Matematisk-fysiske Meddelelser
udgivet af
Det Kongelige Danske Videnskabernes Selskabs
Bind **32,** nr. 13

Mat. Fys. Medd. Dan. Vid. Selsk. **32,** no. 13 (1960)

SOUND PROPAGATION IN A DILUTE FERMI GAS AT ZERO TEMPERATURE

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København 1960
i kommission hos Ejnar Munksgaard

CONTENTS

Synopsis

LANDAU has shown that a non-ideal Fermi gas at zero temperature can sustain collective oscillations of the acoustical type (zeroth sound). The present work concerns itself with the attenuation of zeroth sound in a dilute Fermi gas with repulsive interactions. The problem is formulated in terms of the Green function $\mathfrak{G}$ which describes the propagation of density fluctuations through the system. The simplest approximation to $\mathfrak{G}$ leads to Landau's dispersion law which is analyzed in some detail. The contribution of the phonons' zero point oscillations to the ground state energy is estimated, and shown to lead to a term which has an essential singularity at the origin of the coupling constant plane. The energy and width of the phonon are given by the poles in the spectral representation of $\mathfrak{G}$, and the location of these poles is determined from the Fredholm solution of an approximate integral equation satisfied by $\mathfrak{G}$. In this way it is shown that the width, divided by the displacement of the collective state above the single-particle continuum of the free gas, vanishes linearly in the long wavelength limit. It is also shown that in the limit of extreme dilution the correct damping can be obtained by merely taking the finite lifetime of single-particle excitations into account, and ignoring the dissipative effects of the non-instantaneous interactions between the particles in the medium. Finally, it is also argued that the Fredholm method is the natural tool for discussing many problems in the theory of collective motion.

Printed in Denmark
Bianco Lunos Bogtrykkeri A-S

Introduction

LANDAU has shown [1] that a Fermi gas at the absolute zero of temperature can sustain collective excitations provided the inter-particle forces have a finite range and are predominantly repulsive. In the long wavelength limit he found that these excitations have a phonon spectrum, i. e., an energy proportional to their momentum, but that their velocity exceeds the classical sound velocity by $\sqrt{3}\,(1+\delta)$, with δ depending on the strength of the forces in a very non-analytic fashion. By examining the appropriate density matrix LANDAU has also shown that the mode in question can be described by a rather peculiar distortion of the Fermi sphere which is quite different from what one would expect for an ordinary sound wave, and in order to emphasize these distinctions he has called this motion zeroth sound. If the temperature of the system is raised, the damping of zeroth sound rapidly increases, as ABRIKOSOV and KHALATNIKOV have recently demonstrated [2].

In deriving the results just summarized, LANDAU employed the somewhat semi-classical theory of Fermi liquids which he previously developed [3]. This theory does not readily lend itself to a systematic study of corrections to lowest order results, nor is it clear what approximations have tacitly been made in order to arrive at these results. It is therefore of some interest that Landau's findings have been retrieved from the more general field-theoretic formulation of the many-particle problem by GALITSKII and MIDGDAL [4], and by GLASSGOLD, HECKROTTE and WATSON [5]. From this work it appears at first sight that Landau's approximations are essentially* the same as those invoked in the theory of plasma oscillation of an electron gas [6–9]. As these approximations are known to be strictly valid only in the high density limit for an electron gas [10], it is natural to ask whether Landau's results are characteristic of the approximations used, or whether they will also be found in a more accurate treatment.

In order to answer these questions we have investigated the corrections to Landau's results for the case of a dilute gas. This is the natural system

* We use the qualification "essentially" because Landau's treatment automatically includes a certain class of self-energy effects, and replaces the actual inter-particle potential by a scattering matrix.

to choose for this purpose, since one can hope to make systematic expansions in the small parameter $k_F a_0$ (k_F being the Fermi momentum, and a_0 the scattering length associated with the inter-particle forces). One then finds a complex correction to the phonon energy, the imaginary part representing the decay (or damping) of the highly organized motion into more complicated excitations. This damping is found to vanish in the long wavelength limit, and Landau's conclusions are thereby substantiated.

Unfortunately our considerations cannot be applied to any existing physical system without greatly transcending our basic approximations. He^3 is, from our point of view, a very dense liquid, and a semi-phenomenological theory such as Landau's appears to us the only possible way of describing this system. Nuclear matter is a much more dilute Fermi gas, but in nuclear physics the problem of practical interest is the collective motion of a finite system, with the attendant breakdown of translational invariance. In view of these remarks, the work presented here constitutes a contribution to the mathematical physics, but not the theoretical physics, of many-body systems.

In Section II, the problem is formulated in a general way, and an integral equation for the Green function which describes the propagation of density fluctuations is derived. The phonon's dispersion law in Landau's approximation* is discussed in Section III, and the phonon contribution to the ground state energy is estimated; it is shown that the ground state energy has an essential singularity at $k_F a_0 = 0$. A systematic discussion of damping by means of the Fredholm theory is presented in Section IV. As we shall see, the Fredholm method is the natural tool for investigating the dispersion law of any collective motion, because it leads directly to the poles of Green's function in the complex energy plane. The method is, moreover, very practical because the diagonal terms in the Fredholm determinant exist in the many-body case, and do not have to be eliminated as in three-dimensional scattering problems.

II. Formulation of the Problem

A. The Response to an External Field

The most natural way to formulate the problem at hand is to ask for the response of the system to a time-dependent, externally applied field $F(t)$.** We shall assume that the resulting interaction Hamiltonian $W(t)$ is linear

* For an elementary account of this matter, see ref. [11].

** This approach has, of course, been used by many authors. Cf., e.g., references [12-14].

in $F(t)$, and that $F(t)$ can be made weak enough for the Born approximation to be valid. We shall, furthermore, assume that $F(t)$ interacts with the particles of the target one at a time, that is to say, we assume that $W(t)$ is a sum of one-particle operators. In the case of He^3, for example, an external field satisfying our requirements would be provided by a beam of light[15], a magnetic field, or perhaps by the interaction with cold neutrons.

Since $W(t)$ is assumed to be linear in $F(t)$, we can restrict ourselves to an external field having a single frequency ω and one wave number $\boldsymbol{q}$, in which case $W(t) = \{W_{q,\omega}\, e^{-i\omega t} + \text{h.c.}\}$. In terms of second quantized operators we shall write $W_{q,\omega}$ as*

$$W_{q,\omega} = \int (d^3x)\, e^{i\boldsymbol{q}\cdot\boldsymbol{x}}\, \psi^\dagger(\boldsymbol{x})\, [F^{(0)}_{q,\omega} + \boldsymbol{\sigma}\cdot\boldsymbol{F}^{(1)}_{q,\omega}]\, \psi(\boldsymbol{x}). \tag{1}$$

$F^{(0)}_{q,\omega}$ and $\boldsymbol{F}^{(1)}_{q,\omega}$ are the parameters which characterize the interaction. Thus, if we apply a magnetic field $\{\boldsymbol{H}_{q,\omega}\, e^{i(\boldsymbol{q}\cdot\boldsymbol{x}-\omega t)} + \text{c.c.}\}$ to He^3, $F^{(0)}_{q,\omega} = 0$ and $\boldsymbol{F}^{(0)}_{q,\omega} = \mu \boldsymbol{H}_{q,\omega}$, with μ the magnetic moment of the He^3 nucleus, whereas for neutron diffraction

$$F^{(0)} \sim \frac{1}{4}(\alpha_s + 3\,\alpha_t),$$

$$\boldsymbol{F}^{(1)} \sim \frac{1}{4}(\alpha_t - \alpha_s)\,\sigma_n,$$

where α_s and α_t are the singlet and triplet scattering lengths, and σ_n is the neutron's spin vector. If we choose the spin quantization axis of the target particles parallel to $\boldsymbol{F}^{(1)}$, we can rewrite eq. (1) as

$$W_{q,\omega} = \sqrt{2}\int d^3x\, e^{i\boldsymbol{q}\cdot\boldsymbol{x}}\, [F^{(0)}_{q,\omega}\, \varrho_0(\boldsymbol{x}) + F^{(1)}_{q,\omega}\, \varrho_1(\boldsymbol{x})], \tag{2}$$

with

$$\varrho_0(\boldsymbol{x}) = \frac{1}{\sqrt{2}}\left\{\psi^\dagger_\uparrow(\boldsymbol{x})\,\psi_\uparrow(\boldsymbol{x}) + \psi^\dagger_\downarrow(\boldsymbol{x})\,\psi_\downarrow(\boldsymbol{x})\right\}, \tag{3a}$$

$$\varrho_1(\boldsymbol{x}) = \frac{1}{\sqrt{2}}\left\{\psi^\dagger_\uparrow(\boldsymbol{x})\,\psi_\uparrow(\boldsymbol{x}) - \psi^\dagger_\uparrow(\boldsymbol{x})\,\psi_\downarrow(\boldsymbol{x})\right\}. \tag{3b}$$

The density operators ϱ_0 and ϱ_1 are, respectively, scalars and vectors in the total spin space of the target; if we restrict ourselves to transitions out of

* $\psi_s(\boldsymbol{x})$ destroys a particle with spin s at the point $\boldsymbol{x}$, σ is the Pauli matrix, $\int (d^3x)$ indicates integration over all space, and a sum over spin indices. Frequently, the spin variables are suppressed. Of course, eq. (1) is not the most general form $W_{q,\omega}$ can have, since we assume that the interaction is both local in $\boldsymbol{x}$-space and velocity-independent. A more general form for $W_{q,\omega}$ would only lead to tedious complications which are completely irrelevant to our consideration.

a ground state with total spin zero, this means that the states which can be excited by ϱ_0 and ϱ_1 are, respectively, singlets and triplets, and therefore incoherent.

We define the response of the system, $\Re(q\omega)$, to be the total transition rate out of the ground state $|0\rangle$ induced by the external field. Thus,

$$\Re(q\omega) = \Re_0(q\omega) + \Re_1(q\omega), \tag{4}$$

with $(\hbar = 1)$

$$\Re_\lambda(q\omega) = 4\pi |F^{(\lambda)}_{q,\omega}|^2 \sum_n \delta(\omega - E_n) |\langle n | \int d^3x \varrho_\lambda(\boldsymbol{x}) e^{i\boldsymbol{q}\cdot\boldsymbol{x}} | 0 \rangle|^2. \tag{5}$$

Here, $|n\rangle$ is an exact eigenstate of the system with excitation energy E_n.

For purposes of calculation, and also for intuitive considerations, it is very convenient to express $\Re_\lambda$ as a ground state expectation value. This may be achieved by introducing the autocorrelation function of the density fluctuations

$$\mathfrak{G}'_\lambda(q\omega) = -i\int_{-\infty}^{\infty} dt\, e^{i\omega t} \int d^3x d^3x' e^{-i\boldsymbol{q}\cdot(\boldsymbol{x}-\boldsymbol{x}')} \langle (\delta\varrho_\lambda(\boldsymbol{x}, t)\, \delta\varrho_\lambda(\boldsymbol{x}'))_+ \rangle_0, \tag{6}$$

where $(\ )_+$ is Wick's time ordering symbol. The time-dependent density fluctuation is defined as

$$\delta\varrho_\lambda(\boldsymbol{x}, t) = e^{iHt} \varrho_\lambda(\boldsymbol{x})\, e^{-iHt} - \langle \varrho_\lambda(\boldsymbol{x}) \rangle_0,$$

where H is the Hamiltonian of the target (in the absence of the external field). If $\omega \neq 0$, $\delta\varrho$ may of course be replaced by ϱ itself in (6). Upon reintroducing the eigenstates $|n\rangle$ of H, (6) becomes

$$\mathfrak{G}'_\lambda(q\omega) = \lim_{\eta\to 0} \sum_n \left\{ \frac{|\langle n | \int d^3x e^{i\boldsymbol{q}\cdot\boldsymbol{x}} \varrho_\lambda(\boldsymbol{x}) | 0 \rangle|^2}{\omega - E_n + i\eta} - \frac{|\langle n | \int d^3x e^{-i\boldsymbol{q}\cdot\boldsymbol{x}} \varrho_\lambda(\boldsymbol{x}) | 0 \rangle|^2}{\omega + E_n - i\eta} \right\} + 2\pi i \delta(\omega) |\langle \int d^3x\, e^{i\boldsymbol{q}\cdot\boldsymbol{x}} \varrho_\lambda(\boldsymbol{x}) \rangle_0|^2. \tag{7}$$

Comparing eq. (5) with eq. (7), we have

$$\Re_\lambda(q\omega) = -4 |F^{(\lambda)}_{q,\omega}|^2 \operatorname{Im} \mathfrak{G}'_\lambda(q\omega) \tag{8}$$

if $\omega > 0$.

According to eq. (7), the poles of $\mathfrak{G}'_\lambda(q\omega)$ lie at the system's excitation energies (resonant frequencies), while the residues at these poles are pro-

portional to the transition probabilities for exciting the corresponding states. If the system is a free Fermi gas, these poles are confined to the region of the $q-\omega$ plane bounded by the curves $\omega = (q^2/2\,m) \pm qk_F/m$. The residues at these poles are squares of single-particle matrix elements, and therefore independent of the number of particles N.

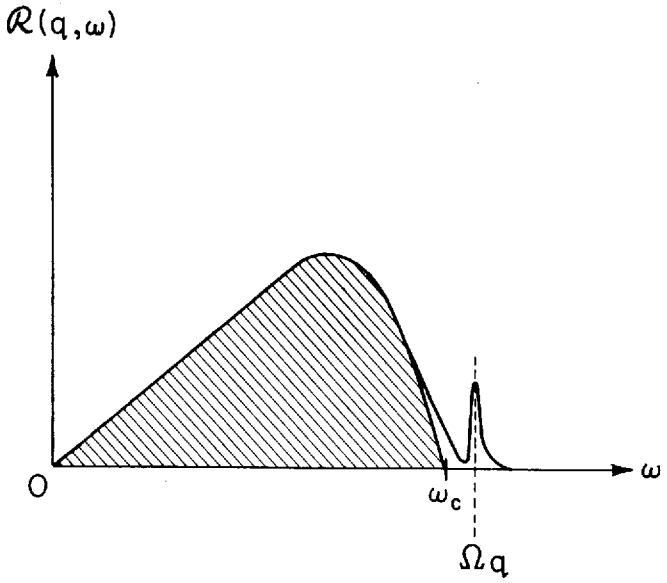

Fig. 1. The response as a function of ω for fixed q is sketched here. The shaded portion show the response in the absence of interaction, and ω_c is defined by eq. (33). The collective resonanc is centered at Ω_q.

When the interactions in the system are turned on, $\mathfrak{G}'_\lambda(q\,\omega)$ acquires singularities throughout the entire $q-\omega$ plane, the residues at these new poles being in general quite small if the interactions are weak. In the approximation of Landau (and Galitskii — Migdal), however, only a line of new singularities appears, but their residues are anomalously large. They therefore correspond to states having transition probabilities out of the ground state which are vastly greater than the single-particle probabilities mentioned earlier, and it is therefore natural to call these collective excitations.

A more accurate treatment exhibits the other singularities, and the the huge residues of order N are spread over many poles in the vicinity

of the line of singularities discovered by Landau. For fixed q and as a function of ω, $\Re(q\omega)$ then has the form shown in Fig. 1, with the sharp resonance centered at Ω_q coming from the collective excitation.

B. The Propagation of Density Fluctuations

We now turn to the evaluation of $\mathfrak{G}'_\lambda(q\omega)$. It is not possible, in general, to find an equation which determines $\mathfrak{G}'_\lambda(q\omega)$ itself*, and it is necessary to consider the most general 4-point propagator**

$$\left.\begin{aligned}\mathfrak{G}(p_1 s_1,\ p_2 s_2;\ p_3 s_3,\ p_4 s_4) = (2\pi)^{-8}\int d^4x_1 \ldots d^4x_4\, e^{-i\,(p_1\cdot x_1 - p_2\cdot x_2 + p_3\cdot x_3 - p_4\cdot x_4)}\\ \times\langle(\psi_{s_1}(x_1)\,\psi^\dagger_{s_2}(x_2)\,\psi_{s_3}(x_3)\,\psi^\dagger_{s_4}(x_4))_+\rangle_0 .\end{aligned}\right\} \quad (9)$$

The physical significance of the expectation value in eq. (9) is quite clear: when $t_1,\ t_2 > t_3,\ t_4$, it represents the probability amplitude for finding a "particle-hole" excitation superimposed on the true ground state at $(x_1,\ x_2)$, if such an excitation was originally prepared at $(x_3,\ x_4)$. The relationship between the function defined by eq. (9) and $\mathfrak{G}'_\lambda(q\omega)$ will now be given. First, note that translation invariance requires eq. (9) to contain the factor $\delta(p_1 - p_2 + p_3 - p_4)$; it is therefore convenient to introduce the variables

$$p = (p_1 + p_2)/2,\quad \bar{p} = (p_3 + p_4)/2,\quad P = p_1 - p_2,\quad P' = p_4 - p_3 . \quad (10)$$

Functions which describe singlet ($\lambda = 0$) and triplet ($\lambda = 1$) excitations can now be defined through

$$\delta(P - P')\,\mathfrak{G}_{1-\lambda}(p\bar{p};P) = \sum_{s_1 \ldots s_4} \langle \lambda 0 \mid -s_1 s_2\rangle\, \mathfrak{G}(p_1 s_1,\ \ldots,\ p_4 s_4)\langle -s_3 s_4 \mid \lambda 0\rangle, \quad (11)$$

where $\langle \lambda 0 \mid ss'\rangle$ is the usual Clebsch-Gordan coefficient for spin 1/2. Again because of translation symmetry, it is more convenient to consider

* In Hubbard' spapers on the electron gas [7], an equation which employs the notion of irreducible polarization parts is derived for $\mathfrak{G}'(q\omega)$ itself. We, however, are concerned with strong, short-range forces, for which direct and exchange diagrams are of the same magnitude, and which, moreover, must be represented by a rather complicated pseudo-potential (see Figs. 3 and 4, and eq. (24)). Therefore, it does not appear to be possible consistently to define simple polarization parts, and a technique such as the one described here seems to be required.

** Our notation is: $x = (\boldsymbol{x}, t)$, $p = (\boldsymbol{k}, \varepsilon)$, $x \cdot p = \boldsymbol{x}\cdot\boldsymbol{k} - \varepsilon t$, and $\psi(x)$ is the Heisenberg operator $e^{iHt}\psi(\boldsymbol{x})\,e^{-iHt}$.

$$\mathfrak{G}_\lambda(\varkappa) = -i(2\pi)^3 \int d^4x\, e^{-i\varkappa\cdot x} \langle (\varrho_\lambda(x)\varrho_\lambda(0))_+ \rangle_0, \qquad (6')$$

$\varkappa = (\boldsymbol{q}, \omega)$ being the 4-vector of momentum transfer; since $x = 0$, the relationship between eqs. (6) and (6′) is simply $\mathfrak{G}'_\lambda(\varkappa) = (V/8\pi^3)\,\mathfrak{G}_\lambda(\varkappa)$, where V is the volume of the system. Upon comparing eq. (6′) with eqs. (9) and (11), we arrive at the required connection

$$\mathfrak{G}_\lambda(\varkappa) = \frac{1}{2\pi i}\int_+ d^4p \int_+ d^4\bar{p}\, \mathfrak{G}_\lambda(p\bar{p};\varkappa). \qquad (12)$$

The subscript "+" on the integral signs indicates that in carrying out hte integrations over the energy components of p and $\bar{p}$ the contours are to be

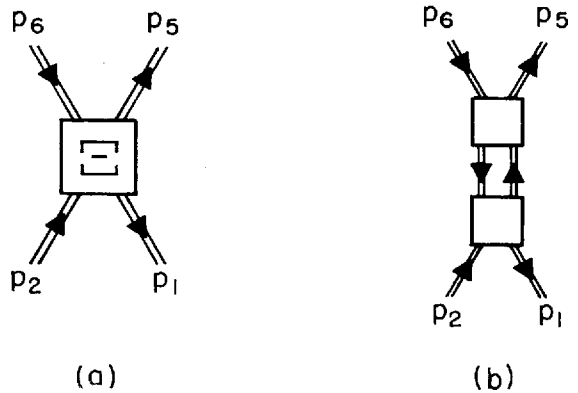

Fig. 2. The definition of $\Xi(15;26)$. Double lines represent the "dressed" propagator $G(p)$.

closed in the upper half plane; this prescription produces the correct sequence of equal-time field operators as demanded by eq. (6′).

As Galitskii and Migdal have noted, the function defined by eq. (9) satisfies the identity[16–18]

$$\mathfrak{G}(12;34) = \mathfrak{G}^0(12;34) + G(1)\,G(2)\int \Xi(15;26)\,\mathfrak{G}(65;34)\,(d^4p_5)\,(d^4p_6); \qquad (13)$$

here $G(p)$ is the one-particle Green function

$$G(p) = -i\int d^4x\, e^{-ip\cdot x} \langle (\psi(x)\,\psi^\dagger(0))_+ \rangle_0, \qquad (14)$$

and

$$\mathfrak{G}^0(12;34) = G(p_2)\,G(p_4)\,[\delta(p_2-p_3)\,\delta(p_1-p_4) - \delta(p_1-p_2)\,\delta(p_3-p_4)].$$

The kernel $\Xi(15;26)$ is the sum of all diagrams of the type shown in Fig. 2(a) which *cannot* be reduced to diagrams like that shown in Fig. 2(b).

If the interparticle force v is weak, the leading contribution to this kernel is just proportional to the matrix element $(15 \mid v \mid 26)$. When the forces are strong, but have a range small compared to the mean interparticle spacing, Ξ can be expanded in powers of the free space scattering amplitudes. We postpone a more detailed discussion of Ξ for a moment, because it is possible to effect a considerable simplification of eq. (13) on general principles*. Invariance considerations immediately lead to the conclusion that**

$$\left.\begin{aligned} &\Xi(p_1 s_1,\ p_5 s_5;\ p_2 s_2,\ p_6 s_6) \\ &= \delta(p_1+p_5-p_2-p_6)\sum_{SM}\langle s_1 s_5 \mid SM\rangle\, \Xi_S(\tilde{p},\ \tilde{p}';\ \tilde{P})\,\langle SM \mid s_2 s_6\rangle, \end{aligned}\right\} \quad (15)$$

with

$$\tilde{p} = (p_1-p_5)/2,\ \tilde{p}' = (p_2-p_6)/2,\ \tilde{P} = p_1+p_5.$$

Therefore, if we change variables according to eq. (10), employ eq. (15) and the inverse of relation (11), and integrate over $\overline{p}'$, we find that eq. (13) reduces to

$$\mathfrak{G}_\lambda(p;\varkappa) = \mathfrak{G}^0(p;\varkappa)\left\{1+\int d^4p'\, R_\varkappa^{(\lambda)}(p \mid p')\,\mathfrak{G}_\lambda(p';\varkappa)\right\}, \quad (16)$$

with

$$\mathfrak{G}_\lambda(p;\varkappa) = \int_+ d^4\overline{p}\,\mathfrak{G}_\lambda(p\overline{p};\varkappa), \quad (17)$$

and $(\varkappa \neq 0)$

$$\mathfrak{G}^0(p;\varkappa) = G\left(p+\frac{1}{2}\varkappa\right)G\left(p-\frac{1}{2}\varkappa\right). \quad (18)$$

The kernel in eq. (16) is related to the functions defined by eq. (15) through

$$\left.\begin{aligned} R_\varkappa^{(0)}(p \mid p') &= 1/2\left\{3\,\Xi_1(Q_1, Q_2; Q)+\Xi_0(Q_1, Q_2; Q)\right\}, \\ R_\varkappa^{(1)}(p \mid p') &= 1/2\left\{\Xi_1(Q_1, Q_2; Q)-\Xi_0(Q_1, Q_2; Q)\right\}, \end{aligned}\right\} \quad (19)$$

with

$$Q_1 = (p-p'+\varkappa)/2,\ Q_2 = (p-p'-\varkappa)/2,\ Q = p+p'. \quad (19')$$

Eq. 16 is our basic equation; its solution $\mathfrak{G}_\lambda(p;\varkappa)$, after integration over p, gives us the desired quantity $\mathfrak{G}_\lambda(\varkappa)$ from which we can immediately deduce the response. The reader will have noted that eq. (16) has the same

* In ref. [4], a homogeneous particle-hole wave equation is derived from eq. (13). As GELL-MANN and LOW [17] have pointed out, such a wave equation exists only if the eigenstate in question is a discrete one (e. g., the deuteron in the meson-nucleon system). In our problem, the collective mode is degenerate with a host of more complex excitations (i. e., it is damped), and so the Galitskii-Migdal equation (their eq. 40) only holds in the no-damping approximation. Our procedure is always valid, and is, in fact, just as easy to work with as the particle-hole wave equation.

** Here we assume that there are no tensor or spin-orbit forces, i. e., that the system's total spin vector is a constant of motion.

structure as the integral equation for a two-particle Green function in normal scattering theory. Loosely speaking, $\mathfrak{G}_\lambda(p;\varkappa)$ describes the propagation through the medium of a particle-hole pair with total 4-momentum $\varkappa$, relative 4-momentum p, and spin λ. It is impossible, however, to extract a complete orthonormal set of particle-hole wave functions from $\mathfrak{G}_\lambda(p;\varkappa)$.

It is of course not possible to solve eq. (16) with arbitrary kernels, and approximate solutions must be sought. But the approximation technique must be capable of producing resonances in the response, and ordinary perturbation theory is therefore ruled out. (In fact, as we shall see in the next section, the ground state energy itself has an essential singularity at the origin of the coupling constant plane.) The Fredholm method[19], on the other hand, constructs the solution as the ratio of two entire functions of the coupling parameter, and is therefore well defined even if $\mathfrak{G}_\lambda(\varkappa)$ is not an analytic function of the coupling constant. The Fredholm solution reads

$$\mathfrak{G}_\lambda(\varkappa) = \Pi(\varkappa) + \frac{1}{2\pi i}\frac{\mathfrak{N}_\lambda(\varkappa)}{\mathfrak{D}_\lambda(\varkappa)}, \tag{20}$$

with

$$\Pi(\varkappa) = \frac{1}{2\pi i}\int_+ d^4p\, G(p+\varkappa/2)\, G(p-\varkappa/2), \tag{21}$$

$$\mathfrak{N}_\lambda(\varkappa) = \sum_{n=1}^{\infty}\mathfrak{N}_\lambda^{(n)}(\varkappa),\quad \mathfrak{D}_\lambda(\varkappa) = 1-\sum_{n=1}^{\infty}\mathfrak{D}_\lambda^{(n)}(\varkappa), \tag{22}$$

$$\mathfrak{N}_\lambda^{(1)}(\varkappa) = \int_+ d^4p\int d^4p'\,\mathfrak{G}^0(p;\varkappa)\,R_\varkappa^{(\lambda)}(p\,|\,p')\,\mathfrak{G}^0(p';\varkappa), \tag{23a}$$

$$\left.\begin{aligned}\mathfrak{N}^{(2)}(\varkappa) = \int_+ d^4p\int d^4p'd^4p''\,\mathfrak{G}^0(p;\varkappa)\,R_\varkappa^{(\lambda)}(p\,|\,p')\,\mathfrak{G}^{(0)}(p';\varkappa)\,R_\varkappa^{(\lambda)}(p'\,|\,p'')\,\mathfrak{G}^0(p'';\varkappa)\\ -\mathfrak{D}_\lambda^{(1)}(\varkappa)\,\mathfrak{N}_\lambda^{(1)}(\varkappa),\end{aligned}\right\} \tag{23b}$$

$$\mathfrak{D}_\lambda^{(1)}(\varkappa) = \int d^4p\,\mathfrak{G}^0(p;\varkappa)\,R_\varkappa^{(\lambda)}(p\,|\,p), \tag{23c}$$

$$\mathfrak{D}_\lambda^{(2)}(\varkappa) = -1/2\int d^4p\,d^4p'\begin{vmatrix}\mathfrak{G}^0(p;\varkappa)\,R_\varkappa^{(\lambda)}(p\,|\,p) & \mathfrak{G}^0(p;\varkappa)\,R_\varkappa^{(\lambda)}(p\,|\,p')\\ \mathfrak{G}^0(p';\varkappa)\,R_\varkappa^{(\lambda)}(p'\,|\,p) & \mathfrak{G}^0(p';\varkappa)\,R_\varkappa^{(\lambda)}(p'\,|\,p')\end{vmatrix}, \tag{23d}$$

etc.

We now return to a closer specification of Ξ, and thereby of the kernels of eq. (16). As stated in Section I, we shall be interested in the dilute gas, and so an expansion of Ξ in terms of free scattering amplitudes is required. For this purpose, we introduce the T-matrix in the ladder approximation (see Fig. 3); this quantity plays the role of a non-singular pseudo-potential

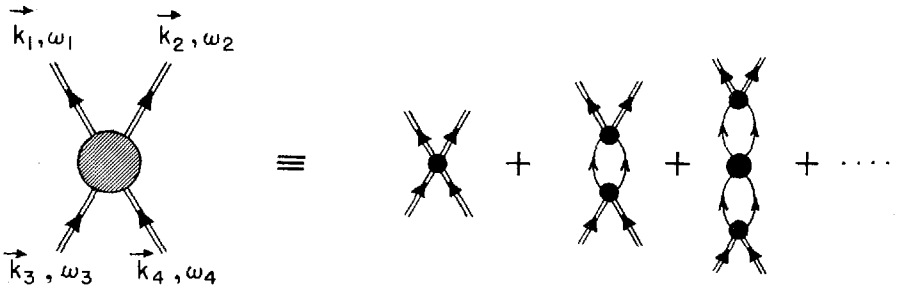

Fig. 3. The T-matrix is represented by the shaded vertex, and the two-particle potential v by the heavy dot. The T-matrix is the sum of all the diagrams shown. The single lines in this figure represent the free-particle propagator $G_0(p)$ defined by eq. (27).

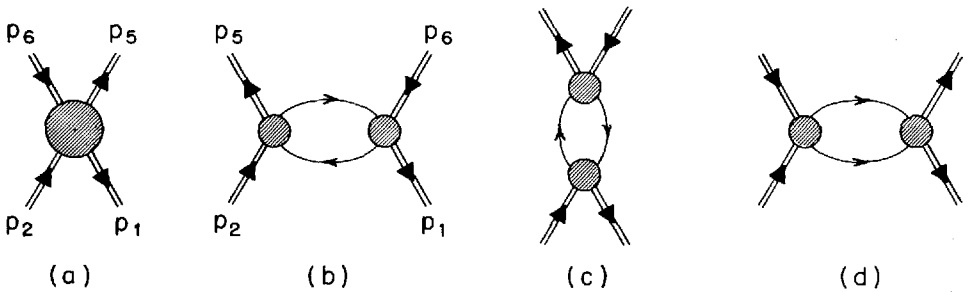

Fig. 4. The two lowest order diagrams contributing to the kernel Ξ are shown in (a) and (b). The second order diagrams (c) and (d) are not to be counted because they are already included by the integral equations for $\mathfrak{G}(p;\varkappa)$ and the T-matrix.

in terms of which expansions can be carried out, the lowest contributions to Ξ being those of Figs. 4(a) and (b). To each T-vertex there corresponds a factor $\langle \boldsymbol{k}_1 s_1 \boldsymbol{k}_2 s_2 \mid T(E) \mid \boldsymbol{k}_3 s_3 \boldsymbol{k}_4 s_4 \rangle$, where $\boldsymbol{k}_1 s_1$, $\boldsymbol{k}_2 s_2$ ($\boldsymbol{k}_3 s_3$, $\boldsymbol{k}_4 s_4$) are the momenta and spins of the lines leaving (entering) the vertex, and $E = \omega_1 + \omega_2 = \omega_3 + \omega_4$ is the sum of the energies entering or leaving the vertex. Now let $\boldsymbol{k} = (\boldsymbol{k}_1 - \boldsymbol{k}_2)/2$, $\boldsymbol{k}' = (\boldsymbol{k}_3 - \boldsymbol{k}_4)/2$, $\boldsymbol{K} = \boldsymbol{k}_1 + \boldsymbol{k}_2$, and

$$\langle \boldsymbol{k}_1 s_1 \boldsymbol{k}_2 s_2 \mid T(E) \mid \boldsymbol{k}_3 s_3 \boldsymbol{k}_4 s_4 \rangle$$
$$= \frac{\delta(\boldsymbol{k} - \boldsymbol{k}_3 - \boldsymbol{k}_4)}{(2\pi)^3} \sum_{SM} \langle s_1 s_2 \mid SM \rangle \langle \boldsymbol{k} \mid T_S(E, \boldsymbol{K}) \mid \boldsymbol{k}' \rangle \langle SM \mid s_3 s_4 \rangle .$$

We wish to express $\langle \boldsymbol{k} \mid T \mid \boldsymbol{k}' \rangle$ in terms of quantities which describe the free-space scattering. The most convenient parameters are the amplitudes $f_S(\boldsymbol{k} \mid \boldsymbol{k}')$, which are related to the center-of-mass differential cross sections for unpolarized particles through

$$d\sigma_{\boldsymbol{k}\leftarrow\boldsymbol{k}'} = \{ 1/4 \mid f_0(\boldsymbol{k} \mid \boldsymbol{k}') \mid^2 + 3/4 \mid f_1(\boldsymbol{k} \mid \boldsymbol{k}') \mid^2 \} \left(\frac{m}{4\pi}\right)^2 d\Omega .$$

If these f's satisfy outgoing wave boundary conditions at infinity, then the relation

$$\langle \boldsymbol{k} \mid T_S(E, \boldsymbol{K}) \mid \boldsymbol{k}' \rangle = f_S(\boldsymbol{k} \mid \boldsymbol{k}') - \frac{m}{2} \int d^3k'' \frac{f_S(\boldsymbol{k} \mid \boldsymbol{k}'') f_S^*(\boldsymbol{k}' \mid \boldsymbol{k}'')}{\boldsymbol{k}'^2 - \boldsymbol{k}''^2 + i\eta}$$

$$+ 1/2 \int m\, d^3k'' \left\{ \frac{\Lambda_{\boldsymbol{K}}^{(+)}(\boldsymbol{k}'')}{mE - \frac{1}{4}K^2 - k''^2 + i\eta} - \frac{\Lambda_{\boldsymbol{K}}^{(-)}(\boldsymbol{k}'')}{mE - \frac{1}{4}K^2 - k''^2 - i\eta} \right\} f_S(\boldsymbol{k} \mid \boldsymbol{k}'') f_S^*(\boldsymbol{k}' \mid \boldsymbol{k}'') \quad (24)$$

gives the T-matrix correctly to second order in $f^{(20)}$. The other new symbols are

$$\Lambda_{\boldsymbol{K}}^{(\pm)}(\boldsymbol{k}) = \Theta_{\pm}(\boldsymbol{K}/2 + \boldsymbol{k})\, \Theta_{\pm}(\boldsymbol{K}/2 - \boldsymbol{k}).$$

with

$$\left.\begin{aligned} \Theta_+(k) &= 1 - \Theta_-(k) \\ &= \begin{cases} 1 & \text{if } k > k_F \\ 0 & \text{if } k < k_F \end{cases} \end{aligned}\right\} \quad (25)$$

The factor 1/2 in front of the integrals in eq. (24) is due to the fact that our amplitudes are already antisymmetrized: if primes denote unsymmetrized amplitudes, then

$$f_S(\boldsymbol{k} \mid \boldsymbol{k}') = f_S'(\boldsymbol{k} \mid \boldsymbol{k}') + (-1)^S f_S'(\boldsymbol{k} \mid -\boldsymbol{k}').$$

We are now in possession of the formal apparatus required for our investigation. We begin by summarizing the situation when damping is ignored.

III. The Undamped Approximation

A. Collective Frequency and Cut-off Momentum

The inhomogeneous term $\mathfrak{S}^0(p; \varkappa)$ of eq. (16) has branch cuts along the entire real axis of ω, and therefore the solution of this equation must be expected to have the same analytic property, no matter what approximation for the kernel R is used. In physical terms, therefore, eq. (16) has no undamped solutions. This is of course due to the fact that the single-particle

propagators G which we have used describe "dressed quasi-particle" excitations which are themselves damped. If we are to get undamped solutions, we must therefore replace $\mathfrak{G}^0(p;\varkappa)$ by

$$\mathfrak{G}_f(p;\varkappa) = G_0\left(p+\frac{1}{2}\varkappa\right) G_0\left(p-\frac{1}{2}\varkappa\right), \tag{26}$$

where $G_0(p)$ is the free-particle propagator,

$$G_0(p) \equiv G_0(\boldsymbol{k}, \varepsilon) = \frac{\Theta_+(k)}{\varepsilon - \frac{1}{2m}|\boldsymbol{k}|^2 + i\eta} + \frac{\Theta_-(k)}{\varepsilon - \frac{1}{2m}|\boldsymbol{k}|^2 - i\eta}. \tag{27}$$

The lowest order contributions to the kernels of eq. (16) are linear in the f's, and therefore arise from Fig. 4(a). They are simply

$$R_\varkappa^{(0)}(p \mid p') = -\frac{1}{2} i (2\pi)^{-4} \left\{ 3 f_1(\boldsymbol{Q}_1 \mid \boldsymbol{Q}_2) + f_0(\boldsymbol{Q}_1 \mid \boldsymbol{Q}_2) \right\}, \tag{28a}$$

$$R_\varkappa^{(1)}(p \mid p') = -\frac{1}{2} i (2\pi)^{-4} \left\{ f_1(\boldsymbol{Q}_1 \mid \boldsymbol{Q}_2) - f_0(\boldsymbol{Q}_1 \mid \boldsymbol{Q}_2) \right\}, \tag{28b}$$

the $\boldsymbol{Q}'_s$ being the spatial parts of the 4-vectors defined by eq. (19′). The momentum transfer involved in these scattering amplitudes is $\boldsymbol{q}$, which is always small compared to k_F (long wavelength disturbances). From eqs. (26) and (27) it follows that $|\boldsymbol{k}|$ and $|\boldsymbol{k}'|$ lie in the vicinity of k_F, and therefore $|\boldsymbol{Q}_1| \lesssim k_F$, $|\boldsymbol{Q}_2| \lesssim k_F$. Hence in the dilute gas, where the effective range is small compared to k_F^{-1}, we can replace the amplitudes in eq. (28) by their zero-energy limits. Since f_1 only contains states with odd angular momentum, we have

$$f_1 \to 0 \tag{29a}$$

in this limit, whereas

$$f_0 \to \frac{8\pi}{m} a_0, \tag{29b}$$

a_0 being the conventional S-wave scattering length[21]. With these simplifications eq. (16) becomes trivially soluble,

$$\mathfrak{G}_\lambda(\varkappa) = \frac{\Pi_0(\varkappa)}{1 - (-1)^\lambda (a_0/2\pi^2 m)\, \Pi_0(\varkappa)}, \tag{30}$$

with

$$\Pi_0(\varkappa) = 1/2\pi i \int d^4p\, G_0(p+\varkappa/2)\, G_0(p-\varkappa/2). \tag{31}$$

Note that eq. (30) has the same structure as the Fredholm solution (20) of the complete equation.

According to our previous discussion, the resonant frequencies of the system are determined by the poles of $\mathfrak{G}_\lambda(\varkappa)$, i. e., in present approximation by the condition

$$1 = (-1)^\lambda \frac{a_0}{2\pi^2 m} \Pi_0(\varkappa). \tag{32}$$

The solution of this equation has been extensively discussed in the literature for the case of Coulomb interactions[7-9]. We recall that $\mathfrak{G}_\lambda(\varkappa)$ is an even function of ω, and so we can confine ourselves to $\omega > 0$; it has a cut along the real ω-axis when $\omega \leqslant \omega_c$, where

$$\omega_c = \frac{q k_F}{m} + \frac{q^2}{2m}. \tag{33}$$

This cut corresponds to a perturbed continuum of simple "particle-hole" excitations which is present whatever the forces, and is of little interest to us. If, however, $(-1)^\lambda a_0 > 0$, another *isolated* solution of eq. (32) can exist for $\omega > \omega_c$. This is because $\Pi_0(q, \omega)$ has the properties

$$\left.\begin{aligned} &\operatorname{Im}\Pi_0(q,\omega) = 0, \quad \Pi_0(q,\omega) > 0, \\ &\frac{\partial \Pi_0(q,\omega)}{\partial \omega} < 0, \quad \Pi_0(q,\omega) \xrightarrow[\omega\to\infty]{} 0, \end{aligned}\right\} \tag{34}$$

when $\omega > \omega_c$. The explicit form of eq. (32) for $\omega > \omega_c$ is

$$1 = (-1)^\lambda (k_F a_0 / \pi) A(q, \omega), \tag{35}$$

with

$$\left.\begin{aligned} A(q,\omega) = -1 + \left\{ \frac{(\omega+\omega_0)(\omega-\omega_c)}{2q^3} \ln\left|\frac{\omega-\omega_c}{\omega+\omega_0}\right| + (\omega \to -\omega) \right\}, \\ \omega_0 = q - q^2/2. \end{aligned}\right\} \tag{36}$$

In eq. (36) and all subsequent formulae of this section, momenta and energies are expressed in units of k_F and (k_F^2/m), respectively.

We are only interested in the case $a_0 > 0$, since for negative scattering lengths superconductivity is to be expected, and then the whole theory developed here is certainly not valid*. Thus we have a collective root only for spin singlet excitations. From eq. (35) we find that this root exists provided $q \leqslant q_c^0$, the cut-off momentum being given by

* Unlike GLASSGOLD et. al. [5], we only find a complex root if $k_F a_0 < -\pi/2$. It is therefore clear that these pathological roots bear no simple relation to the occurrence of superconductivity, since they only appear when the forces are extremely strong.

$$q_c^0 = 0.736\, e^{-1/\xi}, \quad \xi = \frac{k_F a_0}{\pi}, \tag{37}$$

when $\xi \ll 1$. The collective root has the phonon form as $(q/q_c^0) \to 0$:

$$\Omega_q^0 \to q\left\{1 + 0.270\, e^{-1/\xi}\left[1 + 0.309\,(q/q_c^0)^2 + \ldots\right]\right\}. \tag{38}$$

A plot of the phonon energy in the weak coupling limit will be found in Fig. 5.

Another quantity of interest is the excitation probability of the collective state. We recall that this probability is proportional to the residue of $\mathfrak{G}_\lambda(\varkappa)$ at the phonon pole. A simple calculation yields

$$\Re_q \underset{q \to 0}{\longrightarrow} \frac{50\, nq}{\xi^2}\, e^{-1/\xi} \tag{39}$$

for this residue, $n = (k_F^3/3\pi^2)$ being the density. Furthermore, $\Re_q \to 0$ as $q \to q_c^0$, as is to be expected.

A word concerning the accuracy of eqs. (37) and (38) is in order here. It is, of course, an easy matter to compute the leading correction terms to these expressions; for the extreme case $k_F a_0 = 1$ (i. e., $\xi = 1/\pi$), we then find that eqs. (37) and (38) are in error by 9 and 7 per cent, respectively. Furthermore, Fig. 5 reveals that eq. (38) reproduces the actual q-dependence quite well for the entire range below q_c^0. We therefore conclude that eqs. (37) and (38) adequately characterize the collective root in the present approximation for all reasonable values of the scattering length a_0.

On the other hand, once we depart from the extreme low-density limit ($\xi \to 0$), the approximation of eq. (29) must be modified, and then the simple solution (30) is no longer valid. We shall therefore resort to the Fredholm solution as given in Section II. If we retain the kernel given by eq. (28), we find that the $\lambda = 0$ secular equation is

$$1 - \frac{1}{16\pi^3}\left[f_0(\boldsymbol{q}/2 \mid \boldsymbol{q}/2) - 3 f_1(\boldsymbol{q}/2 \mid \boldsymbol{q}/2)\right] \Pi_0(\varkappa) = 0 \tag{40}$$

instead of eq. (32). The contribution from the second Fredholm determinant, eq. (23 d), has been neglected here because it already leads to a result of higher order in ξ than we shall obtain from eq. (40). We now expand these forward amplitudes about $\boldsymbol{q} = 0$. Retaining terms up to q^2, we find that

$$1 = (\xi + q^2 \zeta)\, A(q, \omega) \tag{41}$$

replaces our previous eq. (35). Here,

$$\zeta = \frac{1}{4\pi} k_F^3 \left(\frac{1}{2} r_0 \, a_0^2 - a_0^3 - 3 \, a_1^3 \right), \tag{42}$$

r_0 being the S-wave effective range[21], and a_1 the P-wave scattering length (i. e., the P-wave phase shift $\delta_P \sim -(qa_1)^3$ as $q \to 0$). In deriving eqs. (41) and (42) we have only kept the real parts of f_S ($\boldsymbol{q}/2 \,|\, \boldsymbol{q}/2$). The imaginary parts of the amplitudes should not be taken into account in the present

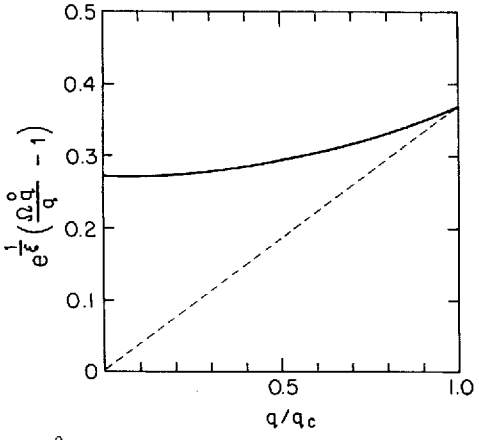

Fig. 5. The phonon energy Ω_q^0 as a function of the momentum q can be obtained from this graph. The abscissa shows q divided by the cut-off momentum (the latter quantity being given by eq. (37)). The ordinate is

$$\left(\frac{\Omega_q^0}{q} - 1 \right) \exp(1/\xi),$$

where Ω_q^0 is the solution of $1 = \xi A\left(q, \Omega_q^0\right)$, i. e. it does not include the parameter ζ appearing in eq. (41). The present curve is therefore "universal" in the sense that it gives Ω_q^0 or all $k_F a_0 \ll \pi$. The broken line indicates the top of the single-particle excitation continuum in these variables, i. e. $\left(\omega_c q^{-1} - 1\right) \exp(1/\xi) = 0.368\left(q/q_c^0\right)$.

no-damping approximation; the reason for this will become clear in the following section (see discussion following eq. (58)).

From (4) we readily find that the cut-off momentum is now

$$q_c = q_c^0 \left\{ 1 + q_c^0 \left(1 + \frac{1}{2\xi} \right) + \frac{\zeta}{\xi^2} (q_c^0)^2 + 0\left((q_c^0)^3\right) \right\}, \tag{43}$$

whereas the collective state lies at

$$\Omega_q^0 = q \left\{ 1 + 2\,\varepsilon_0 \left[1 + \varepsilon_0 \left(5 + \frac{2}{\xi} \right) + \left(\frac{\zeta}{\xi^2} + \frac{0.309}{(q_c^0)^2} \right) q^2 + \ldots \right] \right\}, \tag{44}$$

with $\varepsilon_0 = \exp(-2-\xi^{-1})$. *It is important to note that for all reasonable values of* ξ *and* ζ *the cut-off* q_c *lies well below the Fermi momentum.* For, even if we take a rather dense gas with $k_F a_0 = 1$, we find that $q_c^0 = 0.032$, and so the parameter ζ, which contains the P-wave and the momentum-dependent part of the S-wave, has a very minute effect on eq. (43), the same being true of eq. (44).

B. Phonon Contribution to Ground State Energy

The contribution of the phonons' zero point motion to the ground-state energy can be evaluated by using the variational principle and invoking the well-known connection between the frequency integral over the response and the equal-time pair correlation function in the ground state. The ground-state energy per particle is then[7]

$$(E_0/N) = \frac{i}{4\pi n}\int_0^{\xi}\frac{d\xi'}{\xi'}\int\frac{d^3q}{(2\pi)^3}\int_+ d\omega\ \ \frac{\xi' A(q,\omega)}{1-\xi' A(q,\omega)},$$

whence the contribution of the collective pole is

$$(E_0/N)_{\text{coll.}} = \frac{1}{2n}\int_0^{\xi}\frac{d\xi'}{\xi'}\int\frac{d^3q}{(2\pi)^3}\ \frac{\xi' \Re_q(\xi')}{2\pi k_F m},$$

where $\Re_q(\xi)$ is the residue at this pole. One obtains an upper limit to this integral by assuming eq. (39) to be valid for all q's, and cutting the q-integration off at q_c^0. One then finds

$$(E_0/N)_{\text{coll.}} < 0.005\, e^{-5/\xi}\,(k_F^2/2m).$$

(The ξ-dependence of the exact result for $(E_0/N)_{\text{coll.}}$ is of the same form as this upper bound.) This contribution to the ground-state energy is, of course, completely negligible for all practical purposes. Nevertheless it does show that *the ground state energy of a hard sphere gas has an essential singularity at the origin of the coupling constant plane,* a result which is hardly surprising if one recalls that changing the sign of $\xi = k_F a_0/\pi$ leads to a drastically different physical system — the superfluid.

IV. Attenuation of Zeroth Sound

A. Definition of the Complex Phonon Energy

We now consider $\mathfrak{G}(\varkappa) \equiv \mathfrak{G}(q, z)$ as a function of the complex variable z, with $\omega = \mathrm{Re}\, z$. According to the spectral representation (7), $\mathfrak{G}(q, z)$ has the following properties: it is analytic throughout the plane except for poles and cuts along the real axis; across these cuts $\mathrm{Im}\,\mathfrak{G}$ is discontinuous while $\mathrm{Re}\,\mathfrak{G}$ is continuous; it is an even function of z; for $\omega > 0$, $\mathrm{Im}\,\mathfrak{G} < 0$ as $z \to \omega + i0$. We recall that eq. (7) follows directly from the definition of $\mathfrak{G}$, and therefore the preceding statements are generally valid. It is also clear that the excitation energies E_n entering eq. (7) will extend to infinity, and that for a large system they must be expected to form a continuous spectrum. In our case, therefore, $\mathfrak{G}(q, z)$ should have cuts along the entire real axis*.

In the approximation considered in the last section, $\mathfrak{G}(q, z)$ was found to have a cut in the interval $-\omega_c \leqslant \mathrm{Re}\, z \leqslant \omega_c$, and simple poles at $z = \pm \Omega_q^0$. The manner in which these results will be modified by higher approximations can be seen most easily by retaining the simplest kernel in the low-density limit, eq. (29), but leaving the dressed single-particle propagators in the inhomogeneous term of eq. (16). Instead of eq. (30) we then have

$$\mathfrak{G}(q, z) \simeq \frac{\Pi(q, z)}{1 - (a_0/2\pi^2 m)\, \Pi(q, z).} \tag{45}$$

This solution already has the infinite cut, because $\mathrm{Im}\,\Pi(q, z)$ is discontinuous along the entire real axis. In fact, $\Pi(q, z)$ can be easily evaluated in terms of the spectral densities $\varrho_{\pm}(p, \varepsilon)$ appearing in the representation[4]

$$G(p, \varepsilon) = \int_0^\infty d\varepsilon' \left(\frac{\varrho_+(p, \varepsilon')}{\varepsilon - \mu - \varepsilon' + i\eta} + \frac{\varrho_-(p, \varepsilon')}{\varepsilon - \mu + \varepsilon' - i\eta} \right), \tag{46}$$

where μ is the separation energy (i. e., the chemical potential). According to eq. (21), therefore,

$$\Pi(q, z) = \int d^3p \int_0^\infty d\varepsilon \int_0^\infty d\varepsilon'\, \varrho_+(|\boldsymbol{p} + \boldsymbol{q}|, \varepsilon)\, \varrho_-(p, \varepsilon') \left(\frac{1}{z - \varepsilon - \varepsilon'} - \frac{1}{z + \varepsilon + \varepsilon'} \right). \tag{47}$$

The densities $\varrho_{\pm}(p, \varepsilon)$ are real and positive definite for real ε, and satisfy the relation

* Since we shall only interest ourselves in the singlet excitation ($\lambda = 0$) we drop the subscript λ henceforth.

$$\int_0^\infty d\varepsilon\,[\varrho_+(p,\varepsilon)+\varrho_-(p,\varepsilon)]=1\,.$$

Thus, $\Pi(q,z)$ has the analytic properties listed in the preceding paragraph, and therefore the same holds for eq. (45). In other words, the poles at $z=\pm\Omega_q^0$ have now disappeared. However, since $-\mathrm{Im}\,\Pi(q,\omega+i0)$ is small (actually $0(f^2)$, as we shall see) when $\omega^2>\omega_c^2$, the denominator of eq. (45) nearly vanishes if, for some $\Omega_q^2>\omega_c^2$,

$$1-(a_0/2\pi^2 m)\,\mathrm{Re}\,\Pi(q,\Omega_q)=0\,. \tag{48}$$

When this is the case, the discontinuity of $\mathfrak{G}(q,z)$, and *ipso facto* the response, will have sharp resonances (see Fig. 2) of approximately Lorentzian shape centered at $\mathrm{Re}\,z=\pm\Omega_q$, with Ω_q near Ω_q^0 since $\mathrm{Re}\,(\Pi-\Pi_0)$ is also small. The parameters describing the line shape are then identified in the standard way with the energy Ω_q and width Γ_q of the phonon. The solution $\mathfrak{G}(q,z)$ that we shall find can be used immediately to determine the actual shape of the absorption line (i. e., the deviations from the Lorentz form), but we shall confine ourselves to the evaluation of the parameters Ω_q and Γ_q.

As is well known[4,9,22], the occurrence of a resonance is associated with a pole in the analytic continuation of $\mathfrak{G}(q,z)$ from $\mathrm{Im}\,z>0$, $\mathrm{Re}\,z>0$, onto the next Riemann sheet in the lower half plane, the location of the pole being $z=\Omega_q-{}^1\!/_2\,i\Gamma_q+0\,(\Gamma_q^2/\Omega_q)$. This continuation can be effected easily by means of the Fredholm solution (20). All we need do is to evaluate $\mathfrak{D}(q,z)$ as a power series in z about some point in the first quadrant of z—the obvious choice being the zeroth order collective frequency Ω_q^0+i0—and look for a root of $\mathfrak{D}(q,z)=0$ with $\mathrm{Im}\,z<0$. The Fredholm method has the additional advantage of providing $\mathfrak{D}(q,z)$ as a power series in the coupling parameter, since the kernel $R_\varkappa(p\mid p')$ of eq. (16) is itself such a power series. In this connection it is interesting to note that the Fredholm denominator $\mathfrak{D}(q,z)$ consists of terms which never appear in perturbation theory. The perturbation series is contained in the numerator $\mathfrak{N}$ of eq. (20), which does not enter into the determination of the pole.

The location of this pole will now be given. Let

$$\mathfrak{D}(q,z)=1-\xi A(q,z)-B(q,z), \tag{48}$$

where $A(q,z)$ is the function introduced in Sec. III A (cf. eqs. (35) and (36)), and $B(q,z)$ contains all the corrections, both those arising from higher order kernels and higher order Fredholm determinants, as well as those due to using dressed one-particle Green functions in the inhomogeneous term of eq. (16). Expansion of eq. (48) about Ω_q^0 gives

$$\mathfrak{D}(q,z) \simeq -\xi (z-\Omega_q^0) \left.\frac{\partial A}{\partial z}\right|_{\Omega_q^0+i0} - B(q,\Omega_q^0+i0) + \cdots . \tag{49}$$

This expansion in $(z-\Omega_q^0)$ converges extremely well because $(z-\Omega_q^0)$ is presumably small compared to $(\Omega_q^0-\omega_c)$, the zeroth order level displacement, and this latter quantity, it will be recalled, tends to zero exponentially as $\xi \to 0$. Therefore, if Ω_q^0 is a meaningful approximation to the energy, eq. (49) provides an accurate representation of $\mathfrak{D}(q,z)$ for the z's of interest to us. Except for terms which vanish exponentially, we thus have

$$\Omega_q = \Omega_q^0 - \frac{B(q,\Omega_q^0+i0)}{\xi \left.\dfrac{\partial A(q,\omega)}{\partial \omega}\right|_{\omega=\Omega_q^0}} \tag{50}$$

for the location of the pole. If $q \ll q_c^0$ and $q_c^0 \ll 1$, (we shall confine ourselves to this domain throughout),

$$\left.\frac{\partial A(q,\omega)}{\partial \omega}\right|_{\Omega_q^0} \simeq \frac{1}{q-\Omega_q^0} \equiv -\frac{1}{D_q^0}. \tag{52}$$

Here, D_q^0 is the displacement of the collective state above the continuum of single-particle excitations in the approximation of Section III. Hence the "exact" displacement of this state is

$$\Omega_q - q = D_q^0 \left[1 + \frac{1}{\xi} B(q,\Omega_q^0+i0)\right]. \tag{53}$$

In particular, therefore, the width Γ_q divided by the zeroth order displacement is

$$\frac{\Gamma_q}{D_q^0} = -\frac{2}{\xi} \operatorname{Im} B(q,\Omega_q^0+i0). \tag{54}$$

In the following pages we shall confine ourselves to a calculation of (Γ_q / D_q^0), for which we require only $\operatorname{Im} B$. It is of course possible to consider the real part of the correction to D_q^0, but this is of rather less intrinsic interest, and is also much more difficult to compute in detail. Our calculations of eq. (54) will be found in the following section. Readers who are not interested in such details can, with but little loss of continuity, proceed directly to Section IVC.

B. Corrections to the Fredholm Denominator

The complete Fredholm denominator, $\mathfrak{D}(q, z)$, was given in Section IIB. To second order in the free space amplitude the kernel $R_\varkappa(p \mid p')$ has contributions from only two diagrams, those of Figs. 4(a, b); we call these kernels R' and R'', respectively. Since we shall not go as far as fourth order, we only require the diagonal elements

$$\left.\begin{aligned} R'_{q\omega}(\boldsymbol{p}\varepsilon \mid \boldsymbol{p}\varepsilon) = \frac{1}{(2\pi)^4 i}\, 1/2 \Big\{ & 3 \langle \boldsymbol{q}/2 \mid T_1(2\boldsymbol{p}, 2\varepsilon) \mid -\boldsymbol{q}/2 \rangle \\ & + \langle \boldsymbol{q}/2 \mid T_0(2\boldsymbol{p}, 2\varepsilon) \mid -\boldsymbol{q}/2 \rangle \Big\} \end{aligned}\right\} \quad (55)$$

and

$$R''_{q\omega}(\boldsymbol{p}\varepsilon \mid \boldsymbol{p}\varepsilon) = -\frac{8}{(2\pi)^4 i} \left(\frac{a_0}{m}\right)^2 \Pi_0(0,0) + 0(a_0^3), \quad (56)$$

where $\Pi_0(\boldsymbol{k}, \omega)$ is given by eq. (31).

In computing the first Fredholm trace with eq. (56), we can replace $\mathfrak{G}^0(p; \varkappa)$ by $G_0(p + \varkappa/2)\, G_0(p - \varkappa/2) = \mathfrak{G}_f(p; \varkappa)$: the difference between $\mathfrak{G}^0$ and $\mathfrak{G}_f$ leads to terms of order a_0^4 in $\operatorname{Im}\mathfrak{D}$, and does not concern us. Therefore the contribution of R'' to eq. (23c) is

$$\int d^4p\, R''_\varkappa(p \mid p)\, \mathfrak{G}_f(p; \varkappa) = -\frac{1}{\pi^3}\left(\frac{a_0}{m}\right)^2 \Pi_0(0,0)\, \Pi_0(q, \omega). \quad (57)$$

Since $\omega > \omega_c$, $\operatorname{Im}\Pi_0(q, \omega) = 0$; furthermore, $\operatorname{Im}\Pi_0(\boldsymbol{k}, 0) = \operatorname{Im}\Pi_0(0, \omega) = 0$ for all $\boldsymbol{k}$ and ω. Thus eq. (57) is real, and Fig. 4(b) therefore does not contribute to the damping in the present approximation.

Using the representation of the T-matrix given in eq. (24), the Fredholm trace of eq. (55) is found to be

$$\left.\begin{aligned} \mathfrak{D}^{(1)}(\varkappa) = 1/2 \int_{-\infty}^{\infty} \frac{d\varepsilon}{2\pi i} \int \frac{d^3p}{(2\pi)^3}\, G\left(p + \frac{1}{2}\varkappa\right) G\left(p - \frac{1}{2}\varkappa\right) \\ \times \Bigg\{ \left[f_0(\boldsymbol{q}/2 \mid \boldsymbol{q}/2) - 3 f_1(\boldsymbol{q}/2 \mid \boldsymbol{q}/2) \right] - \frac{1}{2} m \int d^3k \frac{\mid f_0(\boldsymbol{q}/2 \mid \boldsymbol{k}) \mid^2 - 3 \mid f_1(\boldsymbol{q}/2 \mid \boldsymbol{k}) \mid^2}{\frac{1}{4} q^2 - k^2 + i\eta} \\ + \frac{1}{2} m \int d^3k \left[\mid f_0 \boldsymbol{q}/2 \mid \boldsymbol{k}) \mid^2 - 3 \mid f_1(\boldsymbol{q}/2 \mid \boldsymbol{k}) \mid^2 \right] \\ \times \left[\frac{\Lambda_{2\boldsymbol{p}}^{(+)}(\boldsymbol{k})}{2m\varepsilon - p^2 - k^2 + i\eta} - \frac{\Lambda_{2\boldsymbol{p}}^{(-)}(\boldsymbol{k})}{2m\varepsilon - p^2 - k^2 - i\eta} \right] \Bigg\}. \end{aligned}\right\} \quad (58)$$

The imaginary part of the first two terms inside the curly braces of eq. (58) vanishes in virtue of the optical theorem. Because of this, only the real parts of the f's were retained in Section III. The third term of eq. (58) is already $0(f^2)$, and so we can use free propagators G_0 in computing its contribution to $\mathfrak{D}^{(1)}(\varkappa)$. Therefore

$$\left.\mathfrak{D}^{(1)}(q,\omega)=\frac{1}{16\pi^3}\Pi(q,\omega)\,\mathrm{Re}\,[f_0(\boldsymbol{q}/2\,|\,\boldsymbol{q}/2)-3f_1(\boldsymbol{q}/2\,|\,\boldsymbol{q}/2)].\ +2\pi\xi^2 J(q,\omega),\right\}\quad(59)$$

where $\Pi(q,\omega)$ is given by eq. (47), and

$$\left.\begin{aligned}J(q,\omega)=&\int_{-\infty}^{\infty}\frac{d\varepsilon}{2\pi i}\int d^3k\,d^3p\,G_0(|\boldsymbol{p}+\boldsymbol{q}/2|,\,\varepsilon+\omega/2)\,G_0(|\boldsymbol{p}-\boldsymbol{q}/2|,\,\varepsilon-\omega/2)\\&\times\left\{\frac{\Lambda_{2\boldsymbol{p}}^{(+)}(\boldsymbol{k})}{2\varepsilon-p^2-k^2+i\eta}-\frac{\Lambda_{2\boldsymbol{p}}^{(-)}(\boldsymbol{k})}{2\varepsilon-p^2-k^2-i\eta}-\frac{\mathrm{P.V.}}{q^2/4-k^2}\right\}.\end{aligned}\right\}\quad(60)$$

In order to arrive at eq. (60) we have retained only the S-wave scattering in the effective range approximation (eq. (29)), and expressed momenta and energies in units of k_F and k_F^2/m, respectively.

We proceed directly to the evaluation of Im J. After carrying out the ε-integration, we find that

$$-\mathrm{Im}\,J(q,\omega+i0)=I_+(q,\omega)+I_-(q,\omega)\qquad(61)$$

with

$$\left.\begin{aligned}&I_+(q,\omega)\\&=\pi\int d^3p\,d^3k\,\Lambda_{2\boldsymbol{p}}^{(+)}(\boldsymbol{k})\,\Theta_-(\boldsymbol{p}-\boldsymbol{q}/2)\,\frac{\delta((\boldsymbol{p}-\boldsymbol{q}/2)^2-(p^2+k^2)+\omega)}{\omega-\boldsymbol{p}\cdot\boldsymbol{q}}\end{aligned}\right\}\quad(62_+)$$

and

$$\left.\begin{aligned}&I_-(q,\omega)\\&=\pi\int d^3p\,d^3k\,\Lambda_{2\boldsymbol{p}}^{(-)}(\boldsymbol{k})\,\Theta_+(\boldsymbol{p}+\boldsymbol{q}/2)\,\frac{\delta((\boldsymbol{p}+\boldsymbol{q}/2)^2-(p^2+k^2)-\omega)}{\omega-\boldsymbol{p}\cdot\boldsymbol{q}}\end{aligned}\right\}\quad(62_-)$$

provided $\omega>\omega_c$. In proceeding further it is necessary to consider the integrals

$$\int d^3k'\,\delta(k^2-k'^2)\,\Lambda_{2\boldsymbol{p}}^{(\pm)}(\boldsymbol{k}')=2\pi k\,\chi_\pm(p,k).\qquad(63)$$

The functions $\chi_\pm$ vanish in certain domains:

$$\chi_+(p,k)=0\quad\text{if}\quad k^2+p^2<1,\qquad(64_+)$$

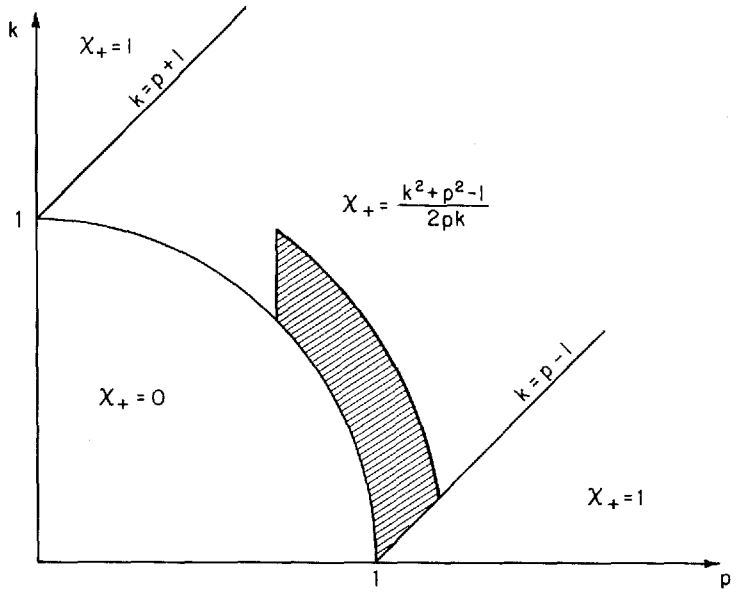

Fig. 6. The function $\chi_+(p, k)$ defined by eq. (63). The functional form of χ_+ changes in the indicated manner upon crossing the heavy solid lines. The integrand of $I_+(q, \omega)$ lies within the shaded domain, which does not intersect the line $k = p+1$ if $\omega_c \ll 1$.

$$\chi_-(p, k) = 0 \quad \text{if} \quad k^2 + p^2 > 1 . \tag{64-}$$

The functional form of χ_+ in the balance of the $k-p$ plane is shown in Fig. 6. Fortunately only one form of χ_+, namely

$$\chi_+(p, k) = \frac{k^2 + p^2 - 1}{2pk}, \tag{65}$$

enters into our calculation if $\omega > \omega_c$, and $\omega_c \ll 1$. This can be proven as follows. From the step and delta functions in I_+ we have

$$1 \leqslant k^2 + p^2 \leqslant 1 + \omega, \tag{66a}$$

$$1 + q/2 \geqslant p \geqslant \sqrt{1-\omega} - q/2 . \tag{66b}$$

A lower bound on k for fixed magnitude of $\boldsymbol{p}$ is

$$k^2 \geqslant (\omega + q^2/4 - pq) \equiv k_0^2 .$$

To show that χ_+ always has the form given by eq. (65) we must have $k_0^2 \geqslant (p-1)^2$. However, by using eq. (66b) we find that

$$k_0^2-(p-1)^2 \geqslant \omega-\omega_c,$$

and so for $\omega > \omega_c$ we have the desired result.

The situation with respect to I_- is quite different. From the δ-function we have

$$q^2/4 - \omega - pq \leqslant k^2 \leqslant q^2/4 - \omega + pq.$$

On the other hand, $p < 1$ according to eq. (64_), and therefore

$$k^2 \leqslant \frac{1}{4} q^2 - \omega_c + q$$

if $\omega \geqslant \omega_c$. Since the right hand side of this inequality is negative, $I_- = 0$.

We therefore conclude that after the $\boldsymbol{k}$-integration eq. (61) reads

$$\left.\begin{aligned} -\operatorname{Im} J(q, \omega+i0) &= \pi^2 \int\limits_{1-\omega \leqslant (\boldsymbol{p}-\frac{1}{2}\boldsymbol{q})^2 \leqslant 1} \frac{(\boldsymbol{p}-\boldsymbol{q}/2)^2+\omega-1}{\omega-\boldsymbol{p}\cdot\boldsymbol{q}} \frac{d^3p}{p} \\ &= \frac{2\pi^3}{p} \int_{1-\omega}^{1} ds \int_{\sqrt{s}-\frac{1}{2}q}^{\sqrt{s}+\frac{1}{2}q} dp \frac{s+\omega-1}{\omega+s-p^2-q^2/4}, \end{aligned}\right\} \quad (67)$$

with $s = (\boldsymbol{p}-\boldsymbol{q}/2)^2$. We are interested in the behaviour of the function defined by eq. (67) in the immediate neighbourhood of $q = 0$, $\omega = 0$, but for finite values of the velocity $v = \omega/q$. It is therefore convenient to make the transformation $y = (1-s)/q$, in which case eq. (67) reduces to

$$\left.\begin{aligned} -\operatorname{Im} J(q, vq+i0) &= \pi^3 q \int_0^v dy \sqrt{\frac{v-y}{1+q(v-y)-q^2/4}} \\ &\times \ln\left(\frac{v+\sqrt{1+q(v-y)-q^2/4}}{v-\sqrt{1+q(v-y)-q^2/4}}\right). \end{aligned}\right\} \quad (68)$$

In the long wavelength limit eq. (68) simply becomes

$$-\operatorname{Im} J(q, vq+i0) \xrightarrow[q\to 0]{} \frac{1}{2}\pi^3 q v^2 \ln\left(\frac{v+1}{v-1}\right), \quad (69)$$

provided $qv < (v-1)$.

Finally we come to the first term of eq. (59), i. e., to the evaluation of $\operatorname{Im} \Pi(q, \omega)$. From eq. (47) we have

$$-\operatorname{Im} \Pi(q, \omega+i0) = \pi \int d^3p \int_0^\omega d\varepsilon \varrho_+(|\boldsymbol{p}+\boldsymbol{q}|, \omega-\varepsilon) \varrho_-(p, \varepsilon). \quad (70)$$

One can readily compute the spectral densities $\varrho_\pm$ in perturbation theory, and for small ω and q the further integrations demanded by eq. (70) can be easily carried through. The resulting expression is, however, linearly divergent as $\omega \to \omega_c$, and it is not possible to tell from the perturbation calculation itself whether this expression is still valid when $\omega = \Omega_q^0$, which, we recall, is extremely close to ω_c (cf. Section III A). A more accurate evaluation of eq. (70) is therefore required. With this purpose in mind, we ecall that

$$\varrho_\pm(p,\varepsilon) = -\mp 1/\pi \operatorname{Im} G(p,\mu\pm\varepsilon) \tag{71}$$

are immediate consequences of eq. (46). Upon introducing the selfenergy $\Sigma(p,\varepsilon)$ in the usual way[4],

$$G(p,\varepsilon) = \frac{1}{G_0^{-1}(p,\varepsilon) - \Sigma(p,\varepsilon)},$$

we see from eq. (71) that

$$\varrho_\pm(p,\varepsilon) = \mp\frac{1}{\pi}\frac{\operatorname{Im}\Sigma(p,\mu\pm\varepsilon)}{|\,G_0^{-1}(p,\,\mu\pm\varepsilon) - \operatorname{Re}\Sigma(p,\,\mu\pm\varepsilon)\,|^2 + |\operatorname{Im}\Sigma(p,\,\mu\pm\varepsilon)\,|^2}. \tag{72}$$

$\Sigma(p,\varepsilon)$ has been studied by GALITSKII[20] to order f^2. For $p \approx k_F$, $\varepsilon \approx \mu$, $\operatorname{Re}\Sigma(p,\varepsilon)$ is found to be a relatively slowly varying function and can, for our purpose, be thought of as having been absorbed into the definition of an effective mass m^* and the energy μ. These changes have no effect on (Γ_q/D_q^0), however, and so we shall retain the "bare" mass m, and put $\mu = \mu_0 = (k_F^2/2\,m)$ hereafter. On the other hand, for $p \approx k_F$,

$$\operatorname{Im}\Sigma(p,\varepsilon) \simeq \left\{\begin{array}{lll} -c^2(\varepsilon-\mu_0)^2\, m/k_F^2 & \text{if} & \varepsilon > \mu_0 \\ c^2(\mu_0-\varepsilon)^2\, m/k_F^2 & \text{if} & \varepsilon < \mu_0 \end{array}\right\}, \tag{73}$$

with

$$c^2 = \pi\xi^2. \tag{74}$$

Putting eq. (73) into eq. (72), and this in turn into eq. (70), we have

$$\left.\begin{array}{l} L(q,\omega) \equiv -(k_F m)^{-1} \operatorname{Im}\Pi(q,\omega+i0) \\ = \dfrac{1}{\pi}\displaystyle\int d^3p \int_0^\omega d\varepsilon \, \frac{c^2\Delta^2}{[1/2+\Delta-(\boldsymbol{p}+\boldsymbol{q})^2/2]^2 + c^4\Delta^4} \cdot \frac{c^2\varepsilon^2}{(1/2-\varepsilon-p^2/2)^2 + c^4\varepsilon^4} \end{array}\right\} \tag{75}$$

where all momenta and energies in eq. (75) are again dimensionless, and

$$\Delta = \omega - \varepsilon.$$

Integrating over the orientation of $\boldsymbol{p}$ we have

$$\left.\begin{aligned} L(q,\omega) &= \frac{2}{q}\int_0^{\omega} d\varepsilon \int_{-\infty}^{\infty} p\,dp\, \frac{c^2\varepsilon^2}{(1/2-\varepsilon-p^2/2)^2+c^4\varepsilon^4}\,\mathrm{arctgt}\left(\frac{(p+q)^2/2-1/2-\Delta}{c^2\Delta^2}\right) \\ &= \frac{2}{q}\int_0^{\omega} c^4\Delta^2\varepsilon^2\,d\varepsilon \int_0^1 d\gamma \int_{-\infty}^{\infty} p\,dp\, \frac{(p+q)^2/2-1/2-\Delta}{[(1/2-\varepsilon-p^2/2)^2+c^4\varepsilon^4]\left\{c^4\varepsilon^4+\gamma^2[(p+q)^2/2-1/2-\Delta]^2\right\}}. \end{aligned}\right\} \quad (76)$$

The p-integral is now elementary, but the resulting expression is rather clumsy, and must moreover be handled with great care if one wishes to evaluate $L(q,\omega)$ when ω is in the immediate neighbourhood of ω_c. If, however, the inequality

$$\frac{c^2\omega^2}{\omega-\omega^c} \ll 1 \quad (77)$$

is satisfied, a considerable simplification of the expression referred to becomes possible. Putting $\omega = \Omega_q^0$ in eq. (77), and using eqs. (37) and $(38)_q$ we have

$$\frac{c^2(\Omega_q^0)^2}{\Omega_q^0-\omega_c} \longrightarrow 2.73\,c^2\left(\frac{q}{q_c^0}\right) \quad \text{as} \quad \frac{q}{q_c^0}\to 0;$$

therefore eq. (77) is satisfied if $q \ll q_c^0$. When $q \to q_c^0$, the inequality (77) will of course break down at $\omega = \Omega_q^0$.

After carrying out the approximations permitted by eq. (77) we find that eq. (76) reduces to

$$\left.\begin{aligned} L(q,\omega) &= \frac{2\pi c^2}{q}\int_0^{\omega}\Delta^2\,d\Delta\int_0^1 d\gamma\left(\frac{\omega+q}{\gamma^2(\omega+q)^2+c^4\Delta^4}-(q\to -q)\right) \\ &\quad -\frac{2\pi c^4}{q}\,\mathrm{Im}\int_0^{\omega}\Delta^2\varepsilon^2\,d\varepsilon\int_0^1 d\gamma\left(\frac{1}{[\gamma(\omega+q)-ic^2\Delta^2]^2}+(q\to -q)\right) \\ &= \frac{8\pi}{3}\,\frac{c^2\omega^3}{\omega^2-q^2}+0(c^6). \end{aligned}\right\} \quad (78)$$

This is just the result one obtains by using perturbation theory for the spectral functions $\varrho_{\pm}$. The only thing that our calculation has achieved is the inequality

$$8.6\,\xi^2(q/q_c^0) \ll 1 \quad (79)$$

which must be satisfied if $L(q,\Omega_q^0)$ is to be given by eq. (78). If one wants

to compute the width as $q \to q_c^0$, a more accurate evaluation of eq. (76) is required.

Collecting our results from eqs. (59), (69), (75), and (78), we finally have

$$-\operatorname{Im} B(q,\ \Omega_q^0 + i0) \simeq \pi^4 \xi^2 q \left(\frac{\Omega_q^0}{q}\right)^2 \ln\left(\frac{\Omega_q^0 + q}{\Omega_q^0 - q}\right) + \frac{2\pi\xi^3 q^2}{3 D_q^0}, \tag{80}$$

and according to eq. (54), the expression for the width follows immediately from this.

C. The Width

In view of our calculation of the preceding subsection, it is instructive to break the leading contributions to Γ_q into two partial widths, Γ_q^{dress} and Γ_q^{int}. The former arises from "dressing" the single particle excitations, the latter from the modifications of the interaction law between the particles due to their immersion in the Fermi sea (cf. eq. (24)). Recalling our results of Sections IIIA and IVB, we find

$$\frac{\Gamma_q^{\text{int}}}{D_q^0} \simeq 2\pi^4 \left(\frac{q}{k_F}\right)(1 + 2\ \xi), \tag{81}$$

whereas

$$\frac{\Gamma_q^{\text{dress}}}{D_q^0} = 11.4\ \xi^2 \left(\frac{q}{q_c^0}\right). \tag{82}$$

At first sight it would appear that $\Gamma_q^{\text{int}} \gg \Gamma_q^{\text{dress}}$. This is somewhat fallacious, however, because q must be less than q_c^0, and this cut-off momentum should be explicitly introduced in (81). When this is done, the more perspicuous expression

$$\frac{\Gamma_q^{\text{int}}}{D_q^0} = 144\ e^{-1/\xi} \left(\frac{q}{q_c^0}\right) \tag{83}$$

emerges.

It is quite clear that higher order corrections to the interaction kernel Ξ (as well as higher order Fredholm determinants) will only append a power series in ξ to eq. (83). The reason for the very great difference in the ξ-dependence of eqs. (82) and (83) is a result of the fact that $\Pi(q, \omega)$, which describes the one-particle damping, must grow very rapidly as ω tends to ω_c, because for $\omega \leqslant \omega_c$ $\operatorname{Im} \Pi(q, \omega)$ does not vanish even if the system is free (see Fig. 1). On the other hand, the contributions resulting from the

corrections to the kernel Ξ naturally do not display any dramatic increase as the single-particle excitation continuum is approached from above.

We therefore conclude that, in the limit $\xi \to 0$, the phonon damping which results from the fact that the one-particle excitations have a finite lifetime completely dominates the dissipation arising from the non-instantaneous nature of the interaction law between one-particle excitations. However, for more moderate values of $\xi = k_F a_0/\pi$ the large numerical coefficient in eq. (83) implies that Γ^{dress} and Γ^{int} can be of quite comparable magnitude.

Another lesson we have learned is that the Fredholm method is ideally suited to an investigation of the present type where one is mainly interested in the isolated poles of a Green function. It may perhaps be fruitful to apply this method to transport problems in a system with non-trivial interactions, such as He^3 [2].

Acknowledgments

We take pleasure in expressing our gratitude to Professor NIELS BOHR and Professor AAGE BOHR for enabling us to enjoy the generous support and warm hospitality of the Institut for Teoretisk Fysik where the work reported herein was initiated. We should also like to thank Professor A. BOHR, Dr. J. GOLDSTONE, and Professor P. C. MARTIN for comments and suggestions. The work of the first-named author was supported in part by the United States Air Force Office of Scientific Research (ARDC), under contract number AF 49(638)—589.

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Indleveret til Selskabet den 8. april 1960.
Færdig fra trykkeriet den 19. november 1960.