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A THEORY OF INTERFERENCE FILTERS

BY

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Introduction.

This paper describes the theory of interference filters of various types (especially interference filters with three and four silver layers, § 4 and § 5).

It will be shown how all the optical properties (such as reflectivity, transmission, phase changes at reflection, transmission, etc.) of an interference filter can be exactly calculated when the indices of refraction $\nu - i\varkappa$, n and the thicknesses t, d of the different thin layers are known as a function of the wavelength λ ($\nu - i\varkappa$ is the index of refraction and t the thickness of a metal layer). Furthermore relations are deduced between the optical constants of the reflective layers which give optimum conditions for the different types of filters.

In a following paper, it will be discussed how it is possible to measure the thicknesses of the dielectric layers on the filter base itself with an accuracy of about 20 Å and how such a filter can be made by means of the high-vacuum evaporation process for a filter area of 22×22 cm.

§ 1. Fresnel's Equations.

Reflection of light from and transmission through a boundary (fig. 1) between two materials 0 and 1 with indices of refraction n_0 and n_1 are determined by Fresnel's equations derived from the Maxwell equations of electrodynamics [1] & [2].

The following notations will be used:

φ angle of incidence, χ angle of refraction, and n index of refraction. s used as index means the component of the electric vector perpendicular to the plane of incidence and p used as index the component parallel to the plane of incidence.

χ is determined by SNELL's law:

$$n_0 \cdot \sin \varphi = n_1 \sin \chi. \tag{1, 1}$$

If (E_s, E_p) are the components of the electric vector of the incident plane light wave, the components of the electric vector

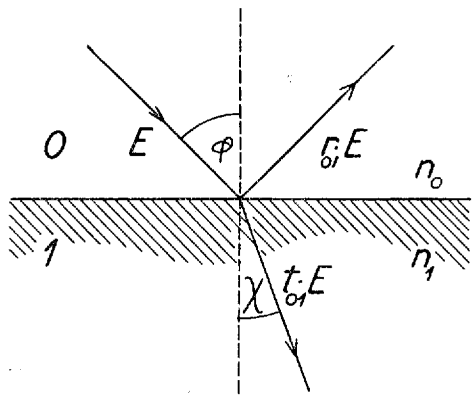

Fig. 1.

φ: Angle of incidence.
χ: Angle of refraction.
E: s or p component of electric vector of incident light wave.

of the reflected light wave $(E_s^{(R)}, E_p^{(R)})$ and of the transmitted wave $(E_s^{(T)}, E_p^{(T)})$ are determined by

$$E_s^{(R)} = E_s \cdot r_s; \quad r_s = \frac{n_0 \cos \varphi - n_1 \cos \chi}{n_0 \cos \varphi + n_1 \cos \chi} \tag{1, 2}$$

$$E_p^{(R)} = E_p \cdot r_p; \quad r_p = \frac{n_1 \cos \varphi - n_0 \cos \chi}{n_1 \cos \varphi + n_0 \cos \chi} \tag{1, 3}$$

$$E_s^{(T)} = E_s \cdot t_s; \quad t_s = 1 + r_s. \tag{1, 4}$$

$$E_p^{(T)} = E_p \cdot t_p; \quad t_p = (1 + r_p) \cdot \frac{n_0}{n_1}. \tag{1, 5}$$

The direction of the light is $0 \rightarrow 1$.

If the direction of the light is the opposite $1 \rightarrow 0$, n_0 must be interchanged with n_1 and φ with χ.

The following relations are satisfied:

$$r_{01} = -r_{10} \tag{1, 6}$$

$$t_{01} \cdot t_{10} - r_{01} \cdot r_{10} = 1 \tag{1, 7}$$

(valid either for the s or the p component).

At a normal angle of incidence ($\varphi = \chi = 0$) only one component of the electric vector is present, and the FRESNEL equations in this special case are the following:

$$r_{01} = \frac{n_0 - n_1}{n_0 + n_1}. \tag{1, 8}$$

$$t_{01} = 1 + r_{01}. \tag{1, 9}$$

Direction of the light: $0 \rightarrow 1$.

(The reason why $r_s = -r_p$ when $\varphi = 0$ is that $E_s = -E_p$ for the incident wave by definition [1]).

All these equations are also valid when the material 1 is absorbent (especially a metal). In this case the index of refraction n_1 is represented by a complex number $n_1 = \nu - i\varkappa$ and χ is a complex angle determined by (1, 1).

In accordance with [3] we define $a - ib = n_1 \cos \chi$; from (1, 1) we get $a - ib = \sqrt{(\nu - i\varkappa)^2 - n_0^2 \sin^2 \varphi} = i\sqrt{g + i \cdot 2\nu\varkappa}$ (with $g = \varkappa^2 + n_0^2 \sin^2 \varphi - \nu^2$), and from this equation we then obtain

$$b = \sqrt{\frac{1}{2}\left(g + \sqrt{g^2 + (2\nu\varkappa)^2}\right)}, \tag{1, 10}$$

and

$$a = \frac{\nu\varkappa}{b}. \tag{1, 11}$$

By introducing $n_1 \cos \chi = a - ib$ into (1, 2) and (1, 3) the FRESNEL equations can be written as follows:

$$r_s = \varrho_s \cdot e^{i\delta_s} = \frac{1 - \dfrac{1}{n_0 \cos \varphi}(a - ib)}{1 + \dfrac{1}{n_0 \cos \varphi}(a - ib)} \tag{1, 12}$$

TABLE 1. Angle of incidence $\varphi = 45°$.

a: $\varkappa$ \ ν	0.1	0.2	b: $\varkappa$ \ ν	0.0	0.1	0.2
1.0	0.08174	0.16402	1.0	1.2247	1.2234	1.2194
1.5	.09048	18114	1.5	1.6583	1.6578	1.6561
2.0	.09429	.18865	2.0	2.1213	2.1211	2.1203
2.5	.09623	.19249	2.5	2.5981	2.5979	2.5975
3.0	.09734	.19468	3.0	3.0822	3.0821	3.0819
3.5	.09803	.19605	3.5	3.5707	3.5707	3.5705
4.0	.09847	.19695	4.0	4,0620	4,0620	4,0619
4.5	.09879	.19759	4.5	4,5552	4,5552	4,5551
5.0	.09902	.19803	5.0	5.0498	5.0497	5.0497
5.5	.09918	.19837	5.5	5.5453	5.5453	5.5453
6.0	.09931	.19863	6.0	6.0415	6.0415	6.0415

c: $\varkappa$ \ ν	0.1	0.2	h: $\varkappa$ \ ν	0.0	0.1	0.2
1.0	0.10893	0.21819	1.0	0.8165	0.8165	0.8166
1.5	.10690	.21378	1.5	1.3568	1.3570	1.3578
2.0	.10476	.20945	2.0	1.8856	1.8858	1.8863
2.5	.10335	.20668	2.5	2.4056	2.4057	2.4061
3.0	.10245	.20490	3.0	2.9200	2.9201	2.9203
3.5	.10185	.20372	3.5	3.4307	3.4307	3.4309
4.0	.10146	.20291	4.0	3.9389	3.9390	3.9391
4.5	.10117	.20232	4.5	4.4455	4.4455	4.4456
5.0	.10096	.20191	5.0	4.9507	4.9508	4.9508
5.5	.10080	.20159	5.5	5.4551	5.4551	5.4551
6.0	.10068	.20134	6.0	5.9588	5.9588	5.9588

$$r_p = \varrho_p \cdot e^{i\delta_p} = -\frac{\left(1 - \dfrac{\cos\varphi}{n_0}(c - ih)\right)}{\left(1 + \dfrac{\cos\varphi}{n_0}(c - ih)\right)} \qquad (1, 13)$$

with

$$c - ih = \left(\frac{2\,b^2 - (\varkappa^2 - \nu^2)}{a^2 + b^2}\right) \cdot a - i\left(\frac{2\,a^2 + \varkappa^2 - \nu^2}{a^2 + b^2}\right) \cdot b. \quad (1, 14)$$

If the light wave with the angle of incidence φ coming from air ($n_0 = 1$) is first to pass under the angle of refraction ψ through

TABLE 2. Angle of incidence $\varphi = 60°$.

	a			b		
$\varkappa$ \ ν	0.1	0.2	$\varkappa$ \ ν	0.0	0.1	0.2
1.0	0.07569	0.15192	1.0	1.3229	1.3213	1.3165
1.5	.08664	.17349	1.5	1.7321	1.7313	1.7292
2.0	.09178	.18365	2.0	2.1794	2.1791	2.1780
2.5	.09450	.18904	2.5	2.6458	2.6456	2.6449
3.0	.09608	.19218	3.0	3.1225	3.1224	3.1220
3.5	.09708	.19416	3.5	3.6056	3.6055	3.6052
4.0	.09774	.19548	4.0	4.0927	4.0926	4.0925
4.5	.09820	.19640	4.5	4.5826	4.5825	4.5824
5.0	.09853	.19707	5.0	5.0744	5.0744	5.0743
5.5	.09878	.19756	5.5	5.5678	5.5678	5.5677
6.0	.09898	.19795	6.0	6.0622	6.0622	6.0621

	c			h		
$\varkappa$ \ ν	0.1	0.2	$\varkappa$ \ ν	0.0	0.1	0.2
1.0	0.10809	0.21680	1.0	0.7559	0.7555	0.7542
1.5	.10826	.21658	1.5	1.2990	1.2992	1.2998
2.0	.10625	.21249	2.0	1.8352	1.8355	1.8361
2.5	.10461	.20920	2.5	2.3623	2.3624	2.3628
3.0	.10346	.20692	3.0	2.8823	2.8823	2.8827
3.5	.10268	.20534	3.5	3.3975	3.3976	3.3978
4.0	.10211	.20422	4.0	3.9094	3.9095	3.9096
4.5	.10170	.20341	4.5	4.4189	4.4190	4.4191
5.0	.10141	.20280	5.0	4.9266	4.9267	4.9268
5.5	.10117	.20235	5.5	5.4331	5.4331	5.4332
6.0	.10099	.20199	6.0	5.9385	5.9385	5.9385

a dielectric layer with the index of refraction n (before reaching the boundary) (a, b) *and* (c, h) *will be unchanged* as g is unchanged. $(n_0 \cdot \sin\varphi = n \cdot \sin\psi)$ and in (1, 12) and (1, 13) we have only to change n_0 to n and $\cos\varphi$ to $\cos\psi$.

In Tables 1—3 (a, b) and (c, h) are given as functions of $(\nu, \varkappa)$ with angles of incidence $\varphi = 45^0, 60^0$, and 75^0, respectively, and with $n_0 = 1{,}0$ (only to be used for silver). From these tables it is apparent that for $\nu < 0.2$ it will be sufficient in most cases to use the approximation:

TABLE 3. Angle of incidence $\varphi = 75°$.

a			b			
$\varkappa \backslash \nu$	0.1	0.2	$\varkappa \backslash \nu$	0.0	0.1	0.2
1.0	0.07202	0.14457	1.0	1.3903	1.3886	1.3834
1.5	.08412	.16846	1.5	1.7841	1.7833	1.7808
2.0	.09006	.18023	2.0	2.2210	2.2206	2.2193
2.5	.09329	.18663	2.5	2.6801	2.6798	2.6791
3.0	.09519	.19041	3.0	3.1517	3.1515	3.1510
3.5	.09640	.19282	3.5	3.6308	3.6307	3.6304
4.0	.09721	.19442	4.0	4.1150	4.1149	4.1147
4.5	.09777	.19555	4.5	4.6025	4.6024	4.6023
5.0	.09819	.19637	5.0	5.0924	5.0923	5.0923
5.5	.09849	.19699	5.5	5.5842	5.5842	5.5841
6.0	.09873	.19746	6.0	6.0773	6.0772	6.0772

c			h			
$\varkappa \backslash \nu$	0.1	0.2	$\varkappa \backslash \nu$	0.0	0.1	0.2
1.0	10677	.21428	1.0	0.7193	0.7185	0.7163
1.5	.10874	.21758	1.5	1.2611	1.2612	1.2616
2.0	.10708	.21415	2.0	1.8010	1.8011	1.8017
2.5	.10538	.21076	2.5	2.3320	2.3321	2.3326
3.0	.10413	.20823	3.0	2.8556	2.8557	2.8560
3.5	.10321	.20641	3.5	3.3739	3.3739	3.3742
4.0	.10256	.20512	4.0	3.8882	3.8883	3.8885
4.5	.10208	.20416	4.5	4.3998	4.3998	4.4000
5.0	.10171	.20344	5.0	4.9092	4.9092	4.9093
5.5	.10144	.20288	5.5	5.4171	5.4171	5.4172
6.0	.10122	.20244	6.0	5.9237	5.9238	5.9238

$$\left.\begin{array}{c} b = \sqrt{g}; \quad a = \dfrac{\nu\varkappa}{\sqrt{g}}; \quad h = \dfrac{\varkappa^2}{\sqrt{g}} \quad \text{and} \quad c = \left(2 - \dfrac{\varkappa^2}{g}\right) a \\ (g = \varkappa^2 + n_0^2 \sin^2\varphi). \end{array}\right\} \tag{1, 15}$$

In the case of normal incidence ($\varphi = 0$) we obtain

$$r_{01} = \varrho_0 \cdot e^{i\delta_0} = \frac{1 - \left(\dfrac{\nu}{n_0} - i\dfrac{\varkappa}{n_0}\right)}{1 + \left(\dfrac{\nu}{n_0} - i\dfrac{\varkappa}{n_0}\right)}. \tag{1, 16}$$

TABLE 4. $180 - \delta_0$ (in degrees).

$\varkappa$ \ ν	0.0	0.1	0.2	0.3
1.0	90.000	89.713	88.850	87.423
1.1	84.547	84.289	83.517	82.235
1.2	79.611	79.380	78.690	77.547
1.3	75.137	74.931	74.317	73.300
1.4	71.075	70.892	70.346	69.444
1.5	67.381	67.218	66.732	65.931
1.6	64.011	63.866	63.435	62.723
1.7	60.931	60.803	60.419	59.785
1.8	58.109	57.995	57.652	57.088
1.9	55.517	55.415	55.109	54.605
2.0	53.130	53.039	52.765	52.314
2.1	50.927	50.845	50.599	50.194
2.2	48.888	48.814	48.594	48.229
2.3	46.997	46.930	46.732	46.403
2.4	45.240	45.180	45.000	44.703
2.5	43.603	43.548	43.386	43.117
2.6	42.075	42.026	41.878	41.634
2.7	40.647	40.601	40.467	40.244
2.8	39.308	39.267	39.144	38.941
2.9	38.051	38.014	37.902	37.716
3.0	36.870	36.836	36.732	36.563
3.1	35.757	35.726	35.632	35.475
3.2	34.708	34.679	34.592	34.449
3.3	33.717	33.690	33.610	33.478
3.4	32.779	32.754	32.680	32.558
3.5	31.891	31.868	31.800	31.686
3.6	31.048	31.027	30.964	30.859
3.7	30.248	30.228	29.170	29.072
3.8	29.487	29.469	29.414	29.324
3.9	28.763	28.746	28.695	28.611
4.0	28.072	28.057	28.009	27.930
4.1	27.414	27.399	27.355	27.281
4.2	26.785	26.771	26.730	26.661
4.3	26.184	26.171	26.132	26.068
4.4	25.609	25.596	25.560	25.500
4.5	25.058	25.046	25.012	24.955
4.6	24.529	24.519	24.487	24.433
4.7	24.023	24.013	23.983	24.932
4.8	23.537	23.527	23.499	23.451
4.9	23.069	23.060	23.033	22.988
5.0	22.620	22.611	22.586	22.544

Table 4 (continued).

$\varkappa$ \ ν	0.0	0.1	0.2	0.3
5.0	22.620	22.611	22.586	22.544
5.1	22.187	22.179	22.155	22.116
5.2	21.771	21.763	21.741	21.703
5.3	21.370	21.363	21.341	21.305
5.4	20.983	20.976	20.956	20.922
5.5	20.610	20.603	20.584	20.552
5.6	20.249	20.243	20.225	20.194
5.7	19.901	19.895	19.878	19.849
5.8	19.565	19.559	19.543	19.515
5.9	19.239	19.234	19.218	19.192
6.0	18.925	18.920	18.905	18.879

The Fresnel equations in reflection (1, 12—13—16) are all written in the following manner: $\varrho \cdot e^{i\delta} = \frac{1-(x-iy)}{1+(x-ix)}$ (x and y are positive numbers).

The reflectivity is

$$R = \varrho^2 = \frac{1+x^2+y^2-2x}{1+x^2+y^2+2x} \qquad (1, 17)$$

and the phase change δ at reflection is determined by

$$tg\,\delta = -\frac{2y}{x^2+y^2-1}. \qquad (1, 18)$$

To calculate (ϱ_0, δ_0) at normal incidence and (ϱ_s, δ_s); (ϱ_p, δ_p) at oblique incidence, mathematical tables of ϱ (1, 17) and δ (1, 18) as a function of (x, y) would have been of great value.

$$(0 < y < 20 \quad \text{and} \quad 0 < x < 2{,}0).$$

By calculation of ϱ intervals in x: 0.01 and in y: 0.1 and by calculation of δ intervals in x: 0.1 and in y: 0.01. However, such mathematical tables are not available.

In this paper only a small table of δ as a function of $(\nu, \varkappa)$ is given (Table 4).

When once r is calculated, $t = \xi \cdot e^{i\beta}$ can most easily be calculated from (1, 4—5—9) by means of *A Table for Use in the Addition of Complex Numbers* calculated by Jørgen Rybner and K. Steenberg Sørensen [4].

§ 2. "Fresnel's Factors" for a System of Thin Layers.

We consider a plane infinite incident wave of light; just before it reaches System I fig. 2 the s or p component of the electric vector at the point A we shall denote E_A (complex number).

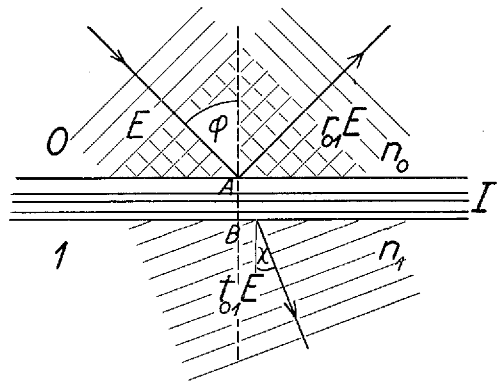

Fig. 2.
A system of thin layers I sandwiched between material 0 and material 1. System I may consist of one or more thin layers, the thickness of each being less than a few wavelengths of light.
E: s or p component of the electric vector.

After reflection from System I the component considered has now at the point A the value $E_A^{(R)}$ and after transmission through System I the value $E_B^{(T)}$ at the point B. We now define the Fresnel factors (r, t) for the system of thin layers I by the following:

$$r_{01} = \frac{E_A^{(R)}}{E_A} \qquad \text{and} \qquad t_{01} = \frac{E_B^{(T)}}{E_A}.$$

Direction of light: $0 \rightarrow 1$ and s and p components still considered separately (indices not written).

When the direction of the light is the opposite: $1 \rightarrow 0$ the Fresnel factors belonging to I are defined by

$$r_{10} = \frac{E_B^{(R)}}{E_B} \qquad \text{and} \qquad t_{10} = \frac{E_A^{(T)}}{E_B}.$$

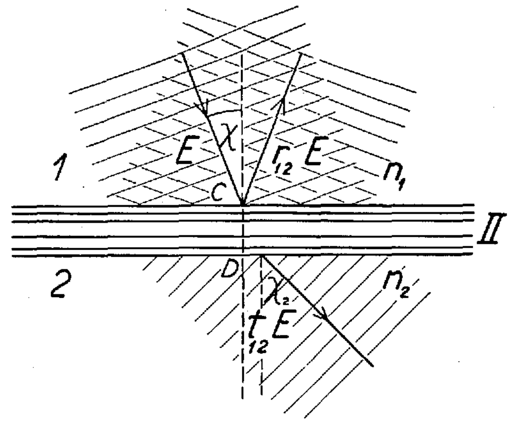

Fig. 3.
Another system of thin layers sandwiched between material 1 and material 2.

Next we consider a second system of thin layers II (fig. 3). The FRESNEL factors of this System II are defined in the same way as for System I by:

$$r_{12} = \frac{E_C^{(R)}}{E_C}; \quad t_{12} = \frac{E_D^{(T)}}{E_C} \quad \text{(direction of light: } 1 \rightarrow 2)$$

and

$$r_{21} = \frac{E_D^{(R)}}{E_D}; \quad t_{21} = \frac{E_C^{(T)}}{E_D} \quad \text{(direction of light: } 2 \rightarrow 1)$$

Now Systems I and II are combined to form a new system of thin layers I + II as shown on fig. 4.

It is now easy to express the FRESNEL factors $r_{02} = \frac{E_A^{(R)}}{E_A}$ and $t_{02} = \frac{E_D^{(T)}}{E_A}$ belonging to I + II by the FRESNEL factors r_{01}, t_{01}; r_{10}, t_{10} and r_{12}, t_{12}; r_{21}, t_{21}, belonging to Systems I and II, respectively.

If we consider the oscillations of the plane (infinite) wave which takes place in the layer between Systems I and II, we find by superposition of the wave systems in reflected light at point A directly from fig. 4 (by considering the plane wave front):

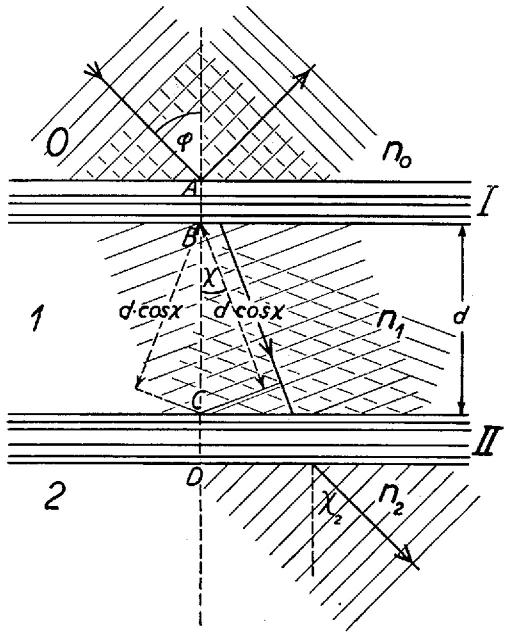

Fig. 4.
System I (fig. 2) and System II (fig. 3) combined to make a new System I + II. Material 1 forms a thin layer with thickness d between I and II.

$$E_A^{(R)} = E_A \cdot r_{02} = E_A \cdot r_{01} + E_A \cdot t_{01} \cdot e^{-i\frac{x}{2}} \cdot r_{12} \cdot e^{-i\frac{x}{2}} \cdot t_{10} +$$

$$E_A \cdot t_{01} \cdot e^{-i\frac{x}{2}} \cdot r_{12} \cdot e^{-i\frac{x}{2}} \cdot r_{10} \cdot e^{-i\frac{x}{2}} \cdot r_{12} \cdot e^{-i\frac{x}{2}} \cdot t_{10}$$

$$+ E_A \cdot t_{01} \cdot e^{-i\frac{x}{2}} \cdot r_{12} \cdot e^{-i\frac{x}{2}} \cdot r_{10} \cdot e^{-i\frac{x}{2}} \cdot r_{12} \cdot e^{-i\frac{x}{2}} \cdot r_{10} \cdot e^{-i\frac{x}{2}} \cdot r_{12} \cdot e^{-i\frac{x}{2}} \cdot t_{10}$$

$$+ \text{etc.} = E_A \cdot r_{01} + E_A \cdot t_{01} \cdot t_{10} \cdot r_{12} \cdot e^{-ix} \cdot \sum_{m=0}^{\infty} (r_{12} \cdot r_{10})^m \cdot e^{-imx}$$

$$= E_A \left(r_{01} + \frac{t_{01} \cdot t_{10} \cdot r_{12} \cdot e^{-ix}}{1 - r_{12} \cdot r_{10} \cdot e^{-ix}} \right).$$

From which follows:

$$r_{02} = \frac{r_{01} - r_{12}\,(r_{01} \cdot r_{10} - t_{01} \cdot t_{10}) \cdot e^{-ix}}{1 - r_{12} \cdot r_{10} \cdot e^{-ix}} \qquad (2, 1)$$

with

$$x = \frac{2\pi n_1}{\lambda} \cdot 2\,d\cos\chi = \frac{4\pi d n_1 \cos\chi}{\lambda} \tag{2, 2}$$

(derived directly from fig. 4).

In transmission we find in the same way by superposition of all the plane waves oscillating between I and II:

$$t_{02} = t_{01} \cdot t_{12} \cdot e^{-i\frac{x}{2}} \sum_{m=0}^{\infty} (r_{12} \cdot r_{10})^m \cdot e^{-imx},$$

from which follows:

$$t_{02} = \frac{t_{01} \cdot t_{12} \cdot e^{-i\frac{x}{2}}}{1 - r_{12} \cdot r_{10} \cdot e^{-ix}}. \tag{2, 3}$$

The reflectivity of System I + II (with direction of the incoming light $0 \to 2$) is

$$R_{02} = r_{02} \cdot \bar{r}_{02} \tag{2, 4}$$

($\bar{r}_{02}$ means the complex conjugate number of r_{02}).

The transmitted energy through I + II can be derived from Poynting's theorem of electrodynamics [1] to be

$$T'_{02} = t_{02} \cdot \bar{t}_{02} \cdot \frac{n_2 \cos\chi_2}{n_0 \cos\varphi}, \tag{2, 5}$$

where χ_2 is the angle of refraction in material 2.

To derive r_{20} and t_{20} we only have to interchange the indices 0 and 2 in (2, 1) and (2, 3).

The following relation is valid:

$$\frac{t_{02}}{t_{20}} = \frac{t_{01} \cdot t_{12}}{t_{10} \cdot t_{21}}. \tag{2, 6}$$

The fundamental formulae (2, 1—3) have been developed by Abelès [5] in much the same way as here by summing an infinite system of interfering wave systems. Recently, however, Ishiguro and Kato [6] have developed (2, 1—3) directly from the boundary conditions of electrodynamics by using a matrix representation. This rigorously proves that (2, 1—3) are valid also when material 1 is absorbent, with an index of refraction

$\nu - i\varkappa$. In this case we have to put

$$x = \frac{4\pi}{\lambda} \cdot d(a - ib),$$

where (a, b) are determined by (1, 10—11), and we obtain

$$e^{-ix} = e^{-\frac{4\pi d \cdot b}{\lambda}} \cdot e^{-i\frac{4\pi d \cdot a}{\lambda}}.$$

First we consider a special case of the fundamental equations (2, 1—3), where System I is only a boundary (all layers in

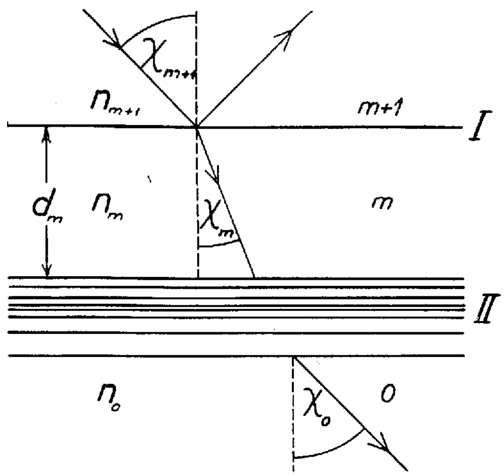

Fig. 5.
The direction of the light is the opposite of that used in fig. 4.

System I have zero thickness) and System II consists of $m - 1$ thin layers (fig. 5). In this special case we obtain from (1, 6—7)

$$r_{m+1,m} = -r_{m,m+1} \quad \text{and} \quad t_{m+1,m} \cdot t_{m,m+1} - r_{m+1,m} \cdot r_{m,m+1} = 1$$

and when this is introduced into (2, 1) and (2, 3) and when the notations $r_{12} = r_{m,0}$ and $t_{12} = t_{m,0}$ are used, we get the following fundamental recurrence formulae: (s or p component)

$$r_{m+1,0} = \frac{r_{m+1,m} + r_{m,0} \cdot e^{-ix_m}}{1 + r_{m+1,m} \cdot r_{m,0} \cdot e^{-ix_m}} \tag{2, 7}$$

$$t_{m+1,0} = \frac{t_{m+1,m} \cdot t_{m,0} \cdot e^{-i\frac{x_m}{2}}}{1 + r_{m+1,m} \cdot r_{m,0} \cdot e^{-ix_m}} \tag{2, 8}$$

$$x_m = \frac{4\pi d_m \cdot n_m \cdot \cos\chi_m}{\lambda}. \tag{2, 9}$$

$r_{m+1,m}$, $t_{m+1,m}$ are determined by (1, 2—3) (when $n_0 = n_{m+1}$ $n_1 = n_m$ $\chi_{m+1} = \varphi$ and $\chi_m = \chi$ are introduced), i. e.

$$r^{(s)}_{m+1,\,m} = \frac{n_{m+1} \cdot \cos\chi_{m+1} - n_m \cdot \cos\chi_m}{n_{m+1} \cdot \cos\chi_{m+1} + n_m \cdot \cos\chi_m} \tag{2, 10}$$

$$r^{(p)}_{m+1,\,m} = \frac{n_m \cdot \cos\chi_{m+1} - n_{m+1} \cdot \cos\chi_m}{n_m \cdot \cos\chi_{m+1} + n_{m+1} \cdot \cos\chi_m} \tag{2, 11}$$

$$t^{(s)}_{m+1,\,m} = 1 + r^{(s)}_{m+1,\,m} \tag{2, 12}$$

$$t^{(p)}_{m+1,\,m} = (1 + r^{(p)}_{m+1,\,m}) \cdot \frac{n_{m+1}}{n_m}. \tag{2, 13}$$

The reflectivity of the system is determined by:

$$R_{m+1,0} = r_{m+1,0} \cdot \bar{r}_{m+1,0}; \tag{2, 14}$$

the energy transmitted through the system is:

$$T'_{m+1,0} = t_{m+1,0} \cdot \bar{t}_{m+1,0} \cdot \frac{n_0 \cos\chi_0}{n_{m+1} \cdot \cos\chi_{m+1}}, \tag{2, 15}$$

and all other optical properties (such as phase change by reflection, transmission, etc.) of a system of m thin layers (absorbent or not) can be calculated when d_m and n_m are known.

From the fundamental formulae (2, 7—15) it is now easy to show that the following relations are valid:

$$\frac{1 + r^{(s)}_{m+1,0}}{t^{(s)}_{m+1,0}} = \frac{1 + r^{(s)}_{m,0} \cdot e^{-ix_m}}{t^{(s)}_{m,0} \cdot e^{-i\frac{x_m}{2}}} \quad (s \text{ and } 0 \text{ components}) \tag{2, 16 a}$$

$$\frac{1+r_{m+1,0}^{(p)}}{t_{m+1,0}^{(p)}} = \frac{1+r_{m,0}^{(p)}\cdot e^{-ix_m}}{t_{m,0}^{(p)}\cdot e^{-i\frac{x_m}{2}}}\cdot\frac{n_{m+1}}{n_m} \quad (p \text{ components}) \qquad (2,\,16\,\text{b})$$

and

$$\frac{1-r_{m+1,0}^{(s)}}{1+r_{m+1,0}^{(s)}} = \frac{n_m\cdot\cos\chi_m}{n_{m+1}\cdot\cos\chi_{m+1}}\cdot\left(\frac{1-r_{m,0}^{(s)}\cdot e^{-ix_m}}{1+r_{m,0}^{(s)}\cdot e^{-ix_m}}\right) \qquad (2,\,17\,\text{a})$$

$$\frac{1-r_{m+1,0}^{(p)}}{1+r_{m+1,0}^{(p)}} = \frac{n_{m+1}\cdot\cos\chi_m}{n_m\cdot\cos\chi_{m+1}}\cdot\left(\frac{1-r_{m,0}^{(p)}\cdot e^{-ix_m}}{1+r_{m,0}^{(p)}\cdot e^{-ix_m}}\right). \qquad (2,\,17\,\text{b})$$

Further, if the upper layer m (with thickness d_m) is transparent (dielectric layer), we get

$$\left.\begin{aligned} & n_{m+1}\cdot\cos\chi_{m+1}\cdot\frac{(1-\bar{r}_{m+1,0}\cdot r_{m+1,0})}{\bar{t}_{m+1,0}\cdot t_{m+1,0}} = \\ & \qquad n_m\cdot\cos\chi_m\cdot\frac{(1-\bar{r}_{m,0}\cdot r_{m,0})}{t_{m,0}\cdot\bar{t}_{m,0}} \end{aligned}\right\} \qquad (2,\,18\,\text{a})$$

(valid both for s and p components).

By means of (2, 14—15) this can be expressed by

$$\frac{1-R_{m+1,0}}{T'_{m+1,0}} = \frac{1-R_{m,0}}{T'_{m,0}}, \qquad (2,\,18\,\text{b})$$

i. e. *if to a system of thin layers (absorbent or not) is added one or more transparent layers,* $\frac{1-R}{T'}$ *remains constant.* This theorem has first been proved to be generally valid by F. Abelès [7].

For only one layer we deduce from (2, 6) and (2, 10—13)

$$t_{20} = t_{02}\cdot\frac{n_2\cos\chi_2}{n_0\cos\chi_0},$$

and by induction we get generally for a system of $m-1$ layers

$$t_{m,0} = t_{0,m}\cdot\frac{n_m\cdot\cos\chi_m}{n_0\cdot\cos\chi_0}. \qquad (2,\,19)$$

The transmission through the system is (2, 15)

$$T'_{m,0} = t_{m,0} \cdot \bar{t}_{m,0} \cdot \frac{n_0 \cos \chi_0}{n_m \cdot \cos \chi_m} = t_{0,m} \cdot \bar{t}_{0,m} \cdot \frac{n_m \cdot \cos \chi_m}{n_0 \cdot \cos \chi_0} = T'_{0,m} \quad (2, 20)$$

i. e. *at a system of thin layers the transmission remains the same if the direction of the light is reversed.* (The layers can be absorbent or not, the material above and below the system of thin layers must not be absorbent). Other general proofs of this theorem have been given by MAYER [2] and ABELÈS [7].

§ 3. Interference Filters with *Two* Systems of Reflective Layers I and II. (Spec. two silver layers).

An interference filter can very generally be defined as a thin dielectric layer enclosed between two strong reflective systems of thin layers I and II (fig. 6).

We now make the assumption that the reflectivity and the transmission (and phase change by reflection or transmission) of each

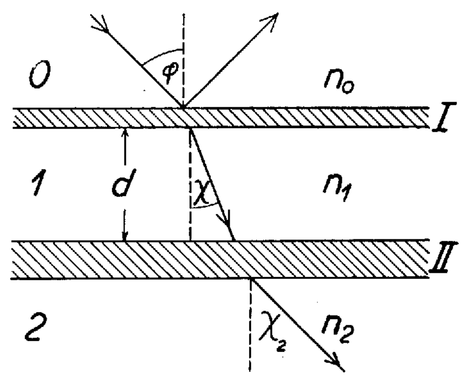

Fig. 6.

of the systems I and II considered separately, only show a small variation with the wavelength $\left(\text{within each spectral region of } \frac{\lambda}{10}\right)$ or expressed more simply: I or II must not be interference filters themselves.

For this simple type of interference filters many general properties can be derived directly from (2, 1—3).

If in (2, 1—3) the following substitutions are made:

$$r_{01} = \sqrt{R_{01}} \cdot e^{i\delta_{01}} \quad r_{10} = \sqrt{R_{10}} \cdot e^{i\delta_{10}} \quad r_{12} = \sqrt{R_{12}} \cdot e^{i\delta_{12}} \quad \text{and}$$

$$t_{01} = \sqrt{T_{01}} \cdot e^{i\beta_{01}} \quad t_{10} = \sqrt{T_{10}} \cdot e^{i\beta_{10}} \quad t_{12} = \sqrt{T_{12}} \cdot e^{i\beta_{12}}$$

(δ is the phase change at reflection and β the phase change at transmission) and if we further introduce

$$\sigma \cdot e^{i\alpha} = 1 - \frac{t_{01} \cdot t_{10}}{r_{01} \cdot r_{10}} = 1 - \sqrt{\frac{T_{01} \cdot T_{10}}{R_{01} \cdot R_{10}}} \cdot e^{i(\beta_{01} + \beta_{10} - \delta_{01} - \delta_{10})}. \tag{3, 1 a}$$

$$\boldsymbol{R} = \sqrt{R_{10} \cdot R_{12}} \tag{3, 1 b}$$

and

$$y = x - \delta_{10} - \delta_{12}, \tag{3, 1 c}$$

we get the following general formulae:

$$r_{02} = \sqrt{R_{02}(\lambda)} \cdot e^{i\delta_{02}(\lambda)} = \frac{\sqrt{R_{01}}\left(1 - \sigma \boldsymbol{R} \cdot e^{-i(y-\alpha)}\right) e^{i\delta_{01}}}{\left(1 - \boldsymbol{R} \cdot e^{-iy}\right)} \tag{3, 2}$$

$$t_{02} = \sqrt{T_{02}(\lambda)} \cdot e^{i\beta_{02}(\delta)} = \frac{\sqrt{T_{01} \cdot T_{12}} \cdot e^{-i\left(\frac{x}{2} - \beta_{01} - \beta_{12}\right)}}{\left(1 - \boldsymbol{R} \cdot e^{-iy}\right)}. \tag{3, 3}$$

The intensity of the reflected light $R_{02}(\lambda)$ and of the transmitted light $I(\lambda)$ (in proportion to the intensity of the incident light $I_s^{(0)} = 1$ or $I_p^{(0)} = 1$; s and p components are treated separately) are from (3, 2—3) and (2, 4—5) determined to be:

$$R_{02}(\lambda) = \frac{R_{01}\left(1 - 2\,\sigma \boldsymbol{R} \cdot \cos(y - \alpha) + (\sigma \boldsymbol{R})^2\right)}{\left(1 - 2\,\boldsymbol{R} \cos y + \boldsymbol{R}^2\right)} \tag{3, 4}$$

(direction of light: $0 \rightarrow 2$),
and if we define

$$\boldsymbol{T}_1 = \frac{n_1 \cdot \cos\chi}{n_0 \cos\varphi} \cdot T_{01} \quad \text{and} \quad \boldsymbol{T}_2 = \frac{n_2 \cdot \cos\chi_2}{n_1 \cos\chi} \cdot T_{12}$$

we get the intensity distribution for the transmitted light

$$I(\lambda) = \frac{\boldsymbol{T}_1 \cdot \boldsymbol{T}_2}{1 - 2\,\boldsymbol{R} \cos y + \boldsymbol{R}^2} \tag{3, 5 a}$$

or written in a more convenient manner

$$I(\lambda) = \frac{\boldsymbol{T}_1 \cdot \boldsymbol{T}_2}{(1 - \boldsymbol{R})^2} \frac{1}{\left(1 + \frac{4\,\boldsymbol{R}}{(1 - \boldsymbol{R})^2} \cdot \sin^2 \frac{y}{2}\right)} \tag{3, 5 b}$$

($I(\lambda)$ will according to (2, 20) be the same if the direction of the light is reversed).

The wavelengths λ_m at which $I(\lambda)$ reach a maximum are determined by

$$y = 360^0\,(m - 1) \qquad m = 1, 2, 3 \,\ldots\ldots$$

and the wavelengths $\lambda_{m+\frac{1}{2}}$ at which $I(\lambda)$ becomes a minimum by

$$y = 180^0\,(2\,m - 1) \qquad m = 1, 2, 3 \,\ldots\ldots$$

The Determination of $y(\lambda)$ or of $\lambda(y)$.

This determination is important in calculating λ_m and $I(\lambda)$ in the neighbourhood of λ_m.

$$y = \frac{360}{\lambda} \cdot 2\,dn_1 \cdot \cos\chi - (\delta_{10} + \delta_{12}) \quad \left\{ \begin{array}{l} \text{(when } y \text{ is mea-} \\ \text{sured in degrees)} \end{array} \right\} \tag{3, 6 a}$$

or

$$\frac{360 + y}{360} = \frac{1}{\lambda} \cdot \left(2\,dn_1 \cos\chi + \frac{(360 - (\delta_{10} + \delta_{12}))}{360} \cdot \lambda\right) \tag{3, 6 b}$$

δ_{01} and δ_{12} are dependent on the wavelength λ.
We now define

$$Z(\lambda) = \left(\frac{360 - \delta_{10}(\lambda) - \delta_{12}(\lambda)}{360}\right) \cdot \lambda. \tag{3, 7}$$

By introducing $Z(\lambda)$ and λ_m (corresponding to $y = 360 \cdot (m - 1)$) into (3, 6 b) we get the following fundamental equations:

$$m\lambda_m = 2\,dn_1 \cdot \cos\chi + Z(\lambda_m) \tag{3, 8}$$

and

$$\frac{360 + y}{360} = \frac{1}{\lambda}\left(m \cdot \lambda_m + Z(\lambda) - Z(\lambda_m)\right). \tag{3, 9}$$

If I and II (fig. 6) each consist of only one silver layer, $Z(\lambda)$ will only show a small dependence upon λ (fig. 8 below). If, however, I and II consist of a combination of one silver layer and several quarter wavelength layers of dielectrics with low and high index of refraction or of $\frac{\lambda}{4}$-dielectric layers alone, the dependence of $\delta_{10} + \delta_{12}$ upon λ in the neighbourhood of λ_m must in each case be calculated by means of (2, 7) and next $y(\lambda)$ by (3, 6 a). (The results of such calculations of $y(\lambda)$ are shown in fig. 49 p. 86 and in fig. 51 p. 90).

From (3, 5—9) we are now able to calculate $I(\lambda)$ in detail. However, it will often be sufficient to describe $I(\lambda)$ by means of the following quantities:

1. The values of λ_m (from (3, 8) or direct from (3, 6 a));

2. The values of

$$I_{\max} = I(\lambda_m) = \frac{\boldsymbol{T}_1(\lambda_m) \cdot \boldsymbol{T}_2(\lambda_m)}{(1 - \boldsymbol{R}(\lambda_m))^2}; \tag{3, 10}$$

3. The contrast factor

$$F = \frac{I_{\max}}{I_{\min}} = \frac{I(\lambda_m)}{I\left(\lambda_{m+\frac{1}{2}}\right)} = \frac{\boldsymbol{T}_1(\lambda_m) \cdot \boldsymbol{T}_2(\lambda_m)}{\boldsymbol{T}_1\left(\lambda_{m+\frac{1}{2}}\right) \cdot \boldsymbol{T}_2\left(\lambda_{m+\frac{1}{2}}\right)} \cdot \left(\frac{1 + \boldsymbol{R}\left(\lambda_{m+\frac{1}{2}}\right)}{1 - \boldsymbol{R}(\lambda_m)}\right)^2. \tag{3, 11 a}$$

If R and T with sufficient accuracy are independent of λ $\left(\text{within the wavelength region } \lambda_{m+\frac{1}{2}} < \lambda < \lambda_{m-\frac{1}{2}}\right)$ the contrast factor is simply expressed by

$$F = \frac{I_{\max}}{I_{\min}} = \left(\frac{1 + \boldsymbol{R}}{1 - \boldsymbol{R}}\right)^2; \quad (\boldsymbol{R} = \sqrt{R_{10} \cdot R_{12}}). \tag{3, 11 b}$$

4. The half intensity band width W_2 defined by

$$I\left(\lambda_m \pm \frac{W_2}{2}\right) = \frac{1}{2} \cdot I(\lambda_m)$$

and the tenth intensity band width W_{10} defined by

$$I\left(\lambda_m \pm \frac{W_{10}}{2}\right) = \frac{1}{10} \cdot I(\lambda_m).$$

If we introduce $y = 360^0 (m-1) + \gamma_k$ (where γ_k is a small angle corresponding to the k'th intensity band width W_k), we find (from (3, 5 b))

$$I\left(\lambda_m \pm \frac{W_k}{2}\right) = \frac{\boldsymbol{T}_1 \cdot \boldsymbol{T}_2}{(1-\boldsymbol{R})^2} \cdot \frac{1}{\left(1 + \frac{4\boldsymbol{R}}{(1-\boldsymbol{R})^2} \sin^2\left(\frac{\gamma_k}{2}\right)\right)}$$

and this is by definition equal to

$$\frac{1}{k} \cdot I(\lambda_m) = \frac{1}{k} \cdot \frac{\boldsymbol{T}_1 \boldsymbol{T}_2}{(1-\boldsymbol{R})^2}.$$

So we obtain

$$\frac{4\boldsymbol{R}}{(1-\boldsymbol{R})^2} \sin^2\left(\frac{\gamma_k}{2}\right) = k-1 \quad \text{and} \quad \sin\frac{\gamma_k}{2} = \frac{(1-\boldsymbol{R})\sqrt{1-k}}{2\sqrt{\boldsymbol{R}}} \qquad (3, 12\,\text{a})$$

or approximately

$$\gamma_k = \frac{180}{\pi} \cdot \frac{(1-\boldsymbol{R})\sqrt{k-1}}{\sqrt{\boldsymbol{R}}} \qquad (3, 12\,\text{b})$$

degree.

In the neighbourhood of $\lambda = \lambda_m$ we have approximately

$$y(\lambda) = 360^0 \cdot \left(m - 1 - f \cdot \frac{(\lambda - \lambda_m)}{\lambda_m}\right), \text{ where } f = -\left(\frac{dy}{d\lambda}\right)_{\lambda=\lambda_m} \cdot \frac{\lambda_m}{360}, \quad (3,13\text{a})$$

and as $y = 360 \cdot (m-1) + \gamma_k$ corresponds to $\lambda_m - \lambda = \frac{W_k}{2}$ we obtain $\frac{\gamma_k}{180} = \frac{f \cdot W_k}{\lambda_m}$, and from (3, 12 b) we finally get

$$W_k = \frac{\lambda_m}{f} \cdot \frac{(1-\boldsymbol{R})}{\pi \cdot \sqrt{\boldsymbol{R}}} \cdot \sqrt{1-k},$$

and in particular we get

$$W_2 = \frac{\lambda_m}{f} \cdot \frac{(1-\boldsymbol{R})}{\pi \cdot \sqrt{\boldsymbol{R}}} \tag{3, 14}$$

and

$$W_{10} = 3 \cdot W_2. \tag{3, 15}$$

When I and II (fig. 6) are silver layers we have approximately $y(\lambda) = 360 \cdot \left(\frac{m \cdot \lambda_m}{\lambda} - 1\right)$ according to p. 21 and in this case we simply get

$$f = m \qquad (m = 1, 2, 3 \ldots\ldots).$$

In case of filters where I and II (fig. 6) consist of several layers f will be different from an integer.

If the mean reflectivity $\boldsymbol{R} = \sqrt{R_{10} \cdot R_{12}}$ is increased W_2 (3, 14) will decrease and $F = \frac{I_{\max}}{I_{\min}}$ (3, 11) will increase. However, because of absorption in I and II (fig. 6) $I_{\max}$ (3, 10) will rapidly decrease. If we assume that the absorption in both I and II is $A = (A_{10} = A_{12})$, $I_{\max}$ is expressed by:

$$I_{\max} = \frac{(1 - R_{10} - A)(1 - R_{12} - A)}{(1 - \sqrt{R_{10} \cdot R_{12}})^2}. \tag{3, 16}$$

It is now easy to show that for a definite (constant) value of $\boldsymbol{R} = \sqrt{R_{10} \cdot R_{12}}$ (i. e. for a definite value of the contrast factor) $I_{\max}$ will reach its highest value when $R_{12} = R_{10} = (\boldsymbol{R})$ (i. e. when I and II have the same reflectivity; for a definite (constant) value of R_{12}, however, $I_{\max}$ will reach its highest value when $R_{10} = R_{12} \cdot (1 - A)^2$).

From the above considerations it is obvious that one of the greatest problems in producing interference filters is that of finding a material (consisting of one or more thin layers) with a sufficiently high reflectivity R throughout a spectral region of reasonable length (as great as possible).

In most applications an interference filter is used at normal incidence ($\varphi = 0$). A small deviation $\Delta\varphi$ from parallelism of the incident light gives rise to a shift, say $\Delta\lambda_m$, towards violet according to (3, 8). If $\dfrac{Z(\lambda_m) - Z(\lambda_m + \Delta\lambda_m)}{m}$ is sufficiently small we get

5. $$\Delta\lambda_m = \frac{1}{2}\frac{(\Delta\varphi)^2}{n_1^2}\cdot\left(\lambda_m - \frac{Z(\lambda_m)}{m}\right). \qquad (3, 17)$$

However, the dependence of Z upon wavelength often makes this calculation of $\Delta\lambda_m$ more difficult.

At an oblique angle of incidence all the formulae (3, 1—16) must be written separately for s and p components. (3, 8) especially will split up into

$$m \cdot \lambda_m^{(s)} = 2\, dn_1 \cos\chi + Z_s(\lambda_m^{(s)}) \qquad (3, 18\text{ a})$$

and

$$m\lambda_m^{(p)} = 2\, dn_1 \cos\chi + Z_p(\lambda_m^{(p)}). \qquad (3, 18\text{ b})$$

Because of the difference in δ_s and δ_p (evident from (1, 12—14)) Z_s and Z_p will usually be unequal and result in a splitting up into two transmission peaks at $\lambda_m^{(s)}$ and $\lambda_m^{(p)}$, respectively, the one polarized perpendicular upon and the other parallel to the plane of incidence.

At an oblique angle of incidence φ a small deviation $\Delta\varphi$ in the angle of incidence will give rise to a shift in wavelength of $\Delta\lambda_m$ determined by means of the derivate of (3, 8):

$$\Delta\lambda_m = -\frac{\left(\lambda_m - \dfrac{Z(\lambda_m)}{m}\right)\sin 2\varphi}{2\left(\left(\dfrac{n}{n_0}\right)^2 - \sin^2\varphi\right)}\cdot\Delta\varphi \qquad (3, 19)$$

(Z regarded as constant and angle of incidence φ in material with index of refraction n_0).

It should be noticed that this is a first order deviation in $\Delta\varphi$ as opposed to (3, 17) at $\varphi = 0$.

The Properties of the Filter (fig. 6) in Reflection.

If no absorption takes place in I and II we have

$$I(\lambda) + R_{02}(\lambda) = 1 \quad \text{(conservation of energy)}$$

(in this special case we have $R_{02} = R_{20}$ according to (2, 20)).

From (3, 4—5 a) we get

$$I(\lambda) + R_{02}(\lambda) = \frac{(1-R_{10})(1-R_{12})}{1-2\boldsymbol{R}\cos y+\boldsymbol{R}^2} + \frac{R_{01}(1-2\sigma\boldsymbol{R}\cdot\cos(y-\alpha)+(\sigma\boldsymbol{R})^2)}{1-2\boldsymbol{R}\cos y+\boldsymbol{R}^2}$$

with $\boldsymbol{R} = \sqrt{R_{10}\cdot R_{12}}$.

This will only be equal to 1 if

$$\alpha = 0 \quad \text{and} \quad \sigma = \frac{1}{R_{10}}\left(= \frac{1}{R_{01}}\right) \tag{3, 20}$$

as $\sigma \cdot e^{i\alpha}$ at the same time must satisfy (3, 1 a).

With a thin metal layer (such as *Al* or *Ag*) absorption takes place in I and then α will no longer be zero, but with layers which are almost opaque α will only have a small negative value (less than one degree). In this case we get as a first approximation

$$\sigma = \frac{1-A}{R_{01}} \quad (A \text{ absorption in I}). \tag{3, 21}$$

The condition for obtaining $R_{02}(\lambda) = 0$ at a definite wavelength is (from (3, 4))

$$1 - 2\sigma\boldsymbol{R}\cos(y-\alpha) + (\sigma\boldsymbol{R})^2 = 0$$

or

$$\cos(y-\alpha) = \frac{1+(\sigma\boldsymbol{R})^2}{2\sigma\boldsymbol{R}}.$$

This quantity is always $\geqslant 1$, i. e. $R_{02}(\lambda) = 0$ only if

$$\underline{\sigma\boldsymbol{R} = 1} \qquad (\boldsymbol{R} = \sqrt{R_{10}\cdot R_{12}}), \tag{3, 22}$$

and this takes place at

$$y = \alpha^\circ + 360^\circ(m-1).$$

If I is not absorbent

$$\sigma \boldsymbol{R} = \frac{1}{R_{10}} \cdot \sqrt{R_{10} \cdot R_{12}} = \sqrt{\frac{R_{12}}{R_{10}}};$$

in this case the condition for zero intensity of $R_{02}(\lambda)$ is $R_{12} = R_{10}$ (it is immaterial whether II is absorbent or not).

If I is absorbent and $R_{10} = R_{12}$ no value of λ exist at which $R_{02}(\lambda) = 0$; however, it is possible to find a value of R_{10} at which the equation (3, 22) is satisfied (as a first approximation we get $R_{01} = (1 - A)^2 \cdot R_{12})$. In this case $R_{02}(\lambda) = 0$ is satisfied at wavelengths determined by $y = \alpha + 360 \cdot (m - 1)$ in combination with (3, 9). The wavelength at which the maximum of transmission occurs is determined by $y = 360 \cdot (m - 1)$. Hence it follows that a small difference results between $\lambda_{\max}$ in transmission and $\lambda_{\min}$ in reflection. The same will be the case if $R_{10} = R_{12}$ and I is absorbent.

The FABRY-PEROT *Filter* $ML_{2m}M$.

The simplest of the types of interference filters treated above is the FABRY-PEROT filter which simply consists of two silver layers M with a dielectric layer L_{2m} in between. (L_{2m} means a $2\,m \cdot \frac{\lambda}{4}$ layer). The name of this filter originates from the fact that the filter is a Fabry-Perot interferometer with a very small spacing between the reflecting surfaces. The first production and description of filters of this type are due to GEFFCKEN [8].

It is important to note that the filter blank need not be more accurately polished than an ordinary optical surface as the different thin layers all follow the irregularities of the blank. (It is unnecessary to use an interferometer plate as filter blank).

In order to calculate the properties of this filter the first step is to make numerical calculations of R, T, δ, β, σ, α ....., etc., for silver layers of different thicknesses t and at different wavelengths λ.

The reflectivity R_∞ and the phase change δ_0 at the boundary between an opaque silver layer and a dielectric with an index of refraction n_0 are determined (at normal incidence) by (1, 17—18) when we put $x - iy = \frac{\nu}{n_0} - i\frac{\varkappa}{n_0}$. When n_0 increases R_∞ decreases, which means that only dielectrics with a low index of refraction can be used for interference filters of the type $ML_{2m}M$. (L means a $\frac{\lambda}{4}$-layer with *low* index of refraction).

In the numerical calculations (the results of which are given in Tables 6—13) it is assumed that $n = 1.36$ on both sides of the silver layer (fig. 7). In practice one side of the silver layer is bounded by glass (or cement) with $n = 1.5$; the influence upon R_{10} and T will, however, be small as compared with the experimental uncertainty in determining $\nu - i\varkappa$. $n_0 = 1.36$ has been chosen because this corresponds to the index of refraction obtained by slow evaporation of cryolite. The index of refraction of MgF_2 too is only slightly higher than 1.36.

The different values of $\nu - i\varkappa$ published [2] vary greatly, depending partly upon the conditions by producing the silver layers and partly upon the different optical methods by which $\nu - i\varkappa$ is measured. The most reliable values of $\varkappa$ seem to be those published by SCHULZ [9], which were determined by measurements of δ_0 at the boundary between the air and a nearly opaque silver layer. Furthermore from accurate measurements of R and T at nearly opaque silver layers published by KUHN [10] ν can be calculated from $R_\infty = R + T$. We get

$$\nu = \frac{1.02 + \varkappa^2}{2} \cdot \left(\frac{1 - R_\infty}{1 + R_\infty}\right)$$

when as a first approximation $\nu^2 = 0.02$ is adopted.

The values of $\nu - i\varkappa$ employed in the following numerical calculations are given in Table 5.

TABLE 5.

λ	ν	$\varkappa$	R_∞
3800	0.20	1.77	0.82
4000	0.18	1.95	0.86
4500	0.14	2.42	0.92
5000	0.14	2.89	0.94
5500	0.15	3.36	0.95
6000	0.15	3.82	0.96
6560	0.13	4.27	0.97
7100	0.14	4.68	0.97
7680	0.15	5.11	0.98

TABLE 6. $\lambda = 4000$ Å $\nu - i\varkappa = 0.18 - i\ 1.95$.

t in Å	R	T	A	$\left(\frac{T}{1-R}\right)^2$	$\left(\frac{1+R}{1-R}\right)^2$	$180-\delta^\circ$	β°	σ	$-\alpha^\circ$
150	.1852	.7179	.0969	.7762	2.12	75°13	7°38	4.8495	11°92
200	.2881	.5956	.1163	.6999	3.27	73.44	8.79	3.0427	10.48
250	.3889	.4809	.1302	.6194	5.17	72.16	9.72	2.2145	8.98
300	.4802	.3801	.1397	.5347	8.11	71.23	10.24	1.7719	7.53
350	.5585	.2954	.1461	.4477	12.5	70.56	10.43	1.5118	6.21
400	.6231	.2266	.1503	.3615	18.6	70.11	10.36	1.3489	5.05
500	.7155	.1297	.1548	.2080	36.4	69.61	9.66	1.1706	3.25
550	.7469	.0972	.1559	.1475	47.6	69.49	9.13	1.1211	2.57
600	.7709	.0725	.1566	.1001	59.8	69.42	8.52	1.0865	2.03
∞	.8414	.00	.1586	.00		69.48			

TABLE 7. $\lambda = 4500$ Å $\nu - i\varkappa = 0.14 - i\ 2.42$.

t in Å	R	T	A	$\left(\frac{T}{1-R}\right)^2$	$\left(\frac{1+R}{1-R}\right)^2$	$180-\delta$	β	σ	$-\alpha$
150	.2560	.6662	.0778	.8017	2.85	70.29	14.39	3.5903	7.69
200	.3838	.5266	.0896	.7304	5.04	67.08	17.40	2.3613	4.66
250	.4997	.4042	.0961	.6527	9.00	64.64	19.57	1.7997	5.18
300	.5970	.3038	.0992	.5683	15.7	62.84	21.06	1.5012	4.11
350	.6748	.2250	.1002	.4785	26.5	61.54	22.00	1.3271	3.22
400	.7351	.1649	.1000	.3876	42.9	60.62	22.53	1.2191	2.50
500	.8150	.0869	.0981	.2204	96.2	59.57	22.74	1.1031	1.47
∞	.9060	.00	.0940	.00		58.55			

TABLE 8. $\lambda = 5000$ Å $\nu - i\varkappa = 0.14 - i\ 2.89$.

t in Å	R	T	A	$\left(\frac{T}{1-R}\right)^2$	$\left(\frac{1+R}{1-R}\right)^2$	$180-\delta$	β	σ	$-\alpha$
150	.3290	.5955	.0755	.7878	3.92	64.56	20.62	2.8009	6.21
200	.4717	.4456	.0827	.7114	7.76	60.35	24.63	1.9374	4.88
250	.5902	.3247	.0851	.6279	15.1	57.29	27.34	1.5439	3.81
300	.6825	.2328	.0847	.5375	28.1	55.12	29.12	1.3363	2.825
350	.7516	.1652	.0832	.4422	49.7	53.61	30.49	1.2160	2.119
400	.8022	.1165	.0813	.3470	83.0	52.56	31.17	1.1422	1.582
500	.8649	.0573	.0778	.1799	190.6	51.35	31.57	1.0644	0.873
∞	.9282	.00	.0718	.00		50.32			

TABLE 9. $\lambda = 5500$ Å $\nu - i\varkappa = 0.15 - i\ 3.36$.

t in Å	R	T	A	$\left(\frac{T}{1-R}\right)^2$	$\left(\frac{1+R}{1-R}\right)^2$	$180-\delta$	β	σ	$-\alpha$
150	.4009	.5234	.0757	.7633	5.44	59.11	26.22	2.2978	5.291
200	.5495	.3713	.0792	.6792	11.8	54.29	30.83	1.6697	3.932
250	.6634	.2582	.0784	.5884	24.4	50.95	33.92	1.3847	2.872
300	.7462	.1779	.0759	.4913	47.3	48.67	35.89	1.2349	2.087
350	.8051	.1220	.0729	.3914	85.8	47.13	37.09	1.1488	1.513
400	.8463	.0834	.0703	.2976	144.3	46.10	37.76	1.0965	1.097
500	.8951	.0389	.0660	.1377	326.4	44.94	38.10	1.0422	0.575
∞	9399	.00	.0601	.00		44.00			

TABLE 10. $\lambda = 6000$ Å $\nu - i\varkappa = 0.15 - i\ 3.82$.

t in Å	R	T	A	$\left(\frac{T}{1-R}\right)^2$	$\left(\frac{1+R}{1-R}\right)^2$	$180-\delta$	β	σ	$-\alpha$
150	.4690	.4609	.0701	.7535	7.65	54.47	31.27	1.9771	4.222
200	.6179	.3119	.0702	.6661	17.93	49.28	36.27	1.5007	2.986
250	.7239	.2087	.0674	.5712	38.98	45.82	39.48	1.2853	2.100
300	.7965	.1392	.0643	.4681	77.93	43.54	41.48	1.1726	1.479
350	.8466	.0930	.0604	.3676	144.9	42.03	42.66	1.1082	1.045
400	.8803	.0622	.0575	.2698	246.7	41.04	43.31	1.0693	0.741
500	.9188	.0278	.0534	.1175	558.4	39.96	43.63	1.0296	0.374
∞	.9516	.00	.0484	.00		39.12			

TABLE 11. $\lambda = 6560$ Å $\nu - i\varkappa = 0.13 - i\ 4.27$.

t in Å	R	T	A	$\left(\frac{T}{1-R}\right)^2$	$\left(\frac{1+R}{1-R}\right)^2$	$180-\delta$	β	σ	$-\alpha$
150	.5291	.4149	.0560	.7764	10.5	50.86	35.76	1.7811	2.971
200	.6748	.2709	.0543	.6937	26.5	45.40	40.96	1.3991	2.084
250	.7728	.1763	.0509	.6022	60.9	41.87	44.39	1.2265	1.386
300	.8375	.1152	.0473	.5023	127.9	39.59	46.45	1.1364	0.956
350	.8802	.0756	.0442	.3975	246.3	38.10	47.66	1.0850	0.668
400	.9084	.0498	.0418	.2956	434.1	37.14	48.36	1.0542	0.466
500	.9398	.0218	.0384	.1305	1038.3	36.09	48.80	1.0228	0.230
∞	.9654	.00	.0346	.00		35.31			

Table 12. $\lambda = 7100$ Å $\nu - i\varkappa = 0.14 - i\ 4.68$.

t in Å	R	T	A	$\left(\frac{T}{1-R}\right)^2$	$\left(\frac{1+R}{1-R}\right)^2$	$180-\delta$	β	σ	$-\alpha$
150	.5711	.3739	0550	.7600	13.4	47.63	39.00	1.6518	2.669
200	.7104	.2375	0521	6726	34.9	42.13	44.34	1.3324	1.770
250	.8004	.1517	0479	5776	81.4	38.65	47.62	1.1881	1.187
300	.8582	.0977	0441	4747	171.7	36.43	49.60	1.1128	.808
350	.8957	.0634	0409	3695	330.3	35.00	50.77	1.0701	.558
400	.9202	.0414	0384	2691	579.0	34.09	51.41	1.0445	.386
450	.9363	.0271	0366	1810	924.0	33.50	51.70	1.0286	.269
500	.9470	.0173	.0357	.1066	1349.5	33.11	51.78	1.0180	.182
∞	.9685	.00	.0315	.00		32.38			

Table 13. $\lambda = 7680$ Å $\nu - i\varkappa = 0.15 - i\ 5.11$.

t in Å	R	T	A	$\left(\frac{T}{1-R}\right)^2$	$\left(\frac{1+R}{1-R}\right)^2$	$180-\delta$	β	σ	$-\alpha$
150	.6100	.3363	.0537	.7436	17.0	44.63	42.01	1.5489	2.384
200	.7419	.2085	.0496	.6527	45.5	39.14	47.34	1.2794	1.543
250	.8241	.1310	.0449	.5544	107.5	35.75	50.54	1.1578	1.017
300	.8756	.0834	.0410	.4498	227.3	33.62	52.44	1.0944	.684
350	.9086	.0537	.0377	.3447	436.1	32.26	53.54	1.0585	.468
400	.9299	.0348	.0353	.2464	757.9	31.39	54.14	1.0370	.321
450	.9438	.0227	.0335	.1632	1196.3	30.82	54.40	1.0237	.223
500	.9530	.0148	.0322	·0997	1726.7	30.46	54.45	1.0153	.155
∞	.9713	.00	0.287	.00		29.78			

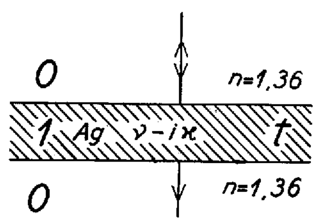

Fig. 7.

Tables 6—13 are calculated by means of (2, 7—9) and (1, 16—18), which in the special case indicated in fig. 7 become:

$$\sqrt{R}\cdot e^{i\delta} = r_{01}\frac{1-e^{-ix}}{1-r_{01}^2\cdot e^{-ix}} \tag{3, 23}$$

$$\sqrt{T}\cdot e^{i\beta} = \frac{(1-r_{01}^2)\,e^{-i\frac{x}{2}}}{1-r_{01}^2\cdot e^{-ix}} \tag{3, 24}$$

$$r_{01} = \frac{1-\left(\frac{\nu}{n_0}-i\frac{\varkappa}{n_0}\right)}{1+\frac{\nu}{n_0}-i\frac{\varkappa}{n_0}} \tag{3, 25}$$

and

$$e^{-ix} = e^{-\frac{4\pi\varkappa t}{\lambda}}\cdot e^{-i\frac{4\pi\nu t}{\lambda}}. \tag{3, 26}$$

To calculate $R(\lambda)$ of a Fabry-Perot filter we must further calculate $\sigma e^{i\alpha} = 1-\frac{T}{R}e^{i(2\beta-2\delta)}$ from the calculated values of (R, δ) and (T, β).

All these calculations have been carried out directly from (3, 23—26) by means of Rybner's tables [4].

In the calculations it is assumed that $\nu - i\varkappa$ at a definite wavelength λ is independent of the thickness t of the silver layer.

The formulae (3, 23—26) depend only upon the variable quantities in the combinations $\frac{\nu(\lambda)-i\varkappa(\lambda)}{n_0}$ and $\frac{\nu(\lambda)-i\varkappa(\lambda)}{\lambda}t$.

If the index of refraction n_0 is changed to n' the tables can still be used if the λ scale is changed to λ' and the t scale to t'; the transformation is determined by

$$\frac{\nu(\lambda')-i\varkappa(\lambda')}{n_0} = \frac{\nu(\lambda)-i\varkappa(\lambda)}{n'} \text{ and } \frac{\nu(\lambda')-i\varkappa(\lambda')}{\lambda'}t' = \frac{\nu(\lambda)-i\varkappa(\lambda)}{\lambda}t,$$

i. e. $t' = \frac{n'}{n_0}\cdot\frac{\lambda'}{\lambda}\cdot t$, and if approximately $\nu(\lambda)-i\varkappa(\lambda) = (k_1-ik_2)\cdot\lambda$, then $\lambda' = \frac{n_0}{n'}\cdot\lambda$ and $t' = t$. (This will be a good approximation for δ because it only slightly depends upon ν).

From Tables 6—13 graphs of $Z(\lambda, t) = \left(\frac{180-\delta_0(\lambda, t)}{180}\right)\cdot\lambda$ can be made for different thicknesses t of the silver layers (fig. 8).

If the filter consists of two silver layers of different thicknesses t' and t''

$$Z_{\text{res}} = \frac{360 - \delta_0(t') - \delta_0(t'')}{360} = \frac{1}{2}(Z(\lambda_1, t') + Z(\lambda_1, t'')).$$

These graphs (fig. 8) are important because it is possible when $Z(\lambda)$ is known to determine the optical thickness nd which

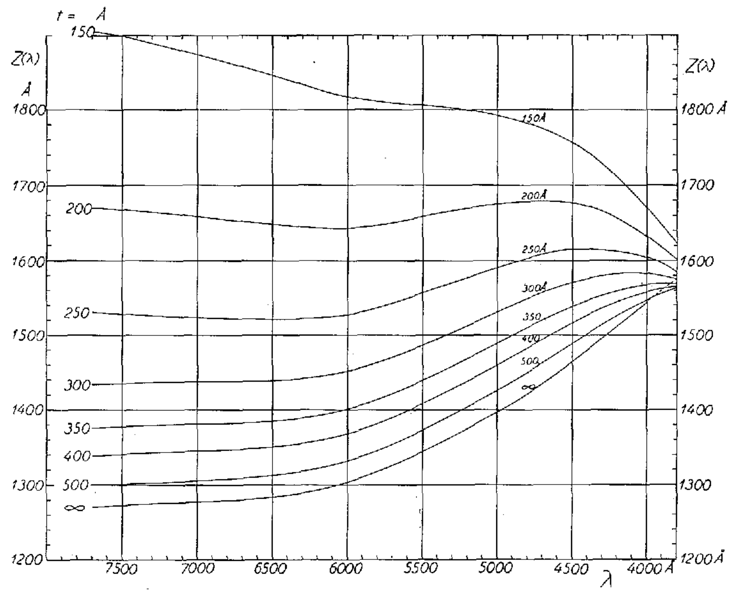

Fig. 8.

corresponds to a definite λ_m. (From (3, 8)). Inversely, if λ_m are measured spectroscopically and d is measured for one definite wavelength, $Z(\lambda)$ can be determined directly by experiment. (This has been done by SCHULZ [9]; with $n = 1$ this gives a determination of $\varkappa(\lambda)$).

The half intensity band width W_2 has not been calculated in the tables for each t, λ value. For this reason a table of

$$W_2(R) = \frac{(1-R)\,5500}{\pi\sqrt{R}\cdot 2}$$

is added (Table 14) corresponding to a filter with $\lambda_m = 5500$ Å and $m = 2$. From this table W_2' corresponding to another λ_m

can easily be calculated by means of

$$W_2'(R) = W_2(R) \cdot \frac{\lambda_m}{m \cdot 2750}.$$

TABLE 14.

R	W_2 in Å	R	W_2 in Å	R	W_2 in Å
0.75	252.7	0.84	152.8	0.93	63.5
.76	241.0	.85	142.4	.94	54.2
.77	229.4	.86	132.2	.95	44.9
.78	218.0	.87	122.0	.96	35.7
.79	206.8	.88	112.0	.97	26.7
.80	195.7	.89	102.1	.98	17.7
.81	184.8	.90	92.3	.99	8.8
.82	174.0	.91	82.6		
.83	163.3	.92	73.0		

When R, T, $y(\lambda)$ are calculated it is possible to calculate line shapes at different wavelengths and thicknesses of the silver layers. In all the following graphs it has simply been chosen to calculate the wavelength scale by means of (3, 9) with the approx. $Z(\lambda) = Z(\lambda_m)$. If $y = 360 \cdot (m-1) + \gamma$ (γ a small angle), we get:

$$\lambda = \frac{360\, m \cdot \lambda_m}{360\, m + \gamma}, \tag{3, 27 a}$$

and this combined with (3, 5 b)

$$I(\lambda) = \frac{\boldsymbol{T}^2}{(1-\boldsymbol{R})^2} \frac{1}{\left(1 + \frac{4\boldsymbol{R}}{(1-\boldsymbol{R})^2} \sin^2 \frac{\gamma}{2}\right)} \tag{3, 27 b}$$

determines the intensity distribution in the neighbourhood of a peak.

Furthermore R and T of the silver layers are regarded as constant in all the following graphs throughout the spectral region considered in the graphs. If the variation of R, T, etc., upon wavelength within the line were taken into account, the calculations would be rather tedious and only result in deviations in

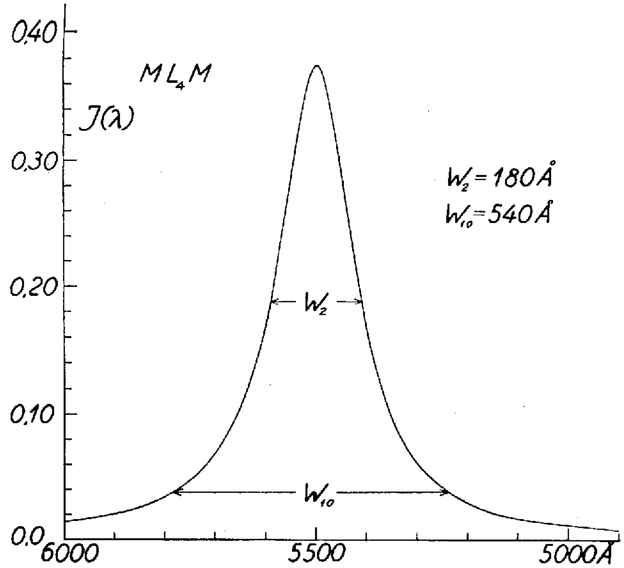

Fig. 9.
2nd order filter. W_2 calculated from (3, 14) is 181 Å and $W_{10} = 3 \cdot W_2 = 543$ Å
$\nu - i\varkappa = 0.15 - i\,3.36$.

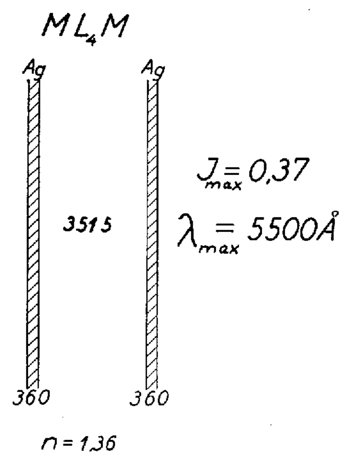

Fig. 10.
(All measures in Å).
Fabry-Perot filter 2nd order. The peak of the 1st order is at about 10800 Å and the peak of the 3rd order at about 3750 Å. The line shape (2nd order) is shown in fig. 9.

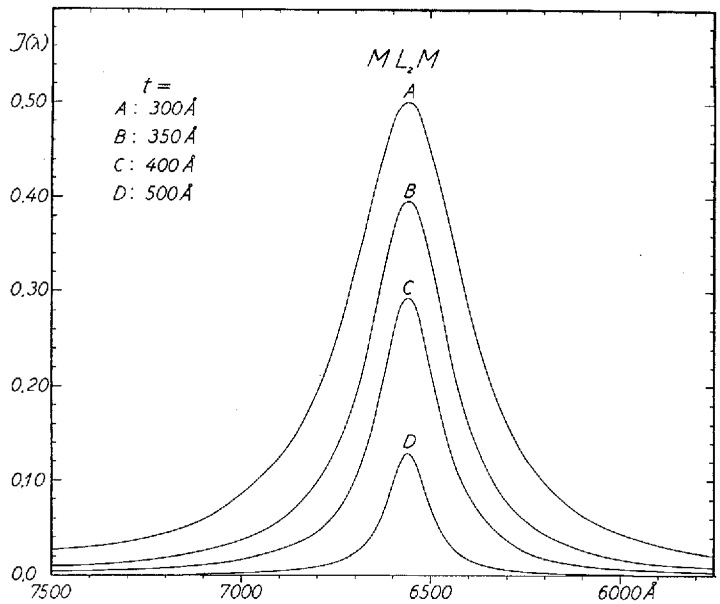

Fig. 11.

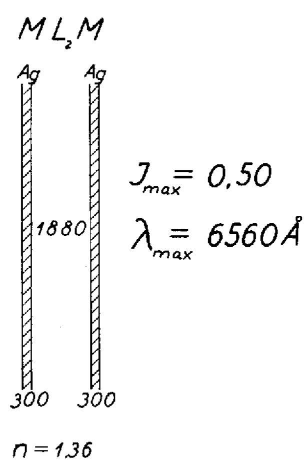

Fig. 12.
1st order filter ($m = 1$) (the peak of the 2nd order is at 3400 Å). (Curve A in fig. 11).

the "wings" of the line and even here the effect is small as compared with the experimental uncertainty in $\nu - i\varkappa$. The deviations will be the greatest for a filter of the first order.

In fig. 9 is shown the line shape of a Fabry-Perot filter of the 2nd order, and fig. 10 shows the relative thicknesses of the thin layers. In fig. 11 line shapes have been calculated with different thicknesses t of the silver layers and with peak transmission at 6560 Å. The rapid decrease in $I_{\max}$ with increasing t is apparent.

Furthermore is it possible to calculate $R(\lambda)$ for a filter of the type $ML_{2m}M$ by means of (3, 2) or (3, 4). (3, 2) becomes

$$R(\lambda) = \frac{R \cdot \left| 1 - \sigma \boldsymbol{R} \cdot e^{-i(y-\alpha)} \right|^2}{\left| 1 - \boldsymbol{R} \cdot e^{-iy} \right|^2}; \; (R = \boldsymbol{R}). \qquad (3, 28)$$

The wavelength scale is calculated by means of (3, 27 a). Fig. 13 and in fig. 14 show the results of such calculations of the intensity distribution $R(\lambda)$ in reflected light (at normal incidence).

In fig. 13 the same filter is considered as in fig. 11 Curve C in transmission. (Each silver layer has a thickness of 400 Å;

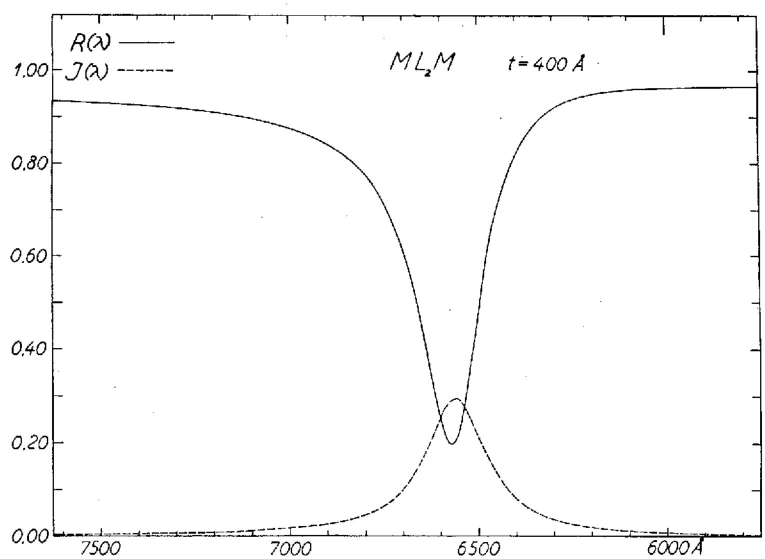

Fig. 13.
$R(\lambda)$ and $I(\lambda)$ for a Fabry-Perot filter with the silver layers of equal thickness ($t = 400$ Å).

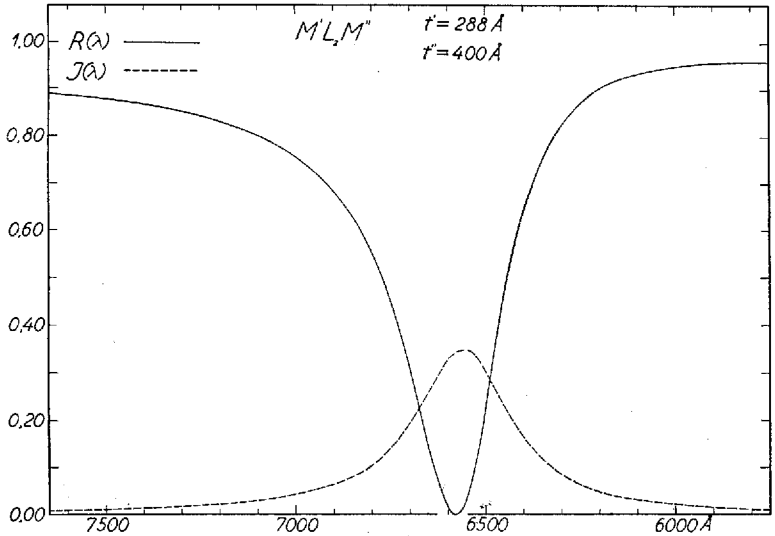

Fig. 14.
$R(\lambda)$ and $I(\lambda)$ for a FABRY-PEROT filter. The silver layers are of unequal thickness ($t'' = 400$ Å and $t' = 288$ Å) t' is determined in such a way that $\sigma \boldsymbol{R} = 1$; ($\boldsymbol{R} = \sqrt{R' \cdot R''}$).

$R = 0.908$). The small negative value of $\alpha = -0^0.466$ (Table 11) gives rise to an asymmetric line shape of $R(\lambda)$. Furthermore it should be noted that the minimum value of $R(\lambda)$ turns out to be as high as 0.20 and the minimum is found at a wavelength a little higher than the wavelength at which $I(\lambda)$ has a maximum in transmission.

In fig. 14 a filter $M'L_2M''$ is considered with the two silver layers of unequal thickness. $t'' = 400$ Å and $t' = 288$ Å is determined in such a way (from Table 11) that $\sigma \boldsymbol{R} = 1$; $\boldsymbol{R} = \sqrt{R' \cdot R''} = 0.864$ (the reflection takes place from the t' side of the filter), $R(\lambda) = 0$ at $y = \alpha = -1^0.060$ (see page 25). This calculation shows that it is possible to extinguish a spectral line by means of a reflection interference filter. $I_{max} = 0.34$ ($I_{max} = 0.45$ of a filter with the two silver layers of equal thickness and with a reflectivity equal to the mean reflectivity of the filter in fig. 14. $R = \sqrt{R' \cdot R''} = 0.86$).

Reflection interference filters with an opaque metal layer at the bottom (e. g. aluminium) have first been treated in theory

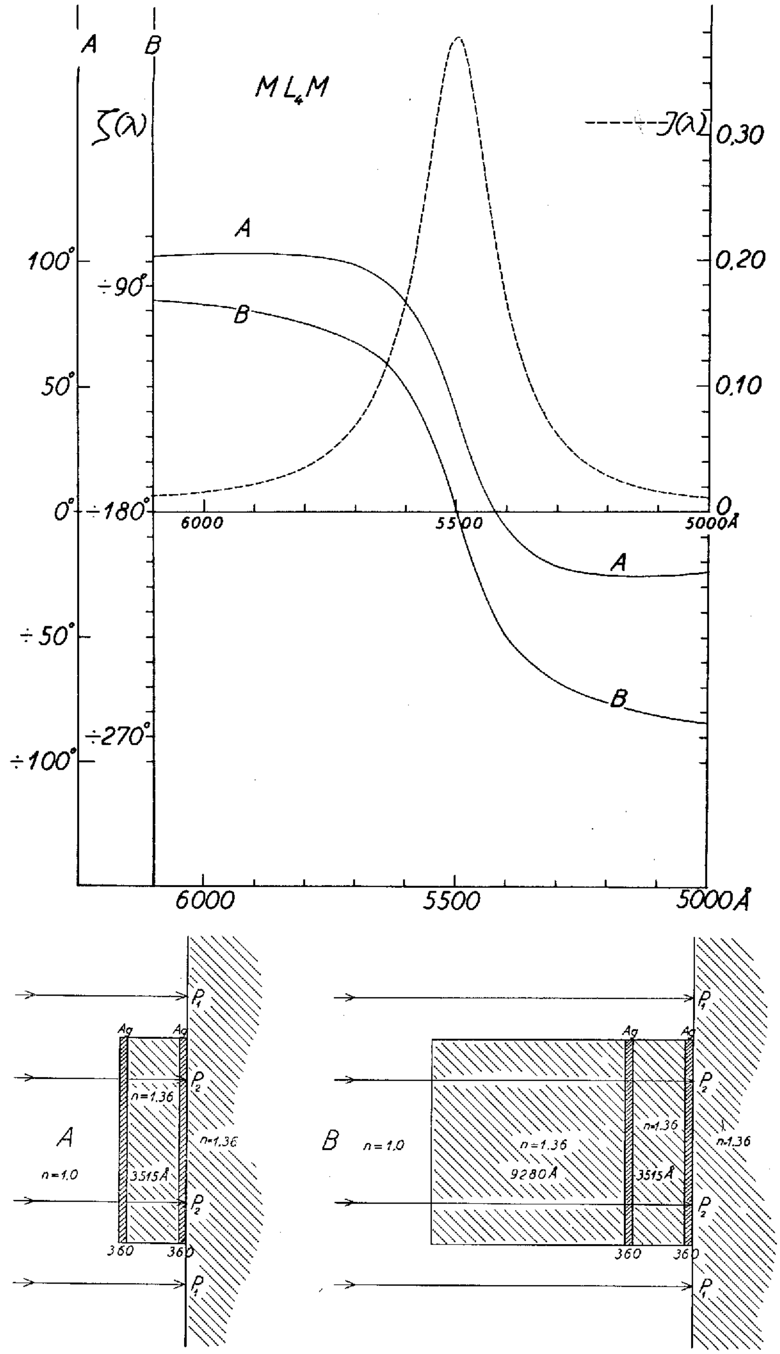

Fig. 15.
A Fabry-Perot interference filter used as phase plate. Unbroken lines: A. The filter is on one side bounded by air. B. To the filter is added a thin dielectric layer in such a way that the phase difference at the peak is —180°. Broken line: $I(\lambda)$ for the filter (the same as the filter in fig. 10).

and practice by HADLEY and DENNISON [11]. These reflection filters have the great advantage of also being applicable to the infrared and ultraviolet spectral region, but have the disadvantage that rather broad spectral regions are reflected.

By means of (3, 2—3) and RYBNER's tables [4] it is furthermore easy to calculate the phase change at reflection and transmission, as a function of the wavelength, in the neighbourhood of a peak. As the phase change at transmission by interference filters is of special interest in the phase contrast microscope as shown by LOCQUIN [12], a calculation has been made in the case of the filter in fig. 10. The results are given in fig. 15. In the case of A the phase difference between P_2 (light passing through the phase plate) and P_1 (light passing outside the phase plate) is (from (3, 3))

$$\zeta(\lambda) = \left((\beta_{01} + \beta_{12}) - \frac{360}{\lambda} \cdot nd - \varepsilon(\lambda)\right) + \frac{360}{\lambda}(2t + d); \qquad (3,29)$$

t is the thickness of the silver layers and $\varepsilon(\lambda)$ is determined from $\varrho \cdot e^{i\varepsilon(\lambda)} = 1 - \boldsymbol{R} \cdot e^{-iy}$. — The phase changes at transmission through the silver layers are determined from Table 9. The approximations $\beta_{01} = \beta_{12} = \beta$ and $\beta = \beta_0 - k \cdot \lambda$ have been made. If, as in the case of B (fig. 15), a thin dielectric layer (with index of refraction n and thickness d_1) is added to the phase plate, the phase difference $-\frac{360}{\lambda} \cdot (n-1) \cdot d_1$ has to be added to $\zeta(\lambda)$ in (3, 29). In fig. 15 curve B, d_1 is determined in such a way that the phase difference is $-180°$ at $\lambda_m = 5500$ Å. The graphs correspond very closely to those previously published by DUFOUR [13]. By the use of a combination of the type B (fig. 15) it is possible to change from a negative to a positive contrast of the image by a variation of wavelength.

Calculation of $I(\lambda)_s$ and $I(\lambda)_p$ at an Oblique Angle of Incidence.

When the angle of incidence φ is increased from $\varphi = 0$, a shift of λ_m towards violet, and a splitting up in two components $\lambda_m^{(s)}$ and $\lambda_m^{(p)}$ result. The first problem now is to calculate $\lambda_m^{(s)}$, $\lambda_m^{(p)}$ when $\lambda_m^{(0)}$, $n(\lambda)$ and $\nu(\lambda) - i\varkappa(\lambda)$ are known. The calculation