{\delta^n \sigma}{\delta Q^{n-m} \delta S^m}$, we can proceed as follows: λ and Δ are expanded as power series in the variables μ and σ. It is then possible to construct the power series for E_ν and V_ν^2 and, therefore, the right-hand side of the basic equations (4) and (12). We must put equal to zero the coefficients of the successive powers of μ and σ in the expressions for G and n, since these quantities are independent of the deformation. This provides us with a set of equations from which the coefficients in the expansions for λ and Δ can be derived. These coefficients are inserted in the power series for V_ν^2. The power series for Q and S can then be immediately obtained by using equation (18). After reversing these last two series and performing the necessary differentations, we obtain the following expression for the energy:

$$
\begin{aligned}
E = E_{(\beta=0)} + \frac{C}{2}\beta^2 \\
+\left(\frac{MW}{\hbar}\right)^2 \frac{G\beta^2}{2\theta_n N^2} \Bigg\{ & -\frac{4\beta}{5(1+\alpha)Q_{\max}} x_n \cos 3\gamma + \left(\frac{4\beta}{5(1+\alpha)Q_{\max}}\right)^2 \left(\frac{1}{2} + \frac{3\theta_n}{64}\right) \\
& - \left(\frac{4\beta}{5(1+\alpha)Q_{\max}}\right)^3 x_n \left(\frac{83}{56} - \frac{\theta_n}{2}\right) \cos 3\gamma \\
& + \left(\frac{4\beta}{5(1+\alpha)Q_{\max}}\right)^4 \left[\frac{3635}{448} - \theta_n \frac{77955}{1792} + \theta_n^2 \frac{5535}{128} \right. \\
& \left. + \left(\frac{223}{56} - \theta_n \frac{4979}{224} + \theta_n^2 \frac{1207}{64}\right) \cos^2 3\gamma \right] + \ldots \Bigg\}
\end{aligned}
\qquad (36)
$$

* More precisely, the seniorities of the states mixed by the quadrupole force into the ground state should be small compared to Ω.

** A more general type of shell has been considered by J. M. ARAÚJO (private communication).

$$C = \left(\frac{MW}{\hbar}\right)^2 \frac{G}{\theta_n N^2} - \varkappa. \tag{36 a}$$

β and γ are defined in (15), M is the mass of the nucleon, and W the frequency of the oscillator field. N, which is assumed to be large compared to unity, is the principal quantum number of the oscillator shell. We also define n_z and $n_\perp$ to be the numbers of oscillator quanta along and perpendicular to the z-axis, respectively.

$Q_{\max}$ is the maximum value of Q which can be obtained with a given number of particles in the shell. One gets this value in the "aligned coupling scheme"[2].

α is defined so that αN is the maximum occupied value of $n_\perp$, for prolate deformation, with a given number of particles $n_\perp$ and no pairing force. In consequence, $0 \leq \alpha \leq 1/\sqrt{2}$. For values of $n > \Omega$ ($\alpha > 1/\sqrt{2}$), $Q_{\max}$ occurs for oblate deformation. In this case the previous expression also holds, holes playing the role of our previous particles.

A few comments can be made on equation (36).

1) The γ-dependence of the terms of a given order in β can be understood on the basis of general invariance arguments. The energy of the system must be invariant with respect to rotations. Therefore, it can be expressed as a linear superposition of the solutions of the five-dimensional quadrupole oscillator corresponding to zero total angular momentum. The γ-dependent part of these solutions can be expressed in terms of Legendre polynomials in the variable $\cos 3\gamma$[16]. The solutions for $I = 0$ can be characterized[21] by the quantum numbers (n_β, l), where n_β is the number of quanta for the β-motion and l is an integer that RAKAVY[22] has called the "seniority". It is related to $\mathfrak{N}$, the total number of phonons, by the equation $\mathfrak{N} = 2n_\beta + 3l$.

The γ-independence of the term proportional to β^2 simply reflects the fact that no function of $\cos 3\gamma$ can be formed from linear combinations of quadratic expressions in $\cos\gamma$ and $\sin\gamma$. The only invariant expression that can be made proportional to β^2 is the β-excitation built on the ground state. This wave function is characterized by the quantum numbers (1,0). For 3 phonons only one solution is possible, and is proportional to $\beta^3 \cos 3\gamma$ (0,1). The only allowed $I = 0$ state with 4 phonons is the second β-excitation of the ground state (2,0), which does not depend on γ. Also in the case of 5 phonons only the solution (1,1) appears. It corresponds to the β-excitation of the (0,1) state and, therefore, has the same γ-dependence, namely $\cos 3\gamma$. Two $\mathfrak{N} = 6$ states appear for $I = 0$. The triple β-excitation of the ground state with no γ-dependence (3,0) and the (0,2) state which is proportional to $\beta^6 \cos^2 3\gamma$.

In general, one can predict that terms which contain odd powers of $\cos 3\gamma$ are multiplied by odd powers of β; even powers of $\cos 3\gamma$ are multiplied by even powers of β.

2) The existence of a negative β^3 term* ensures that, for sufficiently small positive values of C, there is a maximum in the expression for the energy as a function of the axially symmetric deformation. It is situated at

$$\beta_{\max} = \frac{5\,(1+\alpha)}{3 \times \varkappa_n} Q_{\max} C. \tag{37}$$

The smallness of C (and therefore of $\beta_{\max}$) allows us to consider only the β^2 and β^3 terms in (36). One can then easily derive (37).

The existence of a maximum ensures the existence of a second minimum, provided the system does not collapse. Thus, the system has started to deform even before reaching the transition point $C = 0$.

It is interesting to note why there do not occur two minima in the curve which BELYAEV used to illustrate the energy of the system as a function of the axially-symmetric deformation. Let us consider a degenerate shell whose levels are split by a deformation in such a way that the final single-particle spectrum is symmetric with respect to the original energy. This system will have no preference for prolate rather than oblate deformations, or vice versa. Thus, no odd powers of β will appear in an expansion of the energy such as (36), because these terms are associated with odd powers of $\cos 3\gamma$, which can distinguish between $\gamma = 0$ and $\gamma = \pi$. In particular, no β^3 term can occur and therefore the sufficient condition for the existence of two minima no longer holds. BELYAEV has found the ground-state equilibrium deformation for a system of this kind (constant density of levels). One should remember, however, that this system has some kind of γ-unstability, because prolate and oblate deformations are equally favoured. Neither does the energy surface for the γ-deformation of an $n_\perp$-subshell in an axially symmetric harmonic oscillator field present two minima.

The density of states of an axially symmetric harmonic oscillator is proportional to the energy; the density in a deformed j-shell is inversely proportional to the magnitude of the magnetic quantum number. In both cases, the equilibrium deformation is such that the density increases with energy. If the shell is less than half filled, this favours prolate deformation

* A discussion of the β^3 terms, including their effect in the kinetic energy, has been made independently by A. KERMAN (to be published).

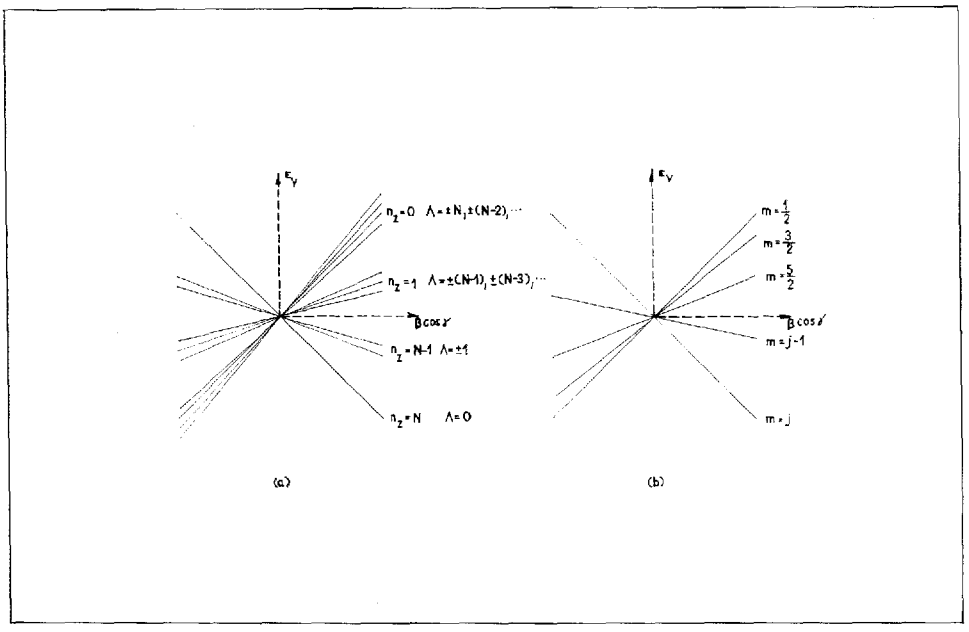

Fig. 1. Level spectra for (a) a harmonic oscillator field of cylindrical symmetry, (b) an axially symmetric field superposed on the spherical field giving rise to a single j-shell.

for the harmonic oscillator and oblate deformation for the j-shell (see Fig. 1). Past the middle of the shell the above arguments apply to the hole states, and thus the roles of prolate and oblate deformations are interchanged.

The most direct consequence of the existence of two minima would be the appearence of a sudden change in the deformation when the second minimum falls below the first. We have seen that the existence of two minima requires a β^3 term, which in turn implies γ-stability. This is consistent with the empirical fact that the transition to deformed nuclei is more abrupt at the beginning of the rare-earth region where the nuclei are γ-stable, than at the end where they approach γ-unstability. More accurate predictions cannot be given at present, because neither the harmonic oscillator nor the j-shell provides a realistic description of the actual single-particle spectra.

3) In the spherically symmetric harmonic oscillator, the consequences of the terms proportional to $\cos 3\gamma$ and $\cos^2 3\gamma$ have been studied by constructing their matrices and diagonalizing them in perturbation theory. The necessary γ-dependent part of the wave functions is given in reference[20]. The term $\cos 3\gamma$ shifts the first 2+ and 4+ states towards the positions that they would occupy in a rotational band. The second 2+ state is pushed

rather high. On the contrary, the $\cos^2 3\gamma$ term tends to bring the second 2+ state below the first 4+ state.

The discussion of the influence of these terms on the transition rates is simplified by the existence of a "γ-parity"[20] which is equal to the parity of l. Any interaction which can be expanded in even powers of $\cos 3\gamma$ preserves a selection rule which forbids the transition from the second 2+ state to the ground state; the odd powers of $\cos 3\gamma$ violate this selection rule.

Most non-deformed, even mass nuclei[23] have their second 2+ level below their first 4+ level; in addition, the transition from the second 2+ level to the ground state is strongly retarded. The previous arguments suggest that both these features can be attributed to the effect of a term proportional to $\beta^6 \cos^2 3\gamma$. One can imagine situations in which the coefficient of the β^3 term would be reduced, for example if the single-particle spectrum is intermediate between those of the harmonic oscillator and the j-shell, or if protons and neutrons are filling opposite ends of similar shells (sec 2). The main effect of a β^4 term would be on the position of the second 0+ state, about which very little is known experimentally.

We have considered only the β- and γ-dependence of the nuclear surface energy. Similar terms in the mass parameter should also be taken into account in a more detailed study of nuclear vibrations.

IV. Gamma vibrations in a deformed harmonic oscillator field

We assume that the system has a prolate axially symmetric equilibrium deformation ($\gamma = 0$), and we study the change in the potential energy for small changes in γ. In this chapter we consider the case of a harmonic oscillator field. Because of the very particular degeneracies associated with this field, we do not expect quantitative agreement with actual nuclei. However, the oscillator gives a first qualitative picture of a realistic nuclear shell, and has the advantage that closed expressions for the vibrational parameters can be obtained. In addition, we assume that $N(= n_z + n_\perp)$ is much greater than unity.

The operator corresponding to the γ-deformation has only diagonal matrix elements in a single-particle representation characterized by the quantum numbers N, $n_\perp$ and n_y.

$$s_{\gamma\gamma} = \frac{\hbar}{MW_\perp} \sqrt{3}\,(n_\perp - 2\,n_y). \tag{38}$$

Here, $W_\perp$ is the characteristic frequency for oscillations perpendicular to the z-axis.

We can therefore apply equations (34). One can easily evaluate the vibrational coefficients in two simple cases.

a) The deformation is so great compared to the pairing force that the problem reduces to coupled particles in the $n_\perp$-subshells. The necessary condition for the validity of this approximation is that the two quasi-particle energies are small compared to the distance between 2 subshells, i. e.,

$$G_\perp n_\perp \ll 3\varkappa Q_{\text{eq.}} \frac{\hbar}{MW}, \tag{39}$$

where $G_\perp$ is the effective strength of the pairing force which acts between particles belonging to the $n_\perp$-subshell.

Due to renormalization effects of the other $n_\perp$-subshells, $G_\perp$ is greater than the G to be used if the entire N-shell is treated. We can calculate the renormalization by means of a procedure similar to those employed in[7] and [10] to account for the influence on a particular unfilled shell of the presence of other shells. Let us call $G_{\nu\omega}$ $(= G)$ the pairing force matrix element corresponding to a scattering of a pair of particles from the states $(\nu+,\ \nu-)$ to the states $(\omega+,\ \omega-)$. According to[10],

$$\left.\begin{aligned}(G_\perp)_{\nu\nu'} &= G_{\nu\nu'} + \sum_{\omega=n_\perp+1}^{\omega=N} \frac{G_{\nu\omega}G_{\omega\nu'}}{2(\varepsilon_\omega-\varepsilon_\nu)} + \sum_{\omega=n_\perp-1}^{\omega=0} \frac{G_{\omega\nu'}G_{\nu\omega}}{2(\varepsilon_\nu-\varepsilon_\omega)} + \dots \\ &= G_{\nu\nu'}\left\{\frac{1}{1-\dfrac{G}{6\varkappa Q_{\text{eq}}}\dfrac{MW}{\hbar}\left[\sum_{\omega=1}^{\omega=N-n_\perp}\dfrac{\omega+n_\perp}{\omega} + \sum_{\omega=N_\perp-1}^{\omega=n_\perp}\dfrac{n_\perp-\omega}{\omega}\right]}\right\}.\end{aligned}\right\} \tag{40}$$

By performing the above summations, and using condition (39), we get

$$G_\perp = G\left\{\frac{1}{1-\dfrac{Gn_\perp}{6\varkappa Q_{\text{eq}}}\dfrac{MW}{\hbar}ln\left[n_\perp(N-n_\perp)\right]}\right\}. \tag{41}$$

In this case, a) the simple expressions (35), corresponding to the "degenerate model", can be used for the vibrational parameters

$$\left.\begin{aligned}C &= \frac{G_\perp}{2\theta'_n n_\perp^2}\left(\frac{MW_\perp}{\hbar}\right)^2 - \varkappa \\ B &= \frac{\hbar^2}{2}\frac{1}{G_\perp n_\perp^4 \theta'_n}\left(\frac{MW_\perp}{\hbar}\right)^2,\end{aligned}\right\} \tag{42}$$

where $\theta'_n = \frac{n'}{n_\perp}\left(2 - \frac{n'}{n_\perp}\right)$, and n' is the number of particles in the $n_\perp$ subshell. If $\theta_n^{0'}$ is the value of θ'_n for which the axial shape is no longer stable, (42) and (36) imply that

$$\theta^0_{n'} = \frac{G_\perp}{2\varkappa n_\perp^2}\left(\frac{MW_\perp}{\hbar}\right)^2 = \frac{\theta^0_n}{2}\frac{G_\perp}{G}\frac{N^2}{n_\perp^2}\left(\frac{W_\perp}{W}\right)^2. \tag{43}$$

Neglecting the renormalization effect expressed by (41), we see that $\theta^0_{n'}$ and θ^0_n are roughly of the same order of magnitude. This implies that the fraction of nuclei with axially symmetric equilibrium deformation is of the same order of magnitude as the fraction of nuclei which are spherical.

Nuclei with $0 < \theta'_n < \theta^0_{n'}$ have an axially symmetric stable deformation. The ratio between the frequency of the γ-vibrations and the gap is

$$\frac{\hbar\omega_\gamma}{G_\perp n_\perp} = \left[1 - 2\varkappa\left(\frac{\hbar}{MW_\perp}\right)^2 n_\perp^2 \frac{\theta'_n}{G_\perp}\right]^{1/2}. \tag{44}$$

Here the adiabatic condition implies that $\frac{\hbar\omega_\gamma}{G_\perp n_\perp} \ll 1$.

For nuclei in the region of transition between axially symmetric and γ-deformed nuclei, the potential energy surface does not exhibit two minima (cf. p. 14).

b) We can also easily treat the deformed harmonic oscillator field if we replace summations over the variable $n_\perp$ by integrations, using a level density proportional to the single-particle energies (see p. 15). This is a particular case of the level density used by BELYAEV in his investigation of axial deformations. Equations (46)-(54) are a transcription of some of his results into our notation.

The single-particle energies ε_ν can be labelled by $n_\perp$. With a convenient choice of the zero-point energy, they are given by

$$\varepsilon_\nu = \varepsilon_{n_\perp} = 3\mu\frac{\hbar}{MW}n_\perp - \lambda. \tag{45}$$

W is the frequency of the harmonic oscillator. In neglecting the difference between $W_\perp$ and W_z we make an error of the order of the deformation, i. e., of order $A^{-1/3}$ or N^{-1} for the equilibrium deformation. This can be neglected in our limit $N \gg 1$.

A new parameter η characterizing the deformation is introduced:

$$\eta = \frac{3\,\hbar\mu}{MWG\bar{\varrho}}, \tag{46}$$

where $\bar{\varrho}$ is defined by the condition

$$\int_0^N \frac{\varrho(n_\perp)\,dn_\perp}{E_{n_\perp}} = \bar{\varrho}\int_0^N \frac{dn_\perp}{E_{n_\perp}}. \tag{47}$$

In the axially symmetric harmonic oscillator, the level density, $\varrho(n_\perp)$, equals $n_\perp$. Therefore, the parameters used by BELYAEV in order to characterize the level density are here

$$\varrho_0 = \frac{N}{2}; \qquad \xi = 1. \tag{48}$$

The parameters λ and Δ are always determined from (4) and (12):

$$\lambda = \frac{3\,\hbar\mu N}{2\,MW}(1 - x_n \coth\eta), \tag{49}$$

$$\Delta^2 = \left(\frac{3\,\hbar\mu N}{2\,MW}\right)^2 \frac{(1-x_n^2)}{\sin\hbar^2\eta}, \tag{50}$$

where x_n measures the number of particles in the shell and is also a slow-varying function of η

$$x_n \equiv 2\left\{\frac{1-\frac{n}{\Omega}-\frac{\gamma}{2}}{1+\left[1-2\,\gamma\left(1-\frac{n}{\Omega}\right)+\gamma^2\right]^{1/2}}\right\} \tag{51}$$

with

$$\gamma \equiv \coth\eta\left(1-\frac{2\,\eta}{\sinh 2\,\eta}\right). \tag{52}$$

One can eliminate $\bar{\varrho}$ in (46) by its expression as a function of ϱ_0, ξ, x_n and η

$$\frac{\mu\hbar}{MW} = \frac{GN}{6}\left[\eta - x_n(\eta\coth\eta - 1)\right]. \tag{53}$$

The quadrupole moment Q is given by

$$\left.\begin{aligned} Q &= \frac{\hbar}{MW}\sum_{n_\perp}(2N-3n_\perp)\,n_\perp\,2V_{n_\perp}^2 \\ &= \frac{\hbar}{MW}\sum_{n_\perp}(-2N+3n_\perp)\,n_\perp\frac{\varepsilon_{n_\perp}}{E_{n_\perp}} \\ &\simeq \frac{\hbar}{MW}\frac{N^3}{4}(1-x_n^2)\left[\coth\eta - x_n\left(1+\frac{3}{\sinh^2\eta}\right)+\frac{\eta}{\sinh^2\eta}(3\,x_n\coth\eta-1)\right]. \end{aligned}\right\} \tag{54}$$

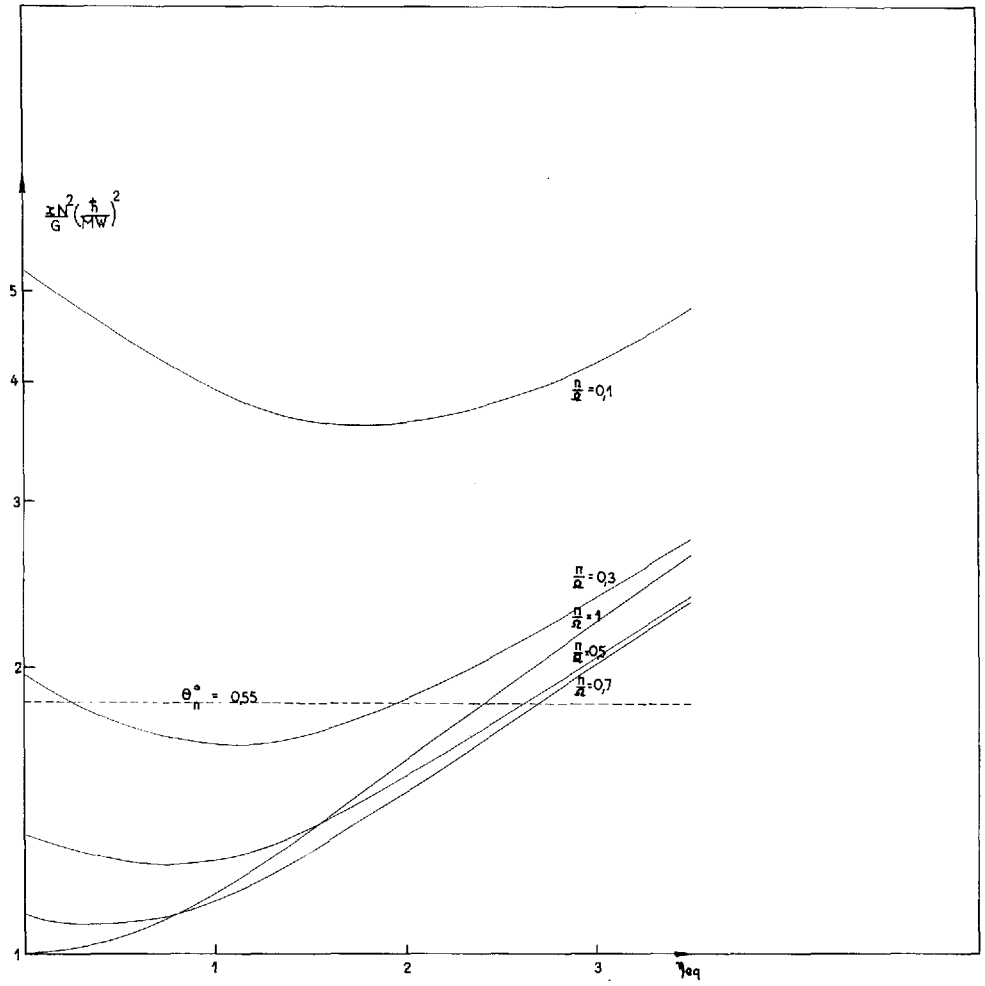

Fig. 2. The right-hand side of equation (55) represented as a function of $\eta = \eta_{\mathrm{eq}}$, for several values of $\frac{n}{\Omega}$. The intersection of these curves with the horizontal dashed line yields the values of η for which the energy is a minimum if the $\frac{\varkappa N^2}{G}\left(\frac{\hbar}{MW}\right)^2$ is such that $\theta_n^0 = 0.55$.

Equations (53) and (54), plus the condition of the vanishing of the Lagrange multiplier at equilibrium ($\mu = \varkappa Q$), define an implicit equation for the equilibrium value of η.

$$\frac{\varkappa N^2}{G}\left(\frac{\hbar}{MW}\right)^2 = \frac{4\sinh^2\eta\,[\eta - x_n(\eta\coth\eta - 1)]}{3\,(1-x_n^2)\,[\sinh 2\eta - 2x_n(3+\sinh^2\eta) + 2\eta\,(3x_n\coth\eta - 1)]}. \quad (55)$$

At $\eta = 0$ the right-hand side becomes indeterminate. However, (36) shows that in this case there is always an extremum in the potential energy surface. The right-hand side of (55) is plotted as a function of η in Fig. 2 for $\frac{n}{\Omega} =$ 0.1, 0.3, 0.5, 0.7, and 1. It is seen that, for a particular $\frac{n}{\Omega}$ and sufficiently small $\frac{\varkappa N^2}{G}\left(\frac{\hbar}{MW}\right)^2$, there are no solutions to (55) and thus the only extremum is the minimum at $\eta = 0$. For larger $\frac{\varkappa N^2}{G}\left(\frac{\hbar^2}{MW}\right)$, there are two values of η satisfying (55), the lower corresponding to a maximum and the upper to a second minimum (cf. discussions on p. 14). For still larger $\frac{\varkappa N^2}{G}\left(\frac{\hbar^2}{MW}\right)^2$, there is only one solution to (55), and thus only one minimum (the extremum at $\eta = 0$ is now a maximum). One can also see that for each $\frac{n}{\Omega} \neq 1$ there is a minimum value for stable deformation.

The expressions $\sum_\nu \frac{s_{\nu\nu}^2}{E_\nu^3}$ and $\sum_\nu \frac{s_{\nu\nu}^2}{E_\nu^5}$, needed in (34) for the evaluation of the restoring force and mass parameter, are to be calculated for the value of η which corresponds to the equilibrium situation. We find

$$\left.\begin{aligned}\sum_\nu \frac{s_{\nu\nu}^2}{E_\nu^a} &= \left(\frac{\hbar}{MW}\right)^2 3\sum_{n_\perp=0}^{n_\perp=N} \sum_{n_y=0}^{n_y=n_\perp} \frac{(n_\perp - 2\,n_y)^2}{E_{n_\perp}^a} \\ &\simeq \left(\frac{\hbar}{MW}\right)^2 \sum_{n_\perp=0}^{n_\perp=N} \frac{n_\perp^3}{E_{n_\perp}^a} \\ &\simeq \left(\frac{\hbar}{MW}\right)^2 \int_0^N \frac{n_\perp^3\, dn_\perp}{E_{n_\perp}^a}\end{aligned}\right\} \quad (56)$$

$$\left.\begin{aligned}\sum_\nu \frac{s_{\nu\nu}^2}{E_\nu^3} = \left(\frac{\hbar}{MW}\right)^2 &\frac{8}{N^2 G^3 [\eta - x_n(\eta\coth\eta - 1)]^3 [1 + (1 - x_n^2)\sinh^2\eta]} \\ &\{x_n [3 + (4 - 3\,x_n^2)\sinh^2\eta - (3 + x_n^2)\sinh^4\eta] \\ &+ \sinh\eta\cosh\eta\,(-3 + (1 + 3\,x_n^2)\sinh^2\eta] \\ &+ 3\,\eta\,(1 - x_n\coth\eta)\,[1 + (1 - x_n^2)\sinh^2\eta)]\}\end{aligned}\right\} \quad (57)$$

$$\left.\begin{aligned}\sum_\nu \frac{s_{\nu\nu}^2}{E_\nu^5} = \left(\frac{\hbar}{MW}\right)^2 &\frac{256\sinh^5\eta}{3N^6 G^5 [\eta - x_n(\eta\coth\eta - 1)]^5 [1 + (1 - x_n^2)\sinh^2\eta]^3} \\ &\{\cosh\eta\,[3 + (4 + 8\,x_n^2)\sinh^2\eta + (1 + 2\,x_n^2 - 3\,x_n^4)\sinh^4\eta] \\ &- x_n \sinh\eta\,[9 + 12\sinh^2\eta + (3 - 2\,x_n^2 - x_n^4)\sinh^4\eta]\}.\end{aligned}\right\} \quad (58)$$

In the limit of vanishing G, $\eta \to \infty$ and equations (50), (51), (55), (57), and (58) reduce to

$$\Delta^2 = \frac{N^4 G^2}{16} \frac{\eta^2}{\sinh^2 \eta} (1 - x_n)^3 (1 + x_n). \tag{50^1}$$

$$x_n = \frac{1 - \frac{2n}{\Omega}}{1 + \left[\frac{2n}{\Omega}\right]^{1/2}} \tag{51^1}$$

$$\lim_{\eta \to \infty} G[\eta - x_n(\eta - 1)] = \frac{3 \varkappa N^2}{2} \left(\frac{\hbar}{MW}\right)^2 (1 - x_n)^2 (1 + x_n) \tag{55^1}$$

$$\sum_\nu \frac{s_{\nu\nu}^2}{E_\nu^3} = \frac{8}{N^2 G^3} \left(\frac{\hbar}{MW}\right)^2 \frac{\sinh^2 \eta}{\eta^3 (1 - x_n^2)} \tag{57^1}$$

$$\sum_\nu \frac{s_{\nu\nu}^2}{E_\nu^5} = \frac{256}{3 N^6 G^5} \left(\frac{\hbar}{MW}\right)^2 \frac{\sinh^4 \eta}{\eta^5 (1 - x_n)^2 (1 - x_n^2)^2}. \tag{58^1}$$

According to (34), the restoring force for γ-vibrations is then given by

$$C = 2 \varkappa \frac{(1 + 2 x_n)}{1 - x_n}, \tag{59}$$

which implies that the axial symmetry is preserved until $x_n = -1/2$, which corresponds to $\frac{n}{\Omega} = \frac{9}{8}$. Thus the first half of the shell $\left(\frac{n}{\Omega} < 1\right)$ has stable prolate deformation.

The mass parameter for the γ-vibrations diverges as $\eta \to \infty$. However, the significant quantity is the ratio between the energy of the γ-vibration and 2Δ. This ratio remains finite, and is given by

$$\frac{\hbar w_\gamma}{2\Delta} = \left[\frac{1 + 2 x_n}{1 + x_n}\right]^{1/2}. \tag{60}$$

For non-zero G, we must solve (55) for η and then evaluate $\frac{\hbar w_\gamma}{2\Delta}$, using (50), (51), (57), and (58). The constant $\frac{\varkappa N^2}{G} \left(\frac{\hbar}{MW}\right)^2$ has been chosen so that $\theta_n^0 = 0.55$, corresponding to a situation in which the spherical shape becomes unstable when the $\frac{n}{\Omega} = \frac{1}{3}$. The results are shown in Fig. 3. The corresponding curve for oblate deformation is obtained by reflecting the curve for prolate deformation about the line $\frac{n}{\Omega} = 1$.

It is seen that the adiabatic condition for γ-vibrations is satisfied for $0.5 < \frac{n}{\Omega} < 1.5$. For prolate (oblate) deformation the frequency of the vibration decreases as the number of particles (holes) increases. Fig. 3 also

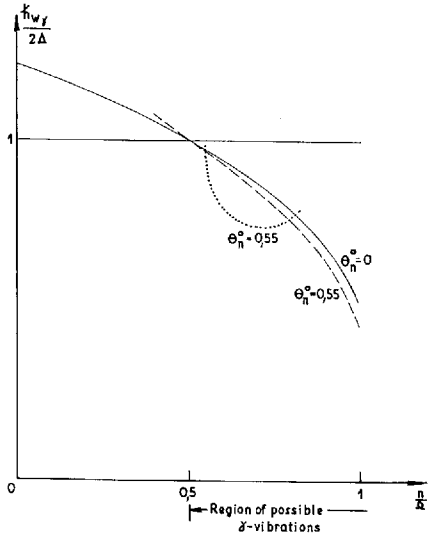

Fig. 3. The ratio between the energy of the γ-vibrations and twice the value of s plotted as a function of $\frac{n}{\Omega}$, for $\theta_n^0 = 0$ ($G = 0$) and $\theta_n^0 = 0.55$. The full and dashed lines represent the two cases in which the calculations were done by replacing the summations by integrations. The dotted line represents the result of the calculations done without this approximation, for $\theta_n^0 = 0.55$ and n such that $n_\perp = 4$ is at the Fermi surface for $G = 0$.

shows that the ratio $\frac{\hbar\omega_\gamma}{2\Delta}$ is not significantly affected by the presence of the pairing force.

If $G \to 0$, so that $\eta \to \infty$, we might expect to approach the situation dealt with in a) above. Nevertheless, the fact that Fig. 3 shows no subshell effects implies that the two methods do not lead to the same result. In fact, for fixed N, the validity of method a) places an upper limit on G (see (39)), whereas the validity of method b) places a lower limit on G. Evidently these regions of validity do not overlap. It is probable that the actual nuclear case is better represented by method b). On the one hand, the reduction in the observed moments of inertia compared to the rigid values implies a mixing by the pairing force of different $n_\perp$-subshells. On the other hand, performing the sums in (4), (12), (54), (56), and (57) exactly for $\theta_n^0 = 0.55$,

$N = 7$, and n such that $n_\perp = 4$ is at the Fermi surface for $G = 0$, leads to the dotted curve in Fig. 3. Although subshell effects do appear, the average ratio agrees well with the result given by method b).

V. Gamma vibrations in a realistic shell model

In the previous section we found that the occurrence of low-energy γ-vibrations requires states with high values of $n_\perp$. However, the oscillator model has very special features, in particular the degeneracy of the $n_\perp$-subshell. Before attempting a detailed comparison with experiment we must give up these special features and make the single-particle Hamiltonian more realistic.

The Nilsson model[24] has been very succesful in explaining the properties of odd-particle states in deformed nuclei[15]. The nucleons are supposed to be in states very similar to those of a deformed harmonic oscillator. The states are labelled by $(N, n_z, \Lambda, \Lambda+\Sigma)$. N and n_z have the same meaning as before, and Λ and Σ are the components, along the symmetry axis, of the orbital and spin angular momenta, respectively. However, this model differs essentially from the harmonic oscillator in that states with the same n_z are no longer degenerate.

We must now consider nuclei with both neutrons and protons outside closed shells. Apart from some very exceptional cases, there are no nuclei in the deformed region in which an external neutron and proton are occupying time-reversed states. Thus, the pairing force we have been using (2) will not couple the neutrons and protons. They will, however, be coupled by the deformed field. The matrix for the single-particle neutron Hamiltonian is, in the Nilsson representation,

$$(\varepsilon_n)_{\nu\omega} = (\varepsilon_n^N)_\nu \, \delta_{\nu\omega} - (\varkappa_n S_n + \varkappa_{np} S_p)\,(s_n)_{\nu\omega}. \tag{61}$$

A corresponding expression holds for protons. The coupling constants $\varkappa_n$, $\varkappa_p$, $\varkappa_{np}$ and $\varkappa_{pn}$ are to be determined, in principle, by the isotopic spin dependence of the nuclear force plus renormalization effects. It will be assumed in the following that $\varkappa_n = \varkappa_p$ and $\varkappa_{np} = \varkappa_{pn}$. The ε_ν^N are the single-particle energies calculated by NILSSON*. They already contain the terms depending on the axially symmetric part of the deformation. The subscripts n and p indicate neutrons and protons, respectively.

* Relatively small shifts will be made in the energy of some of Nilsson's levels in order to get closer agreement with empirical level ordering in odd-mass nuclei (cf. p. 30).

One has then to solve the equations (4) and (12) for neutrons and protons separately*. The neutron-proton coupling is expressed by the terms $-\varkappa_{pn} S_n (S_p)_{op}$ and $-\varkappa_{np} S_p (S_n)_{op}$. In the absence of these terms, we can calculate the vibrational parameters for separate neutron and proton γ-vibrations**. The problem is then equivalent to that of two coupled harmonic oscillators

$$H_\gamma = \frac{B_n}{2}\dot{S}_n^2 + \frac{B_p}{2}\dot{S}_p^2 + \frac{C_n}{2}S_n^2 + \frac{C_p}{2}S_p^2 - \varkappa_{np} S_p S_n . \tag{62}$$

The last term contains the usual factor of 1/2.

We can now decouple the oscillators by transforming to normal coordinates. The lower eigenfrequency is given by

$$\hbar w_\gamma = \left[\frac{1}{2}\left\{\frac{C_p}{B_p} + \frac{C_n}{B_n} - \left[\left(\frac{C_p}{B_p} - \frac{C_n}{B_n}\right)^2 + \frac{4\varkappa_{np}^2 \hbar^2}{B_p B_n}\right]^{1/2}\right\}\right]^{1/2}. \tag{63}$$

We can also calculate the probability of the electric quadrupole transition connecting the first γ-vibrational state with the ground state. For this purpose, it is convenient to regard the γ-vibration as a superposition of two travelling waves[22], Q_2 and Q_{-2}, with definite angular momentum projections along the symmetry axis, and with the same vibrational parameters. The operator $\mathfrak{M}(E2,\mu)$ responsible for the $E2$ transitions[14] is related to Q_μ by

$$\mathfrak{M}_{n,p}(E2,\mu) = \frac{1}{4}\sqrt{\frac{5}{\pi}}\, e_{np} Q_\mu^* , \tag{64}$$

where $e_{n,p}$ is the effective electrical charges carried by the neutron or the proton, respectively. Using eq. (V.34) of ref.[14] the square of the transition matrix element for a single oscillator is found to be

$$|\langle 2|\mathfrak{M}(E2,2)|0\rangle|^2 = \frac{5e^2}{16\pi}\frac{\hbar}{2[BC]^{1/2}}. \tag{65}$$

For the coupled harmonic oscillators it is

$$|\langle 2|\mathfrak{M}(E2,2)|0\rangle|^2 = \frac{5}{64\pi}\frac{\hbar}{w_\gamma}\left[\frac{e_p^2}{B_p}(1-\cos b) + \frac{e_n^2}{B_n}(1+\cos b) + \frac{e_p e_n}{\sqrt{B_p B_n}}\sin b\right], \tag{66}$$

where

$$\operatorname{tg} b = \frac{2\varkappa_{np}\sqrt{B_p B_n}}{(B_n C_p - B_p C_n)}. \tag{67}$$

* Some of the wave functions were kindly supplied by S. G. Nilsson; others were derived in collaboration with Z. Szymański.

** As the operator x^2-y^2 is not diagonal in the Nilsson representation, we have to use (29) and (33).

The reduced transition probability from the ground state to the first γ-vibrational level may be written (cf. eq. V.33 and note 175 of ref. 14)

$$B(E2; 0 \to 2) = 2 \,|\, \langle 2 \,|\, \mathfrak{M}(E2, 2) \,|\, 0 \rangle \,|^2. \tag{68}$$

We must now discuss the choice of the parameters G_n, G_p, $\varkappa_n$, $\varkappa_{np}$, e_n and e_p*.

In the deformed region, there is no clear distinction between filled and unfilled shells. Therefore, one has no definite prescription for the states into which the pairing force is allowed to scatter. However, states which lie far from the Fermi level contribute to the wave function only through a renormalization of G[7] [10]. Therefore, we have chosen to allow the pairing force to scatter only amongst the 24 states nearest to the Fermi level.

One has to choose an effective value of G_n and G_p such that $2\Delta_n$ and $2\Delta_p$ reproduce the average differences between the neutron and proton binding energies of even and odd-mass nuclei. Furthermore, the predicted quasi-particle excitations should be tested with experimental data. However, one should expect some shiftings due to quasi-particle interactions, to the blocking of some states near the Fermi surface, etc. Therefore, the empirical quasi-particle energies give only a lower limit on the value of G.

Finally we have chosen a value of $G_n = \frac{26.5}{A}$ Mev and of $G_p = \frac{32.1}{A}$ Mev.

The values of $\varkappa_n$ and $\varkappa_{np}$ enter into the calculation of the ground state quadrupole moment. The method used here is analogous to that used in the derivation of (55).

The part of Nilsson's potential responsible for the deformation is

$$-\frac{\delta M W^2}{3}(2z^2 - x^2 - y^2). \tag{69}$$

Comparing (69) with the corresponding term in our single-particle Hamiltonian (13[1]), we get

$$\frac{\delta M W^2}{3} = \mu = \varkappa Q + \hat{\mu}. \tag{70}$$

Using the Nilsson single-particle energies corresponding to a given value of δ, we solve (4) and (12) and thus obtain the electric Q_e and mass Q quadrupole moments as a function of δ. The inverse of the first function enables us to determine δ_{eq} from the observed equilibrium electric quadrupole

* The determination of these constants is only outlined here. It is given with more details in[25].

moment. At equilibrium, $\hat{\mu}$ vanishes and thus the value of $\varkappa$ which would yield these values of δ_{eq} and $Q(\delta_{\text{eq}})$ is given by

$$\varkappa = \frac{MW^2}{3} \frac{\delta_{\text{eq}}}{Q(\delta_{\text{eq}})}. \tag{71}$$

In the rare-earth region, the observed electric quadrupole moments are reproduced by*

$$\varkappa = 122 A^{-5/3} \left(\frac{MW}{\hbar}\right)^2 \text{Mev}. \tag{72}$$

In the calculation of Q_e and Q, V_ν has been chosen to be unity for the states below the selected 24, and zero for those above. The single-particle mass quadrupole moment is given by**

$$\left.\begin{aligned} q_{\nu\omega} &= \frac{\hbar}{M} \langle \nu \mid \left(\frac{2z'^2}{W_z} - \frac{x'^2 + y'^2}{W_\perp}\right) \mid \omega \rangle \\ &= \frac{\hbar}{MW} \left\{ \langle \nu \mid (2z'^2 - x'^2 - y'^2) \mid w \rangle \right. \\ &\left. + \frac{\delta}{3} \left[\left(1 + \frac{3}{2}\delta + \ldots\right) \langle \nu \mid (2z'^2 - x'^2 - y') \mid w \rangle + (2 + \delta + \ldots) \left(N + \frac{3}{2}\right) \right] \right\} \end{aligned}\right\} \tag{73}$$

where $X'^2 = \frac{MW_\perp}{\hbar} x^2$; $y'^2 = \frac{MW_\perp}{\hbar} y^2$; $z'^2 = \frac{MW_z}{\hbar} z^2$.

Use has been made of the relations[24]

$$W_z^2 = W^2 \left(1 - \frac{4}{3}\delta\right) \qquad W_\perp^2 = W^2 \left(1 + \frac{2}{3}\delta\right). \tag{74}$$

It has been verified that the contribution to the total quadrupole moment from the terms multiplied by δ in (73) is equal to the contribution due to the first term. In other words, the same results could be obtained by using a renormalized value of $\varkappa$ equal to twice[9] the value given in (72), and using for the single-particle quadrupole matrix elements the value given by the first term in (73). The coupling parameter $\varkappa$, calculated in this way, is to be considered here as an average value of $\varkappa_n$ and $\varkappa_{np}$.

We are going to calculate the energy and transition probability for the γ-vibrations in three cases, namely $\varkappa_n = \varkappa_{np} = x$; $6\varkappa_n = 2\varkappa_{np} = 3\varkappa$ and

* The $A^{-7/3}$ dependence of $\varkappa$ has been pointed out by BELYAEV[7].

** The single-particle matrix elements have been obtained using the expression of the wave functions in terms of the asymptotic representation[26].

$\varkappa_n = \varkappa_{np} = 1.3\,\varkappa$. The value of x to be used is equal to twice that given in (72). The single-particle matrix element will be

$$s_{\nu\omega} = \frac{\hbar}{MW}\sqrt{3}\,\langle \nu \mid (x'^2 - y'^2) \mid \omega \rangle. \tag{75}$$

Determination of e_n and e_p. It was found above that renormalization effects doubled the mass quadrupole moment. We have used for the additional charge due to these effects a value of Z/A times the additional mass. Accordingly,

$$e_p = e\,(1 + Z/A) \qquad e_n = e\,Z/A, \tag{76}$$

where e is the charge of a free proton.

Results of the calculations. Table I a contains the value of the summations $\sum_1 = \sum_{\nu,\,\omega} \frac{s^2_{\nu\omega}\,(U_\nu V_\omega + U_\omega V_\nu)^2}{E_\nu + E_\omega}$ and $\sum_3 = \sum_{\nu,\,\omega} \frac{s^2_{\nu\omega}\,(U_\nu V_\omega + U_\omega V_\nu)^2}{(E_\nu + E_\omega)^3}$, calculated for some neutron numbers and for the deformation listed in column 1. Columns 4, 5 and 6 contain the restoring force for the neutron vibration, assuming $\varkappa_n = \varkappa_{np} = \varkappa$; $6\,\varkappa_n = 2\,\varkappa_{np} = 3\,\varkappa$ and $\varkappa_n = \varkappa_{np} = 1.3\,\varkappa$, respectively. Column 7 lists the mass parameter. Table I b is the analogous table corresponding to protons.

Table II contains the predicted energy of the first γ-vibrational level and the experimental value.

The two first calculations show that the value of the energy of the first vibrational level does not depend on the ratio $\varkappa_n/\varkappa_{np}$. At the beginning of the deformed region the predicted energies are about 80 per cent greater than the empirical ones. There is, however, a correlation between the empirical and theoretical trends (i. e., decrease in the energy for Er^{166}). This decrease is due mainly to the relative large values of $n_\perp$ for the states which come near to the Fermi energy.

At the end of the deformed region, the predicted trends and order of magnitude of the energy are in good agreement with the experimental values. However, a detailed comparison is hindered in the region of W and Os by the uncertainty in the parameters used. The predicted energies are rather sensitive to the position of the (5101/2) and (5123/2) neutron levels. In the calculations, these levels have been depressed by 250 kev in order to fit the spectrum of W^{183}. Calculations with the original Nilsson energies would decrease the energy of the γ-vibrations for 112 neutrons and increase it for 110 neutrons (keeping $\delta = 0.20$) and would thus give a somewhat better fit. Furthermore, the experimental evidence on the value of δ is not

TABLES I a and I b. The rare-earth region. The units of $\sum_1$ are $\left(\frac{\hbar}{MW}\right)\times \mathrm{Mev}^{-1}$. Those of $\sum_3$ are $\left(\frac{\hbar}{MW}\right)^2\times \mathrm{Mev}^{-3}$. The units of C, C' and C'' are the inverse of those of $\sum_1$. Column C refers to the case $\varkappa_n = \varkappa_{np} = \varkappa$; column C' to the case $3\varkappa_n = \varkappa_{np} = \frac{3\varkappa}{2}$, and column C'' to the case $\varkappa_n = \varkappa_{np} = \varkappa\, 1.3$. The units of B are $(MW)^2\times \mathrm{Mev}^{-1}$.

TABLE I a.

N	δ	Σ_1	Σ_3	C_n	C_n'	C_n''	B_n
90	0.25	39.8	3.38	0.0070	0.0098	0.0053	0.00107
	0.30	38.1	3.47	75	103	58	119
92	0.30	34.3	3.03	92	119	76	129
94	0.30	34.5	3.94	93	119	77	165
98	0.30	38.6	5.29	81	105	66	177
100	0.30	37.2	4.32	88	111	74	156
106	0.25	54.5	9.22	48	70	36	156
108	0.20	60.6	10.81	41	62	28	147
110	0.20	67.7	14.0	33	53	21	153
112	0.15	75.7	15.8	26	46	15	138
	0.20	68.2	14.2	34	53	22	152
114	0.15	74.1	14.4	28	48	17	131

TABLE I b.

Z	δ	Σ_1	Σ_3	C_p	C_p'	C_p''	B_p
62	0.25	24.8	2.01	0.0145	0.0173	0.0128	0.00163
	0.30	22.2	1.74	169	197	151	176
64	0.30	24.9	2.85	147	173	131	230
66	0.30	31.1	5.74	109	135	94	297
68	0.30	30.8	5.65	114	138	99	298
70	0.30	21.0	1.95	191	215	177	221
72	0.25	26.2	1.98	147	169	133	144
74	0.15	40.0	5.14	84	105	72	161
	0.20	34.3	3.82	105	126	93	162
76	0.15	44.2	6.95	74	93	62	178
	0.20	43.2	6.65	76	95	64	176

TABLE II. The rare-earth region. Columns 2, 3 and 4 are in Mev. They correspond to the case $\varkappa_n = \varkappa_{np} = \varkappa$; $3\,\varkappa_{np} = \frac{3\,\varkappa}{2}$ and $\varkappa_n = \varkappa_{np} = \varkappa\ 1.3$, respectively. Column 6 lists the experimental energy.

Nucleus	δ	$(\hbar w_\gamma)_{\text{th}}$	$(\hbar w'_\gamma)_{\text{th}}$	$(\hbar w''_\gamma)_{\text{th}}$	$(\hbar w_\gamma)_{\text{exp.}}$ [27) 28)]
$_{62}Sm^{152}_{90}$	0.25	1.79	1.84	0.77	1.092
	0.30	1.91	1.97	1.18	
Sm^{154}_{92}	0.30	2.11	2.14	1.50	
$_{64}Gd^{156}_{92}$	0.30	1.90	1.91	1.30	1.152
Gd^{158}_{94}	0.30	1.81	1.82	1.26	1.182
$_{66}Dy^{160}_{94}$	0.30	1.45	1.46	0.89	0.964
Dy^{164}_{98}	0.30	1.38	1.38	0.78	
$_{68}Er^{166}_{98}$	0.30	1.43	1.43	0.88	0.787
Er^{168}_{100}	0.30	1.52	1.52	1.03	0.822
$_{70}Yb^{170}_{100}$	0.30	2.05	2.10	1.64	
$_{72}Hf^{178}_{106}$	0.25	1.44	1.56	0.71	
$_{74}W^{182}_{108}$	0.20	1.16	1.24	—	1.222
W^{184}_{110}	0.20	0.92	1.09	—	0.903
W^{186}_{112}	0.15	0.65	0.76	—	0.730
	0.20	1.02	1.10	—	
$_{76}Os^{186}_{110}$	0.20	0.75	0.81	—	0.768
Os^{188}_{112}	0.15	0.53	0.64	—	0.628
	0.20	0.80	0.85	—	
Os^{190}_{114}	0.15	0.66	0.73	—	0.558

so precise for the W and Os isotopes as in other rare-earth nuclei. Table II indicates good agreement for W^{184}, using $\delta = 0.20$, and for W^{186} using a value of δ intermediate between 0.15 and 0.20.

In addition, the restoring force becomes very small, so higher order terms in S could become more important.

We have also performed the calculations using a coupling constant $\varkappa$ which is 30 per cent greater than the one determined by considerations on the axially symmetric equilibrium deformations. These calculations give good agreement for the γ-energies at the beginning of the deformed region and they lead to γ-instability in W and Os. If such would be the case, the γ-vibrations in Hf should be especially low. However, this fact does not appear to be supported by experimental data.

One can estimate roughly the effect of the neglected Coulomb interaction by assuming an ellipsoid with constant density of charge[24]. The Coulomb energy is

$$E_c = \frac{3}{5}\frac{Z^2 e^2}{R}\left[1 - \frac{5}{36}\frac{(Q^2+S^2)}{A^2R^4}\right]. \tag{77}$$

The ratio $\frac{1}{12}\left(\frac{Z}{A}\right)^2 \frac{e^2}{R^5}\cdot\frac{2}{\varkappa} = 1.40\times 10^{-3}A^{2/3}$ is 0.04 if $A = 170$. $\left(R = 1.2\times 10^{-13}\right.$ $\left. A^{1/3}\,\mathrm{cm};\ \frac{Z}{A} = 0.4\right)$. Thus the effect of the Coulomb interaction amounts only to a 4 per cent change in $\varkappa$ in the middle of the first deformed region.

Table III contains the reduced transition probabilities calculated by means of (68) and parameters determined above. In the Gd, Dy, Er, and W-isotopes, the predicted values of the reduced transition probabilities are in agreement with experiment. In Os^{188} and Os^{190}, however, the predicted transition rates are about three times the experimental values. The disturbing aspect of the discrepancy is the fact that, experimentally, no increase in the transition rate occurs as the energy of the γ-vibrations decreases. We expect such an increase since the decrease in the γ-energy is principally due to a reduction in the restoring force, which should lead to oscillations of greater amplitude. On the other hand, if the restoring force goes to zero (γ-unstable oscillations[21]) the transition from the second 2+ state to the ground state is completely forbidden. This reveals an incompleteness in the present treatment, due to the fact that our wave functions do not have the required symmetry properties[16]. This symmetrization would give rise to interference terms which are responsible for the cancellation of the above-mentioned matrix element as the system approaches γ-instability. But these interference effects should be small if the root mean square value of γ is small compared with $\pi/3$. Estimated values for this quantity are also listed in Table III. They have been calculated by means of

$$\gamma_{\mathrm{r.m.s}} = \sqrt{\frac{2\overline{Q_2^2}}{Q_0^2}} = \sqrt{\frac{\overline{B(E2;\,00\to 22)_m}}{B(E2;\,00\to 20)_m}}, \tag{78}$$

where $B(E2)_m$ is the usual reduced transition probability calculated, assuming the same charge for neutrons as for protons. However, the ratio (77) can be well approximated by the ratio between the reduced transition probabilities obtained with the effective charges (76).

It seems that for Os^{188} and for Os^{190} the above-mentioned interference effects could begin to be important. The inclusion of higher-order terms which may have "γ-parity" would increase these interference effects.

The present estimates of $\gamma_{\mathrm{r.m.s}}$ may provide also a test about the validity

TABLE III. Reduced transition probabilities, in units af $10^{-48}\, e^2\,\mathrm{cm}^4$, for the transition from the ground state to the first γ-vibrational level. The last column lists estimated root-mean-square γ-values in units of $\pi/3$. The interference effects neglected here are unimportant if $\gamma_{\mathrm{r.m.s.}} \ll \frac{\pi}{3}$.

Nucleus	δ	$B(E2;0\to\gamma)_{\mathrm{th}}$	$B(E2;0\to\gamma)'_{\mathrm{th}}$	$B(E2;0\to\gamma)''_{\mathrm{th}}$	$B(E2;0\to\gamma)_{\mathrm{exp}}$ [28) 29)]	γ_{rms}
$_{62}Sm^{152}_{90}$	0.25	0.12	0.12	0.30	—	0.20
	0.30	0.10	0.10	0.17	—	0.20
Sm^{154}_{92}	0.30	0.09	0.10	0.13	—	0.17
$_{64}Gd^{156}_{92}$	0.30	0.10	0.11	0.16	~ 0.16	0.17
Gd^{158}_{94}	0.30	0.11	0.11	0.14	~ 0.16	0.16
$_{66}Dy^{160}_{94}$	0.30	0.13	0.13	0.24	—	0.18
Dy^{164}_{98}	0.30	0.12	0.12	0.22	—	0.16
$_{68}Er^{166}_{98}$	0.30	0.12	0.12	0.19	~ 0.22	0.16
Er^{168}_{100}	0.30	0.13	0.13	0.18	~ 0.22	0.16
$_{70}Yb^{170}_{100}$	0.30	0.07	0.08	0.09	—	0.13
$_{72}Hf^{178}_{106}$	0.25	0.10	0.10	0.21	—	0.17
$_{74}W^{182}_{108}$	0.20	0.16	0.18	—	~ 0.12	0.23
W^{184}_{110}	0.20	0.17	0.19	—	0.17 ± 0.05	0.26
W^{186}_{112}	0.15	0.37	0.34	—	0.17 ± 0.03	0.40
	0.20	0.17	0.18	—		0.28
$_{76}Os^{186}_{110}$	0.20	0.19	0.22	—	—	0.33
Os^{188}_{112}	0.15	0.58	0.47	—	—	—
	0.20	0.19	0.23	—	0.20 ± 0.06	0.36
Os^{190}_{114}	0.15	0.35	0.37	—	0.14 ± 0.03	0.48

All measurements of ref. 28 carry an experimental uncertainty of a factor of 2.

of the models which take into account only the degrees of freedom associated with an asymmetric rotor.

In order to summarize our results, we can say that, without any free parameter, we have been able to predict energies for the γ-vibrations which are in good agreement with experimental data at the end of the deformed region. At the beginning, the predicted γ-energies are too high, but the structure in the empirical curve is predicted theoretically. The experimental transition rates are also well accounted for, with the exception of the Os isotopes.

Some calculations have still to be performed in order to test the validity of some of our assumptions. For instance, we have to treat the closed shells explicitly in order to check the renormalization idea. In addition, the use of a central potential which is essentially an harmonic oscillator one, may

overestimate the dependence of the matrix elements $s_{\nu\omega}$ on the asymptotic quantum numbers.

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Institute for Theoretical Physics
University of Copenhagen.

Comisión Nacional de Energia Atómica
Buenos Aires, Argentina.

Appendix

General calculation of the restoring force and mass parameters.

We consider a system at its equilibrium deformation. We assume that we already know the values of λ, Δ and the V_ν. We allow a small change, R, in the previous single-particle field. The matrix (13[1]), with $\mu = \varkappa Q_{\text{equilibrium}}$, $\sigma = \varkappa S_{\text{equilibrium}}$, has to be modified by the addition of a term which is generally not diagonal

$$-(\varkappa R + \hat{\varrho})\, r_{\nu\omega} = -\varrho\, r_{\nu\omega}. \tag{1A}$$

$\hat{\varrho}$ is again a Lagrange multiplier.

The quantities denoted by a prime will refer to the new single-particle field. We shall expand in ϱ and keep only linear terms.

$$\left.\begin{aligned} \lambda' &= \lambda + \varrho\lambda_1 \\ \Delta'^2 &= \Delta^2 + \varrho\Delta_1^2 \end{aligned}\right\} \tag{2A}$$

We diagonalize the single-particle Hamiltonian by means of perturbation theory

$$\left.\begin{aligned} \varepsilon'_\nu &= \varepsilon_\nu - \varrho\,(r_{\nu\nu} + \lambda_1) \\ c'_\nu &= c_\nu + \varrho \sum_\omega \frac{r_{\nu\omega}}{\varepsilon_\omega - \varepsilon_\nu}\, c_\omega \end{aligned}\right\} \tag{3A}^*$$

and use a procedure similar to the one outlined in p. 12. The expressions for E_ν and V_ν^2 are

$$\left.\begin{aligned} E_\nu^1 &= E_\nu + \frac{\varrho}{E_\nu}\left[\frac{\Delta_1^2}{2} - (r_{\nu\nu} + \lambda_1)\,\varepsilon_\nu\right] \\ V_\nu'^2 &= V_\nu^2 + \frac{\varrho}{2E_\nu^3}\left[\frac{\Delta_1^2\,\varepsilon_\nu}{2} + (r_{\nu\nu} + \lambda_1)\,\Delta^2\right]. \end{aligned}\right\} \tag{4A}$$

* Since the quadrupole operator is even under time-reversal ($r_{\nu+\omega+} = r_{\nu-\omega-}$) it follows that the + sign in the second equation (3A) holds for both $c_{\nu+}$ and $c_{\nu-}$. In consequence, $H'_{\text{pair}} = H_{\text{pair}} + 0\,(\varrho^2)$. When this condition is not satisfied (i. e., for the Coriolis force) the following treatment may not be valid.

The two basic equations (4) and (12) must be satisfied independently of the value of ϱ. From the requirement that the terms proportional to ϱ vanish, it follows that

$$\left.\begin{aligned} &\lambda_1 \sum_\nu \frac{\varepsilon_\nu}{E_\nu^3} - \frac{\Delta_1^2}{2} \sum_\nu \frac{1}{E_\nu^3} + \sum_\nu \frac{r_{\nu\nu}\,\varepsilon_\nu}{E_\nu^3} = 0 \\ &\lambda_1 \sum_\nu \frac{1}{E_\nu^3} + \frac{\Delta_1^2}{2\Delta^2} \sum_\nu \frac{\varepsilon_\nu}{E_\nu^3} + \sum_\nu \frac{r_{\nu\nu}}{E_\nu^3} = 0\,. \end{aligned}\right\} \quad (5\,\mathrm{A})$$

The solution of this system of equations is

$$\left.\begin{aligned} \lambda_1 &= \frac{1}{\left[\left(\sum_\nu \frac{\varepsilon_\nu}{E_\nu^3}\right)^2 + \Delta^2\left(\sum_\nu \frac{1}{E_\nu^3}\right)^2\right]} \left[\left(\sum_\nu \frac{\varepsilon_\nu}{E_\nu^3}\right)\left(\sum_\nu \frac{r_{\nu\nu}\,\varepsilon_\nu}{E_\nu^3}\right) + \Delta^2\left(\sum_\nu \frac{1}{E_\nu^3}\right)\left(\sum_\nu \frac{r_{\nu\nu}}{E_\nu^3}\right)\right] \\ \Delta_1^2 &= \frac{2\Delta^2}{\left[\left(\sum_\nu \frac{\varepsilon_\nu}{E_\nu^3}\right)^2 + \Delta^2\left(\sum_\nu \frac{1}{E_\nu^3}\right)^2\right]} \left[\left(\sum_\nu \frac{1}{E_\nu^3}\right)\left(\sum_\nu \frac{r_{\nu\nu}\,\varepsilon_\nu}{E_\nu^3}\right) - \left(\sum_\nu \frac{\varepsilon_\nu}{E_\nu^3}\right)\left(\sum_\nu \frac{r_{\nu\nu}}{E_\nu^3}\right)\right]. \end{aligned}\right\} \quad (6\,\mathrm{A})$$

Using equations (3A), (4A) and (6A), the new values of V_ν and U_ν can be calculated. Then the new ground state wave function $|\,0'\,\rangle$ can be expressed in the representation corresponding to the equilibrium deformation

$$\left.\begin{aligned} |\,0'\,\rangle &= \prod_\nu \left[U'_\nu + V'_\nu\, c^{+\prime}_{\nu+}\, c^{+\prime}_{\nu-}\right] |\,\text{vacuum}\,\rangle \\ &= |\,0\,\rangle \\ &+ \varrho \sum_\nu \frac{1}{4\,U_\nu V_\nu} \left[(r_{\nu\nu} + \lambda_1)\frac{\Delta^2}{E_\nu^3} + \frac{\Delta_1^2}{2}\frac{\varepsilon_\nu}{E_\nu^3}\right] \alpha_\nu^+ \beta_\nu^+ |\,0\,\rangle \\ &+ \varrho \sum_\nu \sum_{\omega \neq \nu} \frac{V_\nu U_\omega}{\varepsilon_\omega - \varepsilon_\nu}\, r_{\nu\omega} \left(\alpha_\nu^+ \beta_\omega^+ + \alpha_\omega^+ \beta_\nu^+\right) |\,0\,\rangle\,. \end{aligned}\right\} \quad (7\,\mathrm{A})$$

Because of the identity $(E_\nu + E_\omega)(U_\omega V_\nu - U_\nu V_\omega) = (\varepsilon_\omega - \varepsilon_\nu)(U_\omega V_\nu + U_\nu V_\omega)$ the third term in (7A), which contains non-diagonal single-particle matrix elements $r_{\nu\omega}$, can be cast into the form

$$\varrho \sum_{\nu,\,\omega} \frac{r_{\nu\omega}\,(U_\nu V_\omega + U_\omega V_\nu)}{(E_\nu + E_\omega)}\, \alpha_\nu^+ \beta_\omega^+ |\,0\,\rangle\,. \quad (8\,\mathrm{A})$$

Using the relations $\langle\, 0 \,|\, R_{o\nu}\, \alpha_\nu^+ \beta_\omega^+ \,|\, 0\,\rangle = r_{\nu\omega}\,(U_\nu V_\omega + U_\omega V_\nu)$ and $2\,U_\nu V_\nu = \dfrac{\Delta}{E_\nu}$, and equations (7A) and (8A), we find

$$\left.\begin{aligned} R &= \langle 0' \mid R_{o\varrho} \mid 0' \rangle \\ &= \langle 0 \mid R_{o\varrho} \mid 0 \rangle \\ &+ \varrho \left[2 \sum_{\nu,\omega} \frac{r_{\nu\omega}^2 (U_\nu V_\omega + U_\omega V_\nu)^2}{E_\nu + E_\omega} + \lambda_1 \Delta^2 \sum_\nu \frac{r_{\nu\nu}}{E_\nu^3} + \frac{\Delta_1^2}{2} \sum_\nu \frac{r_{\nu\nu}\varepsilon_\nu}{E_\nu^3} \right] \end{aligned}\right\} \quad (9\,\mathrm{A})$$

from which the derivative $\frac{d\varrho}{dR}$, and thus the restoring force (eq. 29), can be immediately obtained.

The calculation of the mass parameter is done according to the prescriptions of the "crancking model":

$$\left.\begin{aligned} B &= 2\hbar \sum_{\nu,\omega} \frac{|\langle \nu\omega \mid \frac{\delta}{\delta R} \mid 0 \rangle|^2}{E_\nu + E_\omega} \\ &= 2\hbar^2 \left(\frac{d\varrho}{dR}\right)^2 \sum_{\nu,\omega} \frac{|\langle \nu\omega \mid \frac{d}{d\varrho} \mid 0 \rangle|^2}{E_\nu + E_\omega} \\ &= 2\hbar^2 \left(\frac{d\varrho}{dR}\right)^2 \left\{ 2 \sum_{\nu,\omega} \frac{r_{\nu\omega}^2 (U_\nu V_\omega + U_\omega V_\nu)^2}{(E_\nu + E_\omega)^3} \right. \\ &\left. + \frac{1}{8} \sum_\nu \frac{1}{E_\nu^5} \left(\Delta^2 \lambda_1^2 - \frac{\Delta_1^4}{4} + \frac{\Delta_1^4 E_\nu^2}{4\Delta^2} + \lambda_1 \Delta_1^2 \varepsilon_\nu + 2\lambda_1 \Delta^2 r_{\nu\nu} + \Delta_1^2 r_{\nu\nu}\varepsilon_\nu \right) \right\}. \end{aligned}\right\} \quad (10\,\mathrm{A})$$

We see that if $\sum_\nu \frac{r_{\nu\nu}}{E_\nu^3} = \sum_\nu \frac{r_{\nu\nu}\varepsilon_\nu}{E_\nu^3} = 0$, it follows that $\lambda_1 = \Delta_1^2 = 0$. In this case, formulae (29) and (33) are correct. The first condition $\sum_\nu \frac{r_{\nu\nu}}{E_\nu^3} = 0$ is related to the non-conservation of the particle number (31). To interpret the second condition $\sum_\nu \frac{r_{\nu\nu}\varepsilon_\nu}{E_\nu^3} = 0$ we construct the two quasi-particle perturbed wave function in analogy to (27).

$$|(\nu\omega)'\rangle = |\nu\omega\rangle - \varrho \frac{\langle \nu\omega \mid R_{o\varrho} \mid 0 \rangle}{E_\nu + E_\omega} |0\rangle + \varrho \sum_{\xi\eta} \frac{\langle 0 \mid R_{o\varrho} \mid \xi\eta \rangle}{E_\xi + E_\eta} |\xi\eta\nu\omega\rangle, \quad (11\,\mathrm{A})$$

where $|\xi\eta\nu\omega\rangle = \alpha_\xi^+ \beta_\eta^+ \alpha_\nu^+ \beta_\omega^+ |0\rangle$ represents a four quasi-particle wave function. We require that the matrix element of H_{int} between $|\langle \nu\omega)'\rangle$ and $|0'\rangle$ contains no linear term in ϱ, and therefore can lead to no quadratic term in the expression for the total ground-state energy. From Appendix A of BELYAEV[7] we have

$$\sum_{\xi\eta} \langle \xi\eta\, \nu\omega \mid H_{\text{int}} \mid 0 \rangle = \frac{G}{2} \delta_{\nu\omega} \sum_{\xi} (U_\nu^2 V_\xi^2 + U_\xi^2 V_\nu^2) \tag{12A}$$

$$\sum_{\xi\eta} \langle \xi\eta \mid H_{\text{int}} \mid \upsilon\omega \rangle = -\frac{G}{2} [\delta_{\nu\omega} \sum_{\xi} (U_\nu^2 U_\xi^2 + V_\nu^2 V_\xi^2) + 2\, U_\nu V_\nu U_\omega V_\omega]. \tag{13A}$$

The last term in (13A) is of order $\frac{1}{\Omega}$ compared to the first, as it contains no summation over all the single-particle states. Neglecting it and using (27), (11A), (12A), and (13A),

$$\left.\begin{aligned} &\langle 0' \mid H_{\text{int}} \mid (\nu\omega)' \rangle \\ &= \varrho \sum_{\xi\eta} \frac{\langle \xi\eta \mid R_{op} \mid 0 \rangle \langle 0 \mid H_{\text{int}} \mid \xi\eta\nu\omega \rangle}{E_\xi + E_\eta} + \varrho \sum_{\xi\eta} \frac{\langle 0 \mid R_{op} \mid \xi\eta \rangle \langle \xi\eta \mid H_{\text{int}} \mid \nu\omega \rangle}{E_\xi + E_\eta} \\ &= -\varrho \frac{G}{4} \delta_{\nu\omega} (U_\nu^2 - V_\nu^2) \sum_{\xi} \frac{\langle 0 \mid R_{op} \mid \xi\xi \rangle}{E_\xi} (U_\xi^2 - V_\xi^2). \end{aligned}\right\} \tag{14A}$$

Thus the validity of (27) implies the vanishing of (14A). This in turn requires

$$\sum_{\xi} \frac{\langle 0 \mid R_{op} \mid \xi\xi \rangle}{E_\xi} (U_\xi^2 - V_\xi^2) = \Delta \sum_{\xi} \frac{r_{\xi\xi}\, \varepsilon_\xi}{E_\xi^3} = 0. \tag{15A}$$

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