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# ON THE THEORY OF MULTIPLE COULOMB EXCITATION WITH HEAVY IONS

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## CONTENTS



## Synopsis


The present paper contains formulae and tables for the evaluation of multiple Coulomb excitation cross sections of rotational and vibrational states. For other cases, general calculational procedures have been developed and these are illustrated through examples. For the larger part of the work, the collision time is assumed to be short compared to the nuclear period. The investigation is furthermore simplified by an approximate treatment of the dependence of the cross section on the deflection angle of the projectile. The accuracy of the approximations is also discussed.



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## 1. Introduction

In the last few years, the Coulomb excitation process has become a valuable tool for the investigation of low lying nuclear states. Several review articles on the experimental and theoretical aspects of Coulomb excitation have appeared, [1], [2], [3], [4] which contain bibliographies of the earlier work on this subject*.

The Coulomb excitation process has certain advantages over other nuclear reactions. The fact that the forces responsible for the process are well understood, and the theory is well developed, allows one from a careful analysis of the reaction to determine a number of quantities characteristic of the nuclear states. The main approximation in the existing calculations is the use of perturbation theory which is valid if the probability for nuclear excitation in a single encounter is small. If protons or $\alpha$-particles are used as projectiles, and if the bombarding energy is kept so low that no nuclear reactions take place, this criterion will always be fulfilled. In these cases there is, however, a strong limitation on the number of states which can be investigated. The limitation lies, firstly, in the selection rules for the low multipole interactions which are important for the excitation process. Secondly, only low lying states are accessible, since the reaction for higher excitation energies soon becomes adiabatic. A way to overcome these difficulties is to use heavier ions as projectiles. The electric field exerted on the nucleus then becomes so large that higher order processes occur. While, e. g., a state with spin $4^+$ in first-order perturbation treatment can only be reached from a ground state of spin $0^+$ through an $E4$ interaction, it might already in second order be excited through a state of spin $2^+$ by means of quadrupole interactions. In many cases, one might still use the perturbation expansion to calculate the excitation probabilities (see ref. 1, Chapt. II D, and ref. 2). If, however, the interaction becomes so strong that many levels are actively involved in the excitation process, one has to solve directly the set of coupled

* In the following, the notation of ref. 1 will always be used.

equations which describes the population of the nuclear states during the collision.

The feasibility of such multiple excitations with heavy ions has recently been proved and, from these experiments as well as from the following calculations, it seems that a number of new possibilities are opened for the investigation of nuclear states[5], [6].

In the following we shall consider such multiple excitations. In Section 2, a discussion is given of the parameters which are important for the process. Section 3 contains the general formalism, while the following sections are concerned with special models and numerical tables.

## 2. Characteristic Parameters

The Coulomb excitation process is characterized by a number of parameters. These quantities describe the kind of approximations which are appropriate for the process in question.

A parameter which describes the motion of the projectile in the Coulomb field of the nucleus is $\eta$ defined by

$$\eta = \frac{Z_1 Z_2 e^2}{\hbar v}, \tag{2.1}$$

where $Z_1$ and $Z_2$ are the charge numbers of the projectile and the target nucleus, respectively, while $v$ is the relative velocity of the incident particle and the nucleus. While for protons this parameter may be as small as two, it is, for the heavy ions which are being considered in the following, always much larger than one. Since, furthermore, the projectile in a collision loses only a small part of its energy, one may to a very good approximation use a classical description for its path. The hyperbolic orbit of the particle will be described by the deflection angle $\vartheta$ (see Fig. 1).

The Coulomb interaction between the projectile and the nucleus is given by (see ref. 1, Eqs. (II A. 8) to (II A. 11))

$$\mathfrak{H}_E(t) = Z_1 e \int \varrho(\vec{r}) \frac{d\tau}{|\vec{r} - \vec{r}_p(t)|}, \tag{2.2}$$

where $\varrho(\vec{r})$ is the charge density operator at the position $r$ of the nucleus and $\vec{r}_p(t)$ is the position vector of the projectile, which for a given hyperbolic orbit is a known function of time. The interaction can be expanded in multipole components

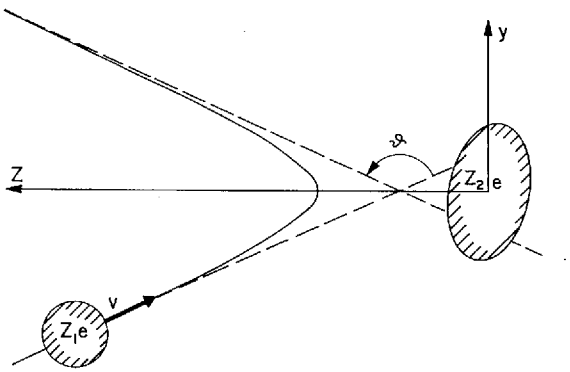


Fig. 1. Classical picture of the projectile orbit in the Coulomb field of the nucleus. The hyperbolic orbit of the projectile is shown in the frame of reference where the nucleus is at rest. The coordinate system which is employed in the present paper, with the $z$-axis along the axis of symmetry, is indicated. The charges of nucleus and projectile are denoted by $Z_2e$ and $Z_1e$, respectively, $v$ is the initial relative velocity, and $\vartheta$ is the deflection angle.

$$\mathfrak{H}_E(t) = 4\pi Z_1 e \sum_{\lambda=1}^{\infty} \sum_{\mu=-\lambda}^{+\lambda} \frac{1}{2\lambda+1} r_p^{-\lambda-1} Y_{\lambda\mu}(\vartheta_p, \varphi_p) \mathfrak{M}^*(E\lambda, \mu), \tag{2.3}$$

where $\mathfrak{M}(E\lambda, \mu)$ is the electric multipole moment of order $\lambda$ of the nucleus defined by

$$\mathfrak{M}(E\lambda, \mu) = \int r^\lambda Y_{\lambda\mu}(\vartheta, \varphi) \varrho(\vec{r}) \, d\tau. \tag{2.4}$$

In first order perturbation treatment, one finds the following expression for the total probability $P$ for the transition from the nuclear state 1 to the state 2 in a given collision with deflection angle $\vartheta$ (see ref. 1, Eqs. (II A. 4), (II A. 28), and (II A. 29)):

$$P_{1\to 2} = \frac{Z_1^2 \cdot e^2}{\hbar^2 v^2} \sum_{\lambda=1}^{\infty} \frac{16\pi^2}{(2\lambda+1)^3} \cdot \frac{1}{a^{2\lambda}} \cdot B(E\lambda) \sum_{\mu=-\lambda}^{+\lambda} \left| Y_{\lambda\mu}\left(\frac{\pi}{2}, 0\right) \right|^2 \cdot |I_{\lambda\mu}(\vartheta, \xi)|^2. \tag{2.5}$$

Here, $a$ is half the distance of closest approach in a head-on collision

$$a = \frac{Z_1 Z_2 e^2}{m v^2}, \tag{2.6}$$

where $m$ is the reduced mass of the projectile and the target nucleus. The reduced transition probability $B(E\lambda, I_1 \to I_2)$ is defined by

$$\left.\begin{aligned} B(E\lambda, I_1 \to I_2) &= \sum_{\mu M_2} |\langle I_1 M_1 | \mathfrak{M}(E\lambda, \mu) | I_2 M_2 \rangle|^2 \\ &= \frac{1}{2I_1+1} |\langle I_1 || \mathfrak{M}(E\lambda) || I_2 \rangle|^2. \end{aligned}\right\} \quad (2.7)$$

For two states with spins $I_1$ and $I_2$, practically only one value of $\lambda$ will give a contribution to the sum in (2.5).

The orbital integrals [7] $I_{\lambda\mu}(\vartheta, \xi)$ depend on the deflection angle and on the parameter $\xi$ which is defined by

$$\xi_{1\to 2} = \eta_f - \eta_i \simeq \frac{Z_1 Z_2 e^2}{\hbar v} \frac{E_2 - E_1}{2E}. \quad (2.8)$$

The quantities $\eta_i$ and $\eta_f$ are given by (2.1), substituting for the velocity the initial and final velocities, respectively. Similarly, $E_1$ and $E_2$ denote the energies of the nucleus in the states 1 and 2, while $E$ is the energy of the projectile. The parameter $\xi$ measures the suddenness of a head-on collission. In general, the suddenness is measured by the quantity

$$\xi(\vartheta) = \frac{\xi}{\sin\frac{\vartheta}{2}}. \quad (2.9)$$

If $\xi(\vartheta)$ is large, the process is essentially adiabatic and the excitation probability small. If $\xi(\vartheta)$ is small, the process has the character of a sudden impact and one may use a sudden approximation.

In the case of multiple Coulomb excitation, the parameter $\xi$ is no more a characteristic of a nuclear state. A definite nuclear state can in this case be populated in different ways. The $\xi$ which is important for the excitation of the state in question need not be the one corresponding to the excitation from the ground state, but is rather a set of $\xi$'s corresponding to the transitions through which it is populated.

The validity of the perturbation treatment which leads to the result (2.5) is guaranteed if $P$ is small compared to one. We may introduce the square root of the contribution to $P$ from a definite multipole order as a measure of the strength with which the state 2 is coupled to the state 1 through the interaction with the projectile

$$\chi^{(\lambda)}_{1\to 2}(\vartheta, \xi) = \pm\sqrt{P^{(\lambda)}_{1\to 2}(\vartheta, \xi)}. \quad (2.10)$$

The sign is to be the same as the sign of the reduced matrix element $\langle I_1 || \mathfrak{M}(E\lambda) || I_2 \rangle$.

If all the parameters $\chi$ which connect the states of the nucleus are small compared to unity, one may use a first order perturbation treatment. This will practically always be the case when protons are used as projectiles. For $\alpha$-particles and heavier ions, the $\chi$'s will also mostly be smaller than one if the matrix elements are of the order of the single-particle value (see ref. 1, Chapt. II A). One may, in such cases, still use the perturbation treatment, when necessary, to second or third order (see ref. 1, Chapt. II D). If, however, the nucleus possesses excited states of collective type[8] with strongly enhanced $B(E2)$ transition probabilities, $\chi^{(2)}$ might be as large as 5. Then, one has to use an approach which avoids the perturbation expansion. On the other hand, states with large quadrupole transition probabilities have usually small excitation energies, and one may use an expansion appropriate for small $\xi$.

The parameter $\chi(\vartheta, \xi)$ attains its largest value for $\xi = 0$ and $\vartheta = \pi$. It will be useful to introduce this value as the fundamental parameter, in the same way as $\xi$ is used instead of $\xi(\vartheta)$. We thus define (see ref. 1, Chapt. II E. 4)

$$\chi_{1\to 2}^{(\lambda)} = \sqrt{16\pi}\,\frac{(\lambda-1)!}{(2\lambda+1)!!}\cdot\frac{Z_1 e}{\hbar v}\cdot\frac{\langle I_1 \| \mathfrak{M}(E\lambda) \| I_2\rangle}{a^{\lambda}\sqrt{2I_1+1}}. \tag{2.11}$$

It will also be convenient sometimes to introduce the value of $\chi(\vartheta, \xi)$ for $\xi = 0$, but arbitrary $\vartheta$ as a parameter. We call this $\chi(\vartheta)$ and, according to (2.5), (2.10), and (2.11), it is defined by

TABLE 1

A survey of different limiting cases of the characteristic parameters $\eta$, $\xi$, and $\chi$. In the table is indicated the kind of approximation which is appropriate for the different cases and the values of $\lambda$ for which computations have been performed. The calculations mentioned under the heading "$\xi$ arbitrary" are quoted in ref. 1. The computations for arbitrary $\eta$ and $\xi \ll 1$ are given in ref. 11, while those mentioned in the last entry refer to the present work.

| | $\eta$ arbitrary | $\eta \gg 1$ semiclassical |
|---|---|---|
| $\xi$ arbitrary | $\chi \ll 1$<br>1. order perturbation<br>$\lambda = 1, 2$ | $\chi < 1$<br>1. and 2. order perturbation<br>$\lambda = 1, 2, 3, 4$ |
| $\xi \ll 1$<br>sudden<br>approximation | $\chi \ll 1$<br>1. order perturbation<br>$\lambda = 3, 4$ | $\chi$ arbitrary<br>multiple excitation<br>$\lambda = 2$ |

$$\chi^{(\lambda)}_{1\to 2}(\vartheta) = \chi^{(\lambda)}_{1\to 2}\frac{(2\lambda-1)!!}{(\lambda-1)!}\left[\frac{\pi}{(2\lambda+1)}\sum_{\mu}{}' \left| Y_{\lambda\mu}\left(\frac{\pi}{2},0\right) I_{\lambda\mu}(\vartheta,0)\right|^2\right]^{\frac{1}{2}}. \qquad (2.12)$$

In Table I, a survey is given of the different limiting cases for which computations on Coulomb excitation have until now been made, including the present work. In the following, we shall limit ourselves mainly to the case of quadrupole excitations ($\lambda = 2$).

## 3. General Theory

In this section, we investigate the equations which determine the multiple Coulomb excitation and discuss some general approximation methods. It will be shown that the special solution for $\xi = 0$ and $\vartheta = \pi$ is a convenient basic solution by means of which the excitation probability for small values of $\xi$ and arbitrary angles may be expressed.

### A. Expansion Methods

The Schrödinger equation for the nuclear state vector $|\psi\rangle$ is

$$i\hbar\frac{\partial}{\partial t}|\psi\rangle = [\mathfrak{H}_0 + \mathfrak{H}_E(t)]\,|\psi\rangle, \qquad (3.1)$$

where $\mathfrak{H}_0$ is the Hamiltonian of the free nucleus and $\mathfrak{H}_E(t)$ the interaction energy given by (2.2), (2.3), and (2.4). It will be useful to introduce a new state vector $|\Phi\rangle$ defined by

$$|\psi\rangle = e^{-\frac{i}{\hbar}\mathfrak{H}_0 t}\,|\Phi\rangle. \qquad (3.2)$$

Before and after the collision this state vector is time-independent and it satisfies the equation

$$i\hbar\frac{\partial}{\partial t}|\Phi\rangle = \tilde{\mathfrak{H}}(t)\,|\Phi\rangle, \qquad (3.3)$$

where

$$\tilde{\mathfrak{H}}(t) = e^{\frac{i}{\hbar}\mathfrak{H}_0 t}\cdot\mathfrak{H}_E(t)\,e^{-\frac{i}{\hbar}\mathfrak{H}_0 t}. \qquad (3.4)$$

The equation (3.3) may also be formulated as a set of coupled differential equations for the amplitudes on the nuclear eigenstates. If we thus define

$$a_n(t) = \langle n\,|\,\Phi\rangle, \qquad (3.5)$$

where $|n\rangle$ is the time-independent eigenstate belonging to the eigenvalue $E_n$ of $\mathfrak{H}_0$,

$$\mathfrak{H}_0 \mid n \rangle = E_n \mid n \rangle, \tag{3.6}$$

we obtain

$$i\hbar \dot{a}_n = \sum_m \langle n \mid \mathfrak{H}_E(t) \mid m \rangle e^{\frac{i}{\hbar}(E_n - E_m)t} a_m(t). \tag{3.7}$$

The solution of (3.3) and (3.7) can often conveniently be expressed as a series in powers of $\mathfrak{H}_E(t)$. This is the usual perturbation expansion which can be obtained by an iteration procedure. It can be written in a closed form due to Dyson

$$\mid \Phi(t) \rangle = T e^{-\frac{i}{\hbar}\int_{-\infty}^{t} \tilde{\mathfrak{H}}(t)\, dt} \mid \Phi(t = -\infty) \rangle, \tag{3.8}$$

where the symbol $T$ stands for the time ordered product, i. e.,

$$\left.\begin{aligned} T e^{-\frac{i}{\hbar}\int_{-\infty}^{t} \tilde{\mathfrak{H}}(t')\, dt'} &= 1 - \frac{i}{\hbar}\int_{-\infty}^{t} \tilde{\mathfrak{H}}(t')\, dt' \\ &+ \left(\frac{-i}{\hbar}\right)^2 \int_{-\infty}^{t} \tilde{\mathfrak{H}}(t')\, dt' \int_{-\infty}^{t'} \tilde{\mathfrak{H}}(t'')\, dt'' + \left(\frac{-i}{\hbar}\right)^3 \int_{-\infty}^{t} \tilde{\mathfrak{H}}(t')\, dt' \int_{-\infty}^{t'} \tilde{\mathfrak{H}}(t'')\, dt'' \int_{-\infty}^{t''} \tilde{\mathfrak{H}}(t''')\, dt'''. \end{aligned}\right\} \tag{3.9}$$

If the nucleus before the collision is in the ground state $\mid 0 \rangle$, the solution (3.8) leads to the following expression for the amplitudes on the different excited states after the collision:

$$a_n(+\infty) = \langle n \mid T e^{-\frac{i}{\hbar}\int_{-\infty}^{+\infty} \tilde{\mathfrak{H}}(t)\, dt} \mid 0 \rangle. \tag{3.10}$$

When one inserts the series (3.9), one obtains exactly the usual perturbation expansion for the excitation amplitudes.

As has been mentioned above, the case where $\xi = 0$ for all states involved is of special importance for the problem of multiple Coulomb excitations. In this case, one has $E_n = E_m$ and $\tilde{\mathfrak{H}} = \mathfrak{H}$, and one can then leave out the time ordering in (3.8). The expression (3.10) now takes the simple form

$$a_n(+\infty) = \langle n \mid e^{-\frac{i}{\hbar}\int_{-\infty}^{\infty} \mathfrak{H}(t)\, dt} \mid 0 \rangle \tag{3.11}$$

characteristic of the sudden approximation. This formula is also applicable in cases where $\int_{-\infty}^{\infty} \mathfrak{H}(t)\, dt \gg \hbar$ since, for the evaluation, it is not necessary to perform a series expansion of the exponential function.

By means of formula (3.11) one may thus avoid the perturbation expansion. On the other hand, the effect of the motion of the nucleus during

the collision has been neglected. The sudden approximation, however, forms a convenient starting point for a series expansion in powers of $\mathfrak{H}_0$. Such a series expansion is generated by the following substitution:

$$|\Phi\rangle = e^{-\frac{i}{\hbar}\int_{-\infty}^{t}\mathfrak{H}_E(t)\,dt}\,|\varphi\rangle. \tag{3.12}$$

The Schrödinger equation (3.3) then takes the form

$$i\hbar\frac{\partial|\varphi\rangle}{\partial t} = e^{\frac{i}{\hbar}\int_{-\infty}^{t}\mathfrak{H}_E(t)\,dt}\,[\tilde{\mathfrak{H}}(t)-\mathfrak{H}_E(t)]\,e^{-\frac{i}{\hbar}\int_{-\infty}^{t}\mathfrak{H}_E(t)\,dt}. \tag{3.13}$$

In this expression, one can expand $\tilde{\mathfrak{H}}(t)$ in powers of $\mathfrak{H}_0$ in the following way:

$$\tilde{\mathfrak{H}}(t) = \mathfrak{H}_E(t)+\frac{i}{\hbar}t\,[\mathfrak{H}_0,\mathfrak{H}_E(t)]+\frac{1}{2}\left(\frac{i}{\hbar}t\right)^2[\mathfrak{H}_0\,[\mathfrak{H}_0,\mathfrak{H}_E(t)]]+\cdots \tag{3.14}$$

and (3.13) takes the form

$$i\hbar\frac{\partial|\varphi\rangle}{\partial t} = \left\{\frac{i}{\hbar}t\,[\overline{\mathfrak{H}}_0,\mathfrak{H}_E(t)]+\frac{1}{2}\left(\frac{i}{\hbar}t\right)^2[\overline{\mathfrak{H}}_0\,[\overline{\mathfrak{H}}_0,\mathfrak{H}_E(t)]]+\cdots\right\}|\varphi\rangle, \tag{3.15}$$

where

$$\overline{\mathfrak{H}}_0 = e^{\frac{i}{\hbar}\int_{-\infty}^{t}\mathfrak{H}_E(t)\,dt}\,\mathfrak{H}_0\,e^{-\frac{i}{\hbar}\int_{-\infty}^{t}\mathfrak{H}_E(t)\,dt}. \tag{3.16}$$

The expression on the right-hand side of (3.15) is a series in powers of nuclear energy differences times the collision time, i. e., it is a series in powers of the $\xi$'s involved. If we express the solution in a similar way,

$$|\varphi\rangle = |\varphi_0\rangle+|\varphi_1\rangle+|\varphi_2\rangle+\cdots, \tag{3.17}$$

where $|\varphi_1\rangle$ is of the order $\xi$ times $|\varphi_0\rangle$, $|\varphi_2\rangle$ of the order $\xi^2$ times $|\varphi_0\rangle$, etc., we obtain the following set of differential equations for these $|\varphi_n\rangle$'s:

$$\left.\begin{aligned}
&i\hbar\frac{\partial|\varphi_0\rangle}{\partial t} = 0\\
&i\hbar\frac{\partial|\varphi_1\rangle}{\partial t} = \frac{i}{\hbar}t\,[\overline{\mathfrak{H}}_0,\mathfrak{H}_E(t)]\,|\varphi_0\rangle\\
&i\hbar\frac{\partial|\varphi_2\rangle}{\partial t} = \frac{i}{\hbar}t\,[\overline{\mathfrak{H}}_0,\mathfrak{H}_E(t)]\,|\varphi_1\rangle+\frac{1}{2}\left(\frac{i}{\hbar}t\right)^2[\overline{\mathfrak{H}}_0\,[\overline{\mathfrak{H}}_0,\mathfrak{H}_E(t)]]\,|\varphi_0\rangle\\
&\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}\right\} \tag{3.18}$$

The initial condition determines $|\varphi_0\rangle$ to be

$$|\varphi_0\rangle = |0\rangle. \tag{3.19}$$

From $|\varphi_0\rangle$ one may determine $|\varphi_1\rangle, |\varphi_2\rangle$, etc. by means of quadratures

$$\left.\begin{aligned} |\varphi_1\rangle &= \frac{1}{\hbar^2}\int_{-\infty}^{t} dt'\, t' \,[\bar{\mathfrak{H}}_0(t'), \mathfrak{H}_E(t')]\,|0\rangle \\ |\varphi_2\rangle &= \frac{1}{\hbar^4}\int_{-\infty}^{t} dt'\, t' \,[\bar{\mathfrak{H}}_0(t'), \mathfrak{H}_E(t')]\int_{-\infty}^{t'} dt''\, t'' \,[\bar{\mathfrak{H}}_0(t''), \mathfrak{H}_E(t'')]\,|0\rangle \\ &\quad + \frac{1}{2}\frac{i}{\hbar^3}\int_{-\infty}^{t} dt'\, t'^2\,[\bar{\mathfrak{H}}_0(t'), [\bar{\mathfrak{H}}_0(t'), \mathfrak{H}_E(t')]]\,|0\rangle. \end{aligned}\right\} \tag{3.20}$$

If the interaction energy $\mathfrak{H}_E(t)$ tends to zero sufficiently rapidly, all integrals converge, and (3.20) offers a systematic expansion in powers of the $\xi$'s. In the case of quadrupole Coulomb excitation, however, $\mathfrak{H}_E(t)$ is of the order of $|t|^{-3}$ for large times, and already the second term in $|\varphi_2\rangle$ diverges. This difficulty is also encountered if one tries to expand the orbital integrals $I_{2,\mu}(\vartheta, \xi)$ in powers of $\xi$. The exact expression for these quantities in terms of confluent hypergeometric functions (see ref. 1, Eq. (II E. 50)) shows that the correct expansion is of the form

$$I_{2,\mu}(\vartheta, \xi) \approx a + b\xi + c\xi^2 + d\xi^2 \log \xi + \cdots. \tag{3.21}$$

In the following, we shall calculate only the first order terms of (3.20). For the evaluation of the higher terms a cut-off procedure might be used.

### B. Choice of Coordinate System

In earlier calculations of Coulomb excitation, the orbital integrals were evaluated in the so-called focal system. In this coordinate system the $z$-axis is perpendicular to the plane of the orbit, and the $x$-axis is along the symmetry axis of the hyperbola. In this paper, another system will be used, where the $z$-axis is along the symmetry axis (see Fig. 1). This system is of special convenience for head-on collisions ($\vartheta = \pi$), where the invariance of the entire Hamiltonian for rotations around the $z$-axis ensures the conservation of the magnetic quantum number during the excitation process.

The time-dependence of the interaction energy (2.3) is, for $\lambda = 2$, given by the collision functions

$$\bar{S}_{2,\mu}(t) = r_p^{-3}(t)\, Y_{2,\mu}[\vartheta_p(t), \varphi_p(t)]. \tag{3.22}$$

In the new coordinate system these collision functions are explicitly given by

$$\left.\begin{aligned}
\bar{S}_{2,2}(t) = \bar{S}_{2,-2}(t) &= -\sqrt{\frac{15}{32\pi}}\frac{1}{r_p(t)^3}\cdot\frac{y_p(t)^2}{r_p(t)^2} \\
\bar{S}_{2,1}(t) = \bar{S}_{2,-1}(t) &= -i\sqrt{\frac{15}{8\pi}}\frac{1}{r_p(t)^3}\cdot\frac{z_p(t)\cdot y_p(t)}{r_p(t)^2} \\
\bar{S}_{2,0}(t) &= \sqrt{\frac{5}{16\pi}}\frac{1}{r_p(t)^3}\cdot\frac{3\,z_p(t)^2 - r_p(t)^2}{r_p(t)^2}
\end{aligned}\right\} \tag{3.23}$$

For the perturbation treatment, the important quantities are the orbital integrals defined by

$$S_{E2,\mu}(\vartheta, \xi) = \int_{-\infty}^{+\infty} \bar{S}_{2,\mu}(t)\, e^{i\xi\frac{v}{a}t}\, dt. \tag{3.24}$$

In the old focal system these orbital integrals were expressed by means of the tabulated functions $I_{2,\mu}$ (see ref. 7) in the following way:

$$S^{\text{old}}_{E2,\mu} = \frac{1}{va^2}\, Y_{2,\mu}\left(\frac{\pi}{2}, 0\right) I_{2,\mu}(\vartheta, \xi). \tag{3.25}$$

In the new coordinate system one can again express the orbital integrals in terms of the $I_{2,\mu}$.

$$\left.\begin{aligned}
S_{E2,0}(\vartheta, \xi) &= \frac{1}{va^2}\sqrt{\frac{5}{16\pi}}\left\{\frac{1}{2}I_{2,0}(\vartheta, \xi) + \frac{3}{4}I_{2,2}(\vartheta, \xi) + \frac{3}{4}I_{2,-2}(\vartheta, \xi)\right\} \\
S_{E2,\pm1}(\vartheta, \xi) &= \frac{1}{va^2}\sqrt{\frac{15}{32\pi}}\left\{\frac{1}{2}I_{2,-2}(\vartheta, \xi) - \frac{1}{2}I_{2,2}(\vartheta, \xi)\right\} \\
S_{E2,\pm2}(\vartheta, \xi) &= \frac{-1}{va^2}\sqrt{\frac{15}{32\pi}}\left\{\frac{1}{2}I_{2,0}(\vartheta, \xi) - \frac{1}{4}I_{2,2}(\vartheta, \xi) - \frac{1}{4}I_{2,-2}(\vartheta, \xi)\right\}
\end{aligned}\right\} \tag{3.26}$$

Since $\bar{S}_{2,\pm1}(t)$ is an odd function of time, one sees that $S_{2,\pm1}$ vanishes for $\xi = 0$. The two remaining orbital integrals can be expressed in a way similar to (3.25):

$$S_{E2,\mu}(\vartheta, \xi = 0) = -\frac{1}{va^2}\, Y_{2,\mu}\left(\frac{\pi}{2}0\right) J_{2,\mu}(\vartheta), \tag{3.27}$$

where

$$\left.\begin{aligned} J_{2,\pm 2}(\vartheta) &= \frac{1}{2}\left[I_{2,0}(\vartheta,0) - I_{2,2}(\vartheta,0)\right] \\ J_{2,0}(\vartheta) &= \frac{3}{2} I_{2,2}(\vartheta,0) + \frac{1}{2} I_{2,0}(\vartheta,0). \end{aligned}\right\} \quad (3.28)$$

These quantities can be expressed by elementary functions, since (see ref. 1, Eq. (II E. 71))

$$\left.\begin{aligned} I_{2,\pm 2}(\vartheta,0) &= \frac{2}{3}\sin^2\frac{\vartheta}{2} \\ I_{2,0}\ (\vartheta,0) &= 2\tan^2\frac{\vartheta}{2}\left[1 - \frac{\pi-\vartheta}{2}\tan\frac{\vartheta}{2}\right] \end{aligned}\right\} \quad (3.29)$$

and they are tabulated in Table 2.

TABLE 2.

The classical orbital integrals for $\xi = 0$ in the coordinate system of Fig. 1. In the two first columns, the functions $J_{2,0}(\vartheta)$ and $J_{2,2}(\vartheta)$ (see Eq. (3.28)) are listed as functions of the deflection angle $\vartheta$. The third column shows the ratio $J_{2,2}(\vartheta)/J_{20}(\vartheta)$ which is important for the $\chi(\vartheta)$ approximation, while the last two columns contain the quantities

$$\chi_{\mathrm{eff}}(\vartheta)/\chi = J_{2,0}(\vartheta)\ /\ J_{2,0}(\pi) \text{ and } \chi(\vartheta)\ /\ \chi = \sqrt{(J_{20}(\vartheta))^2 + 3\,(J_{22}(\vartheta))^2}\ /\ J_{20}(\pi).$$

The entries are given in the form of a number followed (in paranthesis) by the power of ten by which it should be multiplied.

| $\vartheta$ | $J_{2,0}(\vartheta)$ | $J_{2,2}(\vartheta)$ | $J_{2,2}/J_{2,0}$ | $\chi_{\mathrm{eff}}/\chi$ | $\chi(\vartheta)/\chi$ |
|---|---|---|---|---|---|
| 0 | 0.0000 | 0.0000 | 3.333 (—1) | 0.00000 | 0.00000 |
| 10 | 1.4257 (—2) | 4.1288 (—3) | 2.896 (—1) | 0.01069 | 0.01196 |
| 20 | 5.3589 (—2) | 1.3386 (—2) | 2.498 (—1) | 0.04019 | 0.04379 |
| 30 | 1.1360 (—1) | 2.4285 (—2) | 2.138 (—1) | 0.08520 | 0.09085 |
| 40 | 1.9054 (—1) | 3.4574 (—2) | 1.814 (—1) | 0.1429 | 0.1498 |
| 50 | 2.8102 (—1) | 4.2878 (—2) | 1.526 (—1) | 0.2108 | 0.2180 |
| 60 | 3.8180 (—1) | 4.8467 (—2) | 1.269 (—1) | 0.2864 | 0.2932 |
| 70 | 4.8973 (—1) | 5.1078 (—2) | 1.043 (—1) | 0.3673 | 0.3732 |
| 80 | 6.0169 (—1) | 5.0792 (—2) | 8.442 (—2) | 0.4513 | 0.4561 |
| 90 | 7.1460 (—1) | 4.7935 (—2) | 6.708 (—2) | 0.5360 | 0.5396 |
| 100 | 8.2543 (—1) | 4.2997 (—2) | 5.209 (—2) | 0.6191 | 0.6216 |
| 110 | 9.3125 (—1) | 3.6570 (—2) | 3.927 (—2) | 0.6984 | 0.7000 |
| 120 | 1.0293 | 2.9301 (—2) | 2.847 (—2) | 0.7720 | 0.7729 |
| 130 | 1.1170 | 2.1831 (—2) | 1.954 (—2) | 0.8378 | 0.8382 |
| 140 | 1.1921 | 1.4772 (—2) | 1.239 (—2) | 0.8941 | 0.8943 |
| 150 | 1.2527 | 8.6659 (—3) | 6.918 (—3) | 0.9395 | 0.9396 |
| 160 | 1.2971 | 3.9600 (—3) | 3.053 (—3) | 0.9728 | 0.9728 |
| 170 | 1.3242 | 1.0393 (—3) | 7.848 (—4) | 0.9932 | 0.9932 |
| 180 | 1.3333 | 0.0000 | 0.000 | 1.0000 | 1.0000 |

For the special case $\vartheta = \pi$, one has $y_p = 0$, and the only non-vanishing collision function is $\bar{S}_{2,0}(t)$. In this case, $J_{2,\mu}(\pi) = 4/3\,\delta_{\mu,0}$. The special simplification for backward scattering is connected with the symmetry of the problem around the $z$-axis.

Also for $\vartheta \neq \pi$ one can obtain some general rules by symmetry considerations. The Hamiltonian is thus always invariant under a reflection in the plane of the orbit. This reflection brings a state vector $|I, M\rangle$ with spin $I$ and magnetic quantum number $M$ into a state $|I, -M\rangle$. One finds

$$|I, M\rangle \to (-1)^{p+M+I}\,|I, -M\rangle, \tag{3.30}$$

where $p$ is the parity of the state. This rule implies that the excitation probabilities of states with magnetic quantum numbers $M$ and $-M$ are equal, if the initial state is unoriented. The equality of $\bar{S}_{2,\mu}$ and $\bar{S}_{2,-\mu}$ follows also from this symmetry.

For $\xi = 0$, one has the additional symmetry that the Hamiltonian is invariant under a rotation of 180 degrees around the $z$-axis. This rotation gives rise to the following transformation:

$$|I, M\rangle \to e^{i\pi M}\,|I, M\rangle, \tag{3.31}$$

which implies

$$(-1)^{M_f - M_i} = 1, \tag{3.32}$$

where $M_f$ and $M_i$ are the magnetic quantum numbers in the final and initial states, respectively. The disappearance of $S_{1,\pm 1}$ for $\xi = 0$ is also a consequence of this symmetry.

### C. Dependence on Deflection Angle

In the sudden approximation, the interaction energy $\mathfrak{H}_E(t)$ only enters through the expression (see Eqs. (2.3), (3.22), and (3.27))

$$\frac{1}{\hbar}\int_{-\infty}^{+\infty} \mathfrak{H}_E(t)\,dt = -\frac{4\pi Z_1 e}{5\,\hbar v a^2} \sum_{\mu}{}' Y_{2,\mu}\left(\frac{\pi}{2}, 0\right) J_{2,\mu}(\vartheta)\,\mathfrak{M}^*(E2, \mu). \tag{3.33}$$

In this expression it will be convenient to collect the dependence on $Z_1$, $v$, and $a$ in the parameter $\chi$, which corresponds to the excitation from the ground state with spin $I_0$ to one of the excited states with spin $I_1$.

$$\chi_{0\to 1} = \frac{\sqrt{16\pi}}{15}\,\frac{Z_1 e}{\hbar v a^2} \cdot \frac{\langle I_0 \| \mathfrak{M}(E2) \| I_1 \rangle}{\sqrt{2I_0 + 1}}. \tag{3.34}$$

The expression (3.33) then takes the form

$$\frac{1}{\hbar}\int_{-\infty}^{+\infty}\mathfrak{H}_E(t)\,dt = -\chi_{0\to 1}\sqrt{9\pi}\sum_{\mu} Y_{2,\mu}\left(\frac{\pi}{2},\,0\right) J_{2,\mu}(\vartheta)\,\frac{\mathfrak{M}^*(E\,2,\mu)\sqrt{2\,I_0+1}}{\langle I_0\|\mathfrak{M}(E\,2)\|I_1\rangle}. \quad (3.35)$$

The relative order of magnitude of the terms with $|\mu| = 2$ and $\mu = 0$ is given by the ratio $J_{2,2}(\vartheta)\,/\,J_{2,0}(\vartheta)$. This ratio, which vanishes for $\vartheta = \pi$, is given numerically in Table 2 and it is seen that it is very small for most angles.

This observation gives rise to the convenient approximation of neglecting the terms with $|\mu| = 2$. In this approximation, (3.35) has the same form for all angles and one may write it as follows:

$$\frac{1}{\hbar}\int_{-\infty}^{+\infty}\mathfrak{H}_E(t)\,dt = -\chi_{\mathrm{eff}}(\vartheta)\sqrt{9\pi}\,Y_{2,0}\left(\frac{\pi}{2},\,0\right) J_{2,0}(\pi)\,\frac{\mathfrak{M}^*(E\,2{,}0)\sqrt{2\,I_0+1}}{\langle I_0\|\mathfrak{M}(E\,2)\|I_1\rangle}, \quad (3.36)$$

where

$$\chi_{\mathrm{eff}}(\vartheta) = \chi_{0\to 1}\frac{J_{2,0}(\vartheta)}{J_{2,0}(\pi)} = \frac{3}{4}J_{2,0}(\vartheta)\,\chi_{0\to 1}. \quad (3.37)$$

If one uses the approximate interaction Hamiltonian (3.36), the final state vector for arbitrary deflection angles $|\Phi(\vartheta,\chi)\rangle$ is simply related to the state vector for backward scattering, i. e.,

$$|\Phi(\vartheta,\chi)\rangle \approx |\Phi(\pi,\chi_{\mathrm{eff}}(\vartheta))\rangle. \quad (3.38)$$

The accuracy of this approximation can easily be estimated by writing the state vector $|\Phi(\vartheta,\chi)\rangle$ in the form

$$|\Phi(\vartheta,\chi)\rangle = e^{i\chi_{\mathrm{eff}}(\vartheta)\sqrt{\frac{15}{2}}\frac{J_{22}(\vartheta)}{J_{20}(\vartheta)}\frac{\sqrt{2\,I_0+1}}{\langle I_0\|\mathfrak{M}(E\,2)\|I_1\rangle}\sum_{\mu=\pm 2}\mathfrak{M}^*(E\,2,\mu)}\,|\Phi(\pi,\chi_{\mathrm{eff}}(\vartheta))\rangle, \quad (3.39)$$

which follows from (3.11) and (3.35). In this expression, a series development of the exponential function may be performed, and one is thus led to the following expansion which contains (3.38) as the first term:

$$\left.\begin{aligned}
&|\Phi(\vartheta,\chi)\rangle = |\Phi(\pi,\chi_{\mathrm{eff}}(\vartheta))\rangle \\
&+ i\chi_{\mathrm{eff}}(\vartheta)\sqrt{\frac{15}{2}}\frac{J_{22}(\vartheta)}{J_{20}(\vartheta)}\frac{\sqrt{2\,I_0+1}}{\langle I_0\|\mathfrak{M}(E\,2)\|I_1\rangle}\sum_{\mu=\pm 2}\mathfrak{M}^*(E\,2,\mu)\,|\Phi(\pi,\chi_{\mathrm{eff}}(\vartheta))\rangle \\
&-(\chi_{\mathrm{eff}}(\vartheta))^2\frac{15}{4}\left(\frac{J_{22}(\vartheta)}{J_{20}(\vartheta)}\right)^2\frac{2\,I_0+1}{|\langle I_0\|\mathfrak{M}(E\,2)\|I_1\rangle|^2}\sum_{\mu\mu'}\mathfrak{M}^*(E2,\mu)\mathfrak{M}^*(E2,\mu')\,|\Phi(\pi,\chi_{\mathrm{eff}}(\vartheta))\rangle \\
&+\cdots
\end{aligned}\right\} \quad (3.40)$$

An indication of the accuracy of the approximation (3.36) can be obtained by applying it to the old perturbation calculation. For $\xi = 0$ one thus finds, by considering only the term with $\mu = 0$, a total cross section which only differs 5 per cent from the correct one, even though the forward angles, where the approximation is worst, here play a rather important role.

In the following, we shall apply the approximation (3.38) and in a number of cases also investigate the accuracy by calculating the correction terms in (3.40).

We have earlier, in Section 2, introduced a quantity $\chi(\vartheta)$ (see Eq. (2.12)) which is not very different from $\chi_{\text{eff}}(\vartheta)$; the connection between them is given by

$$\left.\begin{aligned} \chi(\vartheta) &= \chi \frac{3}{4} \sqrt{(J_{20}(\vartheta))^2 + 3(J_{22}(\vartheta))^2} \\ &\approx \chi_{\text{eff}}(\vartheta) \left\{ 1 + \frac{3}{2} \left( \frac{J_{22}(\vartheta)}{J_{20}(\vartheta)} \right)^2 + \cdots \right\}. \end{aligned}\right\} \tag{3.41}$$

As can be seen from Table 2, the two quantities $\chi(\vartheta)$ and $\chi_{\text{eff}}(\vartheta)$ differ at the most by 15 per cent, but for most angles the difference is much smaller. For foreward angles where the difference is largest, the excitation process can essentially be treated by the first order perturbation theory, where the excitation probability is $|\chi(\vartheta)|^2$. If we thus substitute $\chi(\vartheta)$ for $\chi_{\text{eff}}(\vartheta)$ in the approximation (3.38), we have made a change only of the order of $(J_{2,2}(\vartheta)/J_{2,0}(\vartheta))^2$, but on the other hand obtained an expression which leads to the correct result for the excitation probability for foreward angles.

In the more general case where the sudden approximation is not applicable, the interaction energy $\mathfrak{H}_E(t)$ enters in a more complicated way into the problem. For $\vartheta = \pi$, again only the term with $\mu = 0$ will appear. For other angles, however, the order of magnitude of the terms with $\mu \neq 0$ depends directly on the collision functions (see Eq. (3.23)). These, especially $\bar{S}_{2,1}(t)$, are in general not very small compared to $\bar{S}_{2,0}(t)$, and the approximation is thus only valid in the neighbourhood of $\vartheta = \pi$. One may here investigate the angular dependence by considering the terms with $\mu \neq 0$ in $\mathfrak{H}_E(t)$ as a perturbation in the Hamiltonian. In the sudden approximation, this method would just lead to the result (3.40).

For the sake of completeness it should be mentioned that, once the final amplitudes $a_{I,M}(\infty)$ on the states with spin $I$ and magnetic quantum number $M$ are known, one may easily obtain all the quantities which are important for the experiments. Thus, the total excitation probability of a level of spin $I_f$ is given by

$$P_{I_f I_i} = \frac{1}{2 I_i + 1} \sum_{M_i, M_f} | a_{I_f M_f} |^2. \tag{3.42}$$

The differential cross section $d\sigma$ is obtained by multiplying $P$ with the Rutherford cross section, i. e.,

$$\left.\begin{aligned} d\sigma &= P_{I_f I_i} d\sigma_R \\ &= \frac{a^2}{4} P_{I_f I_i} \sin^{-4}\left(\frac{\vartheta}{2}\right) d\Omega. \end{aligned}\right\} \tag{3.43}$$

The angular distribution of $\gamma$-quanta emitted after the excitation is also calculable from the amplitudes. One must here take into account that a level which emits the $\gamma$-quantum under consideration may be populated not only through an excitation, but also through the deexcitation by cascade $\gamma$'s from higher excited states.

## 4. Diagonalization Method

In this section, we shall discuss a method of evaluating the multiple Coulomb excitation which does not use any specific nuclear model. We shall thus consider the properties of the nuclear states, i. e., energies and transition matrix elements as empirically determined quantities. Since, in this case, we have a very large number of parameters in the problem, it is not practically possible to give a systematic numerical tabulation of cross sections, etc., and we shall therefore confine ourselves to a few numerical examples which illustrate some important aspects of the problem.

### A. Sudden Approximation

In the sudden approximation (3.11), we have the following expression for the final amplitude $a_n$ on the state $| n \rangle$:

$$a_n = \langle n | e^{-\frac{i}{\hbar}\int_{-\infty}^{\infty} \mathfrak{H}_E(t)\, dt} | 0 \rangle, \tag{4.1}$$

where the exponent is given by (3.33) or (3.35). If the wave functions of the nuclear states are known, the problem is reduced to calculate matrix elements of a known operator. Usually one will be interested, however, in calculating the cross sections from a knowledge of the matrix elements of

the multipole operators themselves. These matrix elements enter in other processes also and are often determined from nuclear spectroscopy.

In order to perform this calculation, we introduce a unitary transformation $U$ which diagonalizes the hermitian operator (3.33) and which is thus defined by the equations

$$U^\dagger U = UU^\dagger = 1 \tag{4.2}$$

and

$$\left.\begin{aligned} &\langle n \mid U^\dagger \frac{1}{\hbar}\int_{-\infty}^{+\infty} \mathfrak{H}_E(t)\, dt\, U \mid q\rangle = \delta_{nq}\cdot\lambda_q \\ &= \sum_{m,p} \langle n \mid U^\dagger \mid m\rangle\langle m \mid \frac{1}{\hbar}\int_{-\infty}^{+\infty} \mathfrak{H}_E(t)\, dt \mid p\rangle\langle p \mid U \mid q\rangle. \end{aligned}\right\} \tag{4.3}$$

The result (4.1) can then be expressed in terms of $U$ and the eigenvalues $\lambda_q$ in the following way:

$$\left.\begin{aligned} a_n &= \langle n \mid UU^\dagger e^{-\frac{i}{\hbar}\int_{-\infty}^{\infty} \mathfrak{H}_E(t)\, dt} UU^\dagger \mid 0\rangle \\ &= \sum_m \langle n \mid U \mid m\rangle e^{-i\lambda_m}\langle m \mid U^\dagger \mid 0\rangle \\ &= \sum_m \langle n \mid U \mid m\rangle\langle 0 \mid U \mid m\rangle^* e^{-i\lambda_m}. \end{aligned}\right\} \tag{4.4}$$

The determination of $U$ requires the knowledge of the matrix elements of the operator (3.33). If we specify the nuclear states by means of the spin $I_n$ and magnetic quantum number $M_n$, these matrix elements are expressible by the reduced multipole matrix elements*[9] defined by

$$\langle I_m \| \mathfrak{M}(E2) \| I_n\rangle = (-1)^{I_m - M_m}\begin{pmatrix} I_m & 2 & I_n \\ -M_m & \mu & M_n \end{pmatrix}^{-1}\langle I_m M_m \mid \mathfrak{M}(E2,\mu) \mid I_n M_n\rangle. \tag{4.5}$$

For the diagonalization it will be convenient to apply the $\chi(\vartheta)$ approximation. In this approximation, the operator $\int_{-\infty}^{\infty} \mathfrak{H}_E(t)\, dt$ (see Eq. (3.36)) is diagonal in $M$, and one may write the matrix elements in the form

$$\langle I_m M \mid \frac{1}{\hbar}\int_{-\infty}^{+\infty} \mathfrak{H}_E(t)\, dt \mid I_n M'\rangle \approx \chi_{\mathrm{eff}}(\vartheta)\, \varrho^M_{mn}\delta_{MM'}, \tag{4.6}$$

where the (symmetric) matrix $\varrho^M_{mn}$ is defined by

* For the angular momentum algebra we use throughout this paper the notation of ref. 9.

$$\varrho_{mn}^{M} = (-1)^{I_m - M} \sqrt{5\,(2\,I_0+1)} \begin{pmatrix} I_m & 2 & I_n \\ -M & 0 & M \end{pmatrix} \frac{\langle I_m \| \mathfrak{M}(E2) \| I_n \rangle}{\langle I_0 \| \mathfrak{M}(E2) \| I_1 \rangle}. \qquad (4.7)$$

The number of states which have to be included in the matrix (4.6) by the diagonalization depends of course on the $\chi$'s. Only those states which are mutually connected with large (collective) matrix elements must be taken into account. One may furthermore classify these in different groups where states within a group are strongly coupled, while states from two different groups are weakly coupled. A group consists, e. g., of the states in a rotational band, and the different groups are the bands belonging to different single-particle states. For each group one must perform the diagonalization and must here take into account a number of those states which are most directly coupled to the ground state. This number will depend on the $\chi$'s and must be determined so that the inclusion of still more states would not change the result. The weak interplay between the groups can be treated by a perturbation calculation.

Since the energy of the projectile and the deflection angle enter only through the common factor $\chi_{\text{eff}}(\vartheta)$, the diagonalization can be used for all energies and all $\vartheta$'s.

The deviation from the $\chi_{\text{eff}}(\vartheta)$ approximation is given by the expression (3.40) which we may write explicitly in the form

$$\left.\begin{aligned} a_{I_f M_f} &= a^{(0)}_{I_f M_f}(\chi_{\text{eff}}(\vartheta))\,\delta_{M_i M_f} \\ &+ i\chi_{\text{eff}}(\vartheta)\frac{J_{22}(\vartheta)}{J_{20}(\vartheta)} \sum_{z\,\mu=\pm 2} \sqrt{\frac{3}{2}} \begin{pmatrix} I_z & 2 & I_f \\ -M_i & \mu & M_f \end{pmatrix} \begin{pmatrix} I_z & 2 & I_f \\ -M_i & 0 & M_i \end{pmatrix}^{-1} \varrho_{zf}^{M_i}\, a^{(0)}_{I_z M_i}(\chi_{\text{eff}}(\vartheta)) \\ &- \left(\chi_{\text{eff}}(\vartheta)\frac{J_{22}(\vartheta)}{J_{20}(\vartheta)}\right)^2 \sum_{zz'\,\mu\mu'=\pm 2} \frac{3}{4} \begin{pmatrix} I_z & 2 & I'_z \\ -M_i & \mu & M'_z \end{pmatrix} \begin{pmatrix} I_z & 2 & I'_z \\ -M_i & 0 & M_i \end{pmatrix}^{-1} \varrho_{zz'}^{M_i} \\ &\quad \times \begin{pmatrix} I'_z & 2 & I_f \\ -M'_z & \mu & M_f \end{pmatrix} \begin{pmatrix} I'_z & 2 & I_f \\ -M_i & 0 & M_i \end{pmatrix}^{-1} \varrho_{z'f}^{M_i}\, a^{(0)}_{I_z M_i}(\chi_{\text{eff}}(\vartheta)) \\ &+ \cdots, \end{aligned}\right\} \qquad (4.8)$$

where $a^{(0)}$ indicates the amplitude in the $\chi_{\text{eff}}(\vartheta)$ approximation. Since, in this approximation, $M_i = M_f$, it is seen that, while the first term only contributes to this substate, the second term proportional to $J_{22}(\vartheta) / J_{20}(\vartheta)$ only contributes to the states with $M_f = M_i \pm 2$. The third term proportional to $(J_{22}(\vartheta) / J_{20}(\vartheta))^2$ contributes to both $M_f = M_i$ and $M_f = M_i \pm 4$. The excitation probability will thus contain no terms linear in $J_{22}(\vartheta) / J_{20}(\vartheta)$. The

terms quadratic in this quantity will arise partly from the square of the second term, and partly from an interference between the first and third term.

One may also avoid the $\chi(\vartheta)$ approximation and the expansion (4.8) by directly diagonalizing the matrix of the complete Hamiltonian (3.35). This matrix is no more diagonal in the magnetic quantum number $M$ and is essentially different for different angles so that the diagonalization will have to be performed for all angles.

For not too large values of $\chi(\vartheta)$ it may be advantageous to use the perturbation expansion to higher order instead of the diagonalization method. The power series expansion of (3.11) leads to the following expression for the excitation amplitude:

$$\left.\begin{aligned} a_n &= \langle I_n M \mid e^{-\frac{i}{\hbar}\int_{-\infty}^{\infty} \mathfrak{H}_E(t)\,dt} \mid I_0 M\rangle \\ &= \delta_{no} - i\chi(\vartheta)\varrho^M_{no} - \frac{1}{2!}\chi(\vartheta)^2 \sum_r \varrho^M_{nr}\varrho^M_{ro} \\ &\quad + \frac{i}{3!}\chi(\vartheta)^3 \sum_{rs} \varrho^M_{nr}\varrho^M_{rs}\varrho^M_{so} + \cdots . \end{aligned}\right\} \tag{4.9}$$

This expansion can also be useful for the discussion of small changes in the matrix elements, e. g., from a rotational model.

### B. Examples

In this section, we shall consider some examples of the methods discussed above. They will mainly be given in order to illustrate how many levels one has to take into account in the diagonalization method, and secondly to illustrate the accuracy of the perturbation expansion and the $\chi(\vartheta)$ approximation. For the sake of comparison with the exact treatment (see Section 5), we shall use the matrix elements characteristic of a rotational band.

For a pure rotational band, one may express the reduced matrix elements entering in (4.7) by means of the constant intrinsic quadrupole moment $Q_0$. One finds

$$\left.\begin{aligned} \langle I_m \| \mathfrak{M}(E2) \| I_n \rangle = \sqrt{\frac{5}{16\pi}} (-1)^{I_m - K} (2I_m + 1)^{1/2} (2I_n + 1)^{1/2} \\ \begin{pmatrix} I_m & 2 & I_n \\ -K & 0 & K \end{pmatrix} e\, Q_0 , \end{aligned}\right\} \tag{4.10}$$

where $K$ is the (constant) projection of the total angular momentum on the nuclear symmetry axis. We shall consider only the case of an even-even nucleus with ground state spin $I_0 = K = 0$. In this case, the matrix $\varrho_{mn}$ (see Eq. (4.7)) takes the form

$$\varrho_{mn} = \sqrt{5}\,(2I_m+1)^{1/2}(2I_n+1)^{1/2}\begin{pmatrix} I_m & 2 & I_n \\ 0 & 0 & 0 \end{pmatrix}^2. \tag{4.11}$$

We shall now successively take more and more states into account. If we include only the ground state and the first excited state $I_1 = 2$, we have to diagonalize the matrix

$$\varrho_{mn} = \left\{ \begin{array}{cc} 0 & 1 \\ 1 & \frac{2\sqrt{5}}{7} \end{array} \right\}. \tag{4.12}$$

The eigenvalues of this matrix are

$$\lambda_0 = \frac{\sqrt{5}-3\sqrt{6}}{7} \quad \text{and} \quad \lambda_1 = \frac{\sqrt{5}-3\sqrt{6}}{7}. \tag{4.13}$$

The unitary matrix which diagonalizes (4.12) is then found to be

$$\langle m \mid U \mid n \rangle = \left\{ \begin{array}{cc} \frac{\sqrt{18+\sqrt{30}}}{6} & \frac{\sqrt{18-\sqrt{30}}}{6} \\ -\frac{\sqrt{18-\sqrt{30}}}{6} & \frac{\sqrt{18+\sqrt{30}}}{6} \end{array} \right\}. \tag{4.14}$$

According to (4.4) we thus obtain the result

$$\left.\begin{aligned} a_0 &= \frac{18+\sqrt{30}}{36}\, e^{-i\chi(\vartheta)\frac{\sqrt{5}-3\sqrt{6}}{7}} + \frac{18-\sqrt{30}}{36}\, e^{-i\chi(\vartheta)\frac{\sqrt{5}+3\sqrt{6}}{7}} \\ &= e^{-i\frac{\sqrt{5}}{7}\chi(\vartheta)}\left[\cos\frac{3\sqrt{6}}{7}\chi(\vartheta) + i\sqrt{\frac{5}{54}}\sin\frac{3\sqrt{6}}{7}\chi(\vartheta)\right] \\ a_1 &= -\frac{7\sqrt{6}}{36}\, e^{-i\chi(\vartheta)\frac{\sqrt{5}-3\sqrt{6}}{7}} + \frac{7\sqrt{6}}{36}\, e^{-i\chi(\vartheta)\frac{\sqrt{5}+3\sqrt{6}}{7}} \\ &= i\frac{7\sqrt{6}}{18}\, e^{-i\frac{\sqrt{5}}{7}\chi(\vartheta)}\sin\frac{3\sqrt{6}}{7}\chi(\vartheta). \end{aligned}\right\} \tag{4.15}$$

The excitation probability $P_2 = |a_1|^2$ is then

$$P_2 = \frac{49}{54} \sin^2 \frac{3\sqrt{6}}{7} \chi(\vartheta). \tag{4.16}$$

This quantity and the probability that the nucleus is left in the ground state $P_0 = |a_0|^2 = 1 - P_2$ are illustrated in Fig. 2 as a function of $\chi(\vartheta)$.

In Fig. 3 and Fig. 4 are shown the extensions of the above calculation to include two and four excited states in the rotational band. The matrix $\varrho_{mn}$, in the latter case, is explicitly given by

$$\varrho_{mn} = \left\{ \begin{array}{ccccc} 0 & 1.0000 & 0 & 0 & 0 \\ 1.0000 & 0.6389 & 0.8571 & 0 & 0 \\ 0 & 0.8571 & 0.5808 & 0.8457 & 0 \\ 0 & 0 & 0.8457 & 0.5692 & 0.8423 \\ 0 & 0 & 0 & 0.8423 & 0.5649 \end{array} \right\} \tag{4.17}$$

In the case that one includes only two of the excited states, one finds the eigenvalues

$$\lambda_0 = -0.9270, \quad \lambda_1 = 0.3484 \quad \text{and} \quad \lambda = 1.7984 \tag{4.18}$$

and the matrix $U$ is then

$$\langle m | U | n \rangle = \left\{ \begin{array}{ccc} 0.6840 & -0.6006 & 0.4139 \\ -0.6341 & -0.2093 & 0.7444 \\ 0.3605 & 0.7717 & 0.5240 \end{array} \right\} \tag{4.19}$$

The final amplitudes on the three states are thus, according to (4.4),

$$\left. \begin{array}{l} a_0 = 0.4679\, e^{i0.9270\chi} + 0.3608\, e^{-i0.3484\chi} + 0.1713\, e^{-i1.7984\chi} \\ a_1 = -0.4338\, e^{i0.9270\chi} + 0.1257\, e^{-i0.3484\chi} + 0.3081\, e^{-i1.7984\chi} \\ a_2 = 0.2466\, e^{i0.9270\chi} - 0.4635\, e^{-i0.3484\chi} + 0.2169\, e^{-i1.7984\chi} \end{array} \right\} \tag{4.20}$$

Similarly, one finds for the complete matrix (4.17) the eigenvalues

$$\left. \begin{array}{l} \lambda_0 = -1.0437 \\ \lambda_1 = -0.4880 \\ \lambda_2 = 0.4302 \\ \lambda_3 = 1.3920 \\ \lambda_4 = 2.0633 \end{array} \right\} \tag{4.21}$$

and the matrix U

$$\langle m \mid U \mid n \rangle = \begin{Vmatrix} 0.5436 & -0.5189 & 0.4681 & -0.3866 & 0.2582 \\ -0.5674 & 0.2532 & 0.2014 & -0.5381 & 0.5328 \\ 0.4795 & 0.2725 & -0.5951 & -0.0218 & 0.5841 \\ -0.3461 & -0.6010 & -0.0981 & 0.5245 & 0.4840 \\ 0.1812 & 0.4808 & 0.6137 & 0.5342 & 0.2721 \end{Vmatrix} \quad (4.22)$$

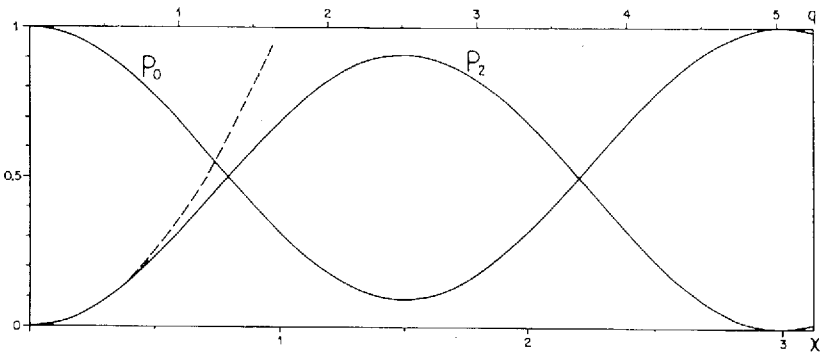


Fig. 2. The result of the two-state calculation for a rotational band on a $0^+$ ground state. The probability for the excitation in the $2^+$ state, $P_2$, and the probability for no excitation, $P_0$, are given as functions of $\chi_{0\to 2}(\vartheta)$ and as a function of the parameter $q(\vartheta)$ characteristic of the rotational model (see Eq. (5.11)). The broken curve shows the result of the first order perturbation calculation.

Figs. 2, 3, and 4 show a very general feature of the multiple excitation process. The excitation probability for a definite state has a maximum as a function of $\chi$. Where this maximum is reached depends on how directly the state is connected with the ground state. The more intermediate states that have to be passed, the higher is the value of $\chi$ for which the maximum is attained. For the rotational band on the $0^+$ ground state, the $2^+$ state is maximally excited for $\chi \cong 1$, the $4^+$ for $\chi \cong 2$, the $6^+$ for $\chi \cong 3$, etc. The heights of the maxima decrease as one passes to higher excited states, partly because a small tail is left in the excitation probability of the lower states. If the band is broken off as in the above calculation, the maximum in the excitation of the last state is much higher than that of any of the others.

A comparison of the curves shows that the deletion of higher states practically does not change the excitation probability of the lower states. This is true at least as long as the last state included is not strongly excited. In Fig. 2, the curve for $P_0$ is thus essentially correct until $\chi = 0.8$. In Fig. 3, the curves for $P_0$ and $P_2$ are similarly correct until $\chi = 1.5$ and, in Fig. 4, one expects $P_0$, $P_2$, $P_4$, and $P_6$ to be correct until $\chi = 3$.

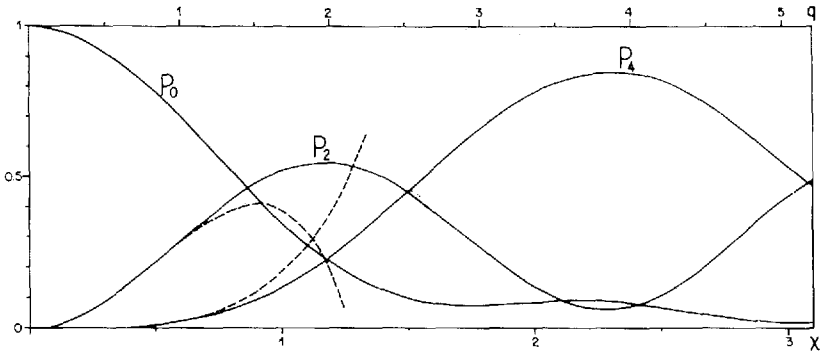


Fig. 3. The result of the three-state calculation for a rotational band on a $0^+$ ground state. The excitation probability in the $2^+$ and $4^+$ states, $P_2$ and $P_4$, and the probability for no excitation, $P_0$, are given as a function of $\chi_{0\to 2}(\vartheta)$ and as a function of the parameter $q(\vartheta)$ characteristic of a rotational model (see Eq. (5.11)). The broken curves show the result of the second order perturbation calculation.

It is interesting to compare the above results with the perturbation calculation. According to the equation (4.9) one finds

$$\left.\begin{aligned}
a_0 &= 1 \qquad -0.5000\,\chi(\vartheta)^2 + i\,0.1065\,\chi(\vartheta)^3 + 0.0893\,\chi(\vartheta)^4 - i\,0.0242\,\chi(\vartheta)^5 \\
&\qquad -0.0092\,\chi(\vartheta)^6 \\
a_1 &= -i\,\chi(\vartheta) - 0.3195\,\chi(\vartheta)^2 + i\,0.3571\,\chi(\vartheta)^3 + 0.1210\,\chi(\vartheta)^4 - i\,0.0551\,\chi(\vartheta)^5 \\
a_2 &= \qquad -0.4286\,\chi(\vartheta)^2 + i\,0.1742\,\chi(\vartheta)^3 + 0.1274\,\chi(\vartheta)^4 \\
a_3 &= \qquad\qquad + i\,0.1208\,\chi(\vartheta)^3
\end{aligned}\right\} \quad (4.23)$$

The power expansion in $\chi$ of the excitation probabilities contains (since $\xi = 0$) only even powers of $\chi$. It is noted that, e. g., a third order perturbation calculation leads to the correct answer for $P_6$ to terms of the order of $\chi^6$, while $P_2$ and $P_4$ are correct only to terms of the order of $\chi^4$, and $P_0$ only to terms of the order of $\chi^2$.

In the comparison of the perturbation expansion with the more exact treatment given above, we have calculated the excitation probabilities to second, fourth, and sixth order. In Fig. 2 is shown the calculation to second order (first order perturbation). This gives a good approximation only up to $\chi \cong 0.4$. In Fig. 3 is shown the calculation to fourth order in $\chi$. This is good up to $\chi \cong 0.7$. Similarly, the calculation to sixth order in $\chi$ shown on Fig. 4 is seen to be correct only up to $\chi \cong 1.0$. It thus seems that the perturbation expansion only offers a poor approximation for large values of $\chi$.

The accuracy of the $\chi(\vartheta)$ approximation which we have used can be

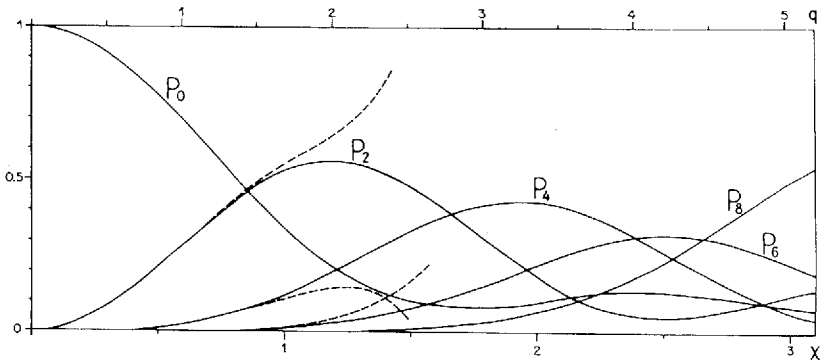


Fig. 4. The result of the five-state calculation for a rotational band on a $0^+$ ground state. The excitation probabilities for the $2^+$, $4^+$, $6^+$ and $8^+$ states $P_2$, $P_4$, $P_6$ and $P_8$, and the probability for no excitation, $P_0$, are given as a function of $\chi_{0\to 2}(\vartheta)$ and as a function of the parameter $q(\vartheta)$ characteristic of a rotational model (see Eq. (5.11)). The broken curves show the result of the third order perturbation calculation.

evaluated by means of Eq. (4.8). We shall only do the explicit calculation in the case of the two-state model (4.12) to (4.16). One finds directly from (4.8) the following expressions to second order in $J_{22}(\vartheta)/J_{20}(\vartheta)$:

$$\left.\begin{aligned}
\iota_{22} &= i\chi_{\mathrm{eff}}(\vartheta)\frac{J_{22}(\vartheta)}{J_{20}(\vartheta)}e^{-i\frac{\sqrt{5}}{7}\chi_{\mathrm{eff}}(\vartheta)}\left[\sqrt{\frac{3}{2}}\cos\frac{3\sqrt{6}}{7}\chi_{\mathrm{eff}}(\vartheta)+i\frac{\sqrt{5}}{2}\sin\frac{3\sqrt{6}}{7}\chi_{\mathrm{eff}}(\vartheta)\right]\\
\iota_{20} &= a_1^{(0)}-\left(\chi_{\mathrm{eff}}(\vartheta)\frac{J_{22}(\vartheta)}{J_{20}(\vartheta)}\right)^2 e^{-i\frac{\sqrt{5}}{7}\chi_{\mathrm{eff}}(\vartheta)}\left[-\frac{3\sqrt{5}}{7}\cos\frac{3\sqrt{6}}{7}\chi_{\mathrm{eff}}(\vartheta)-i\frac{5\sqrt{6}}{14}\sin\frac{3\sqrt{6}}{7}\chi_{\mathrm{eff}}(\vartheta)\right]\\
\iota_{00} &= a_0^{(0)}-\left(\chi_{\mathrm{eff}}(\vartheta)\frac{J_{22}(\vartheta)}{J_{20}(\vartheta)}\right)^2 e^{-i\frac{\sqrt{5}}{7}\chi_{\mathrm{eff}}(\vartheta)}\left[\frac{3}{2}\cos\frac{3\sqrt{6}}{7}\chi_{\mathrm{eff}}(\vartheta)+i\frac{\sqrt{30}}{4}\sin\frac{3\sqrt{6}}{7}\chi_{\mathrm{eff}}(\vartheta)\right].
\end{aligned}\right\}\quad(4.24)$$

In these expressions, $a^{(0)}$ are the amplitudes in the $\chi_{\mathrm{eff}}(\vartheta)$ approximation (4.15).

From (4.24) one obtains the excitation probability

$$P_2(\vartheta,\chi)=\frac{49}{54}\sin^2\frac{3\sqrt{6}}{7}\chi_{\mathrm{eff}}(\vartheta)+\left(\chi_{\mathrm{eff}}(\vartheta)\frac{J_{22}(\vartheta)}{J_{20}(\vartheta)}\right)^2\left[\frac{5}{6}+\frac{13}{6}\cos^2\frac{3\sqrt{6}}{7}\chi_{\mathrm{eff}}(\vartheta)\right].\quad(4.25)$$

One observes that the correction term is an oscillating function of $\chi_{\mathrm{eff}}(\vartheta)$ which has its maxima where $P_2^{(0)}$ shows its minima. The tendency of the correction is thus to fill out the minima of the excitation probability.

To illustrate the magnitude of the correction we have evaluated (4.25)

TABLE 3.

Comparison between the correct excitation probability $P_2(\vartheta, \chi)$ and the $\chi(\vartheta)$ approximation in the two-state model for $\chi = 3$. The quantity $\chi_{\text{eff}}(\vartheta)$ and the probability in the $\chi_{\text{eff}}(\vartheta)$ approximation $P_2(\chi_{\text{eff}}(\vartheta))$ as well as $\chi(\vartheta)$ and the corresponding probability $P_2(\chi(\vartheta))$ are listed for different angles together with $P_2(\vartheta, \chi)$.

| | 180° | 150° | 120° | 90° | 60° | 30° |
|---|---|---|---|---|---|---|
| $P_2(\vartheta, \chi)$ | 0.000 | 0.031 | 0.396 | 0.880 | 0.578 | 0.072 |
| $\chi_{\text{eff}}(\vartheta)$ | 3.000 | 2.820 | 2.315 | 1.698 | 0.859 | 0.256 |
| $P_2(\chi_{\text{eff}}(\vartheta))$ | 0.000 | 0.030 | 0.387 | 0.868 | 0.558 | 0.064 |
| $\chi(\vartheta)$ | 3.000 | 2.820 | 2.318 | 1.709 | 0.880 | 0.272 |
| $P_2(\chi(\vartheta))$ | 0.000 | 0.030 | 0.384 | 0.863 | 0.578 | 0.072 |

numerically in the case of $\chi = 3$, and the result is given in Table 3. One observes here that the maximum correction ($\sim 0.020$) appears for angles between 60 and 90 degrees. This is connected with the fact that $J_{22}(\vartheta)$ is maximal in this range. In the two last rows we have made a comparison with the approximation where $\chi(\vartheta)$ is used instead of $\chi_{\text{eff}}(\vartheta)$. It is seen that this approximation reproduces the correct excitation probability for small angles until the angle which gives the maximum probability. On the other side of the maximum, the $\chi(\vartheta)$ approximation is no improvement over the $\chi_{\text{eff}}(\vartheta)$ approximation.

### C. First Order Expansion in $\xi$

In this paragraph, we shall consider the first order corrections in $\xi$ to the results which were derived earlier in this section.

In Section 3A, we obtained the following expression for the amplitude to first order in $\xi$:

$$\left.\begin{aligned} a_n &= a_n(\xi = 0) \\ &+ \frac{1}{\hbar^2}\langle n \mid e^{-\frac{i}{\hbar}\int_{-\infty}^{\infty}\mathfrak{H}_E(t)\,dt}\int_{-\infty}^{\infty} dt\, t\, e^{\frac{i}{\hbar}\int_{-\infty}^{t}\mathfrak{H}_E(t')\,dt'}\,[\mathfrak{H}_0, \mathfrak{H}_E(t)]\, e^{-\frac{i}{\hbar}\int_{-\infty}^{t}\mathfrak{H}_E(t')\,dt'} \mid 0\rangle. \end{aligned}\right\} \quad (4.26)$$

We shall here make the simplifying assumption that $\vartheta = \pi$. One may use the result which we shall obtain, for other angles also, by the usual substitution $\chi \to \chi_{\text{eff}}(\vartheta)$. As was discussed earlier, the approximation is here less accurate than it was in the case of the sudden approximation.

The simplification by considering only terms with $\mu = 0$ in $\mathfrak{H}_E(t)$ is that

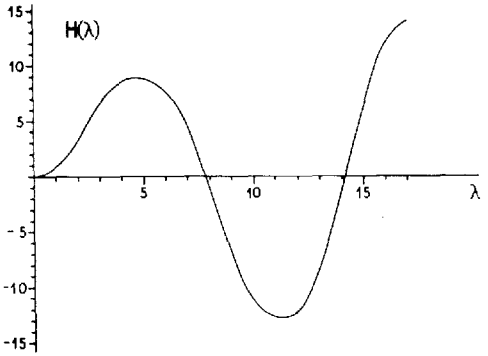


Fig. 5. The function $H(\lambda)$. This function is of importance for the evaluation of the deviation from the sudden approximation in the diagonalization method (see Eq. (4.31)).

the unitary matrix $U$ (see Eqs. (4.2) and (4.3)) diagonalizes not only $\int_{-\infty}^{\infty} \mathfrak{H}_E(t)\, dt$, but also $\int_{-\infty}^{t} \mathfrak{H}_E(t)\, dt$ and $\mathfrak{H}_E(t)$. We thus have

$$\langle n \mid U^\dagger \frac{1}{\hbar} \int_{-\infty}^{t} \mathfrak{H}_E(t)\, dt\, U \mid q \rangle = \lambda_q\, \delta_{qn} \left( \frac{1}{2} + h(t) \right) \tag{4.27}$$

and

$$\langle n \mid U^\dagger \frac{1}{\hbar} \mathfrak{H}_E(t)\, U \mid q \rangle = \lambda_q\, \delta_{qn}\, g(t), \tag{4.28}$$

where

$$h(t) = (S_{E\,2,0}(\vartheta, 0))^{-1} \int_0^t \bar{S}_{20}(t)\, dt \tag{4.29}$$

and

$$g(t) = (S_{E\,2,0}(\vartheta, 0))^{-1}\, \bar{S}_{20}(t). \tag{4.30}$$

The functions $S_{E\,2,0}$ and $\bar{S}_{2,0}$ are defined in Eqs. (3.23) and (3.24), and one sees that $h(t)$ is an odd function while $g(t)$ is an even function of $t$.

By introducing an appropriate number of factors $UU^\dagger$ in (4.26) one may write it in the form

$$\left.\begin{aligned} a_n &= a_n(\xi = 0) \\ &- i \sum_{lm} \langle n \mid U \mid l \rangle\, e^{-i\frac{1}{2}(\lambda_l + \lambda_m)}\, H(\lambda_l - \lambda_m)\, \Xi_{lm} \langle m \mid U^\dagger \mid 0 \rangle, \end{aligned}\right\} \tag{4.31}$$

where the $\Xi$ matrix is the transformed energy matrix

$$\left.\begin{aligned} \Xi_{lm} &= \frac{a}{\hbar v} (U^\dagger \mathfrak{H}_0\, U)_{lm} \\ &= \sum_p \langle l \mid U^\dagger \mid p \rangle \frac{a E_p}{\hbar v} \langle p \mid U \mid m \rangle. \end{aligned}\right\} \tag{4.32}$$

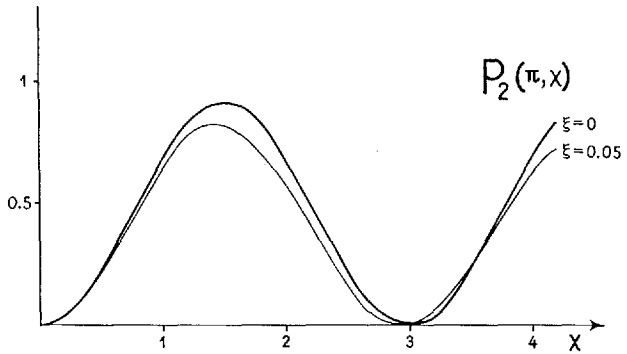


Fig. 6. The excitation probability $P_2(\pi, \lambda)$ for backward scattering in the two-state model is shown as a function of $\chi$ for $\xi = 0$ and $\xi = 0.05$.

Since a constant energy will give no contribution, one may replace the energies in (4.32) by the $\xi$'s corresponding to the excitation of the $p$'s state from the ground state, i. e.,

$$\xi_p = \frac{a(E_p - E_0)}{\hbar v}. \tag{4.33}$$

One thus finds

$$\Xi_{lm} = \sum_p \xi_p \langle p \mid U \mid l \rangle^* \langle p \mid U \mid m \rangle. \tag{4.34}$$

The function $H(\lambda)$ which appears in (4.32) is defined by

$$H(\lambda) = \frac{v}{a}\lambda \int_{-\infty}^{\infty} dt\, t\, g(t) \sin(\lambda h(t)). \tag{4.35}$$

One observes that $H(\lambda)$ is a symmetric function of $\lambda$, i.e.,

$$H(-\lambda) = H(\lambda) \tag{4.36}$$

and that the $\Xi$ matrix is symmetric also in the indices $l$ and $m$.

The function $H(\lambda)$ has been evaluated numerically and is given in Fig. 5. For small values of $\lambda$ it is quadratic in $\lambda$, as may be seen from (4.35), and one finds

$$H(\lambda) \approx 0.9172\, \lambda^2 \quad (\lambda \ll 1). \tag{4.37}$$

For larger values of $\lambda$, $H(\lambda)$ is an oscillating function whose amplitude increases slowly. From formula (4.31) one may thus draw the general conclusion that the first order correction in $\xi$ is only a slowly increasing (and oscillating) function of $\chi$.

As an illustration we shall apply the result (4.31) to the two-state model.

Using the Eqs. (4.12) to (4.16) one finds the following expression for the excitation probability to first order in $\xi$:

$$P_2 = \frac{49}{54}\sin^2\frac{3\sqrt{6}}{7}\chi - \xi\frac{49\sqrt{30}}{972}\sin\left(\frac{3\sqrt{6}}{7}\chi\right)H\left(\frac{6\sqrt{6}}{7}\chi\right). \tag{4.38}$$

This result is illustrated in Fig. 6, where the excitation probability for $\xi = 0.05$ is compared to the earlier calculated excitation probability for $\xi = 0$.

## 5. Excitation of Rotational States

In this section we shall treat the excitation of a rotational band. It will be shown that, in the sudden approximation, one can obtain a closed expression for the cross section including all (infinitely many) states in the band. The problem is analogous to the classical problem of a charged ellipsoid which is set in motion by a fast projectile. At the end of the section we shall make some comments on this classical treatment.

### A. Sudden Approximation

We shall assume that we have a pure rotational band and that only this band is involved in the excitation process. The Schrödinger equation (3.1) for the rotational motion may then be written in the form

$$i\hbar\frac{\partial\bar{\psi}}{\partial t} = H_0\bar{\psi} + H'(t)\bar{\psi}, \tag{5.1}$$

where $\bar{\psi}$ only depends on the Eulerian angles $\alpha$ and $\beta$ describing the orientation of the nuclear symmetry axis. The complete wave function $\psi$ is connected with $\bar{\psi}$ through the equation

$$\psi = e^{-\frac{i}{\hbar}E_{\mathrm{intr}}t}\bar{\psi}(\alpha, \beta, t)\chi(x'), \tag{5.2}$$

where $\chi(x')$ is the intrinsic wave function and $E_{\mathrm{intr}}$ is the intrinsic energy. The free Hamiltonian for the rotation $H_0$ is given by

$$H_0 = -\frac{\hbar^2}{2\Im}\left\{\frac{\partial^2}{\partial\beta^2} + \cot\beta\frac{\partial}{\partial\beta} + \frac{1}{\sin^2\beta}\frac{\partial^2}{\partial\alpha^2}\right\}, \tag{5.3}$$

where $\Im$ is the moment of inertia. Since the quadrupole operator can be expressed in terms of the intrinsic quadrupole moment $Q_0$ in the following way:

$$\mathfrak{M}(E2, \mu) = \frac{1}{2} Q_0 Y_{2\mu}(\beta, \alpha), \tag{5.4}$$

the interaction Hamiltonian $H'(t)$ is given by

$$H'(t) = \frac{2\pi Z_1 e^2}{5} Q_0 \sum_{\mu} \bar{S}_{2\mu}(t) Y^*_{2\mu}(\beta, \alpha). \tag{5.5}$$

The evaluation of the excitation amplitudes in the sudden approximation has now been reduced to the calculation of matrix elements of a known operator. We shall specify the eigenstates of $H_0$ by means of the spin $I$, the magnetic quantum number $M$, and the (constant) projection $K$ of the total angular momentum on the nuclear symmetry axis. The wave function may then be written

$$\bar{\psi}_{IMK} = \sqrt{\frac{2I+1}{4\pi}} D^I_{MK}(\alpha, \beta, 0), \tag{5.6}$$

where $D^I_{MK}$ is the rotation matrix. The excitation amplitude on the state specified by $I_f$, $M_f$, and $K$ is then, according to (3.11), (3.24), and (3.27),

$$\left.\begin{aligned} a_{I_f M_f} = \sqrt{\frac{(2I_i+1)(2I_f+1)}{(4\pi)^2}} \int_0^{2\pi} d\alpha \int_0^{\pi} d\beta \sin\beta \, (D^{I_f}_{M_f K}(\alpha, \beta, 0))^* \\ \times D^{I_i}_{M_i K}(\alpha, \beta, 0) \exp\left\{ i \frac{2\pi Z_1 e Q_0}{5 \hbar v a^2} \sum_{\mu} Y_{2\mu}\left(\frac{\pi}{2}, 0\right) J_{2\mu}(\vartheta) Y^*_{2\mu}(\beta, \alpha) \right\}. \end{aligned}\right\} \tag{5.7}$$

We shall now first show that the excitation of any rotational band with ground state spin $I_i$ and final state spin $I_f$ can be expressed by means of the amplitudes for the excitation of a band with ground state spin 0. This follows from (5.7) by expanding the product of the two $D$-functions on $D$-functions. The amplitude (5.7) may then be expressed in the following way:

$$\left.\begin{aligned} a_{I_f M_f} = \sum_I (2I_i+1)^{1/2} (2I_f+1)^{1/2} (2I+1) (-1)^{M_i - K} \\ \times \begin{pmatrix} I_f & I_i & I \\ -M_f & M_i & M_f - M_i \end{pmatrix} \begin{pmatrix} I_f & I_i & I \\ -K & K & 0 \end{pmatrix} A_{I, M_f - M_i}(\vartheta, q), \end{aligned}\right\} \tag{5.8}$$

where we have introduced the functions

$$\left.\begin{aligned} A_{IM}(\vartheta, q) &= [4\pi(2I+1)]^{-1/2} \\ &\times \int_0^{2\pi} d\alpha \int_0^{\pi} d\beta \sin\beta\, Y_{IM}(\beta, \alpha)\, e^{i\frac{8\pi}{5} q \sum_{\mu} Y_{2\mu}\left(\frac{\pi}{2}, 0\right) J_{2\mu}(\vartheta)\, Y^*_{2\mu}(\beta, \alpha)} . \end{aligned}\right\} \quad (5.9)$$

One observes that these functions are proportional to the amplitudes on the state $I, M$ in a rotational band with ground state spin 0, i. e.,

$$A_{IM}(\vartheta, q) = \frac{1}{\sqrt{2I+1}}\, a_{IM}(I_i = M_i = 0). \quad (5.10)$$

The quantity $q$ is defined by

$$q = \frac{Z_1\, e\, Q_0}{4\hbar\sqrt{\nu a^2}}. \quad (5.11)$$

This quantity is independent of the spins in the rotational band and plays the role of a common $\chi$. It is connected with the $\chi$ corresponding to the first excitation in an even-even nucleus with the same intrinsic quadrupole moment by the relation

$$q = \sqrt{\frac{45}{16}}\, \chi_{0\to 2}. \quad (5.12)$$

The calculation of the excitation cross sections of any rotational band is then reduced to the determination of the functions $A_{I,M}(\vartheta, q)$. From Eq. (5.8) one obtains, e. g., according to (3.42), the following formula for the excitation probability of the state of spin $I_f$:

$$P_{I_f I_i} = (2I_f+1) \sum_{IM} (2I+1) \begin{pmatrix} I_f & I_i & I \\ -K & K & 0 \end{pmatrix}^2 |A_{IM}(\vartheta, q)|^2. \quad (5.13)$$

The functions $A_{I,M}$ can most easily be evaluated in the $\chi(\vartheta)$ approximation where the terms with $|\mu| = 2$ in the exponential function are neglected. The integration over $\alpha$ in (5.9) shows then that $A_{I,M}$ vanishes except for $M = 0$, where one finds

$$A_{I0}(\vartheta, q) \approx A_{I0}(\pi, q_{\text{eff}}(\vartheta)) = e^{i\frac{2}{3} q_{\text{eff}}(\vartheta)} \int_0^1 dx\, P_I(x)\, e^{-2i q_{\text{eff}}(\vartheta)\, x^2}. \quad (5.14)$$

We have here introduced a quantity

$$q_{\text{eff}}(\vartheta) = \frac{3}{4} J_{20}(\vartheta)\, q \quad (5.15)$$

which corresponds to the $\chi_{\text{eff}}(\vartheta)$ introduced in paragraph 3C. The function

$P_I(x)$ is the Legendre polynomial of order $I$. The integral (5.14) can be expressed in terms of a confluent hypergeometric function ${}_1F_1$ with the following result (see ref. 10, Vol. I, p. 171):

$$A_{I0}(\pi, q) = \frac{\Gamma\left(\frac{I+1}{2}\right)}{2\,\Gamma\left(\frac{2I+3}{2}\right)} e^{-i\frac{4}{3}q} (-2iq)^{\frac{I}{2}} {}_1F_1\left(\frac{I+2}{2}, \frac{2I+3}{2}, 2iq\right). \tag{5.16}$$

The confluent hypergeometric function which appears here can always be expressed by means of Fresnel integrals. For $I = 0$,the expression (5.16) thus takes the simple form (see ref. 10, Vol. I, p. 266)

$$A_{00}(\pi, q) = \sqrt{\frac{\pi}{4q}}\, e^{i\frac{2}{3}q} [C(2q) - iS(2q)], \tag{5.17}$$

where $C(x)$ and $S(x)$ are the Fresnel integrals which are tabulated in refs. 12 and 13.

The functions $A_{I0}$ for higher values of $I$ are most easily obtained by means of recursion formulae. The existence of such relations is guaranteed by the theorem that three confluent hypergeometric functions with parameters differing only by integer numbers are linear dependent. Accordingly, one finds the following recursion formula for the functions $A_{I0}$:

$$\left.\begin{aligned}(I+2)(2I-1)A_{I+2,0}(\pi, q) = &\left[\frac{(2I-1)(2I+1)(2I+3)}{4iq} + 2I + 1\right] A_{I0}(\pi, q) \\ &+ (I-1)(2I+3)A_{I-2,0}(\pi, q).\end{aligned}\right\} \tag{5.18}$$

For the application of this formula one needs two consecutive $A$'s. Instead of $A_{2,0}$ it is practical to use the non-physical function $A_{-2,0}$ which, according to (5.16), is a simple exponential function

$$A_{-2,0}(\pi, q) = -\frac{1}{4iq} e^{-i\frac{4}{3}q}. \tag{5.19}$$

The functions $A_{I,0}(\pi, q)$ have been computed numerically in this way. The result is given in Table 4.

The excitation probabilities in a rotational band with ground state spin 0 are easily found from these numbers. They are tabulated in Table 5 and the result is shown in Fig. 7.

TABLE 4.

The functions $A_{I,0}(\pi, q)$ for backward scattering. The real part, Re $A_{I,0}$, and the imaginary part, Im $A_{I,0}$, are tabulated as functions of the parameter $q$ for spin values up to 22.

| $q$ | Re $A_{0,0}$ | Im $A_{0,0}$ | Re $A_{2,0}$ | Im $A_{2,0}$ | Re $A_{4,0}$ | Im $A_{4,0}$ |
|---|---|---|---|---|---|---|
| 0.0 | 1.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 |
| 0.5 | 0.95625 | 0.00276 | —0.01228 | —0.12916 | —0.01236 | 0.00150 |
| 1.0 | 0.83311 | 0.02081 | —0.04427 | —0.23461 | —0.04567 | 0.01123 |
| 1.5 | 0.65268 | 0.06336 | —0.08317 | —0.29888 | —0.08980 | 0.03372 |
| 2.0 | 0.44537 | 0.12960 | —0.11265 | —0.31506 | —0.13173 | 0.06754 |
| 2.5 | 0.24234 | 0.20825 | —0.11852 | —0.28773 | —0.15978 | 0.10530 |
| 3.0 | 0.06838 | 0.28073 | —0.09370 | —0.23062 | —0.16731 | 0.13599 |
| 3.5 | —0.06252 | 0.32687 | —0.04075 | —0.16163 | —0.15447 | 0.14879 |
| 4.0 | —0.14819 | 0.33131 | 0.02890 | —0.09722 | —0.12756 | 0.13695 |
| 4.5 | —0.19556 | 0.28834 | 0.09852 | —0.04786 | —0.09629 | 0.10036 |
| 5.0 | —0.21596 | 0.20372 | 0.15132 | —0.01611 | —0.06998 | 0.04592 |
| 5.5 | —0.22000 | 0.09300 | 0.17549 | —0.00242 | —0.05425 | —0.01454 |
| 6.0 | —0.21388 | —0.02302 | 0.16734 | —0.01578 | —0.04929 | —0.06786 |
| 6.5 | —0.19840 | —0.12396 | 0.13171 | 0.03154 | —0.05024 | —0.10350 |
| 7.0 | —0.17046 | —0.19497 | 0.07968 | 0.05327 | —0.04959 | —0.11644 |
| 7.5 | —0.12642 | —0.22953 | 0.02453 | 0.07913 | —0.04044 | —0.10807 |
| 8.0 | —0.06545 | —0.22954 | —0.02240 | 0.10270 | —0.01943 | —0.08498 |
| 8.5 | 0.00816 | —0.20297 | —0.05466 | 0.11574 | 0.01177 | —0.05610 |
| 9.0 | 0.08504 | —0.16012 | —0.07155 | 0.11168 | 0.04694 | —0.02948 |
| 9.5 | 0.15274 | —0.11018 | —0.07677 | 0.08834 | 0.07745 | —0.00979 |
| 10.0 | 0.19925 | —0.05912 | —0.07582 | 0.04909 | 0.09538 | 0.00255 |

(to be continued)

It is interesting to compare these curves with the excitation probabilities which were obtained for the same situation by means of the diagonalization method. It is seen that the excitation curves for the five-state model (see Fig. 4) are in good agreement with the exact calculation for $\chi$ values up to 3. It is interesting that also the secondary maxima of the excitation curves are present in the calculation with infinitely many levels. These secondary maxima which for $P_2$ appear for $q = 5.5$ and $q = 9$ must be understood as rudiments of the secondary maxima in the calculation with a finite number of states. In the two-state calculation the secondary maxima of $P_2$ appear at $q = 7.5$, 12.5, etc. When more states are introduced, these maxima are decreased (and shifted) due to the possibility of exciting the higher states which are introduced. One must expect that the secondary maxima are rather characteristic of the multiple Coulomb excitation of a pure rotational

TABLE 4 (continued).

| $q$ | Re $A_{6,0}$ | Im $A_{6,0}$ | Re $A_{8,0}$ | Im $A_{8,0}$ | Re $A_{10,0}$ | Im $A_{10,0}$ |
|---|---|---|---|---|---|---|
| 0.0 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 |
| 0.5 | 0.00012 | 0.00087 | 0.00006 | —0.00003 | | |
| 1.0 | 0.00176 | 0.00647 | 0.00074 | —0.00020 | | |
| 1.5 | 0.00813 | 0.01934 | 0.00324 | —0.00143 | —0.00014 | —0.00045 |
| 2.0 | 0.02255 | 0.03862 | 0.00872 | —0.00546 | —0.00104 | —0.00160 |
| 2.5 | 0.04638 | 0.06013 | 0.01724 | —0.01443 | —0.00352 | —0.00400 |
| 3.0 | 0.07751 | 0.07799 | 0.02728 | —0.03002 | —0.00899 | —0.00767 |
| 3.5 | 0.11025 | 0.08694 | 0.03601 | —0.05229 | —0.01881 | —0.01192 |
| 4.0 | 0.13660 | 0.08451 | 0.04021 | —0.07890 | —0.03361 | —0.01518 |
| 4.5 | 0.14878 | 0.07217 | 0.03764 | —0.10527 | —0.05272 | —0.01549 |
| 5.0 | 0.14179 | 0.05491 | 0.02823 | —0.12555 | —0.07385 | —0.01113 |
| 5.5 | 0.11535 | 0.03947 | 0.01454 | —0.13437 | —0.09335 | —0.00157 |
| 6.0 | 0.07426 | 0.03174 | 0.00120 | —0.12857 | —0.10705 | 0.01195 |
| 6.5 | 0.02704 | 0.03448 | —0.00650 | —0.10847 | —0.11151 | 0.02635 |
| 7.0 | —0.01664 | 0.04613 | —0.00457 | —0.07794 | —0.10518 | 0.03733 |
| 7.5 | —0.04883 | 0.06126 | 0.00809 | —0.04338 | —0.08909 | 0.04069 |
| 8.0 | —0.06558 | 0.07250 | 0.02898 | —0.01177 | —0.06674 | 0.03370 |
| 8.5 | —0.06765 | 0.07320 | 0.05252 | 0.01150 | —0.04311 | —0.01632 |
| 9.0 | —0.05967 | 0.05992 | 0.07176 | 0.02413 | —0.02313 | —0.00852 |
| 9.5 | —0.04797 | 0.03368 | 0.08053 | 0.02740 | —0.01011 | —0.03548 |
| 10.0 | —0.03808 | —0.00022 | 0.07552 | 0.02535 | —0.00474 | —0.05831 |

(to be continued)

band, and sensitive to any deviation. The maxima are also, as we shall see, less pronounced for finite $\xi$, and the deviation from the $q(\vartheta)$ approximation will also tend to wash out the oscillations.

The deviation from the $q(\vartheta)$ approximation can be treated by means of the expansion discussed in paragraph 3C. From (3.40), (3.34), (5.4), and (5.12) one finds the amplitude $a_{IM}$ to second order in $J_{22}(\vartheta)/J_{20}(\vartheta)$

$$\left.\begin{aligned} a_{IM} &= a_{IM}^{(0)} \\ &+ i\sqrt{\frac{32\pi}{15}}\, q_{\mathrm{eff}}(\vartheta)\frac{J_{22}(\vartheta)}{J_{20}(\vartheta)} \sum_{I_z M_z \mu=\pm 2}{}' \langle IM \mid Y_{2\mu}^{*}(\beta,\alpha) \mid I_z M_z \rangle a_{I_z M_z}^{(0)} \\ &- \frac{16\pi}{15}\left(q_{\mathrm{eff}}(\vartheta)\frac{J_{22}(\vartheta)}{J_{20}(\vartheta)}\right)^2 \sum_{I_z M_z \mu,\, \mu'=\pm 2}{}' \langle IM \mid Y_{2\mu}^{*}(\beta,\alpha)\, Y_{2\mu'}^{*}(\beta,\alpha) \mid I_z M_z \rangle a_{I_z M_z}^{(0)} \end{aligned}\right\} \quad (5.20)$$

where $a^{(0)}$ are the amplitudes in the $q_{\mathrm{eff}}(\vartheta)$ approximation.

The formula (5.20) for the special case of $I_i = M_i = 0$ may, according

TABLE 4 (continued).

| $q$ | Re $A_{12,0}$ | Im $A_{12,0}$ | Re $A_{14,0}$ | Im $A_{14,0}$ | Re $A_{16,0}$ | Im $A_{16,0}$ |
|---|---|---|---|---|---|---|
| 0.0 | | | | | | |
| 0.5 | | | | | | |
| 1.0 | | | | | | |
| 1.5 | | | | | | |
| 2.0 | —0.00025 | 0.00013 | | | | |
| 2.5 | —0.00074 | 0.00072 | | | | |
| 3.0 | —0.00180 | 0.00220 | 0.00040 | 0.00032 | | |
| 3.5 | —0.00330 | 0.00551 | 0.00137 | 0.00079 | 0.00018 | 0.00031 |
| 4.0 | —0.00478 | 0.01151 | 0.00331 | 0.00130 | 0.00033 | —0.00080 |
| 4.5 | —0.00531 | 0.02089 | 0.00691 | 0.00157 | 0.00041 | —0.00197 |
| 5.0 | —0.00358 | 0.03361 | 0.01263 | 0.00096 | 0.00022 | —0.00406 |
| 5.5 | 0.00172 | 0.04865 | 0.02065 | —0.00145 | —0.00071 | —0.00746 |
| 6.0 | 0.01128 | 0.06390 | 0.03052 | —0.00667 | —0.00305 | —0.01228 |
| 6.5 | 0.02462 | 0.07658 | 0.04106 | —0.01541 | —0.00758 | —0.01833 |
| 7.0 | 0.03983 | 0.08387 | 0.05050 | —0.02764 | —0.01492 | —0.02496 |
| 7.5 | 0.05373 | 0.08389 | 0.05682 | —0.04231 | —0.02530 | —0.03100 |
| 8.0 | 0.06255 | 0.07637 | 0.05836 | —0.05723 | —0.03822 | —0.03508 |
| 8.5 | 0.06303 | 0.06298 | 0.05444 | —0.06944 | —0.05230 | —0.03587 |
| 9.0 | 0.05349 | 0.04698 | 0.04574 | —0.07585 | —0.06541 | —0.03266 |
| 9.5 | 0.03460 | 0.03232 | 0.03437 | —0.07411 | —0.07500 | —0.02564 |
| 10.0 | 0.00945 | 0.02241 | 0.02338 | —0.06344 | —0.07872 | —0.01619 |

(to be continued)

to (5.10), be interpreted as an expansion of the function $A_{I,M}$. The specialization $I_i = M_i = 0$ may thus be done without any loss of generality, since the amplitudes for other ground state spins can be computed by means of (5.8). Introducing this simplification we obtain

$$\left.\begin{aligned} A_{I0}(\vartheta, q) = A_{I0}(\pi, q_{\mathrm{eff}}(\vartheta)) - \frac{8}{3}\left(q_{\mathrm{eff}}(\vartheta)\frac{J_{22}(\vartheta)}{J_{20}(\vartheta)}\right)^2 \sum_{I_z I_z'}{}' (2I_z+1)(2I_z'+1) \\ \times \begin{pmatrix} I & 2 & I_z' \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} I & 2 & I_z' \\ 0 & -2 & 2 \end{pmatrix}\begin{pmatrix} I_z' & 2 & I_z \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} I_z' & 2 & I_z \\ -2 & 2 & 0 \end{pmatrix} A_{I_z 0}(\pi, q_{\mathrm{eff}}(\vartheta)) \end{aligned}\right\} \quad (5.21)$$

$$\left.\begin{aligned} A_{I,\pm 2}(\vartheta, q) = i\sqrt{\frac{8}{3}}\, q_{\mathrm{eff}}(\vartheta)\frac{J_{22}(\vartheta)}{J_{20}(\vartheta)} \sum_{I_z}{}' (2I_z+1) \\ \begin{pmatrix} I_z & 2 & I \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} I_z & 2 & I \\ 0 & -2 & 2 \end{pmatrix} A_{I_z 0}(\pi, q_{\mathrm{eff}}(\vartheta)) \end{aligned}\right\} \quad (5.22)$$

TABLE 4 (continued).

| $q$ | Re $A_{18,0}$ | Im $A_{18,0}$ | Re $A_{20,0}$ | Im $A_{20,0}$ | Re $A_{22,0}$ | Im $A_{22,0}$ |
|---|---|---|---|---|---|---|
| 0.0 | | | | | | |
| 0.5 | | | | | | |
| 1.0 | | | | | | |
| 1.5 | | | | | | |
| 2.0 | | | | | | |
| 2.5 | | | | | | |
| 3.0 | | | | | | |
| 3.5 | | | | | | |
| 4.0 | | | | | | |
| 4.5 | —0.00051 | —0.00009 | | | | |
| 5.0 | —0.00114 | —0.00004 | 0.00000 | 0.00027 | | |
| 5.5 | —0.00236 | 0.00027 | 0.00009 | 0.00069 | | |
| 6.0 | —0.00429 | 0.00116 | 0.00037 | 0.00133 | | |
| 6.5 | —0.00706 | 0.00312 | 0.00111 | 0.00240 | 0.00071 | —0.00034 |
| 7.0 | —0.01056 | 0.00672 | 0.00261 | 0.00392 | 0.00130 | —0.00090 |
| 7.5 | —0.01435 | 0.01248 | 0.00530 | 0.00580 | 0.00209 | —0.00199 |
| 8.0 | —0.01768 | 0.02072 | 0.00958 | 0.00774 | 0.00301 | —0.00389 |
| 8.5 | —0.01958 | 0.03125 | 0.01575 | 0.00922 | 0.00384 | —0.00692 |
| 9.0 | —0.01904 | 0.04330 | 0.02382 | 0.00952 | 0.00420 | —0.01133 |
| 9.5 | —0.01540 | 0.05540 | 0.03334 | 0.00788 | 0.00354 | —0.01718 |
| 10.0 | —0.00863 | 0.06564 | 0.04338 | 0.00371 | 0.00128 | —0.02423 |

and

$$A_{I,\pm 4}(\vartheta, q) = -\frac{4}{3}\left(q_{\text{eff}}(\vartheta)\frac{J_{22}(\vartheta)}{J_{20}(\vartheta)}\right)^2 \sum_{I_z I_z'}{}' (2I_z+1)(2I_z'+1) \times \begin{pmatrix} I & 2 & I_z' \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} I & 2 & I_z' \\ 4 & -2 & -2 \end{pmatrix}\begin{pmatrix} I_z' & 2 & I_z \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} I_z' & 2 & I_z \\ 2 & -2 & 0 \end{pmatrix} A_{I_z 0}(\pi, q_{\text{eff}}(\vartheta)). \quad (5.23)$$

In the excitation probabilities (see Eq. (5.13)) only the squares of the $A_{I,M}$ appear and to second order in $J_{22}(\vartheta)/J_{20}(\vartheta)$ only $A_{I,0}$ and $A_{I,2}$ contribute. The seamplitudes have been calculated numerically for $I = 0,2$, and 4 by means of the known functions $A_{I,0}(\pi, q_{\text{eff}}(\vartheta))$. The probability for the excitation of the states in an even-even nucleus can be written

$$P_I = P_I^{(0)}(q_{\text{eff}}(\vartheta)) + \left(\frac{J_{22}(\vartheta)}{J_{20}(\vartheta)}\right)^2 \Delta_I(q_{\text{eff}}(\vartheta)), \quad (5.24)$$

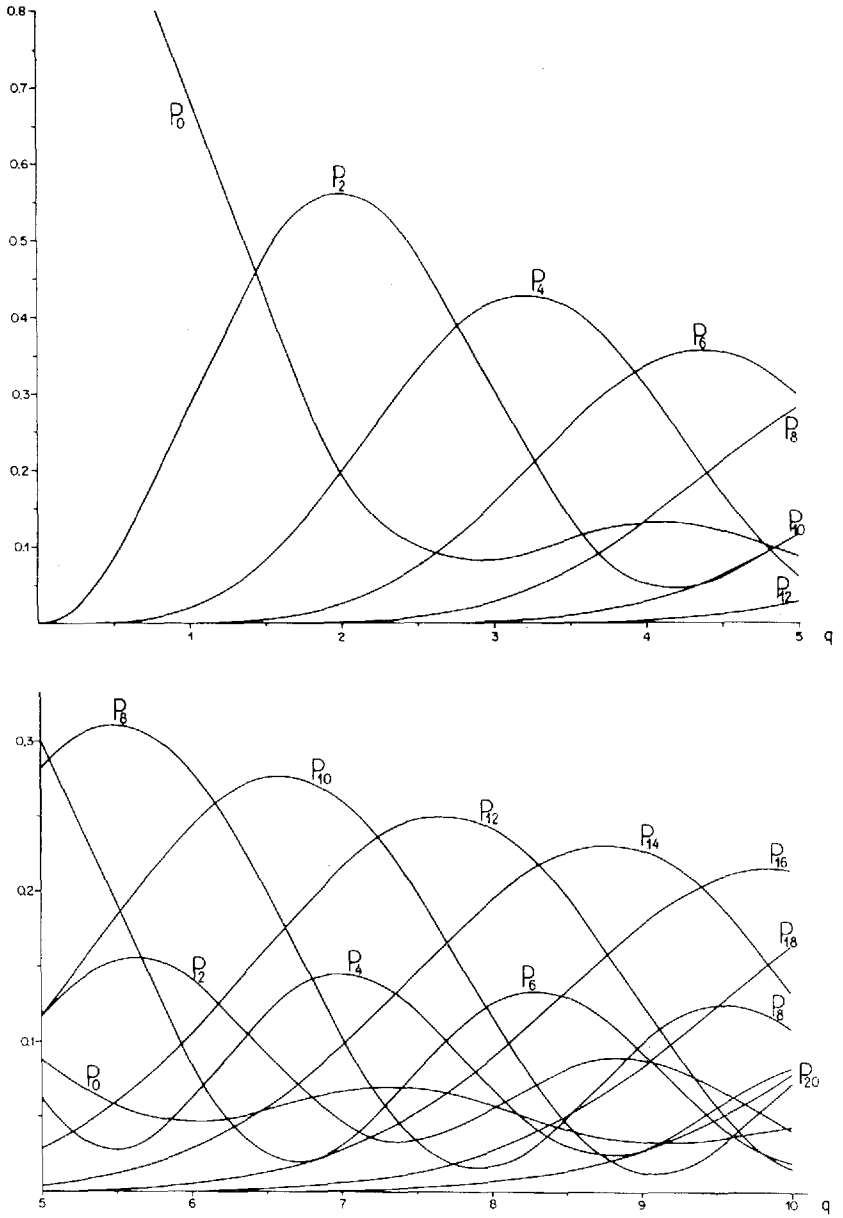


Fig. 7. The multiple Coulomb excitation of a pure rotational band in an even-even nucleus. The excitation probability $P_I$ of the state with spin $I$ is given as a function of the parameter $q$ for backward scattering. The excitation probabilities for other deflection angles and other ground state spins can also be inferred from these curves (see Sect. 5).

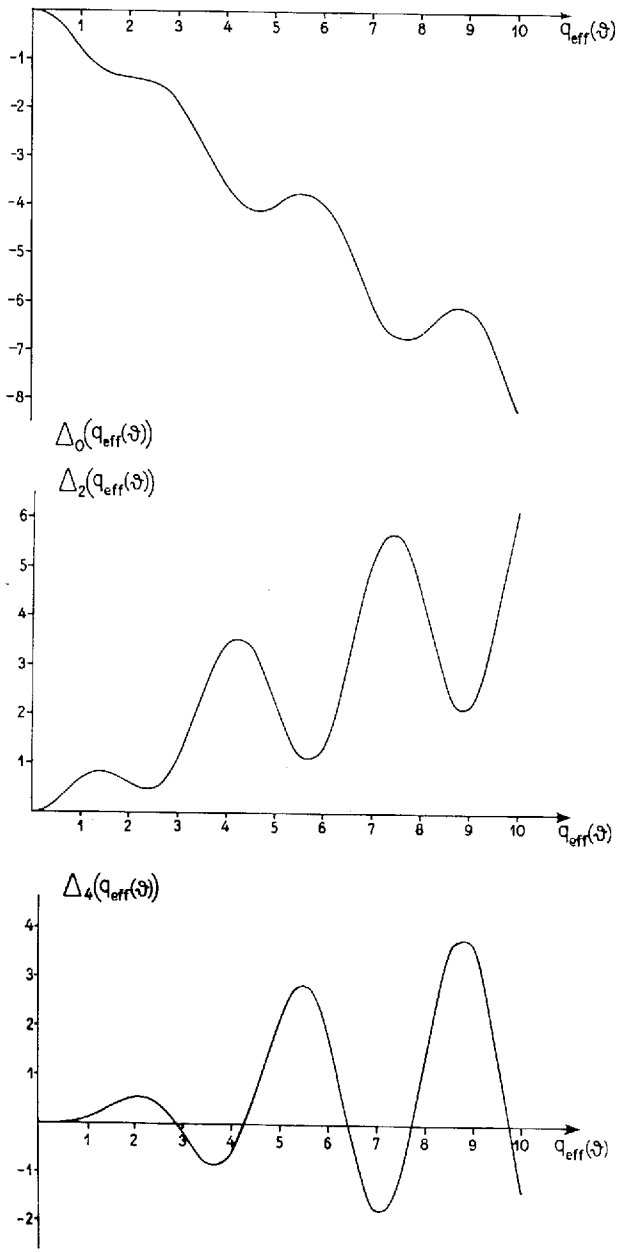


Fig. 8. The coefficients $\Delta_I$ for the correction of the excitation probabilities in the $q_{\text{eff}}(\vartheta)$ approximation. The coefficients $\Delta_I$ (which are defined in Eq. (5.24)) are plotted as functions of $q_{\text{eff}}(\vartheta)$ for the three lowest states $I = 0$, 2, and 4 in a rotational band in an even-even nucleus.

where $P_I^{(0)}$ are the probabilities given in Fig. 7. The coefficients $\Delta_I$ receive a contribution, partly from $A_{I,0}$ partly from $A_{I,2}$. The numerical result for $\Delta_I$ in the cases of $I = 0,2$, and 4 are given in Table 6 and are illustrated in Fig. 8.

It is seen that $\Delta_I$ is an oscillating function of $q_{\text{eff}}(\vartheta)$, and a rough estimate shows that the correction may amount to about 0.1, but usually it will be much smaller. If one compares the curves for $\Delta_I$ and $P_I$, one sees that the tendency of the correction is to fill out the minima of $P_I$. The relative error at the minima of $P_I$ might be rather considerable, as is shown on Fig. 9. The relative error in $P_2$ is here plotted as a function of $q_{\text{eff}}(\vartheta)$ for different angles. The curves end at the value of $q_{\text{eff}}(\vartheta)$ where $q$ reaches the value 10. The maxima on the curves appear as expected at the points where $P_2^{(0)}$ is minimal.

One sees also that for small angles the relative error is rather considerable in the whole range of $q$. This discrepancy can, as was discussed in paragraph 3C, be removed by applying $q(\vartheta)$ defined by

$$q(\vartheta) = q\frac{3}{4}\left[(J_{20}(\vartheta))^2 + 3\,(J_{22}(\vartheta))^2\right]^{1/2} \tag{5.25}$$

instead of $q_{\text{eff}}(\vartheta)$. This approximation will lead to the correct result for angles where the perturbation calculation is applicable. In Fig. 10 we have plotted the relative error of this latter approximation. It is seen that for $q(\vartheta)$ less than 2 one obtains a considerable improvement over the $q_{\text{eff}}(\vartheta)$ approximation (compare Fig. 9). For $q(\vartheta)$ larger than 2, the error is mostly larger than the error of the $q_{\text{eff}}(\vartheta)$ approximation, but it is here not very different from this. As a net result the $q(\vartheta)$ approximation is preferable.

In the case of a pure rotational band, one may calculate the excitation amplitude for arbitrary angles directly from (5.8). This can be done in the following way. We write the exponent in (5.9) explicitly in the form

$$\left.\begin{aligned} &\frac{8\pi}{5} q \sum_{\mu} Y_{2\mu}\left(\frac{\pi}{2}, 0\right) J_{2\mu}(\vartheta)\, Y^*_{2\mu}(\beta, \alpha) \\ &= q\left[\frac{1}{2} J_{20}(\vartheta)\,(1 - 3\cos^2\beta) + \frac{3}{2} J_{22}(\vartheta)\sin^2\beta\cos 2\alpha\right]. \end{aligned}\right\} \tag{5.26}$$

For the spherical harmonics we use the definition

$$Y_{IM}(\beta, \alpha) = (-1)^M\left[\frac{(2I+1)\,(I-M)\,!}{4\pi\,(I+M)\,!}\right]^{\frac{1}{2}} P_I^M(\cos\beta)\, e^{iM\alpha}. \tag{5.27}$$

The integration over $\alpha$ may then be expressed by means of a Bessel function $J_{M/2}$ of order $M/2$ with the following result[10]:

$$\left.\begin{aligned} A_{IM}(\vartheta, q) &= i^{\frac{M}{2}}\left[\frac{(I-M)!}{(I+M)!}\right]^{\frac{1}{2}} e^{i\frac{1}{2}qJ_{20}(\vartheta)} \\ &\times \int_0^1 dx P_I^M(x) J_{\frac{M}{2}}\left(\frac{3}{2}qJ_{22}(\vartheta)(1-x^2)\right) e^{-i\frac{3}{2}qJ_{20}(\vartheta)x^2}. \end{aligned}\right\} \quad (5.28)$$

This formula applies to $I$ and $M$ even. For $I$ or $M$ odd, $A_{I,M}$ vanishes. One observes furthermore the symmetry relation

$$A_{I,-M}(\vartheta, q) = A_{I,M}(\vartheta, q). \quad (5.29)$$

One may then proceed by using the following integral representation of the Bessel function (see ref. 10, Vol. II, p. 81)

$$J_n(y) = \frac{(y/2)^n}{\Gamma\left(n+\frac{1}{2}\right)\sqrt{\pi}} \int_{-1}^{1} dt\, e^{iyt}(1-t^2)^{n-\frac{1}{2}}. \quad (5.30)$$

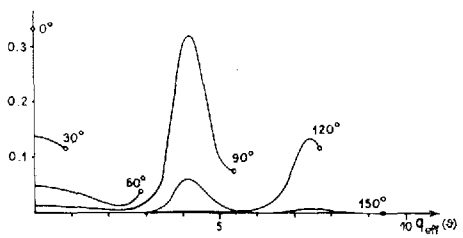


Fig. 9. The relative error of the $q_{\text{eff}}(\vartheta)$ approximation for the excitation of the $2^+$ rotational state in an even-even nucleus. The error $[P_2(\vartheta, q) - P_2(\pi, q_{\text{eff}}(\vartheta))]/P_2(\pi, q_{\text{eff}}(\vartheta))$ is plotted as a function of $q_{\text{eff}}(\vartheta)$ for different angles. The curves end at a value of $q_{\text{eff}}(\vartheta)$ where $q$ reaches the value 10.

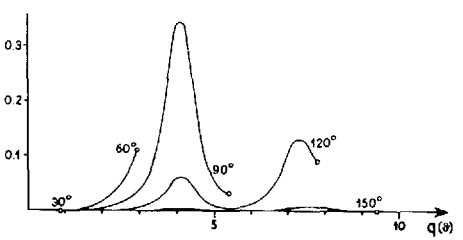


Fig. 10. The relative error of the $q(\vartheta)$ approximation for the excitation of the $2^+$ rotational state in an even-even nucleus. The error $[P_2(\vartheta, q) - P_2(\pi, q(\vartheta))]/P_2(\pi, q(\vartheta))$ is plotted as a function of $q(\vartheta)$ for different angles. The curves end at a value of $q(\vartheta)$ where $q$ reaches the value 10.

The integration over $x$ can be done by expanding the exponential function in power series in $q\,[J_{2,0}(\vartheta)+tJ_{2,2}(\vartheta)]$ (see ref. 10, Vol. I, p. 172). The integration over $t$ can finally be done when the powers of this quantity are expanded according to the binomial formula. The result is a double series

$$\left.\begin{aligned} A_{IM}(\vartheta, q) = &\left[\frac{(I+M)!}{(I-M)!}\right]^{\frac{1}{2}} \frac{\Gamma\left(\frac{I-M+1}{2}\right)(-1)^{\frac{I-M}{2}}}{2^{M+1}\,\Gamma\left(\frac{2I+3}{2}\right)\Gamma\left(\frac{M+2}{2}\right)} e^{-i\frac{4}{3}q_{\mathrm{eff}}(\vartheta)} \\ &\times \sum_{m,n}{}' \frac{\left(\frac{I+M+2}{2}\right)_m \left(-\frac{I-M}{2}-m\right)_{2n}}{\left(\frac{2I+3}{2}\right)_m \left(\frac{M+2}{2}\right)_n} \frac{(2\,iq_{\mathrm{eff}}(\vartheta))^{m+\frac{I}{2}}}{m!} \frac{b^{2n+\frac{M}{2}}}{n!}, \end{aligned}\right\} \quad (5.31)$$

where

$$b = \frac{J_{22}(\vartheta)}{2\,J_{20}(\vartheta)}. \quad (5.32)$$

We have furthermore used the notation

$$\alpha_m = \alpha(\alpha+1)\ldots.(\alpha+m-1). \quad (5.33)$$

The formula (5.31) holds for $M \geqslant 0$. The functions $A_{I,M}$ for negative $M$ are determined by means of (5.29).

It is useful to perform the summation over $M$ whereby (5.31) may be written in the form

$$\left.\begin{aligned} A_{IM}(\vartheta, q) = &\left[\frac{(I+M)!}{(I-M)!}\right]^{\frac{1}{2}} \frac{\Gamma\left(\frac{I-M+1}{2}\right)}{2^{M+1}\,\Gamma\left(\frac{2I+3}{2}\right)\Gamma\left(\frac{M+2}{2}\right)} e^{-i\frac{4}{3}q_{\mathrm{eff}}(\vartheta)} \\ &\times (-2\,iq_{\mathrm{eff}}(\vartheta))^{\frac{I}{2}} (-b)^{\frac{M}{2}} \sum_{n}{}' \frac{b^{2n}}{\left(\frac{M+2}{2}\right)^n n!} (q_{\mathrm{eff}}(\vartheta))^{2n-\frac{I-M}{2}} \\ &\times \frac{d^{2n}}{(dq_{\mathrm{eff}}(\vartheta))^{2n}} \left[(q_{\mathrm{eff}}(\vartheta))^{\frac{I-M}{2}}\, {}_1F_1\left(\frac{I+M+2}{2}, \frac{2I+3}{2}, 2\,iq_{\mathrm{eff}}(\vartheta)\right)\right]. \end{aligned}\right\} \quad (5.34)$$

If one sets $b = 0$ the expressions (5.31) and (5.34) reduce to the simple result (5.16). The expression (5.34) is, similar to (5.20), a systematic expansion in powers of $J_{22}(\vartheta)/J_{20}(\vartheta)$.

### B. First Order Correction in $\xi$

In paragraph 3 A it was outlined how one may calculate the deviation of the excitation amplitudes from the sudden approximation. The result was expressed in a power series in the $\xi$'s which enter into the excitation process. For rotational bands, one may define a common $\xi$ in terms of the moment of inertia in a similar way as, for such spectra, we defined a common $\chi$ in terms of the intrinsic quadrupole moment. We shall use the notation

$$\xi = \frac{3\hbar a}{v\Im}, \tag{5.35}$$

where $\Im$ is the moment of inertia entering in (5.3). The quantity (5.35) is identical to the $\xi$ corresponding to the excitation of the lowest rotational state in an even-even nucleus.

The excitation amplitudes which were evaluated in the previous paragraph are essentially complex numbers. The first order corrections must also be expected to be complex, and it follows therefore that the excitation probabilities have linear terms in $\xi$. This is in contrast to the first order perturbation theory which is independent of $\xi$ to first order in this quantity.

To first order, the excitation amplitude $a_n$ may be written in the form

$$a_n = a_n^{(0)} + a_n^{(1)}, \tag{5.36}$$

where $a_n^{(0)}$ is the amplitude (5.7) in the sudden approximation. The first order correction $a_n^{(1)}$ is, according to (3.20), given by

$$\left.\begin{aligned} a_n^{(1)} &= \langle n | e^{-\frac{i}{\hbar}\int_{-\infty}^{\infty} H'(t)\,dt} | \varphi_1 \rangle \\ &= \frac{\xi v}{6\hbar a} \langle n | e^{-\frac{i}{\hbar}\int_{-\infty}^{\infty} H'(t)\,dt} \int_{-\infty}^{\infty} e^{\frac{i}{\hbar}\int_{-\infty}^{t} H'(t')\,dt'} t\,[L^2, H'(t)] e^{-\frac{i}{\hbar}\int_{-\infty}^{t} H'(t')\,dt'} dt \,|\, 0 \rangle. \end{aligned}\right\} \tag{5.37}$$

In this equation, $H'(t)$ is given by (5.5) and (5.11)

$$H'(t) = \frac{8\pi\hbar v a^2}{5} q \sum_{\mu} \bar{S}_{2\mu}(t)\, Y_{2\mu}^*(\beta, \alpha), \tag{5.38}$$

while $H_0$ (see Eq. (5.3)) has been expressed by means of $\xi$ and the angular momentum operator $L$ through

$$H_0 = \frac{\hbar v}{6a} \xi L^2. \tag{5.39}$$

For the evaluation of (5.37) we shall proceed in the following way: Firstly, the differentiations of the operator $L^2$ are performed. Hereby the two exponential functions in $|\varphi_1\rangle$ will cancel. The expression for $|\varphi_1\rangle$ will then be suitable for an expansion in terms of the eigenfunctions $|m\rangle$ of $H_0$, and the problem is reduced to the already performed calculation of matrix elements in the sudden approximation.

The result of the first step in this program can be written in the form

$$\left.\begin{aligned} |\varphi_1\rangle = &-i\frac{8\pi}{5} q\xi \sum_\mu f_\mu Y^*_{2\mu}(\beta,\alpha)\,|0\rangle \\ &-\frac{8\pi}{15} q\xi \sum_\mu f_\mu \left(\frac{\partial Y^*_{2\mu}}{\partial\beta}\cdot\frac{\partial}{\partial\beta}+\frac{1}{\sin^2\beta}\frac{\partial Y^*_{2\mu}}{\partial\alpha}\frac{\partial}{\partial\alpha}\right)|0\rangle \\ &+i\frac{61\pi^2}{75} q^2\xi \sum_{\mu\mu'} f_{\mu\mu'}\left(\frac{\partial Y^*_{2\mu}}{\partial\beta}\frac{\partial Y^*_{2\mu'}}{\partial\beta}+\frac{1}{\sin^2\beta}\frac{\partial Y^*_{2\mu}}{\partial\alpha}\frac{\partial Y^*_{2\mu'}}{\partial\alpha}\right)|0\rangle, \end{aligned}\right\} \quad (5.40)$$

where the coefficients $f_\mu$ and $f_{\mu\mu'}$ are defined by

$$f_\mu = iv^2 a\int_{-\infty}^{\infty} t\bar{S}_{2\mu}(t)\,dt \quad (5.41)$$

and

$$f_{\mu\mu'} = v^3 a^3 \int_{-\infty}^{\infty}\bar{S}_{2\mu}(t)\,t\int_{-\infty}^{t}\bar{S}_{2\mu'}(t')\,dt'\,dt. \quad (5.42)$$

From the symmetries of the $\bar{S}_{2,\mu}(t)$ (see Eq. (3.23)) one sees immediately that $f_1 = f_{-1}$ is the only non-vanishing $f_\mu$. The first two terms of (5.40) also appear in the first order perturbation treatment while the third, which is proportional to $q^2$, is characteristic of higher order excitations. The second term arises from the initial motion of the rotator, and it disappears for the ground-state spin $I = 0$.

For the evaluation of (5.40) one has to use the properties of the eigenfunctions $|0\rangle$ which are given in terms of the $D$-function in (5.6). We note the following formula

$$\left.\begin{aligned} &2\left[\frac{\partial D^I}{\partial\beta}\frac{\partial D^{I'}}{\partial\beta}+\frac{1}{\sin^2\beta}\frac{\partial D^I}{\partial\alpha}\frac{\partial D^{I'}}{\partial\alpha}\right] \\ &= D_I L^2(D^{I'}) + D^{I'} L^2(D^I) - L^2(D^I D^{I'}) \\ &= (I(I+1)+I'(I'+1)-L^2)\,D^I D^{I'}, \end{aligned}\right\} \quad (5.43)$$

where we have suppressed the lower indices $M$ and $K$ on the $D$ functions. By means of this formula the problem of expanding $|\varphi_1\rangle$ in terms of $|m\rangle$ is reduced to the problem of expanding a product of $D$ functions in terms of $D$ functions.

The result can be expressed by means of the functions $A_{IM}$ defined in (5.9) in the following way:

$$\left.\begin{aligned} a^{(1)}_{I_f M_f} = &-\sqrt{\frac{16\pi}{5}}\, q\,\xi \sum_{II'M\mu=\pm 1}{}' f_\mu(\vartheta)\left[1+\frac{1}{6}I_i(I_i+1)-\frac{1}{6}I'(I'+1)+i\right] \\ &\times (2I_i+1)^{\frac{1}{2}}(2I_f+1)^{\frac{1}{2}}(2I+1)(2I'+1) \\ &\times \begin{pmatrix} 2 & I_i & I' \\ -\mu & M_i & \mu-M_i \end{pmatrix}\begin{pmatrix} 2 & I_i & I' \\ 0 & K-K & \end{pmatrix}\begin{pmatrix} I_f & I' & I \\ -M_f & \mu-M_i & M \end{pmatrix}\begin{pmatrix} I_f & I' & I \\ -K & K & 0 \end{pmatrix} A_{IM}(\vartheta, q) \\ &+ i\frac{8\pi}{15}\, q^2\,\xi \sum_{II'Mlm}{}' (-1)^m f^l_m(\vartheta)(2I_i+1)^{\frac{1}{2}}(2I_f+1)^{\frac{1}{2}}(2I+1)(2I'+1) \\ &\times \begin{pmatrix} l & I_i & I' \\ -m & M_i & m-M_i \end{pmatrix}\begin{pmatrix} l & I_i & I' \\ 0 & K-K & \end{pmatrix}\begin{pmatrix} I_f & I' & I \\ -M_f & m-M_i & M \end{pmatrix}\begin{pmatrix} I_i & I' & I \\ -K & K & 0 \end{pmatrix} A_{IM}(\vartheta, q) \end{aligned}\right\} \quad (5.44)$$

We have here introduced the notation

$$f^l_m(\vartheta) = (2l+1)\begin{pmatrix} 2 & 2 & l \\ 0 & 0 & 0 \end{pmatrix}[12-l(l+1)] \sum_{\mu\mu'}{}' \begin{pmatrix} 2 & 2 & l \\ \mu & \mu' & m \end{pmatrix} f_{\mu\mu'} \quad (5.45)$$

for the tensors of rank 0,2, and 4 which can be built up of the $f_{\mu\mu'}$, given by (5.42).

The coefficients $f_\mu(\vartheta)$ and $f^l_m(\vartheta)$ which are necessary for the evaluation of $a^{(1)}$ have been computed for a few angles and the result is given in Table 7. We note the following property of the $f^l_m$ functions:

$$f^l_{-m}(\vartheta) = f^l_m(\vartheta). \quad (5.46)$$

The $f^l_m$ functions for odd values of $m$ may be expressed by means of $f_1$ and the functions $J_{2,\mu}(\vartheta)$ defined by (3.27):

$$f^2_1(\vartheta) = \quad i\frac{3\sqrt{5}}{28\sqrt{\pi}} f_1[3J_{22}(\vartheta)+J_{20}(\vartheta)] \quad (5.47)$$

$$f^4_1(\vartheta) = -i\frac{3\sqrt{3}}{7\sqrt{2\pi}} f_1[2J_{20}(\vartheta)-J_{22}(\vartheta)] \quad (5.48)$$

$$f^4_3(\vartheta) = \quad i\frac{3\sqrt{3}}{\sqrt{14\pi}} f_1 J_{22}(\vartheta). \quad (5.49)$$

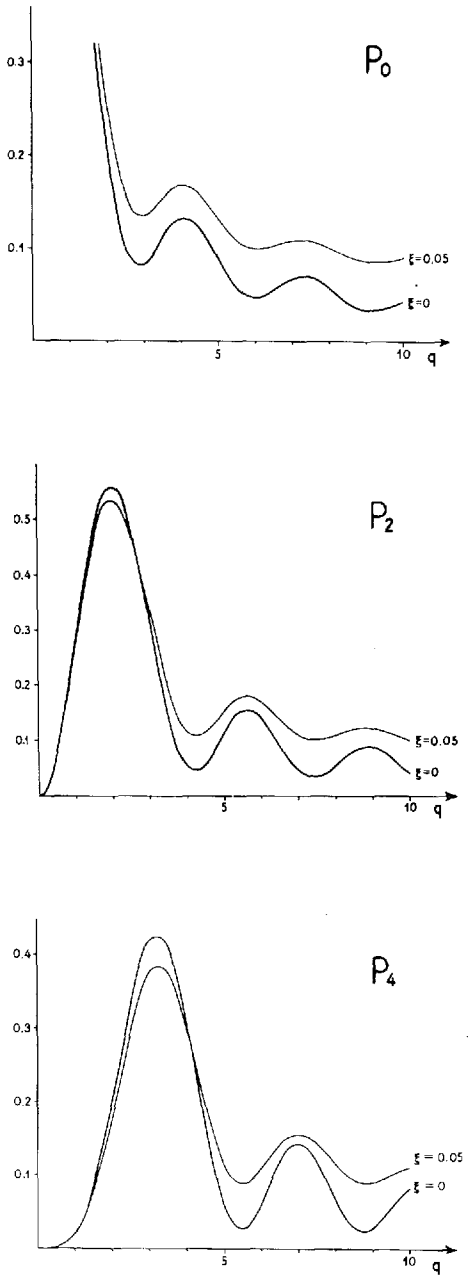


Fig. 11. The first order correction in $\xi$ for the excitation of a rotational band in an even-even nucleus. The curves show the excitation probabilities of the states of spin 2 ($P_2$) and of spin 4 ($P_4$) and the probability that the nucleus stays in its ground state ($P_0$). The probabilities are given as functions of $q$ for backward scattering and for the cases $\xi = 0$ and $\xi = 0.05$.

For backward scattering the only non-vanishing $f$-functions are $f_0^0$, $f_0^2$, and $f_0^4$. This is a consequence of the fact that all $f_\mu$ and $f_{\mu\mu'}$ of Eqs. (5.41) and (5.42) vanish, except $f_{00}$.

We shall here illustrate the first order correction in $\xi$ by considering the special case of backward scattering on an even-even nucleus. In this case, the only non-vanishing amplitude $a^{(1)}$ is

$$\left.\begin{aligned} a_{I_f 0}^{(1)} &= i\frac{8\pi}{15} q^2 \xi (2I_f+1)^{\frac{1}{2}} f_0^0(\pi) \sum_I (2I+1) \\ &\times \left\{ \begin{pmatrix} 0 & I_f & I \\ 0 & 0 & 0 \end{pmatrix}^2 + \begin{pmatrix} 2 & I_j & I \\ 0 & 0 & 0 \end{pmatrix}^2 + \begin{pmatrix} 4 & I_f & I \\ 0 & 0 & 0 \end{pmatrix}^2 \right\} A_{I0}(\pi, q). \end{aligned}\right\} \quad (5.50)$$

We have here used that $f_0^2(\pi) = 5/7\ f_0^0(\pi)$ and $f_0^4(\pi) = -12/7\ f_0^0(\pi)$ which follows from the definition (5.45). The excitation probabilities to first order in $\xi$ may be written in the form

$$P_I(\pi, q) = P_I(\xi = 0) + \Lambda_I(q)\,\xi. \quad (5.51)$$

The coefficient $\Lambda_I(q)$ has been evaluated numerically and is given in Table 8. It is seen that $\Lambda_I$ is an oscillating function of $q$ which is of the order of magnitude 1. The corrections for $\xi \neq 0$ are thus not dominated by the factor $q^2$ in (5.50). The oscillations in $\Lambda_I$ follow the oscillations of $P_I(\xi = 0)$ in such a way that the first maximum of $P_I$ is cut down, while the excitation probability for larger values of $q$ is increased. This increment is largest at the minima of $P_I$ and the effect of $\xi \neq 0$ is thus essentially to smooth out the whole excitation curve. This is clearly seen on Fig. 11 where the excitation probabilities for $I = 0$, 2, and 4 and $\xi = 0.05$ are compared with the excitation probabilities for $\xi = 0$.

For other deflection angles one may use the $q_{\text{eff}}(\vartheta)$ approximation. One must here substitute only the $q$ in $A_{I0}(\pi, q)$ with $q_{\text{eff}}(\vartheta)$. Furthermore, the $f_0^0(\pi)$ should be replaced by $f_0^0(\vartheta)$. As was discussed earlier, this approximation is much less accurate here than in the sudden approximation. An indication of the accuracy can be obtained by comparing the limiting case of (5.50) for $q \ll 1$ with the second order perturbation calculation performed in ref. 14. This comparison shows that the approximation should not be applied for angles less than 90 degrees.

The failure of the $q_{\text{eff}}(\vartheta)$ approximation for $\xi \neq 0$ is due to the fact that the relative importance of the different coefficients $f_m^l(\vartheta)$ in (5.44) for angles smaller than 90 degrees is completely different from the relative importance

in the neighbourhood of 180 degrees where only $f_{00}$ is different from zero. This means that one has to take into account also the amplitudes on the states with magnetic quantum number $M \neq 0$.

For $\xi \neq 0$, one observes from (5.44) that also the states with magnetic quantum numbers which differ by an odd integer from the magnetic quantum number of the ground state are populated. The amplitude on these states will be proportional to $\xi f_1(\vartheta)$ and the excitation probability will thus only receive a contribution of the order $\xi^2$ from such terms.

### C. Numerical Results

In this paragraph we shall collect the numerical results which have been obtained for the excitation of rotational states together with some formulae which facilitate the application of these results to the experiments.

It is thus convenient to write the important parameters directly as functions of the energy of the incident projectile in the laboratory system (see ref. 1, Chapter II C). We shall here quote the expression for half the distance of closest approach in a head-on collision

$$a = 0.07199\left(1+\frac{A_1}{A_2}\right)\frac{Z_1 Z_2}{E_{\mathrm{MeV}}} \times 10^{-12}\,\mathrm{cm}. \tag{5.52}$$

Here, $A_1, Z_1$ and $A_2, Z_2$ are the mass numbers and charges of projectile and target nucleus, respectively. The quantity $E_{\mathrm{MeV}}$ is the bombarding energy expressed in MeV.

The parameter $\xi$ is similarly given by

$$\xi_{1\to 2} = \frac{Z_1 Z_2 A_1^{\frac{1}{2}} \Delta E'_{\mathrm{MeV}}}{12.65\left(E_{\mathrm{MeV}} - \frac{1}{2}\Delta E'_{\mathrm{MeV}}\right)^{3/2}}, \tag{5.53}$$

where $\Delta E'$ is connected with the energy difference $E_2 - E_1$ by the relation

$$\Delta E' = \left(1+\frac{A_1}{A_2}\right)(E_2 - E_1). \tag{5.54}$$

An expression for the parameter $\chi$ (in the case $\lambda = 2$) is found by inserting (5.52) in (2.11), i. e.,

$$\chi_{1\to 2} = 14.52 \frac{A_1^{\frac{1}{2}} [B(E2, I_1 \to I_2)]^{1/2}}{(1 + A_1/A_2)^2 Z_1 Z_2^2} E_{\mathrm{MeV}}^{3/2}. \tag{5.55}$$

The reduced transition probability $B(E2)$ is here measured in units of $e^2 \cdot 10^{-48}\,\mathrm{cm}^4$.

For the excitation of rotational states we have introduced two parameters which are related to $\chi_{1\to 2}$ and $\xi_{1\to 2}$ and are defined in terms of the nuclear moments, so that they are independent of the spin sequence in the rotational bands. We have thus (see Eq. (5.35)) defined a common $\xi$ in terms of the moment of inertia $\Im$ by means of (5.53) where

$$\Delta E' = \left(1 + \frac{A_1}{A_2}\right)\left(\frac{3\hbar}{\Im}\right)_{\mathrm{MeV}}. \tag{5.56}$$

For an even-even nucleus this $\xi$ is identical with the $\xi_{0\to 2}$ for the excitation of the lowest rotational state.

We have furthermore defined a quantity $q$ by means of the intrinsic quadrupole moment $Q_0$ in the following way (see Eq. (5.11)):

$$q = 7.6241 \frac{A_1^{\frac{1}{2}} Q_0}{(1 + A_1/A_2)^2 Z_1 Z_2^2} E_{\mathrm{MeV}}^{3/2}, \tag{5.57}$$

where $Q_0$ is measured in units of $e \cdot 10^{-24}\,\mathrm{cm}^2$. The quantity $q$ is related to the $\chi$ for the excitation of the lowest state in an even-even nucleus by Eq. (5.12).

The differential Coulomb excitation cross section is given by (3.43) through the excitation probability $P_{I_f, I_i}(\vartheta, q, \xi)$ which is the probability that the nucleus is excited from the ground state with spin $I_i$ into the state with spin $I_f$ when the projectile moves in an orbit with deflection angle $\vartheta$ in the center of mass system.

The probabilities $P$ as well as other quantities interesting for the experiments can be obtained from the excitation amplitudes (see Eq. (3.42)). For $\xi \ll 1$ the amplitudes $a_{I_f, M_f}(\vartheta, q, \xi)$ are easily obtainable from the functions $A_{IM}(\vartheta, q)$ (see Eq. (5.8)). These functions can in turn be expressed by the functions $A_{I0}(\pi, q)$ which are related to the amplitudes for the excitation of rotational states in an even-even nucleus for $\vartheta = \pi$ and $\xi = 0$ by means of (5.10). These fundamental quantities have been calculated according to the formulae given in paragraph 5 A, and the result is given in Table 4.

For other deflection angles, the functions $A_{IM}(\vartheta, q)$ can be obtained to a good approximation from those tabulated by means of Eqs. (5.21) to (5.23) which we shall quote here for the cases $M = 0$ and $M = \pm 2$:

$$\begin{aligned} A_{I0} = A_{I0}(\pi, q_{\text{eff}}) - \left(q_{\text{eff}} \frac{J_{22}(\vartheta)}{J_{20}(\vartheta)}\right)^2 \Bigg\{ & \frac{(I-3)(I-2)(I-1)I}{(2I-5)(2I-3)(2I-1)(2I+1)} A_{I-4,0}(\pi, q_{\text{eff}}) \\ & - \frac{4(I-1)I[(I-1)I-4]}{(2I-5)(2I-1)(2I+1)(2I+3)} A_{I-2,0}(\pi, q_{\text{eff}}) \\ & + \frac{2[3I^2(I+1)^2 - 14I(I+1)+12]}{(2I-3)(2I-1)(2I+3)(2I+5)} A_{I,0}(\pi, q_{\text{eff}}) \\ & - \frac{4(I+1)(I+2)[(I+1)(I+2)-4]}{(2I-1)(2I+1)(2I+3)(2I+7)} A_{I+2,0}(\pi, q_{\text{eff}}) \\ & + \frac{(I+1)(I+2)(I+3)(I+4)}{(2I+1)(2I+3)(2I+5)(2I+7)} A_{I+4,0}(\pi, q_{\text{eff}}) \Bigg\} \end{aligned} \tag{5.58}$$

and

$$\begin{aligned} A_{I,\pm 2} = iq_{\text{eff}} \frac{J_{22}(\vartheta)}{J_{20}(\vartheta)} [(I-1)I(I+1)(I+2)]^{\frac{1}{2}} \Bigg\{ & \frac{1}{(2I-1)(2I+1)} A_{I-2,0}(\pi, q_{\text{eff}}) \\ & - \frac{2}{(2I-1)(2I+3)} A_{I,0}(\pi, q_{\text{eff}}) + \frac{1}{(2I+1)(2I+3)} A_{I+2,0}(\pi, q_{\text{eff}}) \Bigg\}. \end{aligned} \tag{5.59}$$

In these equations, $q_{\text{eff}}(\vartheta)$ is given by (5.15). The ratio $q_{\text{eff}}(\vartheta)/q$ is shown in Table 2 where also the ratio $J_{22}(\vartheta)/J_{20}(\vartheta)$ has been tabulated.

The excitation amplitudes for arbitrary spin sequence in the rotational band is given by Eq. (5.8). The first order correction in the amplitude for $\xi \neq 0$ is expressed by means of the $A_{I,M}(\vartheta, q)$ in Eq. (5.44). We shall in this paragraph only consider the application of $A_{I,M}(\vartheta, q)$ for the evaluation of the excitation probability $P_{I_f, I_i}(\vartheta, q, \xi)$.

In the simplest case of the excitation of a rotational band in an even-even nucleus for $\xi = 0$ and $\vartheta = \pi$, the excitation probabilities $P_I(q) = P_{I,0}(\pi, q, 0)$ are given by

$$P_I(q) = (2I+1) \mid A_{I,0}(\pi, q) \mid^2. \tag{5.60}$$

These probabilities have been evaluated in Table 5, and they are plotted in Fig. 7.

For other deflection angles the probability can be obtained from (5.58) and (5.59). To second order in $J_{22}(\vartheta)/J_{20}(\vartheta)$ it may be written

$$P_{I,0}(\vartheta, q, 0) = P_I(q_{\text{eff}}(\vartheta)) + \left(\frac{J_{22}(\vartheta)}{J_{20}(\vartheta)}\right)^2 \Delta_I(q_{\text{eff}}(\vartheta)). \tag{5.61}$$

TABLE 5.

The probabilities for excitation of the rotational states in an even-even nucleus. The result which is given for backward scattering and $\xi = 0$ is tabulated as a function of $q$ and of the spin of the excited state. The excitation probability for other deflection angles and other spins can easily be inferred from these numbers.

| $q$ | $P_0$ | $P_2$ | $P_4$ | $P_6$ | $P_8$ | $P_{10}$ | $P_{12}$ |
|---|---|---|---|---|---|---|---|
| 0.0 | 1.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.5 | 0.9144 | 0.0842 | 0.0014 | | | | |
| 1.0 | 0.6945 | 0.2850 | 0.0199 | 0.0006 | | | |
| 1.5 | 0.4300 | 0.4812 | 0.0828 | 0.0057 | 0.0002 | | |
| 2.0 | 0.2152 | 0.5597 | 0.1972 | 0.0260 | 0.0018 | 0.0001 | |
| 2.5 | 0.1021 | 0.4842 | 0.3296 | 0.0750 | 0.0086 | 0.0006 | |
| 3.0 | 0.0835 | 0.3098 | 0.4184 | 0.1572 | 0.0280 | 0.0029 | 0.0002 |
| 3.5 | 0.1108 | 0.1389 | 0.4140 | 0.2563 | 0.0685 | 0.0104 | 0.0010 |
| 4.0 | 0.1317 | 0.0514 | 0.3152 | 0.3354 | 0.1333 | 0.0286 | 0.0039 |
| 4.5 | 0.1214 | 0.0600 | 0.1741 | 0.3555 | 0.2125 | 0.0634 | 0.0116 |
| 5.0 | 0.0881 | 0.1158 | 0.0630 | 0.3006 | 0.2815 | 0.1171 | 0.0285 |
| 5.5 | 0.0571 | 0.1540 | 0.0284 | 0.1932 | 0.3105 | 0.1831 | 0.0593 |
| 6.0 | 0.0463 | 0.1412 | 0.0633 | 0.0848 | 0.2810 | 0.2436 | 0.1053 |
| 6.5 | 0.0547 | 0.0917 | 0.1191 | 0.0250 | 0.2007 | 0.2757 | 0.1618 |
| 7.0 | 0.0671 | 0.0459 | 0.1442 | 0.0312 | 0.1036 | 0.2616 | 0.2155 |
| 7.5 | 0.0687 | 0.0343 | 0.1198 | 0.0798 | 0.0331 | 0.2014 | 0.2481 |
| 8.0 | 0.0570 | 0.0552 | 0.0684 | 0.1242 | 0.0166 | 0.1174 | 0.2436 |
| 8.5 | 0.0413 | 0.0819 | 0.0296 | 0.1292 | 0.0491 | 0.0446 | 0.1985 |
| 9.0 | 0.0329 | 0.0879 | 0.0276 | 0.0930 | 0.0974 | 0.0128 | 0.1267 |
| 9.5 | 0.0355 | 0.0685 | 0.0549 | 0.0447 | 0.1230 | 0.0286 | 0.0560 |
| 10.0 | 0.0432 | 0.0408 | 0.0819 | 0.0189 | 0.1079 | 0.0719 | 0.0148 |

(to be continued)

The coefficient $\Delta_I(\vartheta)$ has been evaluated numerically for $I = 0,2$, and 4 and the result is given in Table 6 and Fig. 8.

In many cases a simpler approximation for $P_{I,0}(\vartheta, q, 0)$ will be quite adequate, namely the $q(\vartheta)$ approximation. In this approximation the excitation probability is given by

$$P_{I,0}(\vartheta, q, 0) \approx P_I(q(\vartheta)), \tag{5.62}$$

where the quantity $q(\vartheta)$ is defined by (5.25). The ratio $q(\vartheta) / q$ is given in Table 2. The accuracy of the approximation (5.62) is illustrated in Fig. 10.

As an illustration of the application of (5.62) the differential and total cross sections have been evaluated for the case $q = 3$, and the result is given in Fig. 12. While the cross sections for all higher states tend towards zero for small deflection angles, the excitation of the $I = 2$ state reaches a finite

TABLE 5 (continued).

| $q$ | $P_{14}$ | $P_{16}$ | $P_{18}$ | $P_{20}$ | $P_{22}$ | $P_{24}$ | $P_{26}$ |
|---|---|---|---|---|---|---|---|
| 0.0 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.5 | 0.0000 | | | | | | |
| 1.0 | 0.0000 | | | | | | |
| 1.5 | 0.0000 | | | | | | |
| 2.0 | 0.0000 | | | | | | |
| 2.5 | 0.0000 | | | | | | |
| 3.0 | 0.0000 | | | | | | |
| 3.5 | 0.0001 | | | | | | |
| 4.0 | 0.0004 | | | | | | |
| 4.5 | 0.0014 | 0.0001 | | | | | |
| 5.0 | 0.0046 | 0.0007 | | | | | |
| 5.5 | 0.0124 | 0.0018 | 0.0002 | | | | |
| 6.0 | 0.0283 | 0.0053 | 0.0007 | 0.0001 | | | |
| 6.5 | 0.0558 | 0.0130 | 0.0022 | 0.0003 | | | |
| 7.0 | 0.0961 | 0.0279 | 0.0058 | 0.0009 | 0.0001 | | |
| 7.5 | 0.1455 | 0.0529 | 0.0134 | 0.0025 | 0.0004 | | |
| 8.0 | 0.1938 | 0.0888 | 0.0274 | 0.0062 | 0.0011 | 0.0002 | |
| 8.5 | 0.2258 | 0.1327 | 0.0503 | 0.0136 | 0.0028 | 0.0004 | 0.0001 |
| 9.0 | 0.2275 | 0.1764 | 0.0828 | 0.0270 | 0.0066 | 0.0013 | 0.0002 |
| 9.5 | 0.1935 | 0.2073 | 0.1223 | 0.0481 | 0.0138 | 0.0031 | 0.0005 |
| 10.0 | 0.1326 | 0.2132 | 0.1622 | 0.0777 | 0.0265 | 0.0069 | 0.0014 |

TABLE 6.

The coefficient $\Delta_I(q)$ for the correction of the $q_{\text{eff}}(\vartheta)$ approximation (see Eq. (5.61)), in the case of a rotational band in an even-even nucleus for $\xi = 0$. The result is given for the states of spin $I = 0$, 2, and 4 as a function of $q$.

| $q$ | $\Delta_0$ | $\Delta_2$ | $\Delta_4$ | $q$ | $\Delta_0$ | $\Delta_2$ | $\Delta_4$ |
|---|---|---|---|---|---|---|---|
| 0.0 | 0.000 | 0.000 | 0.000 | 5.5 | —3.756 | 1.178 | 2.844 |
| 0.5 | —0.247 | 0.239 | 0.008 | 6.0 | —3.960 | 1.387 | 1.766 |
| 1.0 | —0.787 | 0.678 | 0.103 | 6.5 | —4.841 | 3.078 | —0.349 |
| 1.5 | —1.225 | 0.821 | 0.347 | 7.0 | —5.993 | 5.048 | —1.746 |
| 2.0 | —1.381 | 0.585 | 0.561 | 7.5 | —6.712 | 5.700 | —1.046 |
| 2.5 | —1.477 | 0.490 | 0.404 | 8.0 | —6.654 | 4.538 | 1.409 |
| 3.0 | —1.887 | 1.120 | —0.206 | 8.5 | —6.195 | 2.733 | 3.639 |
| 3.5 | —2.712 | 2.390 | —0.796 | 9.0 | —6.120 | 2.156 | 3.664 |
| 4.0 | —3.608 | 3.441 | —0.610 | 9.5 | —6.898 | 3.655 | 1.329 |
| 4.5 | —4.084 | 3.787 | 0.630 | 10.0 | —8.213 | 6.188 | —1.379 |
| 5.0 | —4.000 | 2.263 | 2.214 | | | | |

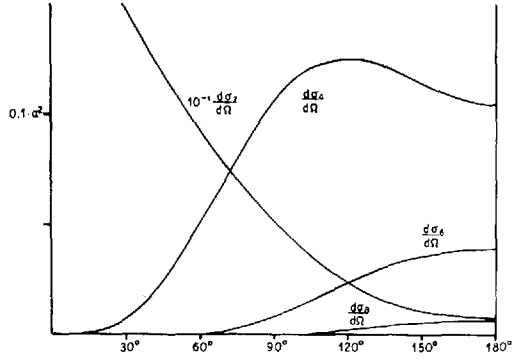


Fig. 12. The differential cross sections for multiple Coulomb excitation of a rotational band in an even-even nucleus for $q = 3$. The curves show the cross sections $d\sigma_I/d\Omega$ for the excitation of the state with spin $I$ in the sudden approximation in units of $a^2$. The curve for the first excited state has been scaled down by a factor 10.

value for $\vartheta = 0°$. This is seen from the perturbation expression which is valid in this region. One thus finds

$$\left(\frac{d\sigma_2}{d\Omega}\right)_{\vartheta=0} = \frac{4}{15} q^2 a^2. \tag{5.63}$$

From the differential cross sections the following values for the total cross sections have been obtained

$$\left.\begin{aligned} \sigma_{I=2} &= 7.93 \quad a^2 \\ \sigma_{I=4} &= 1.06 \quad a^2 \\ \sigma_{I=6} &= 0.160\, a^2 \\ \sigma_{I=8} &= 0.016\, a^2. \end{aligned}\right\} \tag{5.64}$$

For other ground state spins the excitation probabilities of the rotational band can be obtained by means of (5.13). Since $K = I_i$ this equation may be written

$$\left.\begin{aligned} P_{I_f I_i}(\vartheta, q, 0) &= (2I_f+1) \sum_I \begin{pmatrix} I_f & I_i & I \\ -I_i & I_i & 0 \end{pmatrix}^2 P_{I,0}(\vartheta, q, 0) \\ &= \frac{(2I_f+1)!\,(2I_i)!\,(I_i+I_f)!}{(I_f-I_i)!} \sum_I \frac{(I_f-I_i+I)!\, P_{I,0}(\vartheta, q, 0)}{(I_i+I_f+I+1)!\,(I_i-I_f+I)!\,(I_i+I_f-I)!} \end{aligned}\right\} \tag{5.65}$$

As an illustration, the case of $I_i = 5/2$ is shown in Fig. 13 for $\vartheta = 180°$.

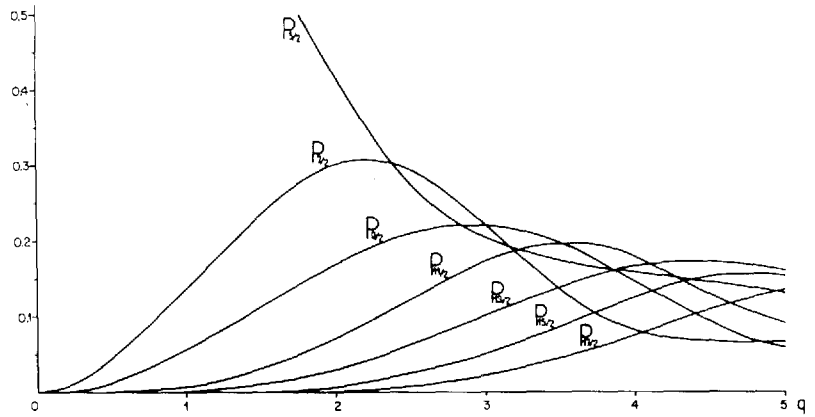


Fig. 13. The multiple Coulomb excitation of a pure rotational band in an odd $A$ nucleus with ground state spin 5/2. The excitation probability $P_I$ of the state with spin $I$ is given as a function of the parameter $q$ for backward scattering.

TABLE 7.

The coefficients $f_1(\vartheta)$ and $f_m^l(\vartheta)$ for the first order corrections in $\xi$ to the excitation of rotational states (see Eq. (5.44)). The coefficients are given as functions of the deflection angle $\vartheta$ (in degrees) for even values of $m$. For odd values of $m$ the $f_m^l(\vartheta)$ are easily obtained from $f_1(\vartheta)$ and the functions $J_{2,\mu}(\vartheta)$ given in Table 2 by means of the Eqs. (5.47) to (5.49). The entries are given in the form of a number followed by the power of ten by which it should be multiplied.

| $\theta$ | $f_1$ | $f_0^0$ | $f_0^2$ | $f_2^2$ |
|---|---|---|---|---|
| 180 | 0.000 | 3.893 (—1) | 2.781 (—1) | 0.000 |
| 150 | 1.917 (—1) | 3.423 (—1) | 2.423 (—1) | 9.405 (—3) |
| 120 | 3.242 (—1) | 2.188 (—1) | 1.495 (—1) | 2.672 (—2) |
| 90 | 3.586 (—1) | 9.615 (—2) | 5.915 (—2) | 3.181 (—2) |
| 60 | 2.904 (—1) | 2.370 (—2) | 9.662 (—3) | 1.920 (—2) |
| 30 | 1.513 (—1) | 1.442 (—3) | —1.047 (—3) | 4.078 (—3) |

| $\theta$ | $f_0^4$ | $f_2^4$ | $f_4^4$ |
|---|---|---|---|
| 180 | —6.673 (—1) | 0.000 | 0.000 |
| 150 | —5.935 (—1) | 1.015 (—3) | —4.062 (—5) |
| 120 | —3.942 (—1) | 4.773 (—3) | —4.713 (—4) |
| 90 | —1.880 (—1) | 8.893 (—3) | —1.373 (—3) |
| 60 | —5.511 (—2) | 8.255 (—3) | —1.654 (—3) |
| 30 | —5.787 (—3) | 2.628 (—3) | —6.296 (—4) |

TABLE 8.

The coefficients $A_I(q)$ for the first order correction in $\xi$ to the excitation probability of even-even nuclei. The coefficient which is defined in Eq. (5.51) is given for backward scattering on even-even nuclei. An approximate expression for other deflection angles can be obtained from the table by means of Eq. (5.67).

| $q$ | $A_0$ | $A_2$ | $A_4$ | $A_6$ | $A_8$ | $A_{10}$ |
|---|---|---|---|---|---|---|
| 0.0 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| 0.5 | 0.030 | —0.029 | —0.001 | | | |
| 1.0 | 0.204 | —0.186 | —0.017 | | | |
| 1.5 | 0.537 | —0.422 | —0.106 | —0.008 | | |
| 2.0 | 0.890 | —0.512 | —0.325 | —0.050 | —0.004 | |
| 2.5 | 1.084 | —0.250 | —0.636 | —0.175 | —0.022 | —0.002 |
| 3.0 | 1.044 | 0.337 | —0.861 | —0.427 | —0.083 | —0.009 |
| 3.5 | 0.861 | 0.963 | —0.773 | —0.775 | —0.234 | —0.038 |
| 4.0 | 0.714 | 1.282 | —0.278 | —1.073 | —0.509 | —0.117 |
| 4.5 | 0.728 | 1.159 | 0.451 | —1.107 | —0.880 | —0.288 |
| 5.0 | 0.880 | 0.783 | 1.053 | —0.735 | —1.224 | —0.578 |
| 5.5 | 1.033 | 0.510 | 1.216 | —0.025 | —1.344 | —0.966 |
| 6.0 | 1.061 | 0.580 | 0.919 | 0.721 | —1.080 | —1.341 |
| 6.5 | 0.955 | 0.930 | 0.465 | 1.136 | —0.433 | —1.526 |
| 7.0 | 0.821 | 1.266 | 0.249 | 1.039 | 0.370 | —1.351 |
| 7.5 | 0.780 | 1.321 | 0.456 | 0.579 | 0.969 | —0.779 |
| 8.0 | 0.865 | 1.082 | 0.922 | 0.148 | 1.087 | 0.031 |
| 8.5 | 0.995 | 0.785 | 1.278 | 0.099 | 0.728 | 0.751 |
| 9.0 | 1.055 | 0.701 | 1.259 | 0.480 | 0.199 | 1.063 |
| 9.5 | 0.996 | 0.901 | 0.920 | 1.006 | 0.089 | 0.855 |
| 10.0 | 0.880 | 1.204 | 0.580 | 1.282 | —0.094 | 0.322 |

(to be continued)

The first order corrections for $\xi \neq 0$ must be calculated by means of Eq. (5.44) from the quantities $A_{IM}(\vartheta, q)$ and from the functions $f_m^l(\vartheta)$ and $f_1(\vartheta)$. The latter have been evaluated numerically for some angles and the result is given in Table 7.

The excitation probability may be written in the form (5.51)

$$P_{I_f I_i}(\vartheta, q, \xi) \approx P_{I_f I_i}(\vartheta, q, 0) + A_{I_f I_i}(\vartheta, q)\, \xi. \tag{5.66}$$

The functions $A$ have been calculated for the special case of $\vartheta = \pi$ and $I_i = 0$, and the result is given in Table 8. The effect of the correction in the excitation probability is illustrated in Fig. 11. The result (5.66) may be applied for angles in the neighbourhood of 180 degrees by the following substitution:

TABLE 8 (continued).

| $q$ | $\Lambda_{12}$ | $\Lambda_{14}$ | $\Lambda_{16}$ | $\Lambda_{18}$ | $\Lambda_{20}$ |
|---|---|---|---|---|---|
| 0.0 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| 0.5 | | | | | |
| 1.0 | | | | | |
| 1.5 | | | | | |
| 2.0 | | | | | |
| 2.5 | | | | | |
| 3.0 | —0.001 | | | | |
| 3.5 | —0.004 | | | | |
| 4.0 | —0.017 | —0.001 | | | |
| 4.5 | —0.055 | —0.007 | | | |
| 5.0 | —0.149 | —0.025 | —0.003 | | |
| 5.5 | —0.336 | —0.073 | —0.016 | | |
| 6.0 | —0.640 | —0.180 | —0.034 | —0.005 | |
| 6.5 | —1.038 | —0.380 | —0.092 | —0.016 | —0.001 |
| 7.0 | —1.436 | —0.694 | —0.210 | —0.044 | —0.007 |
| 7.5 | —1.669 | —1.101 | —0.421 | —0.110 | —0.021 |
| 8.0 | —1.571 | —1.516 | —0.744 | —0.238 | —0.055 |
| 8.5 | —1.072 | —1.788 | —1.163 | —0.459 | —0.128 |
| 9.0 | —0.284 | —1.753 | —1.586 | —0.789 | —0.266 |
| 9.5 | 0.511 | —1.324 | —1.888 | —1.210 | —0.495 |
| 10.0 | 0.981 | —0.571 | —1.907 | —1.648 | —0.832 |

$$\Lambda_{I_f 0}(\vartheta, q) \approx \frac{q^2 f_0^0(\vartheta)}{(q_{\text{eff}}(\vartheta))^2 f_0^0(\pi)} \Lambda_{I_f 0}(\pi, q_{\text{eff}}(\vartheta)). \tag{5.67}$$

This equation only holds as long as $f_0^0$, $f_0^2$, and $f_0^4$ dominate over the coefficients $f_2^2$, $f_2^4$, and $f_4^4$.

The collision between the target nucleus and the projectile may also lead to an excitation of the projectile. The results which we have obtained for target excitation can also be used for projectile excitations, since we have worked in a relative coordinate system. The parameter $\xi$ (see Eq. (2.8)) is thus given by the Eqs. (5.53) and (5.54) where one must insert for $E_2 - E_1$ the excitation energy of the projectile. Similarly the expression for $\chi$ (see Eq. (2.11)) is given by

$$\chi_{1\to 2}^{\text{proj}} = 14.52 \frac{A_1^{1/2} [B(E2, I_1 \to I_2)]^{1/2}}{(1 + A_1/A_2)^2 Z_2 Z_1^2} E_{\text{MeV}}^{3/2}, \tag{5.68}$$

where $B(E2)$ now refers to the projectile. The formula for $q^{\text{proj}}$ has the same relation to $q$ in (5.57) as $\chi^{\text{proj}}$ has to $\chi$ in (5.55).

### D. Classical Treatment

We shall make a few comments about the classical limit of the excitation of rotational states, which can be used for large angular momenta and large $q$. The classical problem of a collision between a charged particle and a charged symmetric top leads to a non-linear equation of motion which, like in the quantum mechanical problem, can only be solved in closed form in the limit where the collision time is short compared to the time of rotation of the top.

The classical Hamiltonian can be written in the form

$$H = \frac{v}{6\,a\,\hbar}\,\xi\left[p_\beta^2 + \frac{1}{\sin^2\beta}\,p_\alpha^2\right] + \frac{8\,\pi}{5}\,\hbar\,v\,a^2\,q\sum_\mu \bar{S}_{2\,\mu}(t)\,Y_{2\,\mu}^*(\beta,\,\alpha), \tag{5.69}$$

where $p_\beta$ and $p_\alpha$ are the momenta which are conjugate to the Eulerian angles $\beta$ and $\alpha$, describing the orientation of the axis of the top.

We shall here consider only the case where one may neglect the terms with $\mu \neq 0$, i.e., we limit ourselves to the case of backward scattering or the $q\,(\vartheta)$ approximation. In this case the angle $\alpha$ is a cyclic variable. For the angle $\beta$ one obtains from (5.69) the following equation of motion:

$$\ddot{\beta} = \sqrt{\frac{4\,\pi}{5}}\,q\,\xi\,a v^2\,\bar{S}_{20}(t)\sin 2\beta\,. \tag{5.70}$$

In the sudden approximation one assumes $\beta$ on the right-hand side to be unchanged (equal to $\beta_0$) during the collision, and the final angular velocity $\dot{\beta}_f$ is thus given by

$$\dot{\beta}_f = \dot{\beta}_i + \frac{2}{3}\,\frac{v}{a}\,q_{\mathrm{eff}}(\vartheta)\,\xi\sin 2\beta_0. \tag{5.71}$$

We have here used Eqs. (5.15), (3.24), and (3.27), and have denoted the angular velocity $\dot{\beta}$ before the collision by $\dot{\beta}_i$.

From (5.71) we obtain the following simple expression for the transfer of angular momentum $\Delta L_\perp$ perpendicular to the symmetry axis of the orbit.

$$\Delta L_\perp = 2\,q_{\mathrm{eff}}(\vartheta)\,\hbar\sin 2\beta_0, \tag{5.72}$$

while the component of $L$ parallel to the axis is unchanged.

In the classical treatment one thus finds that the angular momentum transfer depends on the initial orientation of the top and one sees that the projectile can transfer at most (for $\beta_i = \pi/4$) an angular momentum of magnitude

$$\Delta L_{\max} = 2\,q_{\mathrm{eff}}(\vartheta)\,\hbar. \tag{5.73}$$

If one considers all initial orientations of the top to be equal probable, one may evaluate the classical energy distribution of the top after the collision. In the simplest case where the nucleus is at rest before the collision, one finds corresponding to (5.73) also a maximum energy transfer $E_{\max} = 2(q_{\text{eff}}(\vartheta))^2 \hbar^2/\Im$. In this case, the energy distribution can be written in the form

$$P(E)\,dE = \frac{d\varepsilon}{4\sqrt{\varepsilon}\sqrt{1-\sqrt{\varepsilon}}}, \tag{5.74}$$

where

$$\varepsilon = \frac{E}{E_{\max}} = \frac{\Im}{2(q_{\text{eff}}(\vartheta))^2 \hbar^2} E. \tag{5.75}$$

This energy distribution (5.74) is illustrated in Fig. 14.

The classical treatment gives a qualitative understanding of the result of the quantum mechanical calculations of Fig. 7. In the classical limit, the excitation probability of a state of spin $I$ is zero until $q$ reaches the value $I/2$. As a function of $q$ the excitation probability thereafter goes through a maximum and finally decreases slowly. The quantum mechanical energy distribution is a function of both $q$ and the discrete excitation energies. For a fixed value of $q$ the points corresponding to the different energies oscillate around the classical curve. On Fig. 14 we have illustrated the case of $q = 10$, and we have here, for illustrative purposes, connected the points (indicated by circles) by a smooth curve. It is seen that the result is still far from the classical limit.

Like in the quantum mechanical treatment, the case of $\xi \neq 0$ can be solved for small values of $\xi$. One must then take into account that the nucleus is moving during the collision time. In first order one may consider the change in the right-hand side of (5.70) to be linear in $\beta$. One is thereby led to a hypergeometric differential equation which can be solved explicitly.

The result for the angular momentum transfer perpendicular to the $z$ axis can be written in the form

$$\Delta L_{\perp} = 2\, q_{\text{eff}}(\vartheta)\, \hbar \sin 2\beta_0\, F(q\xi \cos 2\beta_0), \tag{5.76}$$

where the correction factor $F$ to the result for $\xi = 0$ is given by

$$F(x) = \frac{3\cos\left(\pi\sqrt{\frac{9}{4} - 4x}\right)}{4\pi x(2x-1)} \approx 1 + \frac{22}{9}x \quad \text{for} \quad x \ll 1. \tag{5.77}$$

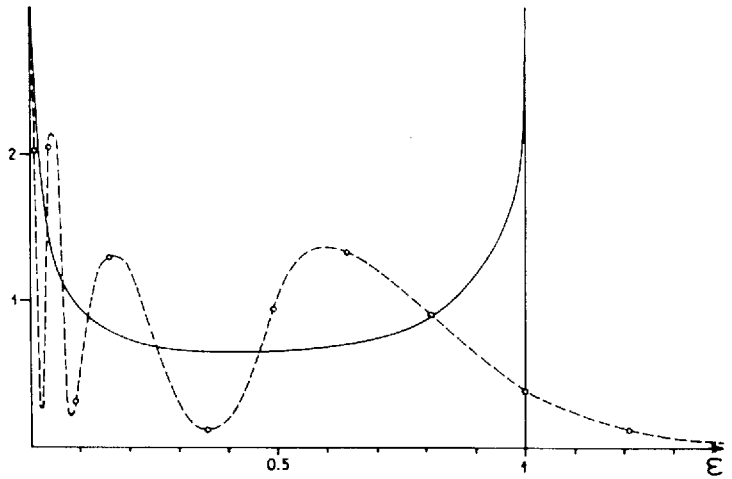


Fig. 14. The classical energy distribution of a charged symmetric top after a head-on impact of a charged particle (full drawn curve). The top is assumed to be at rest before the collision and the impact is assumed to be sudden. The scale of the abscissa is the ratio between the energy of the top $E$ and the maximum energy $E_{max}$ which can be transferred. The circles, which are connected by a broken curve, show the corresponding quantum mechanical result for $q = 10$.

It is illustrative to evaluate the average excitation energy of the nucleus after the collision. If we assume an isotropic distribution for the orientation of the nucleus before the collision, we find from (5.76) to first order in $\xi$ (and backward scattering)

$$\langle E \rangle = \frac{16}{15} \frac{q^2 \hbar^2}{\Im} \left(1 - \frac{44}{63} q \xi\right). \tag{5.78}$$

This result must be correct also in the quantum mechanical treatment. In the limit of $q \ll 1$ where only the lowest state in a rotational band in even-even nuclei is excited, one may use it to calculate the excitation probability. Since the energy of the state of spin 2 is $E_2 = 3\hbar^2/\Im$, one finds

$$P_2 = \frac{16}{45} q^2 \left(1 - \frac{44}{63} q \xi\right). \tag{5.79}$$

If one compares this result with the result (5.51) in the limit of $q \ll 1$, one finds

$$f_0^0(\pi) = \frac{11}{9\pi} = \frac{7}{5} f_0^2(\pi) = -\frac{7}{12} f_0^4(\pi) \tag{5.80}$$

in agreement with Table 7.

## 6. Excitation of Vibrational States

Another important kind of collective excitations in nuclei is that connected with the vibrational degree of freedom. In even-even nuclei a number of low-lying states have been identified as vibrational levels but, in general, the spectra are not as well understood as the corresponding rotational states in deformed nuclei. A survey of the experimental and theoretical status is given in ref. 1, Chapt. VC.

The excitation of pure vibrational states can be solved exactly not only in the sudden approximation, but also for arbitrary $\vartheta$, $\xi$, and $\chi$. The problem is analogous to the classical problem of a forced vibration which can also be solved in an explicit form.

For a pure quadrupole vibration, the Hamiltonian of the free nucleus is given by

$$H_0 = \frac{1}{2} B \sum_{\mu} |\dot{\alpha}_{2\mu}|^2 + \frac{1}{2} C \sum_{\mu} |\alpha_{2\mu}|^2, \tag{6.1}$$

where $B$ is the inertial parameter and $C$ the restoring force. The parameters $\alpha_{2,\mu}$, where $\mu = -2, -1, 0, 1, 2$ describe the shape of the nuclear surface. In the idealized case where the surface is sharply defined and where the nuclear density is constant, the nuclear shape is given by

$$R(\vartheta, \varphi) = R_0 \left[1 + \sum_{\mu} \alpha_{2\mu} Y_{2\mu}(\vartheta, \varphi)\right]. \tag{6.2}$$

The eigenstates of the Hamiltonian (6.1) can be classified according to the five vibrational quantum numbers $n_\mu$, where $n_\mu = 0, 1, 2 \ldots$. The energy of a state $|\, n_\mu\rangle$ can thus be written in the form

$$E = \hbar\omega \sum_{\mu} \left(n_\mu + \frac{1}{2}\right) = \hbar\omega \left(N + \frac{5}{2}\right). \tag{6.3}$$

Here, the frequency $\omega$ is given by

$$\omega = \sqrt{\frac{C}{B}}, \tag{6.4}$$

while the principal quantum number $N$ is defined by

$$N = \sum_{\mu} n_\mu. \tag{6.5}$$

The degenerate nuclear states can also be labelled by this principal quantum number together with the total angular momentum $I$ and the mag-

netic quantum number $M$. For $N \leqslant 3$ these numbers are sufficient to specify the state completely, while for $N > 3$ one needs additional quantum numbers[17]. The connection between the two labellings $n_\mu$ and $N, I, M$ is given in refs. 15 and 16 for a number of cases.

In the following, it will be convenient to introduce a dimensionless coordinate $x_\mu$ defined by the equation

$$\alpha_{2\mu} = \sqrt{\frac{\hbar}{\sqrt{BC}}}\, x_\mu \tag{6.6}$$

instead of $\alpha_{2,\mu}$. If we introduce furthermore $\xi$ by means of the equation

$$\xi = \omega \frac{a}{v}, \tag{6.7}$$

we may write $H_0$ in the form

$$H_0 = \frac{\hbar v \xi}{2a} \sum_\mu [\,|\, p_\mu/\hbar \,|^2 + |\, x_\mu \,|^2\,], \tag{6.8}$$

where the momenta $p_\mu$ are defined by

$$p_\mu = -B\dot{\alpha}^*_{2\mu} = -\sqrt{\hbar \sqrt{\frac{B}{C}}}\, \dot{x}^*_\mu. \tag{6.9}$$

The nuclear multipole moments $\mathfrak{M}(E\,2,\mu)$ are related to the deformation parameters $\alpha_{2,\mu}$ by the following expression (see ref. 1, Eq. (V. 24)),

$$\mathfrak{M}(E\,2,\mu) = \frac{3}{4\pi} Z_1 e R_0^2 \alpha^*_{2\mu}. \tag{6.10}$$

By evaluating the reduced matrix element of (6.10) between the ground state and the first excited state one finds the following expression for the parameter $\chi$ (see Eq. (2.12)):

$$\chi = \frac{Z_1 Z_2 e^2 R_0^2}{v a^2 \sqrt{10\pi\hbar\sqrt{BC}}} \tag{6.11}$$

and one may therefore write the interaction Hamiltonian (2.3) in the following form:

$$H'(t) = \sqrt{\frac{18\pi}{5}}\, \hbar v a^2 \chi \sum_\mu \bar{S}_{2\mu}(t)\, x_\mu, \tag{6.12}$$

where $\bar{S}_{2\mu}(t)$ is given by (3.23).

The eigenstates of the free Hamiltonian $H_0$ are given in terms of the Hermite polynomials $H_n(x)$, i.e.,

$$\left.\begin{aligned}\psi_{n_{-2}\, n_{-1}\, n_0\, n_1\, n_2}(x) &= \prod_\mu \psi_{n_\mu}(x_\mu) \\ &= \prod_\mu \frac{1}{\sqrt{\sqrt{\pi}\, 2^{n_\mu}\, n_\mu!}}\, e^{-\frac{1}{2}x_\mu^2} H_{n_\mu}(x_\mu).\end{aligned}\right\} \tag{6.13}$$

The excitation amplitude in the sudden approximation (3.11) is now very easily found.

The result for the distribution on the different energy levels ($N$) turns out to be a Poisson distribution, where the mean excitation energy is the same as that one would find in the perturbation calculation (see Eq. (6.27) below). This result can be understood by noting that the excitation of an harmonic oscillator can always be interpreted as the collective motion of a large number of mutually uncoupled harmonic oscillators which are, each of them, only weakly excited. The weak excitation of these oscillators can be treated by a perturbation calculation and, since they are mutually uncoupled, the resulting total energy distribution must be a Poisson distribution.

Since the above argument is independent of the sudden approximation, we shall in the following give the details of the calculation in the more general case of $\xi \neq 0$.

The first step in the program will be to introduce a number of auxiliary variables $x^{(i)}$, where $i = 1, 2, \ldots\ldots \mathfrak{N}$ and where $x_\mu^{(1)} = x_\mu$. The 5 $(\mathfrak{N}-1)$ new degrees of freedom are supposed to be coordinates for free vibrations which have the same frequency $\omega$ as the $x_\mu$ oscillators. Furthermore, we take them to be coupled, neither to each other nor to the old $x_\mu$. Under these circumstances they will be left undisturbed in the Coulomb excitation process and will only change the problem in a trivial way. The total Hamiltonian will thus be

$$H = \frac{\hbar v \xi}{2a} \sum_{\mu,\, i} \left[ \left| \frac{p_\mu^{(i)}}{\hbar} \right|^2 + |\, x_\mu^{(i)}\, |^2 \right] + \sqrt{\frac{18\pi}{5}}\, \hbar v a^2 \chi \sum_\mu \bar{S}_{2\mu}(t)\, x_\mu^{(1)}, \tag{6.14}$$

while the eigenstates which are of physical interest will be

$$\psi(x) = \psi_{n_{-2},\, n_{-1},\, n_0, n_1,\, n_2}(x^{(1)}) \prod_{\substack{\mu \\ i+1}} \psi_0(x_\mu^{(i)}) \tag{6.15}$$

where $\psi_0$ is the ground state wave function (6.13) with $n_\mu^{(i)} = 0$.

We perform now a linear transformation on the coordinates $x_\mu^{(i)}$ and introduce hereby new coordinates $Q_\mu^{(i)}$ and new momenta $P_\mu^{(i)}$. The transformation matrix $U$ is supposed to be unitary, i. e.

$$UU^\dagger = 1 \tag{6.16}$$

and is assumed to be diagonal in $\mu$, i. e.,

$$Q_\mu^{(i)} = \sum_j U_{ij}^{(\mu)} x_\mu^{(j)}. \tag{6.17}$$

Furthermore, we prescribe the first row of $U^{(\mu)}$ for all values of $\mu$ to be given by

$$U_{1,j}^{(\mu)} = \frac{1}{\sqrt{\mathfrak{N}}}. \tag{6.18}$$

The new Hamiltonian in the variables $Q_\mu^{(i)}$ and $P_\mu^{(i)}$ is found from (6.14) (6.16), and (6.18) to be

$$H = \frac{\hbar v \xi}{2a} \sum_{\mu i} [\,|\, P_\mu^{(i)}/\hbar \,|^2 + |\, Q_\mu^{(i)} \,|^2] + \sqrt{\frac{18\pi}{5}} \hbar v a^2 \frac{\chi}{\sqrt{\mathfrak{N}}} \sum_\mu \bar{S}_{2\mu}(t) \sum_i Q_\mu^{(i)}. \tag{6.19}$$

The Schrödinger equation for the variables $Q_\mu^{(i)}$ is again separable and the eigenstates of the free Hamiltonian are

$$\varphi_n(Q) = \prod_{i\mu} \varphi_{n_\mu^{(i)}}(Q_\mu^{(i)}), \tag{6.20}$$

where all $n_\mu^{(i)}$ can take the values $n_\mu^{(i)} = 0, 1, 2 \ldots\ldots$

In the new Hamiltonian, however, the interaction term can be made very small for all values of $\mu$ and $i$ by choosing $\mathfrak{N}$ to be a large number. In the new variables, the excitation process can therefore be treated by a perturbation calculation, and one obtains for each of the oscillators $(\mu, j)$ only a very small probability that the oscillator is excited. In the perturbation treatment one finds, for each oscillator the following excitation amplitude on the first excited state:

$$a_\mu^{(j)} = \frac{1}{i\hbar} \langle 1 \,|\, \int_{-\infty}^{\infty} \sqrt{\frac{18\pi}{5}} \hbar v a^2 \frac{\chi}{\sqrt{\mathfrak{N}}} \bar{S}_{2\mu}(t)\, Q_\mu^{(j)} e^{i\omega t} dt \,|\, 0 \rangle. \tag{6.21}$$

Since the matrix element of $Q_\mu^{(j)}$ between the states of one phonon $|\, 1 \rangle$ and the ground state $|\, 0 \rangle$ is given by

$$\langle 1 \mid Q_\mu^{(j)} \mid 0 \rangle = \frac{(-1)^\mu}{\sqrt{2}}, \tag{6.22}$$

one may write (6.21) in the form

$$a_\mu^{(j)} = -\frac{i}{\sqrt{\mathfrak{N}}} \chi_\mu(\vartheta, \xi), \tag{6.23}$$

where (see also Eq. (3.24))

$$\chi_\mu(\vartheta, \xi) = \sqrt{\frac{9\pi}{5}} v a^2 (-1)^\mu S_{E2,\mu}(\vartheta, \xi). \tag{6.24}$$

We can now easily determine the total excitation probability. We ask first for the probability that any one of the five oscillators belonging to a definite value of $j$ is excited. This probability is

$$P_{(j)} = \sum_\mu \mid a_\mu^{(j)} \mid^2 = \frac{1}{\mathfrak{N}} \mid \chi(\vartheta, \xi) \mid^2, \tag{6.25}$$

where $\chi(\vartheta, \xi)$ is given by Eq. (2.12), i. e.,

$$\chi(\vartheta, \xi) = \chi \frac{9\pi}{5} \sum_\mu \mid Y_{2\mu}\left(\frac{\pi}{2}, 0\right) I_{2\mu}(\vartheta, \xi) \mid^2. \tag{6.26}$$

We can then calculate the probability that all the $\mathfrak{N}$ groups of five oscillators have together the total excitation energy $N\hbar\omega$. Since all groups have the same probability (6.25) of having the energy $\hbar\omega$, we obtain in the limit of $\mathfrak{N} \to \infty$ a Poisson distribution for this probability $P_N$

$$P_N = \frac{1}{N!} e^{-[\chi(\vartheta, \xi)]^2} [\chi(\vartheta, \xi)]^{2N}. \tag{6.27}$$

In the old variables $x_\mu^{(i)}$, this result must be interpreted as the total excitation probability of the vibrational state with principal quantum number $N$.

We shall be interested also in the amplitudes on the eigenstates (6.13). We shall evaluate these by calculating the amplitudes $\langle \varphi_n(Q) \mid \Phi(Q) \rangle$ of the final wave function, $\Phi$, on the eigenstates (6.20) as well as the amplitudes $\langle \psi(x(Q)) \mid \varphi_n(Q) \rangle$ of (6.13) on these eigenstates.

From (6.23) one finds directly the amplitude on the states (6.20)

$$\langle \varphi_n(Q) \mid \Phi(Q) \rangle = \prod_{\mu, i} [a_\mu^{(i)}]^{n_\mu^{(i)}} [1 - \mid a_\mu^{(i)} \mid^2]^{(1-n_\mu^{(i)})/2}, \tag{6.28}$$

where the quantum numbers $n_\mu^{(i)}$ are all 0 or 1.

The amplitude of (6.13) on (6.20) follows from the expansion of a Hermite polynomial of a linear function of $Q_\mu^{(i)}$ in terms of a product of Hermite polynomials of $Q_\mu^{(i)}$. One finds from ref. 10 (Vol. 2, p. 196)

$$\langle \psi(x(Q)) \mid \varphi_n(Q) \rangle = \prod_\mu \sqrt{\frac{n_\mu!}{\mathfrak{N}^{n_\mu}(n_\mu^{(1)})!\,(n_\mu^{(2)})!\cdots(n_\mu^{(\mathfrak{N})})!}}. \tag{6.29}$$

According to (6.28) and (6.29), the expression for the excitation amplitude is

$$\left.\begin{aligned} a_{n_{-2}\,n_{-1}\,n_0\,n_1\,n_2} &= \sum_n \langle \psi(x(Q)) \mid \varphi_n(Q) \rangle \langle \varphi_n(Q) \mid \Phi(Q) \rangle \\ &= \prod_\mu \sqrt{\frac{(n_\mu)!}{\mathfrak{N}^{n_\mu}}} \left(\frac{-i\chi_\mu(\vartheta,\xi)}{\sqrt{\mathfrak{N}}}\right)^{n_\mu} \left(1 - \frac{(\chi_\mu(\vartheta,\xi))^2}{\mathfrak{N}}\right)^{\frac{\mathfrak{N}}{2}-\frac{n_\mu}{2}} \binom{\mathfrak{N}}{n_\mu}. \end{aligned}\right\} \tag{6.30}$$

We have here utilized that $n_\mu^{(i)} = 0$ or 1 and have performed the summation over $n_\mu^{(i)}$ with the restriction $\Sigma\, n_\mu^{(i)} = n_\mu$ by multiplying with the number of ways in which $n_\mu$ objects may be chosen among $\mathfrak{N}$ objects. When we let $\mathfrak{N} \to \infty$ the expression (6.30) takes the form

$$a_{n_{-2}\,n_{-1}\,n_0\,n_1\,n_2} = \prod_\mu \frac{(-i)^{n_\mu}}{\sqrt{(n_\mu)!}} (\chi_\mu(\vartheta,\xi))^{n_\mu} e^{-\frac{1}{2}(\chi_\mu(\vartheta,\xi))^2}. \tag{6.31}$$

This equation offers the complete solution of the excitation of pure vibrational states.

It is interesting to observe that the total excitation probability (6.27) depends on $\vartheta$ and $\xi$ only through the quantity $\chi(\vartheta, \xi)$. This means that the excitation probability for arbitrary $\vartheta$ and $\xi$ can be obtained from the probabilities for $\vartheta = \pi$ and $\xi = 0$ by substituting $\chi(\vartheta, \xi)$ for $\chi$, i. e.,

$$P_N(\vartheta, \xi, \chi) = P_N(\pi, 0, \chi(\vartheta, \xi)). \tag{6.32}$$

In the special case of $\xi = 0$ this equation shows that the $\chi(\vartheta)$ approximation (see Eq. (5.62)) in the case of vibrational states is exactly fulfilled.

The function $P_N(\pi, 0, \chi)$ is illustrated on Fig. 15, as a function of $\chi$. It is interesting to compare this result with the corresponding result for the excitation of a rotational band which is illustrated on Fig. 7. The maximum excitation probabilities are larger for rotational states than for vibrational states. However, in the latter case, higher lying states are reached for a definite value of $\chi$.

The ground state and the first excited state have the definite angular momenta 0 and 2. The second excited state, however, is a triplet with spins 0, 2 and 4. Since the vibrational states in nuclei are not pure, the degeneracy

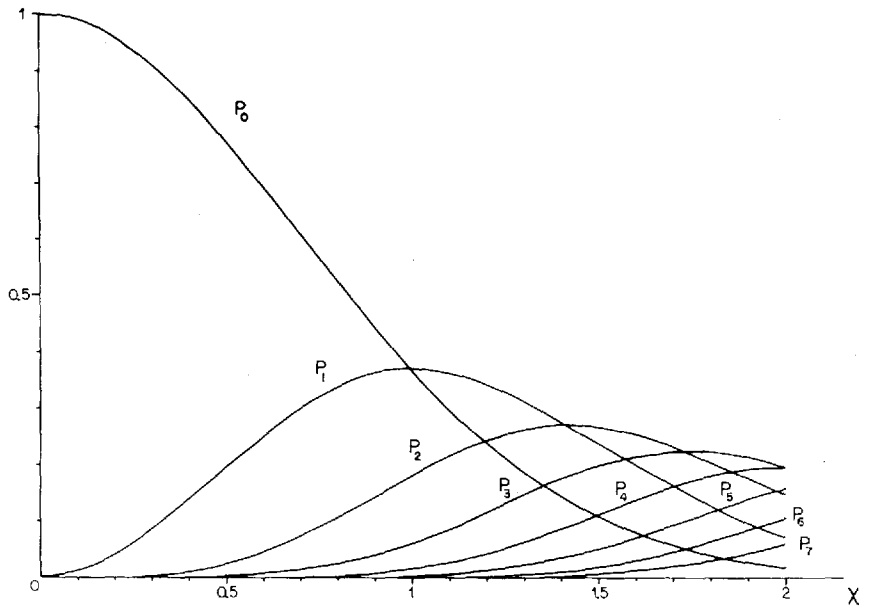


Fig. 15. The multiple Coulomb excitation of a pure vibrational band in an even-even nucleus. The excitation probability $P_N$ of the state with principal quantum number $N$ is given as a function of the parameter $\chi(\vartheta, \xi)$.

is in actual cases removed, and it is thus interesting to find how the total excitation probability (6.27) is distributed on each of the substates.

To perform this calculation one needs the expression (6.31) for the amplitudes on the states specified by $n_{-2}, n_{-1}, \ldots . n_2$. Furthermore, one needs the coefficients for the transformation between the $n_\mu$ and the $N, I, M$ labelling (see refs. 15 and 16).

One finds in the case of $N = 2$ the following result

$$\left.\begin{aligned} P_{2,I=0} &= 1/5 \; P_2 \\ P_{2,I=2} &= 2/7 \; P_2 \\ P_{2,I=4} &= 18/35 \, P_2 \end{aligned}\right\} \qquad (6.33)$$

where $P_2$ is given by (6.27) for $N = 2$. In this case the rule (6.32) thus also holds for the excitation of the substates with $I = 0$, 2 and 4. This is, however, not true any more for the excitation of the substates of the state with principal quantum number $N = 3$.

## 7. Excitation of Coupled Rotational Bands

In the two preceding sections, we have treated the multiple Coulomb excitation of a pure rotational band and of a pure vibrational band, and we assumed there that only the rotational or vibrational degrees of freedom were involved in the excitation process. In actual cases several different degrees of freedom can be excited. One might have cases, such as in most deformed nuclei, where both rotational and vibrational degrees of freedom are involved, or one might have to consider the excitation of the intrinsic degrees of freedom.

In this section, we shall consider a situation where the low energy nuclear spectrum consists of a number of rotational bands which differ in internal (or vibrational) structure. The excitation of these bands can be treated rigorously in the sudden approximation, in some cases when only a finite number of bands have to be taken into account.

In most cases, however, one will find that the parameter $\chi$ (see Eq. (2.12)) which describes the transition between the bands is small, and one may then simplify the calculation by a perturbation expansion for the transition from one band to the others. The transitions within any one of the bands must in any case be treated rigorously.

We shall assume that the nuclear states are described by state vectors of the form

$$|\psi\rangle = |n, K\rangle |I, K, M\rangle, \tag{7.1}$$

where $|I, K, M\rangle$ stands for a rotational wave function of the form (5.6) which only depends on the Eulerian angles describing the orientation of the nuclear axis, while $|n, K\rangle$ describes a state of the intrinsic and vibrational degrees of freedom, which has a component of angular momentum $K$ along the nuclear symmetry axis. The state vector $|n, K\rangle$ depends only on relative coordinates measured with respect to a coordinate system which has its $z$-axis along the nuclear axis.

In actual cases, the nuclear state vector will be a linear combination of state vectors of the form (7.1). Firstly, it will always contain a term identical with (7.1), except for a change of sign on $K$. Secondly, it may often contain admixtures from other bands with different values of $K$ and $n$. The actual excitation probabilities can, however, easily be evaluated once the excitation probabilities for states of the simple type (7.1) are known.

It is convenient to transform the interaction energy to the rotating coordinate system which has its $z$-axis along the nuclear axis.

For the multipole operator (2.4) one finds

$$\mathfrak{M}(E2,\mu) = \sum_{\nu} D^2_{\mu,\nu}(\alpha,\beta,0)\,\mathfrak{M}_{\mathrm{int}}(E2,\nu), \tag{7.2}$$

where the intrinsic multipole operator $\mathfrak{M}_{\mathrm{int}}(E2,\nu)$ is independent of the Eulerian angles $\alpha$ and $\beta$.

If we adopt the $\chi(\vartheta)$ approximation, the excitation amplitude in the sudden approximation (3.11) may be written

$$a^{(f)}_{I_f M_f K_f} = \langle I_f K_f M_f | \langle f K_f | e^{-i\sqrt{\frac{64\pi}{45}}\sum_{\nu} q_\nu(\vartheta)\, Y^*_{2\nu}(\beta,0)} | i K_i\rangle | I_i K_i M_i\rangle \tag{7.3}$$

where the operator $q_\nu(\vartheta)$ is defined by

$$q_\nu(\vartheta) = \sqrt{\frac{\pi}{5}}\,\frac{Z_1 e^2 \mathfrak{M}^*_{\mathrm{int}}(E2,\nu)}{\hbar v a^2}\,\frac{3}{4} J_{2,0}(\vartheta). \tag{7.4}$$

The expectation value of $q_\nu$ for states within a band is exactly the earlier defined quantity of $q_{\mathrm{eff}}(\vartheta)$ for this band (see Eq. (5.15)),viz.,

$$\langle nK | q_\nu(\vartheta) | nK\rangle = q^{(n)}_{\mathrm{eff}}(\vartheta)\,\delta_{\nu 0} \approx q^{(n)}(\vartheta)\,\delta_{\nu 0}. \tag{7.5}$$

If the matrix elements of $q_\nu$ between the bands are small, and if the expectation value of $q_\nu$ or the intrinsic quadrupole moment is not very different in the initial and final band, one may use a perturbation expansion to evaluate (7.3). We may write the amplitude (7.3) in the form

$$\left.\begin{aligned} a^{(f)}_{I_f K_f M_f} &= \langle I_f K_f M_f | e^{-i\sqrt{\frac{64\pi}{45}} q(\vartheta) Y_{2,0}(\beta,0)} \\ &\times \langle f K_f | e^{-i\sqrt{\frac{64\pi}{45}}\sum_{\nu}[q_\nu(\vartheta)-\delta_{\nu 0} q(\vartheta)]\, Y^*_{2\nu}(\beta,0)} | i K_i\rangle | I_i K_i M_i\rangle, \end{aligned}\right\} \tag{7.6}$$

where $q(\vartheta)$ is the $q$ for the initial band. We then perform a series expansion of the second exponential whereby we obtain the following expression for the excitation amplitude for a band different from the ground state band:

$$\left.\begin{aligned} a^{(f)}_{I_f K_f M_i} &\approx -i\sqrt{\frac{64\pi}{45}} \langle f K_f | q_{K_f - K_i}(\vartheta) | i K_i\rangle \\ &\times \langle I_f K_f M_f | Y^*_{2,K_i-K_f}(\beta,0)\, e^{-i\sqrt{\frac{64\pi}{45}} q(\vartheta) Y_{2,0}(\beta,0)} | I_i K_i M_i\rangle. \end{aligned}\right\} \tag{7.7}$$

The excitation probability in the perturbation treatment can thus be written in the form

$$P^{(f)}_{I_f} = r^2_{if} N^{(f)}_{I_f K_f}(q(\vartheta)) \tag{7.8}$$

where we have introduced the notation

$$r_{if} = \frac{\langle fK_f \mid q_{K_f - K_i}(\vartheta) \mid iK_i \rangle}{q(\vartheta)} = \frac{\langle fK_f \mid \mathfrak{M}(E2, \nu) \mid iK_i \rangle}{\langle iK_i \mid \mathfrak{M}(E2, 0) \mid iK_i \rangle}. \tag{7.9}$$

From (7.7), one observes that the total probability of exciting any member of the final band is given by the simple expression

$$P^{(f)} = \sum_I P_{If}^{(f)} = \frac{16}{45} r_{if}^2 [q(\vartheta)]^2. \tag{7.10}$$

This result is identical with the expression which would be obtained in the ordinary perturbation theory for $q \ll 1$, and one sees that the strong coupling within the bands give rise only to a redistribution of the single probabilities $P_{If}^{(f)}$.

In the perturbation treatment which we have used here one may also approximately take into account the effect of finite $\xi$. If the energy difference between the ground states of the two bands is larger than the energy of the lowest states in the bands, the $\xi$ corresponding to the possible transitions between the two bands will be approximately constant. One may then take $\xi$ into account by calculating the total transition probability (7.10) for the finite $\xi$ in the ordinary perturbation treatment and apply the same distribution $P_{I_i}^{(f)}/P^{(f)}$ as for $\xi = 0$.

The matrix element (7.7) can be expressed in terms of the functions $A_{I,M}(\pi, q)$ (see Eq. (5.9)) by expanding the product of $D$-functions in terms of $D$-functions (compare also Eq. (5.20)). The result has been evaluated in the special case of $I_i = 0$. For a band with $K_f = 2$, one finds

$$\left.\begin{aligned} a_{I,2,0}^{(f)} &= -i\frac{2}{3} r_{if}\, q(\vartheta) \sqrt{6(I-1)I(I+1)(I+2)(2I+1)} \\ &\times \left\{ \frac{1}{2(2I+1)(2I+3)} A_{I+2,0}(q(\vartheta)) - \frac{1}{(2I-1)(2I+3)} A_{I,0}(q(\vartheta)) \right. \\ &\left. + \frac{1}{2(2I-1)(2I+1)} A_{I-2,0}(q(\vartheta)) \right\}. \end{aligned}\right\} \tag{7.11}$$

For a band with $K_f = 0$ one finds similarly

$$\left.\begin{aligned} a_{I,0,0}^{(f)} &= -i\frac{2}{3} r_{if} \cdot q(\vartheta) \sqrt{2I+1} \\ &\times \left\{ \frac{3(I+1)(I+2)}{(2I+1)(2I+3)} A_{I+2,0}(q(\vartheta)) + \frac{2I(I+1)}{(2I-1)(2I+3)} A_{I,0}(q(\vartheta)) \right. \\ &\left. + \frac{3(I-1)I}{(2I-1)(2I+1)} A_{I-2,0}(q(\vartheta)) \right\}. \end{aligned}\right\} \tag{7.12}$$

The resulting excitation probabilities are given in Figs. 16 and 17 in terms of the coefficients $N^f_{I_f, K_f}$. This coefficient is plotted for $K_f = 0$, $I = 0, 2$ and 4 and for $K_f = 2$, $I = 2$ and 4. The states with odd spins in the $K = 2$ band are not excited in the $q(\vartheta)$ approximation.

The perturbation treatment (7.7) is only correct if the quantity $r_{if}\, q(\vartheta)$ is smaller than one (compare Fig. 2).

If the bands are strongly coupled through the interaction with the projectile one may evaluate the matrix element (7.3) by a diagonalization method. We thus introduce a unitary transformation $U$ which diagonalizes the matrix elements of the exponent in (7.3), i. e.,

$$\langle aK_a \mid U^\dagger \sum_\nu q_\nu(\vartheta)\, Y^*_{2,\nu}(\beta, 0)\, U \mid bK_b \rangle = \delta_{ab} \cdot \lambda_a. \tag{7.13}$$

The result (7.3) can then be written in the form

$$a_{I_f K_f M_f} = \sum_z \langle I_f K_f M_f \mid \langle f \mid U \mid z \rangle e^{-i\sqrt{\frac{64\pi}{45}}\lambda_z} \langle z \mid U^\dagger \mid i \rangle \mid I_i K_i M_i \rangle. \tag{7.14}$$

Since $\lambda$ and the unitary matrix $U$ in the general case depend on the Eulerian angle $\beta$ in a rather complex way, this result is of practical interest only in some special cases. We shall consider the case where only two bands are involved in the excitation process. We assume that they have the same intrinsic quadrupole moment and that $K_i = K_f$. The matrix diagonalization is then easily performed and one finds the result

$$\left.\begin{aligned} a^{(i)}_{IKM} &= \frac{1}{2}\left\{a_{IM}\left[q(1+r_{if})\right] + a_{IM}\left[q(1-r_{if})\right]\right\} \\ a^{(f)}_{IKM} &= \frac{1}{2}\left\{a_{IM}\left[q(1+r_{if})\right] - a_{IM}\left[q(1-r_{if})\right]\right\}. \end{aligned}\right\} \tag{7.15}$$

It is interesting to compare this result with the result (7.7) from the perturbation treatment. In Fig. 18 we have plotted the excitation probability for the state $\mid 2,0,0 \rangle$ as a function of $q$ for different values of $r$. It is seen that the perturbation treatment is correct if $rq \lesssim 1$, as was to be expected. This condition will in actual cases usually be fulfilled.

If $\mid K_f - K_i \mid$ is larger than 2 the transition between the bands is $K$ forbidden. A possible transition between the bands can then occur only if the wave function (7.1) contains admixtures from other bands. Such admixtures can also play an important role in $K$ allowed transitions[18].

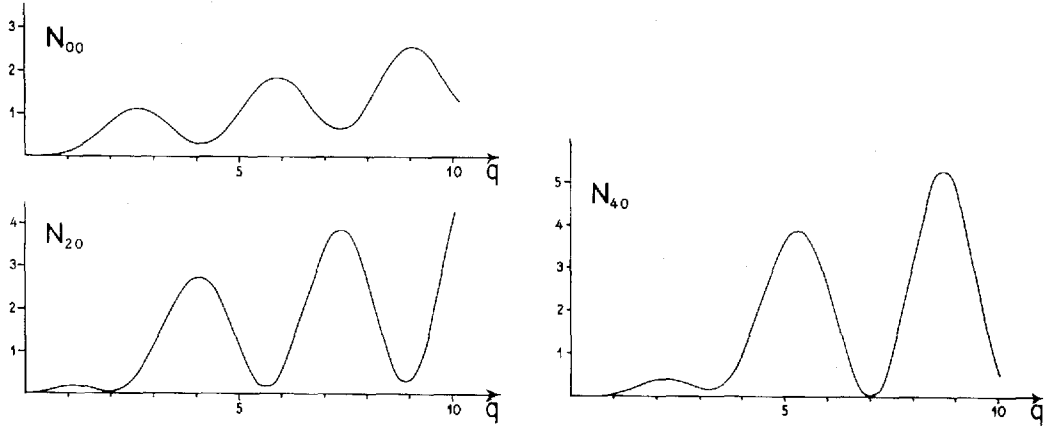


Fig. 16. The excitation probability of a weakly coupled pure rotational band with $K = 0$ in an even-even nucleus. The figure shows the coefficient $N^{(f)}_{I,0}$ in the perturbation treatment (see Eq. (7.8)) for $I = 0,2$ and 4 as a function of $q\,(\vartheta)$, which is assumed to be the same for the two bands.

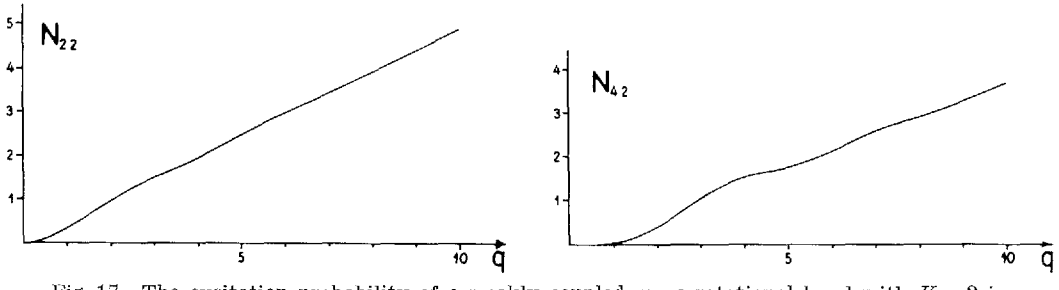


Fig. 17. The excitation probability of a weakly coupled, pure rotational band with $K = 2$ in an even-even nucleus. The figure shows the coefficient $N^{(f)}_{I,2}$ in the perturbation treatment (see Eq. (7.8)) for $I = 2$ and 4 as a function of $q\,(\vartheta)$ which is assumed to be the same for the two bands. In the $q\,(\vartheta)$ approximation, which has been used for the evaluation of $N$, the states with odd $I$ are not excited.

## 8. Conclusion

The multiple Coulomb excitation has until now been observed only in a few cases, but, from these observations (see refs. 5 and 6) as well as from the survey given in the present paper, it seems that a large number of experimental possibilities are offered. Especially it seems promising to investigate the excitation of the vibrational degrees of freedom of the nucleus, since our present knowledge on such states is rather limited. It is known that the vibrational states are mostly rather impure and for a quantitative comparison one may need some modification of the present theory. Simi-

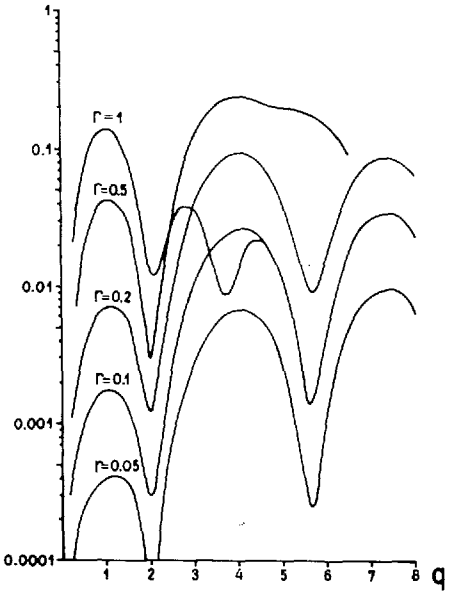


Fig. 18. The excitation probability of a strongly coupled pure rotational band with $K = 0$. The figure shows the excitation probability of the $I = 2$ state for different values of the coupling $r$ (see Eq. (7.9)). The probabilities are given as functions of $q(\vartheta)$ which is assumed to be equal in the two bands.

larly deviations from the pure rotational model have been observed. These deviations introduce a number of new parameters in the theory, and these can, in turn, be determined by a comparison of the experimental cross sections with the theory. On the other hand the increasing number of parameters make a systematic tabulation of cross sections increasingly difficult.

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