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RELATIVISTIC THERMODYNAMICS

A Strange Incident in the History of Physics

Det Kongelige Danske Videnskabernes Selskab
Matematisk-fysiske Meddelelser **36**, 1

Kommissionær: Munksgaard
København 1967

Synopsis

In view of the confusion which has arisen in the later years regarding the correct formulation of relativistic thermodynamics, the case of arbitrary reversible and irreversible thermodynamic processes in a fluid is reconsidered from the point of view of observers in different systems of inertia. Although the total momentum and energy of the fluid do not transform as the components of a 4-vector in this case, it is shown that the momentum and energy of the heat supplied in any process form a 4-vector. For *reversible* processes this four-momentum of supplied heat is shown to be proportional to the four-velocity of the matter, which leads to Ott's transformation formula for the temperature in contrast to the old formula of Planck.

PRINTED IN DENMARK
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Introduction

In the years following Einstein's fundamental paper from 1905, in which he founded the theory of relativity, physicists were engaged in reformulating the classical laws of physics in order to bring them in accordance with the (special) principle of relativity. According to this principle the fundamental laws of physics must have the same form in all Lorentz systems of coordinates or, more precisely, they must be expressed by equations which are form-invariant under Lorentz transformations. In some cases, like in the case of Maxwell's equations, these laws had already the appropriate form, in other cases, they had to be slightly changed in order to make them covariant under Lorentz transformations. This was, for instance, the case with Newton's equations of mechanics which turned out to hold only for phenomena in which the velocities of the particles are sufficiently small with respect to c the velocity of light.

An investigation of the laws of thermodynamics in relation to the relativity principle was carried through by M. PLANCK and others(1) in the years 1907–1908. In all cases the procedure was the following. One starts by assuming that the usual two laws of thermodynamics hold in the rest system of the body concerned. Then one tries to formulate transformation laws for the transferred heat, the entropy, and the temperature in such a way that the usual laws of thermodynamics are valid also for the transformed quantities belonging to an arbitrary system of inertia. If this should turn out not to be possible, one would have to admit that the classical laws of thermodynamics are not generally valid and one would have to modify them similarly as in the case of Newtonian mechanics. However, such a modification of the classical thermodynamical laws is not necessary.

Let us consider a thermodynamic body in thermal equilibrium at rest in a certain system of inertia $\mathfrak{K}^0$. Throughout this paper we shall consider only thermodynamic equilibrium states and (reversible or irreversible) transitions between such equilibrium states. According to the *first law of thermodynamics*, the total energy H^0 in $\mathfrak{K}^0$ is a unique function of the thermodynamical state of the body. In a process which gives rise to a change of the state, the change ΔH^0 in the energy is given by

$$\Delta H^0 = \Delta Q^0 + \Delta A^0, \tag{1}$$

where ΔQ^0 is the amount of heat transferred to the system in the process and ΔA_0 is the mechanical* work performed on the system by its surroundings, all measured in the rest system $\mathfrak{K}^0$ of the body in question. Further, according to *the second law of thermodynamics*, the entropy S^0 in the rest system $\mathfrak{K}^0$ is similarly a function of the thermodynamical state. The change of entropy content by an infinitesimal change of the state is (by definition)

$$dS^0 = \frac{dQ^0_{\text{rev}}}{T^0} = \frac{dH^0 - dA^0_{\text{rev}}}{T^0}, \tag{2}$$

where dQ^0_{rev} and dA^0_{rev} represent the amount of heat and the mechanical work, respectively, in a *reversible* process which brings about the change of the state considered, and T^0 is the Kelvin temperature of the system.

Now, consider the same thermodynamical process from the point of view of an observer in a system of inertia $\mathfrak{K}$ with respect to which the body in question moves with the constant velocity $\boldsymbol{v}$. Then, on account of the relativity principle, we should have the same relations (1) and (2) between the increases of the energy H, the entropy S, and the transferred heat and mechanical work measured in $\mathfrak{K}$, i.e.,

$$\Delta H = \Delta Q + \Delta A \tag{3}$$

$$dS = \frac{dQ_{\text{rev}}}{T} = \frac{dH - dA_{\text{rev}}}{T}. \tag{4}$$

The transformation equations for ΔE and ΔA are known from relativistic mechanics. From (3) and (1) we therefore obtain the transformation law for the transferred heat ΔQ. In this way, Planck found the formula

$$\Delta Q = \Delta Q^0 \sqrt{1-\beta^2}, \quad \beta = \frac{v}{c}. \tag{5}$$

Further he showed that the entropy of a body in thermal equilibrium is a relativistic invariant.

* The word mechanical should not be taken literally, it could also be the work of electromagnetic forces originating from sources in the surroundings.

In order to see this, let us consider a body in some internal equilibrium state which originally is at rest in $\mathfrak{K}$ with an entropy S. If this body is accelerated adiabatically, i.e., infinitely slowly and without heat transfer until it gets the velocity $\boldsymbol{v}$, then its internal state is undisturbed and, on account of (4), it has still the same entropy S as before with respect to $\mathfrak{K}$. On the other hand, it is now with respect to $\mathfrak{K}^0$ in the same situation as it were initially with respect to $\mathfrak{K}$. Its entropy S^0 in $\mathfrak{K}^0$ must therefore be equal to S, i.e. the entropy

$$S = S^0 \tag{6}$$

is a relativistic invariant. From the equations (2)–(6) Planck concluded that the temperature of a body transforms according to the equation

$$T = T^0 \sqrt{1 - \beta^2}. \tag{7}$$

This result has been accepted by all physicists through more than half a century and it is quoted in numerous textbooks including the first edition of my own monograph "The Theory of Relativity"(2). Nevertheless, the equations (5) and (7) are wrong, as has been noticed only quite recently. It is a strange and rather unique incident in the history of physics that a fundamental mistake in the original derivation remained overlooked through such a long period of time.

H. Ott(3) was the first who pointed out that the formulae (5), (7) of Planck in certain cases lead to unreasonable results and he maintained that they had to be replaced by the equations

$$\Delta Q = \frac{\Delta Q^0}{\sqrt{1 - \beta^2}} \tag{8}$$

and

$$T = \frac{T^0}{\sqrt{1 - \beta^2}} \tag{9}$$

in accordance with (6) which was accepted by him.

However, Ott's paper remained unnoticed until quite recently. His treatment was also somewhat special in that he mainly considered systems which are closed before and after the process so that the total momentum and energy of the system transform as the components of a 4-vector under Lorentz transformations and this is generally not the case for the systems with which we are dealing in thermodynamics. Take, for instance, the system considered by Planck in his original paper, which is a fluid enclosed in a container of a changeable volume. Then it is essential that the walls of the container exert a pressure on the fluid before and after the thermodynamical process and, as is well known, the total momentum and energy of the fluid do not transform as the components of a 4-vector in this case.

A few years later, H. ARZÉLIÈS[4] obviously without knowing Ott's work, considered the case of a fluid anew and came to the same formulae (8) and (9) as Ott; but along with the equations (5) and (7) he also discarded the transformation equations for the momentum and energy of the fluid following from the relativistic mechanics of elastic bodies, which undoubtedly are correct. Therefore, the paper by ARZÉLIÈS caused a whole avalanche of mutually contradicting papers[5] on the subject, so that the situation regarding this really quite simple problem has become rather confused. Among all these papers the one by KIBBLE[6] seems to me exceptional. Apart from a few misprints, all his statements seem to be correct. His most important remark is that the work done by the external force in $\mathfrak{K}$, on account of the relativity of simultaneity, may be different from zero even if the volume is not changed, provided that the pressure is changed during the process. However, in his calculation of this effect, he assumes that the pressure rises suddenly at the beginning of the process, and this is not quite in accordance with the assumption of a reversible change which by definition has to be performed "infinitely" slowly.

In view of the principal importance of the question and hopefully in order to finally remove all doubts about the correctness of Ott's equation (9) and also of his equation (8) for reversible processes, I propose once more in all details to consider arbitrary finite reversible and irreversible changes of a fluid enclosed in a container of changeable volume. For irreversible processes it will be shown that the transformation equation (8) is valid only under the condition that the transferred heat does not carry any momentum in the rest system, and we shall find a generalization of (8) for the case where this condition is not satisfied.

Reversible thermodynamical processes

Let us begin by considering a thermodynamic fluid in an *arbitrary state of motion* which exerts a normal pressure on any surface element. According to the relativistic dynamics of continuous media[7] we have, if there is no heat conduction, the following transformation laws connecting the pressure p, the energy density h, and the momentum density $\boldsymbol{g}$ in a fixed system of inertia $\mathfrak{K}$ with the corresponding quantities p^0, h^0, $\boldsymbol{g}^0$ measured in the momentary rest system of the fluid at the point and at the time considered.

The pressure is a relativistically invariant scalar, i.e.,

$$p = p^0. \tag{10}$$

If $\boldsymbol{u}$ is the velocity with respect to $\mathfrak{K}$ of the fluid at the event point considered, we have for the energy- and momentum densities

$$h = \frac{h^0 + p^0 \frac{u^2}{c^2}}{1 - \frac{u^2}{c^2}} \tag{11}$$

$$\boldsymbol{g} = \frac{h + p}{c^2}\boldsymbol{u} = \frac{h^0 + p^0}{c^2 - u^2}\boldsymbol{u}. \tag{12}$$

The latter quantity is connected with the energy current density by Planck's relation

$$\boldsymbol{S} = c^2\boldsymbol{g} = (h+p)\boldsymbol{u}. \tag{13}$$

The equations (10) – (13) are valid only if we can neglect the contributions from heat conduction. If there is heat conduction, like during a process in which heat is supplied from a reservoir, one has to add to (13) the heat flux vector $\boldsymbol{S}^{(h)}$ which gives an extra contribution $\boldsymbol{S}^{(h)}/c^2$ to the momentum density $\boldsymbol{g}$ in (12).

Next, we consider the case where the fluid is in thermodynamic equilibrium in a cylindrical vessel at rest in a system of inertia $\mathfrak{K}^0$, which moves with constant velocity $\boldsymbol{v}$ with respect to $\mathfrak{K}$. Then, $\boldsymbol{u}$ is constant and equal to $\boldsymbol{v}$ for all elements of the fluid. We can obviously arrange it so that the cylinder axis of the container is parallel to the common x- and x^0-axis and that the end walls a and b with area F^0 have the coordinates $x_a^0 = 0$, $x_b^0 = l^0$, respectively, in $\mathfrak{K}^0$. The latter end wall may be a movable piston so that the volume $V^0 = F^0 l^0$ can be changed by varying l^0. The area of the walls a and b is the same in $\mathfrak{K}$ and $\mathfrak{K}^0$, i.e.,

$$F = F^0, \tag{14}$$

but for the volume we have of course

$$V = V^0\sqrt{1 - \beta^2}, \quad \beta = \frac{v}{c}. \tag{15}$$

Since the fluid is in an equilibrium state, there is no heat conduction and the pressures p_a^0 and p_b^0 at a and b are equal in $\mathfrak{K}^0$, i.e.,

$$p_a^0 = p_b^0 = p^0, \tag{16}$$

and the forces exerted by the walls a and b on the fluid are p^0F^0 and $-p^0F^0$, respectively, in the direction of the x^0-axis. In $\mathfrak{K}^0$ they do not perform any work of course for fixed positions of the walls. However, in $\mathfrak{K}$ the wall a performs a work pFv on the fluid pr. unit time while the force of the wall b has a mechanical effect $-pFv$. This is in accordance with the expression (13), according to which the amount of energy which pr. unit time passes through a into the fluid is $F(S_x - hv) = Fpv$ and equal to the rate at which energy is leaving the fluid through the end wall b. The forces exerted by the cylinder walls do not perform any work because they are perpendicular to the direction of $\boldsymbol{v}$.

Since there is no heat conduction in an equilibrium state, we can obtain the total energy $H = hV$ and momentum $\boldsymbol{G} = \boldsymbol{g}V$ by integrating (11) and (12) over the volume of the fluid and, by taking account of (15), we get

$$\left.\begin{aligned} H &= \frac{H^0 + \beta^2 p^0 V^0}{\sqrt{1-\beta^2}} \\ \boldsymbol{G} &= \frac{H^0 + p^0 V^0}{c^2\sqrt{1-\beta^2}}\,\boldsymbol{v}, \end{aligned}\right\} \tag{17}$$

which shows that the total momentum and energy of the fluid do not transform like the components of a 4-vector. (For this particular non-closed system it is the momentum and enthalpy $\mathfrak{E} = H + pV$ which together form a 4-vector).

We shall now consider an arbitrary finite *reversible* change of state in which the volume V^0 and the pressure p^0 are changed by the amounts ΔV^0 and Δp^0. This can be done by keeping the wall a fixed at the position $x_a^0 = 0$ while b is moved from $x_b^0 = l^0$ to $x_b^0 = l^0 + \Delta l^0$, $\Delta l^0 = \Delta V^0/F^0$. If the expansion starts at $t^0 = 0$ and is finished at $t^0 = \tau^0$, the motion of the wall b is described by an equation

$$x_b^0 = \varphi(t^0) \tag{18}$$

where the function $\varphi(t^0)$ increases slowly from the value l^0 for $t^0 \leq 0$ to the value $l^0 + \Delta l^0$ for $t^0 \geq \tau^0$.

Hence,

$$\varphi(t^0) = \begin{cases} t^0, & t^0 \leq 0 \\ \varphi(t^0), & 0 \leq t^0 \leq \tau^0 \\ t^0 + \Delta t^0, & t^0 \geq \tau^0. \end{cases} \tag{19}$$

The velocity with which the wall b moves in $\mathfrak{K}^0$ is then $u_b^0 = \varphi'(t^0)$, which is zero for $t^0 \leq 0$ and for $t^0 \geq \tau^0$. For a reversible change the velocity u_b^0 must be small ("infinitesimal") during the whole process, which means that the time τ^0 must accordingly be large. We shall further assume that simultaneously a certain amount of heat ΔQ^0 is *reversibly* supplied to the fluid through the cylinder walls from a heat reservoir comoving with the container. In order to make this heat supply a reversible process, we have to arrange it so that the temperature of the reservoir at each stage of the process is only infinitesimally higher than the temperature of the fluid. Under these conditions we can assume that during the process the fluid goes through a succession of equilibrium states, which means that the pressures $p_a^0(t^0)$ and $p^0(t^0)$ at the walls a and b are equal for equal times t^0 in $\mathfrak{K}^0$. Hence,

$$p_a^0(t^0) = p_b^0(t^0) = f(t^0), \tag{20}$$

where $f(t^0)$ is a function (depending on the rate of the heat flux through the cylinder and on the function $\varphi(t^0)$) which rises from the value p^0 for $t^0 \leq 0$ to the value $p^0 + \Delta p^0$ for $t^0 \geq \tau^0$, i.e.,

$$f(t^0) = \begin{cases} p^0, & t^0 \leq 0 \\ p^0 + \Delta p^0, & t^0 \geq \tau^0. \end{cases} \tag{21}$$

Since only the wall b is moving in $\mathfrak{K}^0$ during the expansion, the total mechanical work done by the surroundings on the fluid during the process is in $\mathfrak{K}^0$

$$\Delta A^0 = -F^0 \int_0^{\tau^0} p_b^0(t^0) u_b^0 dt^0 = -F^0 \int_0^{\tau^0} f(t^0)\varphi'(t^0) dt^0. \tag{22}$$

We shall now consider the process in question from the point of view of an observer in $\mathfrak{K}$. According to the Lorentz transformations, we have for any event in x^0, y^0, z^0 at the time t^0

$$t = \frac{t^0 + \frac{v x^0}{c^2}}{\sqrt{1-\beta^2}}, \quad y = y^0, \quad z = z^0. \tag{23}$$

For a fixed point on a, where x^0 is constantly equal to zero, we have therefore

$$t_a = \frac{t_a^0}{\sqrt{1-\beta^2}}, \quad dt_a = \frac{dt_a^0}{\sqrt{1-\beta^2}}. \tag{24}$$

On the other hand, for a point on b for which (18) and (19) hold, we get

$$t_b = \frac{t_b^0 + \frac{v}{c^2}\varphi(t_b^0)}{\sqrt{1-\beta^2}}, \quad dt_b = \frac{1+\frac{v\varphi'(t_b^0)}{c^2}}{\sqrt{1-\beta^2}} dt_b^0. \tag{25}$$

From the relativistic formula for the addition of velocities we get for the velocities u_a and u_b of the walls a and b with respect to $\mathfrak{K}$, since $u_a^0 = 0$ and $u_b^0 = \varphi'(t^0)$,

$$\left.\begin{aligned} u_a &= v \\ u_b &= \frac{v+\varphi'(t_b^0)}{1+\frac{v\varphi'(t_b^0)}{c^2}}. \end{aligned}\right\} \tag{26}$$

Hence, by (24) – (26),

$$\left.\begin{aligned} u_a dt_a &= \frac{v\,dt_a^0}{\sqrt{1-\beta^2}} \\ u_b dt_b &= \frac{v+\varphi'(t_b^0)}{\sqrt{1-\beta^2}} dt_b^0. \end{aligned}\right\} \tag{27}$$

Since the pressure is an invariant scalar, we get from (20) for the pressure at the wall a

$$p_a(t_a) = p_a^0(t_a^0) = f(t_a^0), \tag{28}$$

where t_a and t_a^0 are connected by the equation (24). Similarly, we have for the pressure at the wall b

$$p_b(t_b) = p_b^0(t_b^0) = f(t_b^0), \tag{29}$$

where t_b and t_b^0 here are connected by (25).

Now, equal times $t_a = t_b$ in $\mathfrak{K}$ do not correspond to equal times t_a^0 and t_b^0, therefore, in general, $p_a(t) \neq p_b(t)$. From (24), (25), and (19) it follows that

$$t_a = t_b = 0 \quad \text{corresponds to} \quad t_a^0 = 0, \quad t_b^0 = -\frac{vl^0}{c^2}. \tag{30}$$

Therefore, at this time and at earlier times we have, by (28), (29) and (21),

$$p_a(t) = p_b(t) = p, \quad t \leq 0. \tag{31}$$

Similarly, at the time

$$\left.\begin{aligned} t_a &= t_b = \tau, \quad \tau = \frac{\tau^0 + \frac{v}{c^2}(l^0 + \Delta l^0)}{\sqrt{1-\beta^2}} \\ t_a^0 &= \tau_a^0 = \tau^0 + \frac{v(l^0 + \Delta l^0)}{c^2}, \quad t_b^0 = \tau^0. \end{aligned}\right\} \tag{32}$$

we have

Since $\tau_a^0 > \tau^0$, it follows from (28), (29) and (21), that

$$p_a(t) = p_b(t) = p^0 + \Delta p^0 \quad \text{for} \quad t \geq \tau. \tag{33}$$

Further, outside the internal $0 \leq t \leq \tau$, the velocities of the walls a and b are equal, viz., $u_a = u_b = v$. Thus the total mechanical force and the corresponding work on the fluid are zero outside the interval $0 \leq t \leq \tau$.

The work ΔA_a performed by the wall a on the fluid during this interval is now, by (14), (28), (27), (30) and (32),

$$\Delta A_a = \int_0^\tau F p_a(t_a) u_a dt_a = \frac{F^0 v}{\sqrt{1-\beta^2}} \int_0^{\tau_a^0} f(t^0) dt^0. \tag{34}$$

Since $\tau_a^0 - \tau^0 = \dfrac{v(l^0 + \Delta l^0)}{c^2}$ and $f(t^0) = p^0 + \Delta p^0$ in the interval from τ^0 to τ_a^0, we get

$$\Delta A_a = \frac{F^0 v}{\sqrt{1-\beta^2}} \int_0^{\tau^0} f(t^0) dt^0 + \frac{\beta^2 (p^0 + \Delta p^0)(V^0 + \Delta V^0)}{\sqrt{1-\beta^2}}. \tag{35}$$

Similarly, we get for the work performed by the wall b

$$\Delta A_b = -\int_0^\tau F p_b(t_b) u_b dt_b = -\frac{F^0}{\sqrt{1-\beta^2}} \int_{-\frac{vl^0}{c^2}}^{\tau^0} f(t^0)(v + \varphi'(t^0)) dt^0, \tag{36}$$

where we have used (27) and (30). In the interval from $-vl^0/c^2$ to zero we have $f(t^0) = p^0$ and $\varphi'(t^0) = 0$. Thus,

$$\varDelta A_b = -\frac{\beta^2 p^0 V^0}{\sqrt{1-\beta^2}} - \frac{F^0 v}{\sqrt{1-\beta^2}} \int_0^{\tau^0} f(t^0) dt^0 + \frac{\varDelta A^0}{\sqrt{1-\beta^2}} \tag{37}$$

on account of (22). Since again the work performed by the cylinder wall is zero we get in $\mathfrak{K}$ for the total mechanical work performed during the process

$$\varDelta A = \varDelta A_a + \varDelta A_b = \frac{\beta^2 \varDelta (p^0 V^0) + \varDelta A^0}{\sqrt{1-\beta^2}}, \tag{38}$$

where

$$\varDelta (p^0 V^0) = (p^0 + \varDelta p^0)(V^0 + \varDelta V^0) - p^0 V^0 \tag{39}$$

is the increase of the product of pressure and volume in $\mathfrak{K}^0$ during the process. This formula (38) is exact for any finite reversible processes. Note that even if $\varDelta V^0 = 0$, i.e., when $\varDelta A^0 = 0$, we have a finite work in $\mathfrak{K}$, viz.,

$$\varDelta A = \frac{\beta^2 \varDelta (p^0 V^0)}{\sqrt{1-\beta^2}} = \frac{\beta^2 V^0 \varDelta p^0}{\sqrt{1-\beta^2}} \quad \text{for} \quad \varDelta V^0 = 0 \tag{40}$$

which, as we have seen, stems from the relativity of simultaneity.

From (3), (17) and (38) we therefore get, for the amount of heat energy transferred to the fluid in a *reversible* process, since the velocity of the fluid is $\boldsymbol{v}$ before and after the process,

$$\varDelta Q = \varDelta H - \varDelta A = \frac{\varDelta H^0 + \beta^2 \varDelta (p^0 V^0) - \beta^2 \varDelta (p^0 V^0) - \varDelta A^0}{\sqrt{1-\beta^2}}$$

or, on account of (1),

$$\varDelta Q = \frac{\varDelta Q^0}{\sqrt{1-\beta^2}} \tag{41}$$

which is Ott's formula (8). If this formula is applied to an infinitesimal reversible process we also get, by (4), (2) and (6), Ott's formula (9) for the transformation of the temperature. "Infinitesimal" should here be taken in the physical sense, which means that the increments $\varDelta V^0 = dV^0$, $\varDelta p^0 = dp^0$ etc. are so small that we can neglect terms depending on the products or higher powers of these increments. In that case, we can put $\varphi'(t^0) = \dfrac{\varDelta l^0}{\tau^0}$ in the interval $0 < t^0 < \tau^0$ and, neglecting terms of higher order, we get from (22)

$$dA^0 = -F^0 \int_0^{\tau^0} f(t^0)\varphi'(t^0)dt^0 = -\frac{F^0 \Delta l^0}{\tau^0} \int_0^{\tau^0} f(t^0)dt^0 = -p^0 dV^0. \tag{42}$$

Further (38) reduces to

$$dA = \frac{\beta^2(p^0 dV^0 + V^0 dp^0) - p^0 dV^0}{\sqrt{1-\beta^2}} = \frac{\beta^2 V^0 dp^0}{\sqrt{1-\beta^2}} - p^0 dV^0 \sqrt{1-\beta^2} \tag{43}$$

which, by (10) and (15), also may be written

$$dA = \frac{\beta^2 V dp}{1-\beta^2} - pdV. \tag{44}$$

Besides the work $-pdV$ due to the change of volume we have in $\mathfrak{K}$ a work $\frac{\beta^2 V dp}{1-\beta^2}$ which (apart from a minus sign) is in accordance with the result of KIBBLE (loc. cit.[(6)] equation (17)). Since work, energy and time are invariant quantities under purely spatial rotations of the coordinate axes, it is clear that the relations obtained in the preceding developments are independent of the special arrangement of the container with respect to the coordinate axes used in our calculations.

Let us for a moment go back to an arbitrary finite reversible process (with the special arrangements as before). The correct formula (41) or (8) deviates from the earlier equation (5) of Planck and others by a factor $1-\beta^2$. Where lies the error in the earlier derivation? To see this, let us calculate the total mechanical impulse $\Delta \boldsymbol{J}$ in $\mathfrak{K}$ which is the time integral of the resultant mechanical force from the walls on the fluid. Since the fluid at any moment during a reversible process is in an equilibrium state, the pressure $p_c(t)$ at the cylinder walls is at any time t equal for all points with the same x-coordinate. Therefore, the impulse caused by the cylinder wall is zero, i.e.

$$\Delta J_y = \Delta J_z = 0. \tag{45}$$

On the other hand, the x-component of $\Delta \boldsymbol{J}$ is the sum of the impulses ΔJ_a and ΔJ_b of the walls a and b

$$\Delta J_x = \Delta J_a + \Delta J_b. \tag{46}$$

Since the force from a is $K_a(t) = Fp_a(t)$ in the x-direction, we have

$$\Delta J_a = F\int_0^{\tau} p_a(t_a)dt_a = \frac{F^0}{\sqrt{1-\beta^2}}\int_0^{\tau_a^0} f(t^0)dt^0 = \frac{\Delta A_a}{v} \tag{47}$$

on account of (34). Similarly, we have by (29) and (25),

$$\left.\begin{aligned}\Delta J_b &= -F\int_0^{\tau} p_b(t_b)dt_b = -F^0\int_{-\frac{vl^0}{c^2}}^{\tau^0} p_b^0(t_b^0)\frac{dt_b}{dt_b^0}dt_b^0 \\ &= -\frac{F^0}{\sqrt{1-\beta^2}}\int_{-\frac{vl^0}{c^2}}^{\tau^0} f(t^0)\left(1+\frac{v\varphi'(t^0)}{c^2}\right)dt^0\end{aligned}\right\} \tag{48}$$

or, by means of (21) and (22),

$$\Delta J_b = -\frac{vp^0V^0}{c^2\sqrt{1-\beta^2}} - \frac{F^0}{\sqrt{1-\beta^2}}\int_0^{\tau^0} f(t^0)dt^0 + \frac{v\Delta A^0}{c^2\sqrt{1-\beta^2}}. \tag{49}$$

Introducing (47), (49) into (46), and using (35), the x-component of the mechanical impulse becomes

$$\Delta J = \frac{v}{c^2\sqrt{1-\beta^2}}\left[\Delta(p^0V^0) + \Delta A^0\right]. \tag{50}$$

Since with our arrangement, $\boldsymbol{v} = \{v, 0, 0\}$ the three equations (45) and (50) may be comprised in the vector equation

$$\Delta \boldsymbol{J}_x = \frac{\Delta(p^0V^0) + \Delta A^0}{c^2\sqrt{1-v^2}}\boldsymbol{v}. \tag{51}$$

According to the dynamical equations for a continuous medium with heat conduction, the change of momentum of the fluid is determined by the "spatial analogue" of the energy equation (3), i.e.

$$\Delta \boldsymbol{G} = \Delta \boldsymbol{G}^{(h)} + \Delta \boldsymbol{J}, \tag{52}$$

where $\Delta \boldsymbol{G}^{(h)}$ is the increase of momentum due to the conduction of heat to the system. In our case we have, according to (17),

$$\varDelta \boldsymbol{G} = \frac{\varDelta H^0 + \varDelta (p^0 V^0)}{c^2 \sqrt{1-\beta^2}} \boldsymbol{v} \tag{53}$$

which, by (52), (51) and (1), gives

$$\varDelta \boldsymbol{G}^{(h)} = \varDelta \boldsymbol{G} - \varDelta \boldsymbol{J} = \frac{\varDelta H^0 - \varDelta A^0}{c^2 \sqrt{1-\beta^2}} \boldsymbol{v} = \frac{\varDelta Q^0}{c^2 \sqrt{1-\beta^2}} \boldsymbol{v}. \tag{54}$$

The increase of the momentum due to the heat conduction during the reversible process is, therefore, exactly as if we had added a particle of rest mass $\varDelta Q^0/c^2$ and zero velocity in the system $\mathfrak{K}^0$. By means of (41), this part of the momentum increase can also be written

$$\varDelta \boldsymbol{G}^{(h)} = \frac{\varDelta Q}{c^2} \boldsymbol{v} \tag{55}$$

in accordance with Einstein's general relation between energy and inertial mass. The total increase of momentum can now be written as a sum of two parts

$$\varDelta \boldsymbol{G} = \varDelta \boldsymbol{G}^{(h)} + \varDelta \boldsymbol{G}^{(m)} \tag{56}$$

where the second part, according to (52), satisfies the equation

$$\varDelta \boldsymbol{G}^{(m)} = \varDelta \boldsymbol{J}. \tag{57}$$

Thus, $\varDelta \boldsymbol{G}^{(m)}$ is that increase of the momentum which is due to the action of the mechanical forces. Only if there is no heat transfer is the mechanical impuls $\varDelta \boldsymbol{J}$ equal to the total increase of the momentum, and here lies the root of the earlier mistake.

Before we come to a closer scrutiny of this point, let us look at the situation in the system $\mathfrak{K}^0$. The equations (51)–(57) are valid in any system $\mathfrak{K}$. If we let $\mathfrak{K} \to \mathfrak{K}^0$ we have $\boldsymbol{v} \to 0$. From (51) we then get in $\mathfrak{K}^0$, for the mechanical impulse in a reversible process,

$$\varDelta \boldsymbol{J}^0 = 0 \tag{58}$$

as is also seen immediately from (20). Further, we get from (54) in $\mathfrak{K}^0$

$$\varDelta \boldsymbol{G}^{(h)0} = 0 \tag{59}$$

i.e. *in reversible processes the momentum transfer due to the heat supply is zero in the system* $\mathfrak{K}^0$. From (59), (54) and (41) it follows that the quantities

$$\Delta Q_i = \left\{\Delta \boldsymbol{G}^{(h)}, \quad \frac{i}{c}\Delta Q\right\} = \frac{\Delta Q^0}{c^2} V_i \tag{60}$$

transform as the components of a 4-vector in any reversible process (V_i is the constant 4-velocity of the fluid).

The scalar product of the mechanical impulse $\Delta \boldsymbol{J}$ in (51) with the velocity $\boldsymbol{v}$ is

$$(\Delta \boldsymbol{J} \cdot \boldsymbol{v}) = \frac{\beta^2[\Delta(p^0 V^0) + \Delta A^0]}{\sqrt{1-\beta^2}}. \tag{61}$$

If we express the mechanical work ΔA in (38) in terms of this quantity, we get

$$\Delta A = (\Delta \boldsymbol{J} \cdot \boldsymbol{v}) + \Delta A^0 \sqrt{1-\beta^2} \tag{62}$$

or, on account of (57),

$$\Delta A = (\Delta \boldsymbol{G}^{(m)} \cdot \boldsymbol{v}) + \Delta A^0 \sqrt{1-\beta^2}. \tag{63}$$

The mistake made in earlier derivations was to replace $\Delta \boldsymbol{G}^{(m)}$ in this expression by the total increase $\Delta \boldsymbol{G}$ of the momentum of the fluid. The expression ΔA_P for the work obtained in this way, i.e.,

$$\Delta A_P = (\Delta \boldsymbol{G} \cdot \boldsymbol{v}) + \Delta A^0 \sqrt{1-\beta^2} \tag{64}$$

differs from the correct expression (63) by

$$\Delta A_P - \Delta A = (\Delta \boldsymbol{G} - \Delta \boldsymbol{G}^{(m)}) \cdot \boldsymbol{v} = (\Delta \boldsymbol{G}^{(h)} \cdot \boldsymbol{v}) = \frac{\beta^2 \Delta Q^0}{\sqrt{1-\beta^2}}$$

on account of (54). With (3) and (41) one therefore gets

$$\Delta Q_P = \Delta H - \Delta A_P = \Delta Q - \frac{\beta^2 \Delta Q^0}{\sqrt{1-\beta^2}} = \Delta Q^0 \sqrt{1-\beta^2}, \tag{65}$$

which is the equation (5) of Planck.

The philosophy leading to this mistake becomes clearer when we again consider an infinitesimal reversible process. By (42) we get in this case for the mechanical impulse (51)

$$d\boldsymbol{J} = \frac{V^0 dp^0}{c^2\sqrt{1-\beta^2}}\boldsymbol{v} = \frac{V dp}{c^2 - v^2}\boldsymbol{v}. \tag{66}$$

Further, for an infinitesimal change of state, the time τ during which the change is reversibly performed is also physically infinitesimal. If we put $\tau = dt$, the equation (52) formally takes the form of an equation of motion

$$\frac{d\boldsymbol{G}}{dt} = \boldsymbol{K}, \tag{67}$$

where

$$\left.\begin{aligned} \boldsymbol{K} &= \boldsymbol{K}^{(m)} + \boldsymbol{K}^{(h)} \\ \boldsymbol{K}^{(m)} &\equiv \frac{d\boldsymbol{J}}{dt} = \frac{V\dfrac{dp}{dt}}{c^2 - v^2}\boldsymbol{v} \\ \boldsymbol{K}^{(h)} &\equiv \frac{d\boldsymbol{G}^{(h)}}{dt}. \end{aligned}\right\} \tag{68}$$

Let us for simplicity at the moment consider a process in which the volume is unchanged. Then the change of state is due solely to the heat supply which will increase the pressure, and all parts of the fluid have constantly the same velocity $\boldsymbol{v}$ in $\mathfrak{K}$. The old argument was then that $\boldsymbol{K}$ in (67) is the force which is necessary in order to keep this constant velocity in spite of the increase in proper mass of the system due to the heating up and due to the increased pressure which represents an increase in the elastic potential. This force performs a work

$$dA_p = (\boldsymbol{K}\cdot\boldsymbol{v})dt = (d\boldsymbol{G}\cdot\boldsymbol{v}) \tag{69}$$

which is just equal to the expression (64) for an infinitesimal process with $dV^0 = 0$, $dA^0 = 0$.

However, (67) is only *formally* an equation of motion. Actually, according to (57) and (68), it consists of the two equations

$$\frac{d\boldsymbol{G}^{(h)}}{dt} \equiv \boldsymbol{K}^{(h)} \tag{70}$$

$$\frac{d\boldsymbol{G}^{(m)}}{dt} = \boldsymbol{K}^{(m)} \tag{71}$$

of which the first is a *pure identity*. The quantity $\boldsymbol{K}^{(h)}$ is therefore not a real force, it just describes the rate at which the momentum grows on account of the conduction of heat to the fluid. Physically the increase of proper mass due to the heat supply is of exactly the same nature as if we add a particle at rest in $\mathfrak{K}^0$ to the fluid. This particle will then move with the velocity $\boldsymbol{v}$ with respect to $\mathfrak{K}$ and there is no need for any force to keep up this constant velocity.

The only real equation of motion is (71) and the force $\boldsymbol{K}^{(m)}$ given by (68) is just the force which is necessary in order to keep $\boldsymbol{v}$ constant in spite of the increase in elastic potential energy due to the increase of the pressure. There is no mystery about this force $\boldsymbol{K}^{(m)}$, as we have seen it is a force exerted by the walls in the system $\mathfrak{K}$. It is the work of this real force which gives us the correct expression

$$dA = (\boldsymbol{K}^{(m)} \cdot \boldsymbol{v})dt = \frac{\beta^2 V dp}{1-\beta^2}. \tag{72}$$

If we have also a change of V we have to add the "internal" work $-pdV$ in order to get the general expression (44).

Irreversible Processes

In the preceding section we have considered only reversible changes of the state of the fluid, and the main result was the transformation equation (38) for the mechanical work, from which the equation (41) for the transformation of the transferred heat energy followed as a consequence. The problem is now if or under which conditions (38) is valid also for irreversible processes. Let us assume that we have the same special arrangement of the cylindrical container with respect to the common x- and x^0-axis as before. Further, we assume that the change of volume is again obtained by the motion of the piston b which in $\mathfrak{K}^0$ is described by the equations (18) and (19). But now the function $\varphi(t^0)$ can be completely arbitrary in the interval $0 \leqq t^0 \leqq \tau^0$. It may even describe an arbitrary oscillatory motion with an amplitude larger than Δl^0, but we shall assume that it stops at $l^0 + \Delta l^0$ a little time before $t^0 = \tau^0$. Similarly we assume the heat transfer to be stopped a little before τ^0 so as to give the fluid time to reach thermal equilibrium at $t^0 = \tau^0$. As regards the way in which the heat is supplied from the reservoir, we shall not make any assumptions at the moment. The reservoir may for instance have a temperature considerable higher than the temperature of the fluid. Also τ^0 need not be large, and the velocity $\varphi'(t^0)$ of the piston b during the process can be as large as one wants. Since the fluid is in thermal equilibrium before and after the process the equations (17) are valid at $t^0 \leq 0$ and $t^0 \geq \tau^0$ so that we have, as before,

$$\left.\begin{aligned} \Delta H &= \frac{\Delta H^0 + \beta^2 \Delta(p^0 V^0)}{\sqrt{1-\beta^2}} \\ \Delta \boldsymbol{G} &= \frac{\Delta H^0 + \Delta(p^0 V^0)}{c^2\sqrt{1-\beta^2}}\, \boldsymbol{v}. \end{aligned}\right\} \tag{73}$$

Also the equations (23)–(27) are valid here, but the equations (28) and (29) do not hold any more of course. Instead, we have for the pressure $p_a^0(y^0, z^0, t^0)$ in a point at the wall a with coordinates (y^0, z^0) at the time t^0

$$p_a^0 = f(y^0, z^0, t^0), \tag{74}$$

where f is a function about which we know only that it is equal to the constants p^0 and $p^0 + \Delta p^0$ independent of (y^0, z^0) for $t^0 \leqslant 0$ and $t^0 \geqslant \tau^0$, respectively, i.e.,

$$f(y^0, z^0, t^0) = \begin{cases} p^0, & t^0 \leqslant 0 \\ p^0 + \Delta p^0, & t^0 \geqslant \tau^0. \end{cases} \tag{75}$$

Similarly, we have for the pressure at the wall b

$$p_b^0 = g(y^0, z^0, t^0),$$

where

$$g(y^0, z^0, t^0) = \begin{cases} p^0, & t^0 \leqslant 0 \\ p^0 + \Delta p^0, & t^0 \geqslant \tau^0, \end{cases} \tag{76}$$

but we cannot say anything about the functions f and g in the period $0 < t^0 < \tau^0$.

In the system $\mathfrak{K}^0$ the mechanical work on the fluid is obviously

$$\Delta A^0 = -\int_0^{\tau^0}\int_{F^0} p_b^0 dy^0 dz^0 u_b^0 dt_b^0 = -\int_0^{\tau^0}\int_{F^0} g(y^0, z^0, t^0) dy^0 dz^0 \varphi'(t^0) dt^0. \tag{77}$$

Since the pressure is an invariant scalar, we now have in $\mathfrak{K}$

and

$$\left.\begin{aligned} p_a(y, z, t_a) &= p_a^0(y^0, z^0, t_a^0) = f(y^0, z^0, t_a^0) \\ p_b(y, z, t_a) &= g(y^0, z^0, t_b^0) \end{aligned}\right\} \tag{78}$$

with the relations (23)–(27) connecting the variables in $\mathfrak{K}$ and $\mathfrak{K}^0$. Instead of (34) we then get for the mechanical work performed by the wall a in $\mathfrak{K}$ during the process

$$\left.\begin{aligned}\varDelta A_a &= \int\limits_0^{\tau}\int\limits_F p_a(y,z,t_a)dydzu_a dt_a = \int\limits_0^{\tau_a}\int\limits_{F^0} f(y^0,z^0,t_a^0)dy^0dz^0\frac{vdt_a^0}{\sqrt{1-\beta^2}}\\ &= \frac{v}{\sqrt{1-\beta^2}}\int\limits_0^{\tau^0}\int\limits_{F^0} f(y^0,z^0,t^0)dy^0dz^0dt^0\\ &+\frac{\beta^2(p^0+\varDelta p^0)(V^0+\varDelta V^0)}{\sqrt{1-\beta^2}},\end{aligned}\right\}\qquad(79)$$

where we have made use of (27) and (30), (32).

Similarly we get for the work performed by b

$$\left.\begin{aligned}\varDelta A_b &= -\int\limits_0^{\tau}\int\limits_F p_b(y,z,t_b)dydzu_b dt_b\\ &= -\int\limits_{-\frac{vl^0}{c^2}}^{\tau}\int\limits_{F^0} g(y^0,z^0,t_b^0)\frac{v+\varphi'(t_b^0)}{\sqrt{1-\beta^2}}dt_b^0\\ &= -\frac{\beta^2p^0V^0}{\sqrt{1-\beta^2}}-\frac{v}{\sqrt{1-\beta^2}}\int\limits_0^{\tau^0}\int\limits_{F^0} g(y^0,z^0,t^0)dy^0dz^0dt^0+\frac{\varDelta A^0}{\sqrt{1-\beta^2}},\end{aligned}\right\}\qquad(80)$$

where we again have used (27) and the expression (77) for $\varDelta A^0$. The work performed by the cylinder walls is again zero, since the velocity of the fluid at these walls cannot have a component perpendicular to the wall. The total mechanical work in $\mathfrak{K}$ is therefore

$$\left.\begin{aligned}\varDelta A &= \varDelta A_a+\varDelta A_b\\ &= \frac{\beta^2\varDelta(p^0V^0)+\varDelta A^0}{\sqrt{1-\beta^2}}+\frac{v}{\sqrt{1-\beta^2}}\int\limits_0^{\tau^0}\int\limits_{F^0}[f(y^0,z^0,t^0)-g(y^0,z^0,t^0)]dy^0dz^0dt^0,\end{aligned}\right\}\qquad(81)$$

which deviates by the last term from the formula (38) holding for reversible processes.

We shall now calculate the mechanical impulse $\varDelta \boldsymbol{J}$ of the forces from the walls during the period of the irreversible process. The x-component of this vector is composed of the contributions from the walls a and b, i.e.,

$$\varDelta J_x = \varDelta J_a+\varDelta J_b \qquad (82)$$

with

$$\left.\begin{aligned}\Delta J_a &= \int_0^{\tau}\int_F p_a(y,z,t_a)\,dy\,dz\,dt_a\\ &= \int_0^{\tau_a^0}\int_{F^0} f(y^0,z^0,t_a^0)\,dy^0 dz^0 \frac{dt_a^0}{\sqrt{1-\beta^2}} = \frac{\Delta A_a}{v},\end{aligned}\right\} \quad (83)$$

$$\left.\begin{aligned}\Delta J_b &= -\int_0^{\tau}\int_F p_b(y,z,t_b)\,dy\,dz\,dt_b = -\int_{-\frac{vl^0}{c^2}}^{\tau_0}\int_{F^0} g(y^0,z^0,t_b)\,dy^0 dz^0 \frac{dt_b}{dt_b^0}\,dt_b^0\\ &= -\frac{vp^0V^0}{c^2\sqrt{1-\beta^2}} + \frac{v\Delta A^0}{c^2\sqrt{1-\beta^2}}\\ &\quad - \frac{1}{\sqrt{1-\beta^2}}\int_0^{\tau^0}\int_{F^0} g(y^0,z^0,t^0)\,dy^0 dz^0 dt^0.\end{aligned}\right\} \quad (84)$$

Here we have used (78), (76), (24), (25), and (77). Thus,

$$\left.\begin{aligned}\Delta J_x &= \Delta J_a + \Delta J_b\\ &= \frac{v[\Delta(p^0V^0)+\Delta A^0]}{c^2\sqrt{1-\beta^2}}\\ &\quad + \frac{1}{\sqrt{1-\beta^2}}\int^{\tau^0}\int_{F^0} [f(y^0,z^0,t^0) - g(y^0,z^0,t^0)]\,dt^0.\end{aligned}\right\} \quad (89)$$

If we let $\mathfrak{K} \to \mathfrak{K}^0$, (89) shows that

$$\Delta J_x^0 = \int_0^{\tau^0}\int_{F^0} [f(y^0,z^0,t^0) - g(y^0,z^0,t^0)]\,dt^0 \quad (90)$$

so that (89) becomes

$$\Delta J_x = \frac{\frac{v}{c^2}[\Delta(p^0V^0) + \Delta A^0] + \Delta J_x^0}{\sqrt{1-\beta^2}}. \quad (91)$$

The integral appearing in (81) is just the integral (90), and since

$$\boldsymbol{v} = \{v,0,0\}, \quad (92)$$

we have

$$(\boldsymbol{v}\cdot \varDelta \boldsymbol{J}^0) = v\varDelta J_x^0, \quad (v\cdot \varDelta \boldsymbol{J}) = v\varDelta J_x, \tag{93}$$

and (81) may therefore be written in the alternative forms

$$\varDelta A = \frac{\beta^2 \varDelta(p^0 V^0) + \varDelta A^0 + (\boldsymbol{v}\cdot \varDelta \boldsymbol{J}^0)}{\sqrt{1-\beta^2}} \tag{94}$$

or

$$\varDelta A = (\boldsymbol{v}\cdot \varDelta \boldsymbol{J}) + \varDelta A^0 \sqrt{1-\beta^2}. \tag{95}$$

In the latter form it is identical with the equation (62) for reversible processes.

For the components of $\varDelta \boldsymbol{J}$ in the y- and z-directions one easily finds that

$$\varDelta J_y = \varDelta J_y^0, \quad \varDelta J_z = \varDelta J_z. \tag{96}$$

To prove this one only uses that the pressure at the cylinder wall C is a scalar, i.e.,

$$p_c(x,y,z,t) = p_c^0(x^0,y^0,z^0,t^0) = f_c(x^0,y^0,z^0,t^0) \tag{97}$$

for values of $y = y^0$ and $z = z^0$ corresponding to points on C. In (97) the spacetime coordinates are connected by the Lorentz transformation and

$$f_c(x^0,y^0,z^0,t^0) = \begin{cases} p^0, & t^0 \leqslant 0 \\ p^0 + \varDelta p^0, & t^0 \geqslant \tau^0 \end{cases} \tag{98}$$

independently of the spatial coordinates. This has the effect that the integrals over the spacetime coordinates, which define $\varDelta J_y$ and $\varDelta J_y^0$ or $\varDelta J_z$ and $\varDelta J_z^0$, effectively are over the same domain in spacetime and, since the integrands are invariants, (96) follows.

By (92) the equations (91) and (96) may be comprised in the vector equation

$$\left.\begin{aligned} \varDelta \boldsymbol{J} = {} & \frac{\varDelta(p^0 V^0) + \varDelta A^0}{c^2\sqrt{1-\beta^2}}\,\boldsymbol{v} \\ & + \frac{(\boldsymbol{v}/v^2)(\boldsymbol{v}\cdot \varDelta \boldsymbol{J}^0)(1-\sqrt{1-\beta^2}) + \sqrt{1-\beta^2}\,\varDelta \boldsymbol{J}^0}{\sqrt{1-\beta^2}}. \end{aligned}\right\} \tag{99}$$

Now, (52) is of course also valid for irreversible processes with $\varDelta \boldsymbol{G}$ given by (73). In the system $\mathfrak{K}^0$ this gives

$$\varDelta \boldsymbol{J}^0 = -\varDelta \boldsymbol{G}^{(h)^0}. \tag{100}$$

Further, by (52), (73), and (99),

$$\Delta \boldsymbol{G}^{(h)} = \Delta \boldsymbol{G} - \Delta \boldsymbol{J} = \frac{\Delta H^0 - \Delta A^0}{c^2\sqrt{1-\beta^2}}\,\boldsymbol{v} - \Delta \boldsymbol{J}^0 - \boldsymbol{v}\,\frac{(\boldsymbol{v}\cdot\Delta \boldsymbol{J}^0)(1-\sqrt{1-\beta^2})}{v^2\sqrt{1-\beta^2}}$$

or, on account of (100),

$$\Delta \boldsymbol{G}^{(h)} = \Delta \boldsymbol{G}^{(h)^0} + \boldsymbol{v}\,\frac{(\boldsymbol{v}\cdot\Delta \boldsymbol{G}^{(h)^0})(1-\sqrt{1-\beta^2}) + \beta^2 \Delta Q^0}{v^2\sqrt{1-\beta^2}}, \tag{101}$$

where

$$\Delta Q^0 = \Delta H^0 - \Delta A^0 \tag{102}$$

is the transferred heat energy in $\mathfrak{K}^0$ in accordance with the first law (1). When we use the same law (3) in $\mathfrak{K}$, we now get for the transferred heat energy ΔQ in $\mathfrak{K}$, by (73) and (94),

$$\Delta Q = \Delta H - \Delta A = \frac{\Delta H^0 - \Delta A^0 - (\boldsymbol{v}\cdot\Delta \boldsymbol{J}^0)}{\sqrt{1-\beta^2}}$$

or, on account of (100) and (102),

$$\Delta Q = \frac{\Delta Q^0 + (\boldsymbol{v}\cdot\Delta \boldsymbol{G}^{(h)0})}{\sqrt{1-\beta^2}}. \tag{103}$$

Thus, if we want the first law of thermodynamics to be valid in its classical form in every Lorentz system $\mathfrak{K}$, then Ott's formula (8) holds only if we arrange it so that the heat supplied to the fluid does not carry any momentum in the rest system $\mathfrak{K}^0$. For a reversible process this is always the case, as we have seen in the preceding section. However, for an irreversible process Ott's equation (8) has in general to be replaced by the formula (103) which together with (101) has the following satisfactory consequence.

If we define a quantity ΔQ_i with the four components

$$\Delta Q_i = \left\{\Delta \boldsymbol{G}^{(h)}, \frac{i}{c}\Delta Q\right\}. \tag{104}$$

then the equations (101) and (103) show that the ΔQ_i transform as the components of a 4-vector under arbitrary Lorentz transformations[(8)]. This is in accordance with a general argument regarding the equivalence of mass and energy[(9)]. Thus, we have come to the conclusion that the momentum and energy which are transferred to a system by the supply of heat from outside form the components of a 4-vector. This result which has been obtained here by considering a very special system can be shown to be quite general[10].

On the other hand, it would not be in contradiction with the principle of relativity to change the form of the second law in $\mathfrak{K}$ if only it goes over into (1) for $\mathfrak{K} \to \mathfrak{K}^0$. This is actually what H. Ott proposes to do. Instead of defining the transferred heat energy ΔQ by (3), he defines a quantity $\Delta\bar{Q}$ in $\mathfrak{K}$ by the following equation

$$\Delta H = \Delta\bar{Q} + \boldsymbol{v}\left(\Delta \boldsymbol{G}^{(h)} - \frac{\Delta\bar{Q}}{c^2}\boldsymbol{v}\right) + \Delta A \tag{105}$$

which is identical with (1) for $\mathfrak{K} \to \mathfrak{K}^0$, i.e.,

$$\Delta\bar{Q}^0 = \Delta Q^0. \tag{106}$$

By means of (52) and (95), (105) may be written

$$\Delta H - \boldsymbol{v}\Delta\boldsymbol{G} = \Delta\bar{Q}(1-\beta^2) + \Delta A^0\sqrt{1-\beta^2} \tag{107}$$

which, by (73) and (1), gives the transformation formula

$$\Delta Q_o = \frac{\Delta Q^o}{\sqrt{1-\beta^2}} \tag{108}$$

of Ott. For reversible processes and for such irreversible processes where $\Delta\boldsymbol{G}^{(h)0} = 0$, there is no difference between $\Delta\bar{Q}$ and ΔQ in (41), but in general $\Delta\bar{Q}$ differs from ΔQ.

The definition (105) for $\Delta\bar{Q}$ obviously means a different splitting of the total energy into a "heat-part" and a "mechanical part", a procedure which in many cases may lead to ambiguities. One sometimes even speaks of production of heat in a body without any heat transfer from outside.

As an example of this kind of process, let us assume that the piston b in $\mathfrak{K}^0$ is moved violently back and forth until it stops after some time, for instance, in the original position. Then the fluid is "heated up" even if there is no heat supply from outside (compare the heating effect when pumping a bicycle). In this case, we have

$$\left.\begin{aligned} \Delta Q_i &= \Delta Q_i^0 = 0 \\ \Delta H &= \Delta A, \quad \Delta\boldsymbol{G} = \Delta\boldsymbol{J}. \end{aligned}\right\} \tag{109}$$

In a somewhat loose way of speaking, one then says that a certain amount of heat energy ΔQ^0_{prod} is produced in $\mathfrak{K}^0$ which simply is equal to the mechanical work ΔA^0 performed during the process or to the increase of the energy ΔH^0, i.e.,

$$\Delta Q^0_{\text{prod}} = \Delta H^0 = \Delta A^0. \tag{110}$$

What is now the heat produced relative to $\mathfrak{K}$? Besides performing the "internal work" which in $\mathfrak{K}^0$ is identified with ΔQ^0_{prod}, the walls of the container exert extra forces in $\mathfrak{K}$ which are necessary to keep the velocity at the same value $\boldsymbol{v}$ before and after the process. Their impulse $\Delta \boldsymbol{J}$ is equal to the increase $\Delta \boldsymbol{G}$ of the momentum, since we have no supply of heat momentum from outside. These forces perform an "external work" which is equal to $\Delta \boldsymbol{G} \cdot \boldsymbol{v}$. Therefore it is natural to define the heat energy produced in $\mathfrak{K}$ by

$$\Delta Q_{\text{prod}} = \Delta H - (\Delta \boldsymbol{G} \cdot \boldsymbol{v}). \tag{111}$$

By means of (73) this gives

$$\Delta Q_{\text{prod}} = \Delta H^0 \sqrt{1-\beta^2} = \Delta Q^0_{\text{prod}} \sqrt{1-\beta^2}. \tag{112}$$

The same equation follows from (109), (110) and the equation (95) which in our case reads

$$\Delta A = (\boldsymbol{v} \cdot \Delta \boldsymbol{G}) + \Delta Q^0_{\text{prod}} \sqrt{1-\beta^2}. \tag{113}$$

The formula (112) corresponds to the equation (5) of Planck, which up to recently was accepted also for the heat transferred from outside, but as we have seen, it is only valid for the heat produced inside a body under the action of external forces. We have here considered a case where the external forces (from the walls of the container) are mechanical forces, but as mentioned earlier, this is not essential. The main thing is only that they are real forces originating from outside the system. They could also be of electromagnetic nature and it would seem, therefore, that the old point of view of v. Laue[(11)] based on the formula (108) could be maintained in the case of the Joule heat produced in a conducting material under the influence of an external electric field (see also reference 10).

However, it should be emphasized that the notion of heat *produced* in a system during a process in general is a somewhat shady notion. As a matter of fact, it has given rise to many doubtful statements in the past. As is also apparent from the example considered above, the definition of heat production involves a more or less arbitrary splitting up of the increase ΔH

of the total energy in a heat part and a mechanical part. Perhaps it would be better to avoid the use of such ambiguous notions (except in certain well-defined cases). In contrast to the heat produced, the heat transferred to a system during a process, which is clearly defined by the first law of thermodynamics, has an unambiguous meaning, and, as we have seen, the momentum and energy transferred to a system by heat supply transform as the components of a 4-vector under Lorentz transformations.

Acknowledgments

This paper was written during my stay in Leiden as Lorentz Professor in the autumn of 1966. I want to express my sincere thanks to Professor P. Mazur and his collaborators for the most pleasant and inspiring time spent at the Instituut Lorentz.

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8. See f. inst. reference 7, Chap. IV, § 36, Eqs. (29).
9. See f. inst. reference 7, Chap. III, § 30.
10. See a paper by I. BREVIK to be published soon.
11. M. VON LAUE, loc. cit., reference 2.

Indleveret til Selskabet den 31. januar 1967.
Færdig fra trykkeriet den 27. maj 1967.