Det Kongelige Danske Videnskabernes Selskab
Matematisk-fysiske Meddelelser, bind **27**, nr. 8

Dan. Mat. Fys. Medd. **27**, no. 8 (1952)

ON FIELD THEORIES WITH NON-LOCALIZED INTERACTION

BY

CLAUDE BLOCH

København
i kommission hos Ejnar Munksgaard
1952

CONTENTS

Printed in Denmark
Bianco Lunos Bogtrykkeri

Summary.

A relativistic field theory with non-localized interaction is investigated. The field equations are deduced by the variational principle from a Lagrange function containing an interaction term involving a form function. The essential departure from conventional field theory is the lack of causality, or, in other words, the lack of propagation character of the field equations. It is shown, however, that under certain conditions which must be satisfied by the form function this property remains limited to small domains. Similarly there are no continuity equations, but conservation laws hold in the large. The quantization is performed according to an extension of the scheme developed by YANG and FELDMAN making use of the concept of incoming and outgoing fields. It is shown that this procedure is always consistent with the field equations. Assuming that the field equations can be solved by means of power series expansions it is possible to give rules generalizing Feynman's rules giving all the terms of the expansion of an outgoing operator in terms of the incoming operators. Every term is associated with a doubled graph. An investigation is made of the convergence of the integrals obtained in this way. It is shown that many terms converge automatically as soon as the Fourier transform of the form function is supposed to fall off rapidly at large momenta. Some divergences remain in the higher order terms. They can, however, be removed by assuming that the Fourier transform of the form function has only time-like components. It is finally shown that the gauge invariance requires the addition of a new interaction term in the Lagrange function, corresponding to a sort of exchange current.

1. Introduction.

It has been shown by PEIERLS and MACMANUS[1] that it is possible to introduce a smearing function in a field theory in a Lorentz-invariant way. YUKAWA[2], on the other hand, has proposed a theory involving non-local fields, which, as will be shown later, is equivalent to an ordinary field theory with an interaction containing a form function, if one takes a variation principle as a starting point[3]. These theories cannot be put into a Hamiltonian form and, consequently, have met with some difficulty in quantization. Recently, however, a new treatment of conventional field theory was developed by YANG and FELDMAN[4] and by KÄLLÉN[5], which can immediately be applied to field theories involving smearing functions[6, 7]. It has therefore become possible to build a complete Lorentz-invariant quantized field theory with a non-localized interaction, and it may be worth while to investigate the consistency of such a scheme, and the convergence of the results it yields.

If we take, for simplicity, the example of a nucleon field interacting with a neutral scalar meson field, the scalar non-localized interaction term reads(*)

$$L_i = g \int dx' dx'' dx''' F(x', x'', x''') \psi^+(x') u(x'') \psi(x'''), \qquad (1,1)$$

where the form function F must be Lorentz invariant and such that contributions to L_i come only from the volume elements for which the three points x', x'', x''' are very near each other. By points near each other is meant points whose coordinates differ by amounts of the order of a characteristic length λ. The interaction (1,1) is Hermitian if the form function satisfies the condi-

(*) In this formula x stands for x^1, x^2, x^3, $x^4 = t$, and dx for $dx^1 dx^2 dx^3 dx^4$. We shall use units such that $\hbar = c = 1$. We shall write ab for the scalar product $\Sigma a\, b^i$, where $a_i = a^i$ for $i = 1, 2, 3$, and $a_4 = -a^4$. The metric tensor $g_{\mu\nu}$ is defined by $g_{\mu\nu} = 0$ if $\mu \neq \nu$, $g_{\mu\mu} = 1$, if $\mu = 1, 2, 3$, and $g_{44} = -1$.

tion $F(x''', x'', x') = F^*(x', x'', x''')$. The introduction of a form function in L_i corresponds to a kind of interaction which has no propagation character, and it is important that such effects should remain limited to small domains. Because of its Lorentz invariance F will remain finite for arbitrarily large distances of the points x', x'', x''' as long as they remain near the light cones of one another. Under certain conditions, however, compensations can occur in the neighborhood of the light cones in such a way that the corresponding volume elements do not contribute appreciably to the integral (1,1)[1]. A quantitative study of this effect will be made in section 2, and the conditions which F must satisfy will be established.

It may be of some interest to show that YUKAWA's non-local field theory leads to an interaction of the form (1,1) with a particular form function F. We may take, for instance, a non-local field U interacting with a conventional field ψ. The field U is a function of two points x' and x''' in space-time[2], and the field equations can be deduced from a variation principle involving the interaction term

$$L_i = g \int dx' dx''' \psi^+(x') (x' | U | x''') \psi(x'''). \qquad (1,2)$$

The field U can be represented by the Fourier integral

$$(x' | U | x''') = \int dk\, a(k)\, e^{ikX}\, \delta(kr)\, \delta(r^2 - \lambda^2),$$

where $X = (1/2)(x' + x''')$ and $r = x' - x'''$. If we associate with the field U the local field

$$u(x'') = \int dk\, a(k)\, e^{ikx''},$$

we can write (1,2) in the form of (1,1) with

$$F(x', x'', x''') = (2\pi)^{-4} \int dk\, e^{ik(X - x'')}\, \delta(kr)\, \delta(r^2 - \lambda^2).$$

Detailed investigation of this form function, however, shows that it does not yield convergent self-energies[8].

The non-localized interaction is also connected with the field theories involving higher order equations considered by several authors and systematically investigated by PAIS and UHLENBECK[9]. The general type of these equations is

$$f(\Box)u(x) = \varrho(x), \tag{1,3}$$

where $\varrho(x)$ is the source of the field. If f is an analytical function it can be factorized and each factor corresponds to a possible mass of the particles described by the field u. Theories with more than one mass, however, should be rejected because they introduce negative energies. The only acceptable equations (1,3) are then of the form

$$e^{f(\Box)}(\Box - m^2)u(x) = \varrho(x). \tag{1,4}$$

The differential operator $e^{f(\Box)}$ has an inverse and we can write (1,4) in the equivalent form

$$(\Box - m^2)u(x) = e^{-f(\Box)}\varrho(x) = \int dx'\, G(x-x')\varrho(x'), \tag{1,5}$$

where

$$G(x) = (2\pi)^{-4}\int dk\, e^{-f(-k^2)}\, e^{ikx}. \tag{1,6}$$

This is the equation which would be obtained with the interaction (1,1) and a form function $F = G(x'' - x')\,\delta(x' - x''')$. The possibility of transforming an equation such as (1,3) into an equation of the form (1,5) shows that one has to eliminate certain types of form functions corresponding to the introduction of particles with different masses and negative energy. If the Fourier transform of G has poles, G can be written

$$G(x) = (2\pi)^{-4}\int dk\, e^{ikx}\, g(-k^2)\,/\prod_i (k^2 + m_i^2), \tag{1,7}$$

and the equation (1,5) is equivalent to the multi-mass equation

$$\prod_i (-\Box + m_i^2)(\Box - m^2)\,u(x) = g(\Box)\varrho(x).$$

The function (1,7), however, behaves for large x like $|x^2|^{-3/4}$ as does every propagation function (the flux of the square of the function through a given solid angle is independent of the distance). The occurrence of additional masses will then be avoided if it is specified that the form functions should fall off for large x faster than propagation functions.

The field equations are deduced from the Lagrange function by the variation principle. Because of the introduction of a form function in the interaction the field equations are not ordinary differential equations, and the values of the field functions at

$t + dt$ are not simply defined in terms of the values at t. Consequently, the conservation equations do not hold in their differential form. It will be shown, however, that conservation laws hold in the large, in the sense that energy, momentum, angular momentum and electric charge at a time t, before any collision has taken place, are equal to the corresponding quantities after collision.

The quantization can be performed by postulating that the

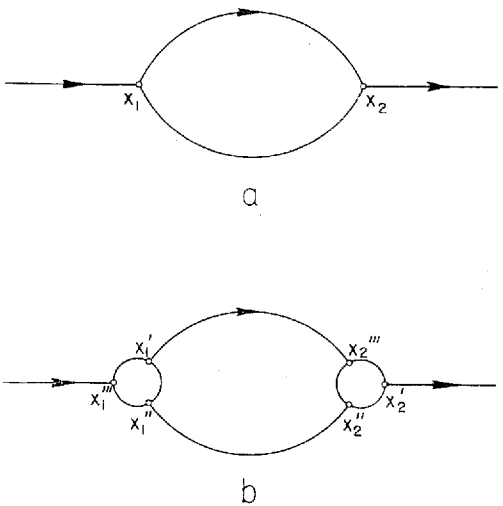

Fig. 1.

asymptotic values of the fields for $t = -\infty$ and $t = +\infty$ (called the incoming and the outgoing fields) satisfy the usual commutation relations of the free fields. It must then be shown that these commutation relations are consistent with the field equations. This can be done by using the fact that the constants of collision: energy, momentum, etc. ... define the infinitesimal canonical transformations corresponding to the infinitesimal translations etc. ... The S-matrix is then defined as the matrix which transforms the incoming fields into the outgoing fields.

Any outgoing operator can in principle be computed from the field equations as a power series of the incoming operators. The calculations are simplified by a set of rules similar to FEYNMAN's rules for electrodynamics[10]. These rules are used for an investigation of the convergence of the self-energies to all orders. The way in which convergence results from the introduction of

a form function in the interaction can easily be seen on the second order self-energy. The graph corresponding to the conventional field theory is represented on Fig. 1 a. To the lines going from x_1 to x_2 correspond functions of $x_1 - x_2$ which are singular on the light cone, and a divergence arises from the fact that the self-energy integral involves a product of two functions becoming singular at the same points. The small circles on Fig. 1 b correspond to the introduction of form functions $F(x', x'', x''')$, and it is seen that the divergence will disappear if F is a smooth function of $x' - x''$ and $x'' - x'''$. A rigorous treatment requires the use of the energy-momentum space. However, it can already be seen that the convergence of the self-energies of both types of particles requires that F be a smooth function of all three variables.

There is then a little difficulty when the interactions with the electromagnetic field are taken into account since the interaction term (1,1) is not gauge invariant. It will be shown, however, that a supplementary interaction term can be added to (1,1) in such a way that the sum is gauge invariant. This term describes the current due to the jumping of the charge between the points x_1''' and x_1', x_2''' and x_2'.

2. The form functions.

In this section we shall investigate under which conditions the non-localizability of the interaction is limited to dimensions of the order of a given length λ. We consider first a simple case:

A. *Functions of two points*[1]. In the conventional theory with a localized interaction the function F is a product of two four-dimensional Dirac functions: $F(x', x'', x''') = \delta(x' - x'')\,\delta(x'' - x''')$. As a first generalization we shall assume that F contains only one four-dimensional Dirac function: $F = \delta(\alpha' x' + \alpha'' x'' + \alpha''' x''')\,G$, where the scalar constants α', α'', α''' satisfy the relation $\alpha' + \alpha'' + \alpha''' = 0$. The factor G can be expressed as a function of two points only, x' and x'', for instance, if $\alpha''' \neq 0$. The invariance under translations and Lorentz transformations requires that G should be a function of $s = (x' - x'')^2$(*).

(*) For $s \leq 0$ the function G can also take two different values for the same value of s depending on whether $x_4' - x_4''$ is positive or negative. We shall come back to this later.

We shall now investigate under which conditions the form factor becomes very small as soon as x' and x'' are not very near one another. More precisely, considering the integral

$$I = \int dx' \, G(s) f(x'), \tag{2,1}$$

where $f(x')$ is an arbitrary smooth function, it should depend only on the values of $f(x')$ for x' very near x''. A first condition to be fulfilled is that $G(s)$ should fall off very rapidly as $|s|$ becomes much larger than λ^2. This condition, however, is not sufficient as $G(s)$ remains finite for x' near the light cone of x''. Thus, the contribution to I coming from the volume elements which are far from x'', but near the light cone of x'', requires a special investigation.

It is convenient to introduce the point x_0 of the light cone of x'' which is near x' and has the same three first coordinates. We call a the three-dimensional length of $x' - x''$ or $x_0 - x''$. We have $x_0^4 - x''^4 = \varepsilon a$, where ε is $+1$ or -1 depending on whether $x_0^4 - x''^4$ is positive or negative. The distance of x' to the light cone is conveniently defined by $\xi = \varepsilon (x_0^4 - x'^4)$. The relation between s and ξ is

$$s = 2 a \xi - \xi^2. \tag{2,2}$$

It shows that for large a a small variation of ξ corresponds to a large variation of s. As G is very small for large values of $|s|$, it follows that for large a we can expand the function f in powers of ξ around the light cone and extend the integration with respect to x'^4 or ξ from $-\infty$ to $+\infty$.

As we are interested in orders of magnitude only we shall omit all numerical coefficients. The Taylor expansion of f around the light cone reads

$$f(x') = f(\boldsymbol{x}_0, x_0^4 - \varepsilon\xi) \cong \sum_0^\infty \xi^k f_0^k,$$

where $f_0^k = (\partial/\partial x'^4)^k f(x')$ taken at the point $x' = x_0$. Finally, we replace the variable of integration x'^4 by s. We have

$$dx'^4 = ds/2\,(a - \xi) \cong (ds/a) \sum_0^\infty (\xi/a)^m,$$

and from (2,2) we deduce

$$\xi = a - (a^2 - s)^{1/2} \cong (s/a) \sum_0^\infty (s/a^2)^n.$$

Using all the preceding expansions the contribution to I of the neighborhood of the light cone of x'' can be written

$$\left.\begin{aligned} &\int dx' G(s) f(x') \cong \int dx' ds \sum_0^\infty \xi^{k+m} G(s) f_0^k / a^{m+1} \cong \\ &\int dx' ds \sum_0^\infty (s/a)^{k+m} (s/a^2)^n G(s) f_0^k / a^{m+1} \cong \int dx' \sum_0^\infty M_{m+k} f_0^k / a^{2m+k+1} \end{aligned}\right\} \quad (2,3)$$

where the M_n are the "moments" of the function G defined by

$$M_n = \int_{-\infty}^{+\infty} ds\, s^n G(s).$$

The formula (2,3) shows how the contributions to I coming from the neighborhood of the light cone of x'' decrease with increasing distance a. Namely, if

$$M_0 = M_1 = M_2 = \cdots = M_{p-1} = 0, \quad M_p \neq 0, \qquad (2,4)$$

the contributions decrease as $(1/a)^{p+1}$. As the volume element dx' is proprotional to $a^2 da$, it is seen that the integral I extended to the whole space-time is convergent for any bounded function f with bounded derivatives if $p \geqslant 3$. The integral $\int dx' G(s)$ is convergent for $p \geqslant 2$.

It should be noted that integrals such as I are usually not absolutely convergent. The convergence is due to cancellations arising within the volume elements which are near the light cone of x''. In calculating such integrals, one must always use a method allowing these cancellations to take place. For instance, one can start by restricting the domain of integration to a finite part of space-time enclosed within a closed surface Σ, and then let Σ go to infinity. It is easily seen that the cancellations which make the integral I convergent will take place if the angle under which Σ cuts the light cone tends nowhere to zero. The possibility of defining in a Lorentz invariant way an integral which is not absolutely convergent clearly comes from the fact that the cancellations making the integral convergent take place within layers along the light cone which become infinitely narrow at infinity.

If for $s \leqslant 0$ the form function takes different values depending on the sign of x^4, it is convenient to write G as a sum of an even

function G_+ which is invariant under the substitution $x \to -x$, and of an odd function G_- which changes sign under the same substitution. It follows from the relativistic invariance that G_- must vanish for $s > 0$. It is easily seen that the functions G_+ and G_- must satisfy the conditions (2,4) independently.

For many calculations it is more appropriate to represent G by a Fourier integral

$$G(x) = \int dk\, e^{ikx} g(k), \tag{2,5}$$

where $g(k)$ is a function of the argument $q = k^2$, and can be represented by a sum of an even function g_+ and an odd function g_-. The Fourier transformation (2,5) gives then G_+ in terms of g_+, and G_- in terms of g_-. We shall now investigate which conditions must be satisfied by g_+ and g_- in order that the corresponding G should be an acceptable form function. This requires a closer investigation of the correspondence between G and g given by (2,5).

The integration in (2,5) with respect to the angular orientation of $\boldsymbol{k}$ yields

$$\begin{aligned} G(x) &= \frac{4\pi}{a} \int_{-\infty}^{+\infty} l\, dl \int_{-\infty}^{+\infty} dk^4 \operatorname{Sin} l a\, e^{-ik^4x^4} g(k) \\ &= -\frac{4\pi}{a} \frac{\partial}{\partial a} \int_{-\infty}^{+\infty} dl \int_{-\infty}^{+\infty} dk^4 \operatorname{Cos} l a\, e^{-ik^4x^4} g(k) \\ &= -\frac{2\pi}{a} \frac{\partial}{\partial a} \int_{-\infty}^{+\infty} dl \int_{-\infty}^{+\infty} dk^4 e^{i(la - k^4x^4)} g(k), \end{aligned}$$

where $a = |\boldsymbol{x}|$, and $|l| = |\boldsymbol{k}|$. Introducing now the decomposition of G and g into even and odd parts we get

$$\left.\begin{aligned} G_+(s) &= -\frac{2\pi}{a} \frac{\partial}{\partial a} \iint_{-\infty}^{+\infty} dl\, dk^4 e^{i(la - k^4x^4)} g_+(q), \\ G_-(s) &= -\frac{2\pi}{a} \frac{\partial}{\partial a} \iint_{-\infty}^{+\infty} dl\, dk^4 e^{i(la - k^4x^4)} \varepsilon(k^4)\, g_-(q), \end{aligned}\right\}$$

where $s = x^2$, $q = k^2$, and $\varepsilon(k^4) = k^4/|k^4|$. It should be noted that the precise definition of $G_-(s)$ is

$$G_-(x) = G_-(s) \text{ if } x^4 > 0, \quad G_-(x) = -G_-(s) \text{ if } x^4 < 0.$$

A similar definition holds for $g_-(q)$. Replacing the variables of integration l and k^4 by

$$\left.\begin{aligned} q = l^2 - (k^4)^2, \text{ and } \alpha = \frac{l - k^4}{a - x^4} \text{ if } a \neq x^4, \\ \text{or } \alpha = \frac{l + k^4}{a + x^4} \text{ if } a = x^4, \end{aligned}\right\}$$

we get after some simple manipulations

$$\left.\begin{aligned} G_+(s) = -\frac{\pi}{a}\frac{\partial}{\partial a}\iint_{-\infty}^{+\infty}\frac{d\alpha\, dq}{\alpha}\,\varepsilon(\alpha)\, e^{(i/2)(\alpha s + q/\alpha)}\, g_+(q), \\ G_-(s) = -\frac{\pi}{a}\frac{\partial}{\partial a}\iint_{-\infty}^{+\infty}\frac{d\alpha\, dq}{\alpha}\, e^{(i/2)(\alpha s + q/\alpha)}\, g_-(q). \end{aligned}\right\}$$

As $\frac{1}{a}\frac{\partial}{\partial a} = 2\frac{\partial}{\partial s}$, we can also write

$$\left.\begin{aligned} G_+(s) = -i\pi\iint_{-\infty}^{+\infty} d\alpha\, dq\, \varepsilon(\alpha)\, e^{(i/2)(\alpha s + q/\alpha)}\, g_+(q), \\ G_-(s) = -i\pi\iint_{-\infty}^{+\infty} d\alpha\, dq\, e^{(i/2)(\alpha s + q/\alpha)}\, g_-(q)^{(*)}. \end{aligned}\right\} \quad (2,6)$$

The formulas (2,6) may be interpreted in the following manner. G is obtained from g by three successive transformations:

a) the Fourier transformation

$$\varphi(\beta) = \int_{-\infty}^{+\infty} dq\, e^{(i/2)\beta q}\, g(q),$$

b) the transformation

$$\psi_+(\alpha) = \varepsilon(\alpha)\varphi_+(1/\alpha), \text{ or } \psi_-(\alpha) = \varphi_-(1/\alpha),$$

c) the Fourier transformation

$$G(s) = -i\pi\int_{-\infty}^{+\infty} d\alpha\, e^{(i/2)\alpha s}\psi(\alpha).$$

(*) In this formula $g_-(q) = 0$ for $q > 0$. It follows then from Cauchy's theorem applied to the integration with respect to α that $G_-(s) = 0$ for $s > 0$.

We shall use this decomposition to find the properties of g sufficient that G defined by (2,5) shall be an acceptable form function.

The conditions which must be satisfied by G are:

$$\left.\begin{array}{l} 1)\ \text{to be continuous;} \\ 2)\ \text{to go to zero as } |s| \to \infty\text{, for instance as } (1/s)^k; \\ 3)\ \int_{-\infty}^{+\infty} ds s^n G(s) = 0, \quad \text{for } n = 0, 1, \ldots, p-1. \end{array}\right\} \quad (2,7)$$

As regards the condition 2), one might require that $G(s)$ should tend to zero much faster than we assume here, exponentially for instance. It seems, however, natural to require only that G behaves for large s in such a way that its integral over the whole space-time is convergent. The condition 2) with $k \geq 3$ is then sufficient. The convergence of the moments involved in the condition 3) requires in fact $k \geq p+1$(*). The condition assumed here seems natural in view of the fact that the contribution to the integral (2,1) coming from the neighborhood of the light cone never decreases faster than an inverse power of the distance.

Sufficient conditions for $\psi(\alpha)$ corresponding to (2,7) are that ψ must have k continuous derivatives such that

$$\left.\begin{array}{l} 1)\ \int_{-\infty}^{+\infty} d\alpha\, |\psi(\alpha)| < \infty\,; \\ 2)\ \int_{-\infty}^{+\infty} d\alpha\, |\psi^{(n)}(\alpha)| < \infty\text{, for } n = 1, 2, \ldots, k^{(**)}; \\ 3)\ \psi^{(n)}(0) = 0, \qquad \text{for } n = 0, 1, \ldots, p-1. \end{array}\right\} \quad (2,8)$$

The derivatives of ψ (with respect to α) are given in terms of the derivatives of φ (with respect to β) by the formula

$$\psi^{(n)}(\alpha) \simeq \sum_{m=1}^{m=n} \beta^{m+n} \varphi^{(m)}(\beta), \quad (2,9)$$

where $\beta = 1/\alpha$, and where the numerical coefficients have been

(*) The expansion (2,3) requires the existence of moments of all orders i. e. an exponential decrease of G for large s. The whole argument, however, can be carried through by means of limited asymptotic expansions only. The condition we have assumed is then sufficient.

(**) Here we make use of a well-known theorem on the aysmptotic value of Fourier integrals; see for instance, S. Bochner, Fouriersche Integrale, Chelsea Publishing Co., New York, p. 11.

omitted. The presence in the relation between φ_+ and ψ_+ of the factor $\varepsilon(\alpha)$ which has a discontinuous variation at $\alpha = 0$ does not modify the equation (2,9) since the function ψ and all its derivatives involved here vanish at $\alpha = 0$ (this follows from 3 for the $p - 1$ first derivatives and for the derivatives of the order $p, p+1, \ldots, k$ from the behavior of $\varphi^{(m)}$ at infinity as indicated below).

It is seen from (2,9) that we must assume that φ has k continuous derivatives. From the condition 1) it follows that φ must be such that

$$\int_{-\infty}^{+\infty} \frac{d\beta}{\beta^2} |\varphi(\beta)| < \infty .$$

This condition is satisfied if we assume that φ is bounded for $\beta = \pm \infty$, is regular at $\beta = 0$, and that $\varphi(0) = \varphi'(0) = 0$. Finally, it is easily seen that the conditions 2) are satisfied if we assume that $\varphi^{(m)}(\beta)$ $(m = 1, 2, \ldots, k)$ behaves at infinity as $(1/\beta)^{m+k}$. The conditions 3) are then automatically satisfied. From the relations $\varphi(0) = \varphi'(0) = 0$ it follows that

$$\int_{-\infty}^{+\infty} dq g(q) = \int_{-\infty}^{+\infty} dq q g(q) = 0 . \tag{2,10}$$

On the other hand,

$$\varphi^{(m)}(\beta) \cong \int_{-\infty}^{+\infty} dq e^{(i/2)\beta q} q^m g(q) ;$$

and this function behaves at infinity as $(1/\beta)^{m+k}$ if $q^m g(q)$ has $m + k$ continuous derivatives absolutely integrable from $-\infty$ to $+\infty$ (BOCHNER loc. cit.). As m takes the values $1, 2, \ldots, k$ we are led to the following conditions:

$$\left.\begin{array}{l} g(q) \text{ is continuous and has } 2k \text{ continuous derivatives;} \\ g^{(n)}(q)\ (n = 0, 1, \ldots, 2k) \text{ goes to zero as } q \to \pm\infty \\ \quad \text{faster than } (1/q)^{k+1} . \end{array}\right\} \tag{2,11}$$

The condtitions (2,10 and 11) are sufficient to insure that $G(s)$ satisfies (2,7). The function $g_-(q)$ vanishes for $q > 0$. It follows then from the continuity of the $2k$ first derivatives that

$$g_-^{(n)}(0) = 0, \text{ for } n = 0, 1, \ldots, 2k. \tag{2,12}$$

The conditions (2, 10 and 11) allow us to choose functions $g(q)$ which vanish outside a certain interval. In such a case, $g(q)$ and its $2k$ first derivatives must vanish at the ends of the interval.

The function $G_-(s)$ can be expressed in terms of the usual function D of field theory. Performing in (2,5) first the integration at q constant, and then the integration over q, we get indeed

$$G_-(s) = (2\pi)^3 i \int_{-\infty}^{0} dq\, g_-(q) D(s, q), \qquad (2,13)$$

where $D(s, q)$ is the function corresponding to the mass $\sqrt{-q}$. If we assume that $g_+(q)$ is different from zero only if $q < 0$ (which implies that $g_+^{(n)}(0) = 0$, for $n = 0, 1, \ldots, 2k$), we can, similarly, express $G_+(s)$ in terms of $D^{(1)(11)}$:

$$G_+(s) = (2\pi)^3 \int_{-\infty}^{0} dq\, g_+(q) D^{(1)}(s, q). \qquad (2,14)$$

The expressions (2,13 and 14) are identical with those used in the theory of regularization[12]. The relations (2,10) also belong to the latter theory. They express the condition that the singularities of the functions D and $D^{(1)}$ at $s = 0$ should not appear in $G(s)$. The conditions (2,11), however, are in contradiction with the limiting process used in the idealistic renormalization, or with the introduction of a discrete set of masses. Consequently, the behavior for large s of a form function is essentially different from that of a regularized function.

It may be noted, finally, that the transformations considered in this section are special cases of the Fourier-Bessel transformation[13]. The transformation of the odd functions, for instance, can be written

$$r G_-(r) = 2i\pi^2 \int_0^{+\infty} \varkappa\, d\varkappa J_1(\varkappa r)\, \varkappa g_-(\varkappa),$$

where $r = \sqrt{-s}$, and $\varkappa = \sqrt{-q}$.

B. *Functions of three points.* As it was shown in section 1, the form function should actually be a smooth function of all three variables. It will then be a function of the invariants(*)

$$s = (x'' - x''')^2, \qquad t = (x''' - x')^2, \qquad u = (x' - x'')^2.$$

(*) These invariants are not entirely independent. No triangle x', x'', x''' exists if s, t and u are negative and if $s^2 + t^2 + u^2 - 2st - 2tu - 2us < 0$.

First of all, the form function should fall off rapidly as s, t, or u becomes large (strictly speaking, one could also require that F falls off as any of two only of the quantities s, t, u becomes large). There are, however, large triangles x', x'', x''' for which s, t and u are small. The contribution to an integral such as

$$I = \int dx'\, dx''\, dx'''\, F\, f(x', x'', x''') \tag{2,15}$$

coming from such triangles can be investigated by the same method as for the functions of two variables. Let a, b and c be the lengths of the space parts of $x'' - x'''$, $x''' - x'$ and $x' - x''$, respectively, and suppose that $a \geqslant b \geqslant c$. For a large triangle a and b at least will be large compared with λ. Let x_0 and x_1 be the points of the light cone of x''' which are near x' and x'', and have the same space coordinates. We have

$$x_1^4 - x'''^4 = \varepsilon a, \qquad x_0^4 - x'''^4 = \varepsilon b, \qquad \varepsilon = \pm 1.$$

Introducing the distances of x' and x'' to the light cone of x''' by

$$\xi = \varepsilon\,(x_1^4 - x''^4), \qquad \eta = \varepsilon\,(x_0^4 - x'^4), \tag{2,16}$$

we have

$$s = 2\,a\xi - \xi^2, \qquad t = 2\,b\eta - \eta^2. \tag{2,17}$$

Again, we can carry out the integration in I with respect to x'^4 and x''^4 using Taylor expansions around the light cone of x'''. It is then convenient to replace the variables x'^4 and x'^4 by s and t with the help of (2,16) and (2,17). An additional complication comes from the fact that u is now a function of s and t since all three quantities are functions of x'^4 and x''^4 (or ξ and η). It is readily found that

$$u = c^2 - (a-b)^2 + (a-b)\left(\frac{s}{a} - \frac{t}{b}\right) + (a-b)\left(\frac{\xi^2}{a} - \frac{\eta^2}{b}\right) - (\xi - \eta)^2,$$

which shows that when the triangle becomes large the quantity $q = c^2 - (a-b)^2$ must remain finite. It is one of the parameters which define the way in which the triangle is increasing. As other parameter we can take $a/b = \mu$, and we have then

$$u = (1-\mu)\,(t - s/\mu) + q + \ldots,$$

where the omitted terms are quadratic in ξ and η. We can then expand F in Taylor series in ξ and η around $u = (1-\mu)(t-s/\mu)+q$. One finally finds for the contribution of the large triangles the expansion

$$I = \int d\boldsymbol{x}'\, d\boldsymbol{x}''\, dx''' \sum_0^\infty \frac{M^{(j+j')/2}_{m+j+k,\, m'+j'+k'}(\mu, q)\, f_0^{kk'}}{a^{2m+j+k+1}\, b^{2m'+j'+k'+1}}, \qquad (2{,}18)$$

where $f_0^{kk'} = (\partial/\partial x'^4)^k (\partial/\partial x'^4)^{k'} f(x', x'', x''')$ taken at $x' = x_0$, $x'' = x_1$, and $M^j_{n,n'}(\mu, q)$ is the derivative of the order j with respect to q of the moment

$$M_{n,n'}(\mu, q) = \int ds\, dt\, s^n\, t^{n'}\, F(s, t, (1-\mu)(t-s/\mu)+q).$$

It is seen from (2,18) that if F satisfies the conditions

$$M_{n,n'}(\mu, q) = 0, \quad \text{for } n + n' \leq p-1,$$

for all relevant values of μ and q, the contribution of the large triangles decreases as $(1/a)^{p+2}$. Thus, we have extended the result obtained for the form functions of two variables.

We shall not go any further into the analysis of the general case since it is much more complicated than for the functions of two points. Moreover, form functions of three points can be built by means of form functions of two points, and this procedure may be sufficient for practical purposes. For instance, we may take

$$F(x', x'', x''') = G^*(x'-x'')\, G(x''-x'),$$

or, more symmetrically,

$$F(x', x'', x''') = \int dx\, G^*(x'-x)\, H(x''-x)\, G(x'''-x). \qquad (2{,}19)$$

The use of such a form function corresponds to replacing the field functions by "smeared fields", as defined by PEIERLS and MACMANUS[1].

The Fourier transform of the function (2,19) is particularly simple. It is the product of the Fourier transforms of the three functions G^*, H and G occurring in F and of the four-dimensional Dirac function $\delta(k'+k''+k''')$.

The results of this section show that the conditions under which a form function behaves like a smeared δ-function have nothing

to do with the condition that its Fourier transform should resemble that of a δ-function. This is due to the non-positive definite character of the distance in space-time. It follows that the behavior for $\lambda \to 0$ of the Fourier transform of a form function gives very little information about the behavior of the function itself. In particular, a form function defined by a Fourier transform $g(k)$ such that $\lim_{\lambda \to 0} g(k) = 1$ may very well give rise for λ small to undesirable interactions transmitted with the velocity of light over large distances, or having no propagation character.

3. Conservation equations.

Taking the usual expressions for the Lagrange functions of the free particles, and the expression (1,1) for the interaction term, we obtain from the variation principle the field equations

$$\left.\begin{aligned}
&\left(\gamma^\mu \frac{\partial}{\partial x^\mu} + M\right)\psi(x) + g\int dx''dx'''F(x,x''x''')u(x'')\psi(x''') = 0,\\
&-\frac{\partial \psi^+(x)}{\partial x^\mu}\gamma^\mu + M\psi^+(x) + g\int dx'dx''F(x',x'',x)\psi^+(x')u(x'') = 0,\\
&(-\Box + m^2)u(x) + g\int dx'dx'''F(x',x,x''')\psi^+(x')\psi(x''') = 0.
\end{aligned}\right\} \quad (3,1)$$

In order to prove the existence of conservation equations it is convenient to introduce the integral L_Ω obtained by restricting the domain of integration in L to a finite part Ω of space-time. In the interaction term L_i it is sufficient to restrict to Ω the integration of only one variable x'' for instance. We thus consider

$$L_\Omega = \int_\Omega dx H(x), \text{ where } H(x) = H_0(x) + H_i(x),$$

in which H_0 is the free particle term and

$$H_i(x) = g\int dx'dx'''F(x',x,x''')\psi^+(x')u(x)\psi(x''').$$

We call E_Ω the field equations deduced from L_Ω by the variational principle. The equations (3,1) will then be called E_∞. The difference between the equations E_Ω and E_∞ is that in the two first equations the integration with respect to x'' is extended to Ω in-

stead of the whole space-time. The difference is very small if x is inside Ω at a distance large compared with λ of the boundary Σ of Ω. This follows from the property of the form function $F(x', x'', x''')$ that it gives contributions to the integral L_i only if the points x', x'', x''' are at distances of the order of λ from one another.

In what follows we shall consider only collision problems. This means that we assume that very far in the past and very far in the future the particles do not interact. The equations E_Ω and E_∞ are then very nearly identical everywhere if Ω is so large that no collision takes place outside Ω or near its boundary.

It should be noted that it is not quite correct to neglect the interaction term outside Ω. Even if the particles do not interact, the existence of the interaction creates self-energies which modify the rest masses of the particles. This can be taken into account by adding to H_i a renormalization term $H_i' = -[\Delta M \psi^+ \psi + (\Delta m^2/2) u^2]$, where ΔM and Δm^2 should be chosen in such a way that the interaction $H_\mathrm{I} = H_i + H_i'$ does not give rise to any self-energy. With this modification the quantities M and m occurring in (3,1) are the real observed masses of the particles, and it is justified to neglect H_I if the particles described by the field are very far from each other.

Finally we see that considering a solution ψ, u of the equations E_∞, it is possible to find a domain Ω such that the equations E_Ω have a solution approximating ψ, u inside Ω and on Σ as closely as required. It follows that if a conservation equation on the boundary Σ of Ω holds for any solution of E_Ω, the same conservation equation will hold for the solutions of E_∞ on Σ if Ω is large enough.

Let us assume now that ψ and u are solutions of the equations E_Ω and consider an arbitrary variation of ψ and u; we have

$$\delta L_\Omega = \frac{1}{2}\int_\Sigma d\sigma_\mu \left(\psi^+ \gamma^\mu \delta\psi - \delta\psi^+ \gamma^\mu \psi + \frac{\partial u}{\partial x_\mu}\delta u + \delta u \frac{\partial u}{\partial x_\mu} \right), \quad (3,2)$$

where $d\sigma_\mu$ is the surface element on Σ pointing toward the outside. If the variations of the fields correspond to a displacement of the fields defined by δx^μ, the variation δL_Ω can be computed directly by making use of the invariance of the Lagrange function under

displacements. The difference between the integral of the displaced fields and the non-displaced fields is readily found

$$\delta L_\Omega = -\int_\Sigma d\sigma_\mu \delta x^\mu H(x). \tag{3,3}$$

From (3,2) and (3,3) we obtain the general conservation equation

$$\int_\Sigma d\sigma_\mu \left[\frac{1}{2}\left(\psi^+\gamma^\mu\delta\psi - \delta\psi^+\gamma^\mu\psi + \frac{\partial u}{\partial x_\mu}\delta u + \delta u\frac{\partial u}{\partial x_\mu}\right) + \delta x^\mu H\right] = 0. \tag{3,4}$$

If the displacement is an infinitesimal translation

$$\delta x_\mu = \varepsilon_\mu, \qquad \delta\psi = -\frac{\partial\psi}{\partial x_\mu}\varepsilon_\mu, \text{ etc.} \cdots\cdots$$

the equation (3,4) becomes

$$\int_\Sigma d\sigma_\nu T^{\mu\nu} = 0, \tag{3,5}$$

where

$$T^{\mu\nu} = \frac{1}{2}\left((\psi^+\gamma^\nu\frac{\partial\psi}{\partial x_\mu} - \frac{\partial\psi^+}{\partial x_\mu}\gamma^\nu\psi\right) + \frac{\partial u}{\partial x_\mu}\frac{\partial u}{\partial x_\nu} - g^{\mu\nu}H \tag{3,6}$$

can be identified as the energy-momentum tensor of the system. The same method applied to the infinitesimal Lorentz transformations leads to the conservation equation of angular momentum. The Lagrange function is also invariant under gauge transformations

$$\psi \to e^{i\alpha}\psi, \quad \psi^+ \to e^{-i\alpha}\psi^+, \quad \alpha = \text{constant}.$$

The corresponding infinitesimal transformation

$$\delta\psi = i\delta\alpha\psi, \quad \delta\psi^+ = -i\delta\alpha\psi^+, \quad \delta u = 0$$

gives $\delta L_\Omega = 0$. From (3,2) we obtain then the conservation equation

$$\int_\Sigma d\sigma_\nu j^\nu = 0, \tag{3,7}$$

where

$$j^\nu = i\varepsilon\psi^+\gamma^\nu\psi \tag{3,8}$$

is the four-vector current-charge.

It should be noted that in the conservation equations (3,5) and (3,7) the surface Σ is not arbitrary; it is the boundary surface

of the volume Ω occurring in L_Ω. It follows that the usual continuity equations

$$\frac{\partial T^{\mu\nu}}{\partial x^\nu} = 0, \qquad \frac{\partial j^\nu}{\partial x^\nu} = 0 \tag{3,9}$$

do not hold if the interaction is non-localized.

It may be of interest to show this more directly. The calculation is simpler for the conservation equation of electric charge. Let us multiply as usual the first of the equations (3,1) on the left-hand side by $\psi^+(x)$, the second on the right-hand side by $\psi(x)$ and subtract. This gives

$$\left.\begin{aligned}\frac{\partial}{\partial x^\nu}(\psi^+\gamma^\nu\psi) = g\Big[&\int dx'dx''F(x', x'', x)\psi^+(x')u(x'')\psi(x)\\ -&\int dx''dx'''F(x, x'', x''')\psi^+(x)u(x'')\psi(x''')\Big].\end{aligned}\right\} \tag{3,10}$$

The right-hand side of (3,10) vanishes of course if F is different from zero only if $x' = x'''$, as in the case of a localized interaction. It does not vanish, however, in general, but if we integrate equation (3,10) over a domain Ω the contribution of the right-hand side reads

$$\left.\begin{aligned}g\Big[&\int_{\Omega'}dx'\int dx''dx'''\\ -&\int_{\Omega'}dx'''\int dx'dx''\Big]F(x', x'', x''')\psi^+(x')u(x'')\psi(x'''),\end{aligned}\right\} \tag{3,11}$$

where Ω' is the part of space-time lying outside Ω. If the interaction term H_I vanishes outside Ω, then (3,11) vanishes and we obtain the conservation equation (3,7).

It will be useful to consider domains Ω limited by two space-like surfaces $\sigma_{(1)}$ and $\sigma_{(2)}$ very far in the past and in the future, respectively. Defining then the quantities

$$G^\mu = \int_\sigma d\sigma_\nu T^{\mu\nu}, \qquad Q = \int_\sigma d\sigma_\nu j^\nu, \tag{3,12}$$

where $d\sigma_\nu$ is the surface element on σ such that $d\sigma_4 > 0$; the equations (3,5) and (3,7) show that G^μ and Q have the same value if $\sigma = \sigma_{(1)}$ or if $\sigma = \sigma_{(2)}$. These quantities are thus constants of collision[14]; they represent the total energy-momentum and electric charge of the system.

4. Quantization.

If all the calculations of the preceding section are carried out in such a way that the order of the factors is always preserved, the results will still hold if the field functions are non-commuting operators. In order to complete the formulation of the theory, we must still define the commutation relations of the field functions. It is not easy to find directly commutation relations which are consistent with the field equations (3,1). In the conventional theory one postulates the commutation relations of the field functions at all points of a space-like surface, and one shows that these relations still hold on any other space-like surface. This is possible because the field equations have one and only one solution for any arbitrary initial conditions given on a space-like surface. It is not easy to see what the corresponding problem is for the field equations (3,1). On the one hand, the knowledge of the field functions on a space-like surface is not sufficient to define the field functions even in the neighborhood of the initial surface. On the other hand, it is not clear that the field functions can be given arbitrary values on a space-like surface. This makes the extension of the canonical method of quantization difficult.

The situation, however, simplifies if one considers a space-like surface very far in the past or very far in the future. Because of the assumption that the interactions are negligible in the distant past and future, the commutation relations on such a surface must in the limit be identical with the conventional commutation relations of free fields. This suggests that the quantization method to be used in the present case will be to postulate the commutation relations for the asymptotic values of the field functions for $x^4 \to \pm \infty$.

The most convenient mathematical method to find the solutions of a differential system with given boundary values is to transform the system into a system of integral equations by means of the Green's functions corresponding to the boundary values considered. The boundary values that we have here are the values for $x^4 \to -\infty$. The corresponding Green's functions are the retarded ones defined by

$$\left.\begin{aligned} &\left(\gamma^\mu \frac{\partial}{\partial x^\mu} + M\right) S_+(x) = -\delta(x), \\ &(-\Box + m^2) D_+(x) = -\delta(x), \\ &\text{and } S_+(x) = D_+(x) = 0 \text{ if } x^4 < 0. \end{aligned}\right\} (4,1)$$

As it will be important in what follows that the interaction term vanishes before and after the collision, it is necessary to use the renormalized interaction

$$L_{\mathrm{I}} = L_i - \int dx \left(\Delta M \psi^+ \psi + \frac{\Delta m^2}{2} u^2 \right);$$

and in order to simplify the writing we shall introduce the notation of variational derivatives. The field equations (3,1) read then

$$\left.\begin{aligned} &\left(\gamma^\mu \frac{\partial}{\partial x^\mu} + M\right) \psi(x) + \frac{\delta L_{\mathrm{I}}}{\delta \psi^+(x)} = 0, \\ &-\frac{\partial \psi^+(x)}{\partial x^\mu} \gamma^\mu + M \psi^+(x) + \frac{\delta L_{\mathrm{I}}}{\delta \psi(x)} = 0, \\ &(-\Box + m^2) u(x) + \frac{\delta L_{\mathrm{I}}}{\delta u(x)} = 0, \end{aligned}\right\} (4,2)$$

where, for instance,

$$\frac{\delta L_{\mathrm{I}}}{\delta \psi^+(x)} = g \int dx'' dx''' F(x, x'', x''') u(x'') \psi(x''') - \Delta M \psi(x), \text{ etc. } \cdots$$

Using now the retarded Green's functions we can transform the system (4,2) into the equivalent system of integral equations

$$\left.\begin{aligned} \psi(x) &= \psi^{\mathrm{in}}(x) + \int dx' S_+(x - x') \frac{\delta L_{\mathrm{I}}}{\delta \psi^+(x')} \\ \psi^+(x) &= \psi^{+\mathrm{in}}(x) + \int dx''' \frac{\delta L_{\mathrm{I}}}{\delta \psi(x''')} S_+^+(x - x''') \\ u(x) &= u^{\mathrm{in}}(x) + \int dx'' D_+(x - x'') \frac{\delta L_{\mathrm{I}}}{\delta u(x'')} \end{aligned}\right\} (4,3)$$

where the fields ψ^{in} and u^{in} satisfy the free field equations. The retarded Green's functions are different from zero only inside the past part of the light cone, and as we assume throughout that the interactions are negligible far in the past, we see that the second

term in equations (4,3) becomes very small as $x^4 \to -\infty$. It follows that the fields ψ^{in} and u^{in} represent asymptotically ψ and u as $x^4 \to -\infty$. They describe the incoming particles.

The integral equations (4,3) can formally be solved by iteration to all orders of approximation for arbitrary incoming fields. It is natural to postulate that the incoming fields satisfy the conventional commutation relations of free fields

$$\left.\begin{aligned} \left[\psi^{\text{in}}(x_1),\ \psi^{\text{in}}(x_2)\right]_+ &= \left[\psi^{+\,\text{in}}(x_1),\ \psi^{+\,\text{in}}(x_2)\right]_+ = 0, \\ i\left[\psi^{\text{in}}_{\varrho}(x_1),\ \psi^{+\,\text{in}}_{\sigma}(x_2)\right]_+ &= S_{\varrho\sigma}(x_1 - x_2), \\ \left[\psi^{\text{in}}(x_1),\ u^{\text{in}}(x_2)\right] &= \left[\psi^{+\,\text{in}}(x_1),\ u^{\text{in}}(x_2)\right] = 0, \\ i\left[u^{\text{in}}(x_1),\ u^{\text{in}}(x_2)\right] &= D(x_1 - x_2), \end{aligned}\right\} \tag{4,4}$$

where $[A, B] = AB - BA$, $[A, B]_+ = AB + BA$. The commutation relations (4,4) are clearly consistent with the field equations (4,3). The commutation relations of the fields ψ and u can in principle be deduced from the relations (4,4) with the help of the field equations (4,3).

The above considerations can be repeated with the boundary conditions for $x^4 \to +\infty$. We introduce the advanced Green's functions S_- and D_- which satisfy the same equations as the retarded Green's functions but vanish for $x^4 > 0$. They lead to the integral equations

$$\left.\begin{aligned} \psi(x) &= \psi^{\text{out}}(x) + \int dx'\, S_-(x - x')\frac{\delta L_{\text{I}}}{\delta \psi^+(x')}, \\ \psi^+(x) &= \psi^{+\,\text{out}}(x) + \int dx'''\, \frac{\delta L_{\text{I}}}{\delta \psi(x''')} S^+_-(x - x'''), \\ u(x) &= u^{\text{out}}(x) + \int dx''\, D_-(x - x'')\frac{\delta L_{\text{I}}}{\delta u(x'')}, \end{aligned}\right\} \tag{4,5}$$

where ψ^{out} and u^{out} are free fields, which are asymptotically identical with ψ and u as $x^4 \to +\infty$, and represent the outgoing particles. The outgoing fields should, of course, satisfy also the free field commutation relations. In fact the commutation relations of the outgoing fields can in principle be deduced from the commutation relations of the incoming fields with the help of the equations (4,3) and (4,5). We have to show that the relations obtained in this way are similar to the relations (4,4).

This can easily be done with the help of the constants of collision defined in the preceding section. These constants can be computed with the incoming or outgoing fields. For instance, the total electric charge before collision is given by

$$Q(\text{in}) = i\varepsilon\int_{\sigma} d\sigma_\nu \psi^{+\text{in}}\gamma^\nu\psi^{\text{in}},$$

where σ is any arbitrary space-like surface. The same applies to the energy-momentum and the angular momentum before and after collision, and the conservation equations read now

$$G^\mu(\text{in}) = G^\mu(\text{out}), \qquad Q(\text{in}) = Q(\text{out}), \cdots \tag{4,6}$$

It is a well known property that the commutator of any incoming field function with an incoming constant of collision is related to the corresponding infinitesimal transformation by the equations

$$\left.\begin{aligned} [A, G^\mu(\text{in})] &= -i\frac{\partial A}{\partial x^\mu} \\ [A, Q(\text{in})] &= -i\frac{\partial A}{\partial \alpha}, \cdots \end{aligned}\right\} \tag{4,7}$$

for any incoming field quantity A. In the second equation (4,7) α is the parameter occurring when use is made of an arbitrary gauge (ψ^{in} is proportional to $e^{i\varepsilon\alpha}$, $\psi^{+\text{in}}$ is proportional to $e^{-i\varepsilon\alpha}$ and u^{in} independent of α). A similar equation connects the angular momentum of the system and the infinitesimal Lorentz transformations. It is easily seen that if two quantities A and B satisfy the relations (4,7), $A + B$ and AB satisfy the same relations. It follows that any quantity which can be built by algebraic operations from quantities satisfying the relations (4,7) also satisfies these relations. In particular the outgoing fields satisfy the relations (4,7), and taking into account the conservation equations (4,6) we get

$$[A, G^\mu(\text{out})] = -i\frac{\partial A}{\partial x^\mu}, \text{ etc.} \cdots \text{ where } A = \psi^{+\text{out}}, u^{\text{out}}, \text{ or } \psi^{\text{out}}. \tag{4,8}$$

From these relations it can be deduced that the commutation relations of the outgoing fields are similar to the relations (4,4). The detailed proof is given in the appendix.

In the present theory we have been using the Heisenberg representation. The situation of the particles before a collision is described by certain operators, the incoming field functions, and by a certain state vector Ψ in Hilbert space. After the collision the operators are changed into the outgoing field functions, but the state vector remains the same. In the interaction representation the initial situation is described by operators which can be identified with the present incoming field functions and by a state vector $\Psi(\text{in})$ which can be identified with Ψ. The situation after the collision is described by the same operators, but by a different state vector $\Psi(\text{out})$. The unitary matrix S which transforms $\Psi(\text{in})$ into $\Psi(\text{out})$ according to $\Psi(\text{out}) = S\Psi(\text{in})$ is the collision matrix, and the squares of the absolute values of its elements give the transition probabilities. The situation after the collision could as well be described by the state vector $S^{-1}\Psi(\text{out}) = \Psi(\text{in})$ and the operators $S^{-1}\psi^{\text{in}}S$, $S^{-1}u^{\text{in}}S$, $\cdots$. In a theory with a localized interaction the formalisms using the Heisenberg representation or the interaction representation are of course equivalent, so we must have

$$\psi^{\text{out}} = S^{-1}\psi^{\text{in}}S, \qquad u^{\text{out}} = S^{-1}u^{\text{in}}S. \tag{4,9}$$

If the interaction is non-localized we do not know the interaction representation. However, as the outgoing fields satisfy the same commutation relations as the incoming fields, we know that there exists a unitary matrix S satisfying the relations (4,9). It is then natural to define this matrix as the collision matrix.

The equations (4,3), (4,4), (4,5) and (4,9) give a complete self-consistent formulation of the theory. These equations can, in principle, be solved by successive approximations. In fact, we need practical rules giving a way of computing any matrix element of S. Such rules will be given in the following sections.

5. Solution of the field equations.

We shall first consider the case where the interaction is a conventional local interaction

$$L_1 = \int dx\, H_{\text{I}}(x),$$

where

$$H_{\mathrm{I}}(x) = g\psi^{+}(x)u(x)\psi(x) - \Delta M\psi^{+}(x)\psi(x) - (\Delta m^2/2)u^2(x).$$

We assume that the solution of the field equations (4,3) can be expanded into powers of the constants g, ΔM and Δm^2, and we set

$$\psi(x) = \sum_0^\infty (-i)^n \psi^{(n)}(x), \qquad u(x) = \sum_0^\infty (-i)^n u^{(n)}(x), \quad (5,1)$$

where $\psi^{(n)}(x)$ and $u^{(n)}(x)$ are of the order n with respect to the constants g, ΔM and Δm^2. The zero order approximation is of course given by the incoming fields. The first order approximation is easily computed, and the value of $\psi^{(1)}$, for instance, is

$$\psi^{(1)}(x) = i\int dx'\, S_{+}(x-x')\frac{\delta L_{\mathrm{I}}^{\mathrm{in}}}{\delta\psi^{+}(x')}, \qquad (5,2)$$

where $L_{\mathrm{I}}^{\mathrm{in}}$ is equal to L_{I} with the field functions replaced by the corresponding incoming field functions.

It is well known that the function S occurring in the commutation relations is connected to the Green's functions by

$$S(x) = S_{-}(x) - S_{+}(x).$$

As $S_{+}(x)$ vanishes if $x^4 < 0$, and $S_{-}(x)$ vanishes if $x^4 > 0$, it follows that

$$\left.\begin{aligned} S_{+}(x) &= -S(x) \quad \text{if } x^4 > 0, \\ &= 0 \qquad \text{if } x^4 < 0. \end{aligned}\right\} \quad (5,3)$$

Similarly

$$\left.\begin{aligned} S_{-}(x) &= 0 \qquad \text{if } x^4 > 0, \\ &= S(x) \quad \text{if } x^4 < 0. \end{aligned}\right\} \quad (5,4)$$

Similar relations hold for the D functions.

The relations (5,3) show that the expression (5,2) can also be written

$$\psi^{(1)}(x) = \int_{x^4 > x'^4} dx' \,[\psi^{\mathrm{in}}(x),\ \psi^{+\mathrm{in}}(x')]_{+}\frac{\delta L_{\mathrm{I}}^{\mathrm{in}}}{\delta\psi^{+}(x')}$$

or still

$$\psi^{(1)}(x) = \int_{x^4 > x'^4} dx' \,[\psi^{\mathrm{in}}(x),\ H_{\mathrm{I}}^{\mathrm{in}}(x')]. \qquad (5,5)$$

Similar formulas hold for $\psi^{+(1)}(x)$ and $u^{(1)}(x)$. The generalization of these formulas to higher orders is obvious. Let $A(x)$ be any field function; the term of order n in its expansion is given by

$$A^{(n)}(x_0) = \int_{x_0 > x_1 > \cdots > x_n} dx_1 dx_2 \cdots dx_n \,[\cdots[[A^{\text{in}}(x_0),\, H_1,],\, H_2],\cdots H_n], \quad (5,6)$$

where H_n has been written for $H_{\text{I}}^{\text{in}}(x_n)$, and $x_i > x_j$ for $x_j^4 > x_j^4$. Formula (5,6) is well known in field theory, and is usually deduced from the Schrödinger equation. It can also be obtained from the field equations (4,3) by induction (see appendix III).

The next step is the calculation of the outgoing fields. We again assume expansions in power series

$$\psi^{\text{out}}(x) = \sum_0^\infty (-i)^n \psi^{\text{out}(n)}(x), \; u^{\text{out}}(x) = \sum_0^\infty (-i)^n u^{\text{out}(n)}(x). \quad (5,7)$$

By subtraction of the equations (4,3) from the equations (4,5) we get for the outgoing fields the expressions

$$\psi^{\text{out}}(x) = \psi^{\text{in}}(x) - \int dx'\, S(x - x') \frac{\delta L_{\text{I}}}{\delta \psi^+(x')}, \text{ etc.} \cdots.$$

The zero order approximation is thus given by the incoming fields, and the first order approximation is readily found to be

$$\psi^{\text{out}(1)}(x) = \int dx' \,[\psi^{\text{in}}(x),\; H_{\text{I}}^{\text{in}}(x')], \quad (5,8)$$

and similar formulas for $\psi^{+\,\text{out}(1)}$ and $u^{\text{out}(1)}$. These formulas can be generalized by induction, and it can be shown that the term of order n of any outgoing field quantity A^{out} is given by

$$A^{\text{out}(n)}(x) = \int_{x_1 > x_2 > \cdots > x_n} dx_1 dx_2 \cdots dx_n \,[[\cdots[A^{\text{in}}(x),\, H_1],\, H_2],\cdots H_n\,]. \quad (5,9)$$

We shall now extend the preceding expressions to the case of a non-localized interaction. It will be convenient to write the interaction term in the form

$$L_{\text{I}} = \int dx' dx'' dx''' H_{\text{I}}(x', x'', x'''),$$

where

$$\left.\begin{aligned} H_{\text{I}}(x', x'', x''') &= gF(x', x'', x''')\psi^+(x')u(x'')\psi(x''') \\ &- \delta(x' - x'')\delta(x'' - x''')\left\{\Delta M \psi^+(x')\psi(x''') - (\Delta m^2/2)u^2(x'')\right\}. \end{aligned}\right\} \quad (5,10)$$

The first order terms can be computed as in the preceding case, and are given by

$$\left.\begin{aligned}
\psi^{(1)}\ (x) &= \int_{x>x'} dx'dx''dx'''\,[\psi^{\text{in}}\ (x),\ H_{\text{I}}(x', x'', x''')],\\
\psi^{+(1)}(x) &= \int_{x>x'''} dx'dx''dx'''\,[\psi^{+\,\text{in}}(x),\ H_{\text{I}}(x', x'', x''')],\\
u^{(1)}\ (x) &= \int_{x>x''} dx'dx''dx'''\,[u^{\text{in}}\ (x),\ H_{\text{I}}(x', x'', x''')].
\end{aligned}\right\} \quad (5{,}11)$$

It should be noted in these formulas that the domain of integration of only one of the three variables occurring in H_{I} is restricted by an inequality. This variable is different depending on the field function which is being computed; it is the variable of the field function which does not commute with the field function which is being computed. This is because the inequalities appear when a Green's function is replaced by a commutator (or anticommutator) of two field functions. This complication makes it impossible to extend directly the formulas (5,6) and (5,9).

It will be convenient in what follows to make use of some conventions and notations. We shall always call x' (sometimes provided with an index) the argument of a function ψ^+, x'' the argument of a function u, x''' the argument of a function ψ. We shall write ψ_n for $\psi^{\text{in}}(x_n''')$, u_n for $u^{\text{in}}(x_n'')$, ψ_n^+ for $\psi^{+\,\text{in}}(x_n')$, and H_n for $H_{\text{I}}^{\text{in}}(x_n', x_n'', x_n''')$. Finally, ξ_n will stand for the three points x_n', x_n'', x_n''', and $d\xi_n$ for $dx_n'dx_n''dx_n'''$.

We shall now try to extend the formulas (5,6) and (5,9) to the case of a non-localized interaction. These formulas are obtained from the field equations by a certain number of algebraic operations: additions, multiplications, integrations. The same operation can be performed as well with a non-localized interaction, and the result should be very similar. The only difference, in fact, lies in the inequalities.

This leads us to consider also in the case of non-localized interaction expressions such as

$$E_n = [\cdots[[\psi_0,\ H_1],\ H_2],\cdots H_n].$$

If we develop E_n by computing first the commutator of ψ_0 with H_1, then the commutator of the result with H_2, and so on, the

result is a sum of terms T, each of which is a product of form functions, of field functions, and of n "elementary" commutators or anticommutators such as $[\psi_i^+, \psi_j]_+$, or $[u_i, u_j]$. These commutators (or anticommutators) associate some of the variables $x_1', x_1'', \cdots x_n'''$ by groups of two. For instance, the two elementary commutators mentioned as examples associate x_i' with x_j''', and x_i'' with x_j'', respectively.

Let us call $I(T)$ the set of inequalities expressing that in all the groups of two variables associated by the elementary commutators or anticommutators of the term T, the variable with the lower index should correspond to a time later than the variable with higher index. There are n inequalities for each term T. They define a domain $D(T)$ which is different for every term. The same decomposition can be performed in the case of a localized interaction, with the only difference that the form functions disappear and that $x_i' = x_i'' = x_i''' = x_i$.

It can easily be seen by analyzing the way in which the successive approximations are obtained from the field equations in the case of a localized interaction, that the terms of the order n appear at first as sums of products of field functions and of n Green's functions. The next step consists in replacing the Green's functions by the corresponding commutators or anticommutators. The domains of integration must then be restricted by certain inequalities. The terms obtained in this way are precisely the terms T obtained by decomposition of E_n, and the inequalities associated with each term are clearly the inequalities $I(T)$. By a further transformation it is possible to replace the inequalities $I(T)$ by the inequalities $\mathrm{I}: x_0 > x_1 > \cdots > x_n$. Since the domain of integration is then the same for all the terms, it becomes possible to put the sum of all the terms T into the compact form E_n, and one gets the final formula (5,6). In the case of a non-localized interaction all the operations can be performed in the same way, except the last transformation. Thus, we must try to extend to the case of a non-localized interaction the expressions of the successive approximations as sums of T terms.

A few definitions will be useful.

We shall call $P(T)$ the set of all the permutations of the indexes 1, 2, . . . n such that in every inequality of $I(T)$ the variable indicated as corresponding to the later time keeps an index lower than the index of the other variable.

$N(T)$ will be the number of the permutations of $P(T)$ (unity included).

Finally we shall call terms equivalent to T the terms deduced from T by a permutation of the indexes belonging to $P(T)$.

These definitions apply as well to localized and non-localized interactions.

As an example, let us consider the following term of E_4 (the form functions have not been written down):

$$T = g^3(-\Delta m^2/2)\,[\psi_0, \psi_1^+]_+\,[\psi_1, \psi_2^+]_+\,[u_1, u_3]\,[u_2, u_4]\,u_3\,\psi_4^+\psi_4\psi_2,$$

which is one of the terms coming from the terms in g of H_1, H_2 and H_4, and from the term in $(-\Delta m^2/2)$ of H_3. The inequalities $I(T)$ are

$$I(T) : x_0''' > x_1', \quad x_1''' > x_2', \quad x_1'' > x_3'', \quad x_2'' > x_4''.$$

The permutations $P(T)$ are besides unity the permutations (2,3) and (3,4) which transform the inequalities $I(T)$ into

$$(2,3) : x_0''' > x_1', \quad x_1''' > x_3', \quad x_1'' > x_2'', \quad x_3'' > x_4'',$$
$$(3,4) : x_0''' > x_1', \quad x_1''' > x_2', \quad x_1'' > x_4'', \quad x_2'' > x_3''.$$

The following properties are easily shown:

a) Equivalent terms integrated over their associated domains give identical results.

b) If T belongs to the development of E_n, all the terms equivalent to T also belong to the development of E_n.

c) In the case of a localized interaction ($x' = x'' = x'''$), the domain $D(T)$ is the sum of the domain D defined by $I : x_0 > x_1 > \cdots > x_n$, and of the domains deduced from D by the permutations of $P(T)$.

As an example of property c), the domain

$$D(T) : x_0 > x_1, \quad x_1 > x_2, \quad x_1 > x_3, \quad x_2 > x_4$$

is the sum of the domains

$$D : x_0 > x_1 > x_2 > x_3 > x_4, \qquad (2,3)D : x_0 > x_1 > x_3 > x_2 > x_4,$$
$$(3,4)D : x_0 > x_1 > x_2 > x_4 > x_3.$$

It will now be possible to make a precise comparison of the expression (5,6) with the development of E_n. According to b)

the terms in the development of E_n can be collected into families of $N(T)$ equivalent terms. From a) and c) it follows that the sum of $N(T)$ equivalent terms integrated over the domain D is equal to any one of the terms of the family integrated over its associated domain. As we know from (5,6) that $\psi^{(n)}(x_0)$ is the sum of all T terms integrated over the domain D, we see that

$$\psi^{(n)}(x_0) = \sum_T (1/N(T)) \int_{D(T)} dx_1 dx_2 \cdots dx_n \; T, \qquad (5,12)$$

where the summation is extended to all the terms T of the development of E_n. One could also omit the factor $1/N(T)$ and instead say that only one term in each family of equivalent terms should be taken into account, and it is easily seen that this is exactly the expression obtained by a straightforward calculation from the field equations.

The extension to the case of a non-localized interaction is now obvious, and we shall write symbolically

$$\psi^{(n)}(x_0) = \int_t d\xi_1 d\xi_2 \cdots d\xi_n [\cdots [[\psi^{\text{in}}(x_0'''), H_1], H_2] \cdots H_n], \quad (5,13)$$

where $\int_t$ is a "time ordered integration" and should be computed in the following way:

a) The integrand must be developed into a sum of T terms.
b) Each term should be integrated over the domain $D(T)$.
c) Each integral should be multiplied by $1/N(T)$.

The formulas (5,11) are clearly particular cases of (5,13). The general formula can be obtained directly from the field equations by induction (see appendix III). The outgoing fields are given by formulas differing from (5,13) only by the fact that all inequalities involving x_0', x_0'', or x_0''' should be omitted. In order that the indexes should remain specifically connected with time ordering, it is convenient then to suppress the index 0 and to write

$$\psi^{\text{out}(n)}(x''') = \int_t d\xi_1 d\xi_2 \cdots d\xi_n [\cdots [[\psi^{\text{in}}(x''), H_1], H_2], \cdots H_n]. \quad (5,14)$$

The domains of integrations are now independent of the points at which the field functions are being computed. This gives the

possibility of a useful generalization of the equations (5,14). Let A^{in} be a polynomial of the incident field functions or, more generally, a power series. The field functions can be taken at different points, and A will depend on a certain number of points in space-time. The value of the same polynomial (or series) of the outgoing field functions taken at the same points is given by

$$A^{\text{out}} = \sum_{n=0}^{\infty} (-i)^n A^{\text{out}(n)}, \tag{5,15}$$

where

$$A^{\text{out}(n)} = \int_t d\xi_1 d\xi_2 \cdots d\xi_n [\cdots [A^{\text{in}}, H_1], H_2], \cdots H_n]. \tag{5,16}$$

The proof is given in appendix III. As an example, let us take for A^{in} the expressions $[\psi^{+\text{in}}(x'), \psi^{\text{in}}(x''')]_+$, $[\psi^{+\text{in}}(x'), u^{\text{in}}(x'')]$, etc. $\cdots$ As these quantities are c-numbers, all terms in the expansions (5,15) vanish, except the first term. Thus, $A^{\text{out}} = A^{\text{in}}$, and this proves again that the commutation relations of the outgoing fields are the same as those of the incoming fields.

A few remarks should be added to the results of this section.

1. *Lorentz invariance.* The domains $D(T)$ are not Lorentz invariant. If the vector joining x_i and x_j is space-like, the time ordering of the two points has no invariant meaning. This time ordering matters for a term T only if T has as a factor a commutator (or anticommutator) of two field functions at the points x_i and x_j. As this commutator (or anticommutator) vanishes if $x_i - x_j$ is space-like, it is seen that the integrated formula is Lorentz invariant.

2. *The Schrödinger equation.* Let us consider in the case of a localized interaction the field functions taken at arbitrary points on a space-like surface σ. In the equations (5,6) the inequality $x_0 > x_1$ can without changing the value of the integral be replaced by the condition that x_1 should be in the past region of space-time with respect to σ. Thus, the domain of integration in (5,6) can be chosen in such a way that it is the same for all the field functions on all points of σ. It follows that if A^{in} is a polynomial of the incoming fields taken at various points of σ, the same polynomial of the fields ψ^+, u and ψ taken at the same points is given by an expansion similar to (5,15), where the term of the order n is given by

$$A^{(n)} = \int_{\sigma > x_1 > x_2 > \cdots > x_n} dx_1 dx_2 \cdots dx_n \, [\cdots [[A^{\text{in}}, H_1], H_2], \cdots H_n].$$

If A is a commutator (or anticommutator) of two field functions, it is seen that $A = A^{\text{in}}$. Thus, the commutation relations of the field functions *on a space-like surface* σ are identical with those of the incoming field functions. Hence, a unitary matrix $S(-\infty, \sigma)$ exists such that

$$A(x_0) = S^{-1}(-\infty, \sigma) A^{\text{in}}(x_0) S(-\infty, \sigma), \qquad (5{,}17)$$

where A is ψ^+, u or ψ and x_0 any point on σ. The Schrödinger equation is (in the Tomonaga—Schwinger form[15]) the differential equation giving the variations of $S(-\infty, \sigma)$ corresponding to infinitesimal variations of the surface σ.

In the case of a non-localized interaction, it is not possible to use the same domains of integration for computing all the field functions on a space-like surface. The equations (5,11), for instance, show that for the first order terms already, the domain of integration unavoidably depends on which field function is being computed. Then the commutation relations of the field functions on a space-like surface are not the same as those of the incoming fields, and there is no matrix satisfying the equations (5,17). This explains why there cannot be any Schrödinger equation if the interaction is non-localized, and shows that one has to use a formalism giving directly the matrix $S = S(-\infty, +\infty)$.

6. Outgoing operators.

Before starting any actual calculation, it is necessary to antisymmetrize the Lagrange function in ψ and ψ^+ so as to introduce the correct interpretation of the negative energy states as antiparticles. Thus, the expression (5,10) for the interaction term should be replaced by

$$\left.\begin{aligned} &H_I(x', x'', x''') = \\ &(g/2) F(x', x'', x''') \, (\psi^+(x') u(x'') \psi(x''') - \psi(x''') u(x'') \psi^+(x')) \\ &- \delta(x' - x'') \delta(x'' - x''') \{(\Delta M/2) \, (\psi^+(x') \psi(x''') - \psi(x''') \psi^+(x')) \\ &- (\Delta m^2/2) u^2(x'')\}. \end{aligned}\right\} \qquad (6{,}1)$$

This clearly does not modify the general conclusions of the preceding sections. In particular the rules given for the calculation of the outgoing fields still apply.

The Fourier expansion of any free field is a superposition of plane waves e^{iKx}, where K is always a time-like vector. Hence, it is possible to split in a Lorentz invariant way any field function in two parts for which K^4 is positive or negative, respectively. Thus, we can write for the incoming fields (in what follows the incoming fields will be called ψ and u without the subscript in)

$$\left.\begin{aligned} \psi(x) &= \psi^{(+)}(x) + \psi^{(-)}(x), \\ \psi^+(x) &= \psi^{+(+)}(x) + \psi^{+(-)}(x), \\ u(x) &= u^{(+)}(x) + u^{(-)}(x). \end{aligned}\right\} \quad (6,2)$$

In the decomposition (6,2), $\psi^{(+)}$ and $\psi^{+(-)}$ are annihilation and creation operators of nucleons, respectively; $\psi^{+(+)}$ and $\psi^{(-)}$ are annihilation and creation operators of antinucleons; $u^{(+)}$ and $u^{(-)}$ are annihilation and creation operators of mesons[11]. These operators are related to one another by the equations

$$(\psi^{(+)})^+ = \psi^{+(-)}, \quad (\psi^{(-)})^+ = \psi^{+(+)}, \quad u^{(+)*} = u^{(-)}. \quad (6,3)$$

The operators introduced in (6,2) commute or anticommute except creation and annihilation operators of the same particles. For these pairs of operators the commutation relations are

$$\left.\begin{aligned} i\left[\psi_\varrho^{(+)}(x_1), \psi_\sigma^{+(-)}(x_2)\right]_+ &= S_{\varrho\sigma}^{(+)}(x_1 - x_2), \\ i\left[\psi_\varrho^{(-)}(x_1), \psi_\sigma^{+(+)}(x_2)\right]_+ &= S_{\varrho\sigma}^{(-)}(x_1 - x_2), \\ i\left[u^{(+)}(x_1), u^{(-)}(x_2)\right] &= D^{(+)}(x_1 - x_2), \\ i\left[u^{(-)}(x_1), u^{(+)}(x_2)\right] &= D^{(-)}(x_1 - x_2), \end{aligned}\right\} \quad (6,4)$$

where $S^{(+)}$ and $S^{(-)}$, $D^{(+)}$ and $D^{(-)}$ are the positive and negative frequency parts, respectively, of S and D.

An important notion in field theoretical calculations is that of "ordered product" of operators[16]. It is defined as follows:

a) The ordered product: $abc\cdots:$ of the creation or annihilation operators $a, b, c\cdots$ is equal to the product $abc\cdots$ re-ordered in such a way that all annihilation operators are at the right-hand

side of the creation operators, multiplied by $(-)^p$, where p is the number of permutations of nucleon operators involved in the reordering procedure.

b) The definition is extended to products of field functions by decomposing the various factors into sums of creation and annihilation operators, and by postulating the distributivity of the ordered product with respect to addition.

The importance of the ordered products comes from the fact that when an ordered product acts on a state vector a particle cannot be created by one of the factors and reabsorbed by another factor. Thus, all virtual particles have been eliminated, and it is easy to select the relevant terms for a particular problem. In this connection it is important to be able to transform any product of operators into a sum of ordered products. This is most conveniently done by introducing the notion of "contractions."

For two field functions a and b the contraction $a^{\cdot}b^{\cdot}$ is defined as the difference between the regular and the ordered products by(*)

$$ab = :ab: + a^{\cdot}b^{\cdot}. \tag{6,5}$$

The only non-vanishing contractions are given by the following relations which are easily deduced from (6,4)

$$\left.\begin{aligned} \psi_\varrho^{\cdot}(x_1)\psi_\sigma^{+\cdot}(x_2) &= (-i)S_{\varrho\sigma}^{(+)}(x_1 - x_2), \\ \psi_\varrho^{+\cdot}(x_2)\psi_\sigma^{\cdot}(x_1) &= (-i)S_{\varrho\sigma}^{(-)}(x_1 - x_2), \\ u^{\cdot}(x_1)u^{\cdot}(x_2) &= (-i)D^{(+)}(x_1 - x_2). \end{aligned}\right\} \tag{6,6}$$

A contraction within an ordered product is defined by

$$:a \cdots bc^{\cdot}d \cdots ef^{\cdot}g \cdots: = (-)^p c^{\cdot}f^{\cdot}\ :a \cdots bd \cdots eg \cdots:, \tag{6,7}$$

where p is the number of permutations of nucleon operators necessary in order to bring the factors c and f beside one another.

The transformation of a product of field functions into a sum of ordered products is now given by the following identity[16]:

$$abcd \cdots = :abcd \cdots: + \sum :a^{\cdot}b^{\cdot}cd \cdots:, \tag{6,8}$$

where the summation is extended to all possible contractions of the factors a, b, c, $\cdots$

(*) This definition is that of HOURIET and KIND[16].

It should be noted finally that the order of the factors in an ordered product can be changed arbitrarily, with only a change of sign if an odd permutation of the nucleon operators has been performed.

The results of the preceding section together with the identity (6,8) make it possible to express any function of the outgoing operators as a sum of ordered products of incoming operators. Each term may be associated with a doubled Feynman graph. The rules will now be given for a single outgoing operator. These rules generalize some of the results obtained by DYSON[17].

Graph. A graph consists of directed lines (nucleon lines), of undirected lines (meson lines), and of vertices of the following types:

a) g-vertices consisting of three points x', x'', x''' on a small circle with an undirected line arriving at x'', and two directed lines arriving at x' and x''', directed away from x' and toward x''';

b) Δm^2-vertices and ΔM-vertices consisting of a single point with two undirected lines or two directed lines of different directions, respectively;

c) one incoming vertex consisting of p points with a line arriving at each of them;

d) one outgoing vertex consisting of one point with one line. The line arriving at the outgoing vertex is an undirected line, a line directed toward the vertex or a line directed away from the vertex, depending on whether the field function which is being computed is u^{out}, ψ^{out} or $\psi^{+\text{out}}$.

Doubled graph. Some of the lines of the graph must be considered as doubled lines. The doubled lines should be drawn in such a way that it is possible to go from any vertex (the incoming vertex excepted) to the outgoing vertex by a uniquely defined path consisting of doubled lines only. This can be pictured by saying that all vertices except the incoming vertex are lying on the branches of a "tree" having its root at the outgoing vertex and forks at some of the g-vertices. It follows that all graphs are connected, and it is easily seen that the number of doubled lines is equal to the number of vertices, incoming and outgoing vertices excluded. The lines arriving at the incoming or outgoing vertices will be called incoming or outgoing lines.

Orientation of the graph. The doubled lines and the incoming lines are oriented toward the outgoing vertex. Every g-vertex should be oriented in the following sense. At every g-vertex arrive one double line oriented away from the vertex and two other lines l_1, l_2. To orient the vertex means to draw an arrow from one of the lines l_1, l_2 toward the other, in an arbitrary way. Consider finally an undoubled line joining two vertices a and b. It is possible to go from a and b to the outgoing vertex by following doubled lines only. The two paths meet at a vertex ξ. The line ab is then oriented according to the orientation of ξ.

Examples of such graphs will be given in the next section (see Fig. 2).

To each graph corresponds a term in the development of the outgoing operator. It is an integral of a product of terms associated with each line and each vertex of the graph. It is convenient to use the energy-momentum variables. A four-vector k is then associated with every line of the graph, and the various factors will be listed now.

We shall write only the factors corresponding to the undirected lines from which the factors corresponding to the directed lines can be deduced by replacing m by M and by multiplying by $(ik^\mu\gamma_\mu - M)$ or $(-ik^\mu\gamma_\mu - M)$ depending on whether the orientation and the direction are parallel or antiparallel.

a) For the doubled lines (except the outgoing line) the factor is

$$D_+(k) = (-1)/(k^2 + m^2), \text{ or } S_+, \text{ or } S_+^+, \tag{6,9}$$

where the integration with respect to k^4 should be taken in the complex plane along a contour passing above the two singularities.

b) For the undoubled lines the factor is

$$D^{(+)}(k) = (-\pi)\,(1 + \varepsilon(k))\,\delta(k^2 + m^2), \text{ etc. } \cdots, \tag{6,10}$$

where $\varepsilon(k)$ is $+1$ or -1 depending on whether k^4 is positive or negative.

c) For the outgoing line the factor is

$$D(k) = (-2i\pi)\,\varepsilon(k)\,\delta(k^2 + m^2), \text{ etc. } \cdots. \tag{6,11}$$

d) Finally, to the incoming lines is associated an ordered product of factors $u(k)\,\delta(k^2 + m^2)$, $\psi^+(k)\,\delta(k^2 + M^2)$, or $\psi(k)\ \delta(k^2 + M^2)$, where

$$u(k)\delta(k^2 + m^2) = \int dx\, u^{\text{in}}(x)\, e^{-ikx}, \text{ etc.} \cdots.$$

To each undirected line, to each directed line with direction and orientation parallel or antiparallel corresponds a factor u, ψ or ψ^+, respectively.

The factors corresponding to Δm^2-, ΔM-, and g-vertices are in the case where all lines are oriented toward the vertex

$$\left.\begin{aligned} &(-\Delta m^2/2)\,\delta(k_1 + k_2), \quad -\Delta M\,\delta(k_1 + k_2), \text{ and} \\ &(g/2)\,\Phi(k', k'', k''')\,\delta(k' + k'' + k''') = \\ &(g/2)\,(2\pi)^{-6}\int d\xi\, F(x', x'', x''')\, e^{i(kx' + \cdots)}, \end{aligned}\right\} \quad (6{,}12)$$

where k', k'', k''' are the vectors associated with the lines arriving at x', x'', x'''. For every line oriented away from the vertex, k should be replaced by $-k$ in (6,12).

Finally, summation should be made over the spinor indexes, the term should be multiplied by $1/N(T)$ and a certain power of i, and integrated over all variables. The outgoing operator is obtained by taking into account all possible graphs and all possible orientations of the vertices.

As for the justification of the preceding rules we only mention that the doubled lines correspond to the elementary commutators, and the undoubled lines to the contractions. The orientation of a vertex corresponds to the effect of the choice of a term $\psi^+ u\psi$ or $\psi u\psi^+$ in the interaction term (6,1) on the order of the operators occurring in the T-term.

The extension to products of outgoing operators is obvious. The only change is that there will be an outgoing line corresponding to every factor of the product. These lines are oriented with respect to one another in the following way: the orientation goes from the line a to the line b when the factor corresponding to a in the product is at the left of the factor corresponding to b.

7. Self-energies.

It has been assumed throughout that ΔM and Δm^2 are chosen in such a way that the interaction term is negligible when the particles described by the field are far apart. We shall now com-

pute the values of ΔM and Δm^2 for which this assumption holds.

An equivalent, but more precise formulation of the same assumption is to state that the interaction term should be rigorously negligible if the system contains zero or one particle. In the interaction representation this fact is described by the equation

$$S|) = |), \tag{7,1}$$

where $|)$ is the vacuum state or a state in which only one particle is present. The corresponding properties of the incoming and outgoing fields follow from

$$A^{\text{out}} = S^* A^{\text{in}} S, \tag{7,2}$$

where A is any field function. In order to avoid complicated notations we shall use a simplified model in which states are characterized only by the number of particles. Moreover, we shall assume the existence of only one kind of particles. The basic vectors may then be represented by $|0), |1), \cdots |n), \cdots$, where $|n)$ is the state in which n particles are present. In this representation the conditions (7,1) read

$$(n|S|0) = \delta_{0n}, \quad (n|S|1) = \delta_{1n}, \tag{7,3a}$$

where δ_{mn} is 0 if $m \neq n$ and 1 if $m = n$. From the unitarity condition of S it follows that

$$(0|S|n) = \delta_{0n}, \quad (1|S|n) = \delta_{1n}. \tag{7,3b}$$

The relation (7,2) can be written

$$(i|A^{\text{out}}|j) = \sum_{m,n} (i|S^*|m)\,(m|A^{\text{in}}|n)\,(n|S|j). \tag{7,4}$$

For $j = 0$, the equation (7,4) specializes into

$$(i|A^{\text{out}}|0) = \sum_{m} (i|S^*|m)\,(m|A^{\text{in}}|0), \tag{7,5}$$

where the conditions (7,3) have been taken into account. If A is a pure annihilation operator $A^{(+)}$, then $A^{(+)\text{in}}|0) = 0$, and from (7,5) it follows that

$$A^{(+)\text{out}}|0) = 0, \tag{7,6}$$

which is the equation of conservation of vacuum. If A is a creation operator $A^{(-)}$, its only non-vanishing matrix element is $(1\,|\,A^{(-)\text{in}}\,|\,0)$, and from (7,3) and (7,5) it follows that

$$A^{(-)\,\text{out}}\,|\,0) = A^{(-)\,\text{in}}\,|\,0). \tag{7,7}$$

Putting now $j = 1$ into (7,4) we get

$$(i\,|\,A^{\text{out}}\,|\,1) = \sum_m (i\,|\,S^*\,|\,m)\,(m\,|\,A^{\text{in}}\,|\,1), \tag{7,8}$$

and a simple relation is obtained only if A is an annihilation operator. It follows then from (7,3) and (7,8) that

$$A^{(+)\,\text{out}}\,|\,1) = A^{(+)\,\text{in}}\,|\,1). \tag{7,9}$$

No simple relation is obtained for $j > 1$. The relation (7,9) is in fact a consequence of (7,6) and (7,7), and of the commutation relations

$$[A^{(+)\,\text{in}},\, A^{(-)\,\text{in}}] = [A^{(+)\,\text{out}},\, A^{(-)\,\text{out}}] = 1. \tag{7,10}$$

On multiplying (7,10) on the right-hand side by $|0)$ and on taking (7,6) into account we get

$$A^{(+)\,\text{out}}\,A^{(-)\,\text{out}}\,|\,0) = A^{(+)\,\text{in}}\,A^{(-)\,\text{in}}\,|\,0). \tag{7,11}$$

The equation (7,9) follows from (7,11) if one takes into account (7,7) and the fact that $A^{(-)\,\text{in}}\,|\,0)$ is a multiple of $|\,1)$. Thus, the basic relations are (7,6) and (7,7). It is in fact a matter of simple algebra to show that, conversely, these relations have the relations (7,3) as consequences (apart from an irrelevant phase factor).

For the actual system, equations similar to (7,6) and (7,7) should hold with $A = \psi^+$, u or ψ. We shall see that the equations (7,6) are identically satisfied, whereas the equations (7,7) define the self-energies ΔM and Δm^2.

Let A be any of the field functions. The Fourier component $A^{\text{out}}(k)$ is, according to the results of the preceding section, given by a sum of terms

$$A^{\text{out}}(k) = \sum_n \int dk_1 \cdots dk_j K_n(k_1, \cdots k_j) : a_1(k_1) \cdots a_j(k_j) :, \tag{7,12}$$

where K_n is a c-number function, and where the k_i satisfy

$$k = \sum k_i, \quad (k_i)^2 + m_i^2 = 0. \tag{7,13}$$

Only the creation parts of the operators a will give contributions to $A^{\text{out}}|0)$. Thus we may in (7,12) restrict the domain of integration to the vectors k such that

$$k_i^4 < 0. \tag{7,14}$$

If $A = A^{(+)}$ is an annihilation operator all its Fourier components are such that $k^4 > 0$. It follows then from (7,13 and 7,14) that $A^{(+)\,\text{out}}|0)$ vanishes identically.

We consider now the case where $A = A^{(-)}$ is a creation operator. Then K_n vanishes except for k such that

$$k^2 + \mu^2 = 0, \quad k^4 < 0,$$

where μ is the mass of the particles described by the field A. From $k = \sum k_i$ it follows that

$$|\boldsymbol{k}| \leqslant \sum |\boldsymbol{k}_i|, \text{ and}$$
$$(k^4)^2 \geqslant \sum (k_i^4)^2 + 2 \sum_{i<k} m_i m_k,$$

where it has been taken into account that all k_i^4 are negative and have m_i as minimum absolute values. From these two inequalities it is easily deduced that

$$\mu \geqslant \sum m_i. \tag{7,15}$$

Finally, we shall also need the remark that if $A = u$, there will be among the operators a as many ψ as ψ^+; if $A = \psi$ (or ψ^+) there will be as many ψ as ψ^+ plus one odd ψ (or ψ^+).

It follows that if $A = \psi$ (or ψ^+), one of the m_i in (7,15) must be equal to $M = \mu$, and we get a contradiction if we assume the existence of more than one m_i. Thus, contributions to $\psi^{(-)\,\text{out}}|0)$ come only from the terms in which there is only one operator $a_1 = \psi$.

If $A = u$ and if we assume that an a is equal to u, the same argument applies and there cannot be any other factor a. The possibility of all a being nucleon operators is ruled out by (7,15) if $m < 2M$(*).

(*) If $m > 2M$, spontaneous decay of a meson into a pair of nucleons becomes possible, and one cannot expect the equation (7,1) to hold for a state in which there is one meson.

Thus, in all cases, the only terms in (7,12) giving contributions to $A^{(-)\,\mathrm{out}}|0)$ are those which contain only one $a = A$. For these terms K_n is a Lorentz invariant function of one argument k only, satisfying the equations

$$(k^2 + m^2)K = 0, \quad \text{or } (ik^\mu \gamma_\mu + M)K = 0, \text{ etc. } \cdots.$$

Hence K_n is proportional to

$$D^{(-)}(k) \cong (1 - \varepsilon(k))\,\delta(k^2 + m^2), \text{ or } S^{(-)}(k), \text{ etc. } \cdots,$$

and if we write $K_n(k) = K_n D^{(-)}(k)$, etc. $\cdots$ we obtain, corresponding to the equation (7,7), the scalar equations

$$\sum K_n = 0. \tag{7,16}$$

As the equations corresponding to ψ and ψ^+ are not distinct, we have two equations defining ΔM and Δm^2.

We shall now investigate the convergence of the integrals occurring in the K_n. Each K_n corresponds to a self-energy graph (graph with one incoming line and one outgoing line) and is given by the rules of paragraph 6.

First of all some of the integrations should be carried out in order to eliminate all the δ functions introduced at the vertices except one $\delta(k^{\mathrm{in}} - k^{\mathrm{out}})$ which expresses the conservation of energy and momentum between the incoming and the outgoing lines. These lines can be considered as associated with a fixed vector $k_0 = k^{\mathrm{in}} = k^{\mathrm{out}}$. There is some arbitrariness in the choice of the variables which should be conserved. We shall show that one can always take as independent variables of integration the vectors p_i associated with the undoubled lines. We have to show

a) that no relation such as $\sum \pm p_i = 0$ can exist,

b) that every vector k associated with a doubled line can be expressed as $k = \sum \pm p_i$, or $k = k_0 + \sum \pm p_i$.

It is easily seen that every relation $\sum \pm k_i \pm p_j = 0$ can be represented on the graph by a closed curve C leaving the incoming and the outgoing vertices outside and cutting the lines associated with the k_i and the p_i involved in the relation.

The assertion a) follows then from the fact that no line C can cut undoubled lines only (the vertices inside could not be connected with the outgoing vertex by means of doubled lines).

The assertion b) follows from the fact that it is always possible to draw a line C which cuts a given doubled line and no other doubled line except, maybe, the incoming line (this is a consequence of the tree-like structure of the doubled lines).

Divergences in the self-energy terms arise from two causes: divergences coming from the large values of the variables of integration and divergences due to the coincidence of several poles of the integrand. The latter type of divergence appears in the terms coming from graphs containing one or more self-energy graphs as sub-graphs. It can be seen that these divergences cancel in the sum (7,16).

The real self-energy divergences come from the large values of the variables p_i. At every g-vertex, the form function introduces a convergence factor Φ (6,12). We can assume that Φ which is a function of k'^2, k''^2 and k'''^2 falls off very rapidly as any of these arguments becomes large. Consequently, we can consider that the domain of integration of the variables p_i is practically limited to the values for which all vectors k associated with the various lines of the graph have bounded four-dimensional lengths k^2.

The following property will be useful: if a time-like vector k has a bounded scalar product with a fixed time-like vector k', its four components are bounded. This is easily seen in a frame of reference where k' reduces to a time component. More precisely, it can be shown that

$$|\boldsymbol{k}| < A(|\boldsymbol{k}'| + |k'^4|)/|(k')^2| \tag{7,17}$$

if $|kk'| < A$.

In every graph the undoubled lines will form a certain number of connected arcs. If there are two such arcs connected with the incoming and the outgoing lines, we shall call them L^{in} and L^{out}; the other arcs will be called L_n.

We shall first consider a term which has no other undoubled lines than L^{in} and L^{out}. Let $p_1, p_2, \cdots$ be the vectors associated with the lines forming L^{in}, starting from the incoming line, and $p'_1, p'_2, \cdots$ be the corresponding quantities associated with L^{out}. The functions Φ limit then the domain of integration to values of the variables p_i and p'_i such that the scalar products

$$k_0 p_1,\ p_1 p_2,\ p_2 p_3, \cdots \qquad k_0 p'_1,\ p'_1 p'_{\text{2}},\ p'_2 p'_3, \cdots$$

are bounded. As k_0 is a fixed time-like vector and the p satisfy equations such as $p^2 + m^2 = 0$, it follows from the inequality (7,17) that the corresponding domain of integration is bounded. Thus, the corresponding terms are convergent, such as, for instance, the second-order self-energies.

We consider now the case where there are other undoubled

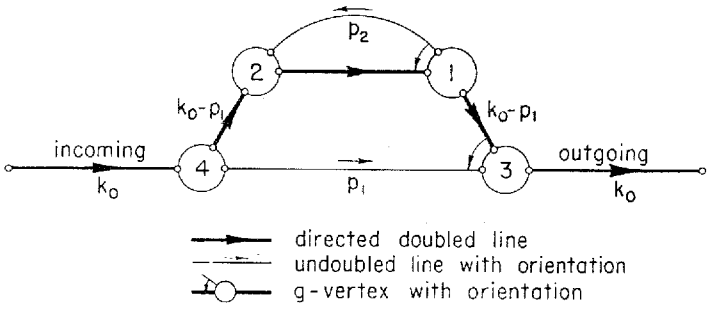

Fig. 2.

lines besides L^{in} and L^{out}. First of all, we know that the vectors associated with L^{in} and L^{out} have bounded components. The same holds for those vectors associated with the doubled lines which are linear combinations of the vectors of L^{in} and L^{out} only. Moreover, if one vector of a line L_n is kept fixed, all other vectors of the same line have bounded components. Some other scalar products will be kept bounded by the effect of the functions Φ. However, for certain graphs these conditions are not sufficient, due to the fact that the vectors associated with the doubled lines are not always time-like vectors. Fig. 2 shows an example of such a case. It is seen that the vertices 1 and 2 give the condition that the scalar product of p_2 with $k_0 - p_1$ should be bounded. The vector $k_0 - p_1$ has bounded components, but as it may be a space-like vector, the components of p_2 are not bounded, and the corresponding divergence remains.

A tentative way out of this difficulty is to assume that the function Φ should be different from zero only if the three vectors k', k'', k''' are time-like vectors. Moreover, we can assume that Φ vanishes if $|k'^2|$, $|k''^2|$ or $|k'''^2|$ are less than a fixed number which may be chosen arbitrarily small. The inequality (7,17) can then be applied to all vectors, and it is clear that all integrals become convergent. The assumption made here does not contradict any

condition previously formulated for the form functions which limits the non-localizability to small domains. It is, however, a very large departure from conventional field theory, especially since it makes many virtual transitions impossible, and its physical consequences should be investigated more completely.

So far, we have been concerned only with the self-energies. However, the matrix elements of any operator lead to integrals very similar to the self-energy integrals, and the investigation of convergence we have made is of quite general validity.

8. Final remarks.

a) *Electromagnetic interactions.*

When the interactions with the electromagnetic field are introduced, new terms have to be added to the Lagrange function so as to make it gauge invariant. The situation in the present theory differs from that in the conventional theory by the fact that the interaction term is not gauge invariant in itself.

A gauge transformation is defined by

$$\bar{A}_\mu(x) = A_\mu(x) + \frac{\partial \Lambda}{\partial x^\mu}, \quad \bar{\psi}(x) = \psi(x) e^{i\varepsilon\Lambda}, \quad \bar{\psi}^+(x) = \psi^+(x) e^{-i\varepsilon\Lambda}, \quad (8,1)$$

where $\Lambda(x)$ is any function such that $\Box\Lambda(x) = 0$. It follows that

$$\bar{\psi}^+(x')\,\bar{\psi}(x''') = \psi^+(x')\,\psi(x''')\,e^{i\varepsilon(\Lambda(x''')-\Lambda(x'))}, \qquad (8,2)$$

which shows the lack of gauge invariance if the form function allows x' to be different from x'''. On using (8,1), however, one can write

$$\Lambda(x''') - \Lambda(x') = \int_C \frac{\partial \Lambda}{\partial x^\mu}\,dx^\mu = \int_C (\bar{A}_\mu - A_\mu)\,dx^\mu, \qquad (8,3)$$

where C is an arbitrary path going from x' to x'''. Substitution of (8,3) into (8,2) yields

$$\bar{\psi}^+(x')\,e^{-i\varepsilon\int_C \bar{A}_\mu\,dx^\mu}\,\bar{\psi}(x''') = \psi^+(x')\,e^{-i\varepsilon\int_C A_\mu dx^\mu}\,\psi(x'''). \qquad (8,4)$$

This equation expresses a gauge invariance property which holds for any path C although the invariant expression depends on the

choice of the path. Considerations of invariance and of simplicity suggest taking as path C the straight line joining x' and x'''. The final expression for the corresponding Hermitian gauge invariant interaction term reads then

$$(g/2)\int dx'dx''dx''' F(x', x'', x''')\psi^{+}(x')\left[u(x''), e^{-i\varepsilon\int_{x'}^{x'''}A_\mu dx^\mu}\right]_{+}\psi(x'''), \quad (8,5)$$

where the integral $\int_{x'}^{x'''}$ is taken along a straight line.

The problem of finding a gauge invariant interaction term which in the limit $A_\mu = 0$ reduces to (1,1) has no unique solution. A very general expression is obtained on replacing in (8,5) the exponential function by an integral in the functional space of all paths going from x' to x''' which may be written

$$\int dC\,\varrho(C)\,e^{-i\varepsilon\int_C A_\mu dx^\mu}. \quad (8,6)$$

In this expression dC is the volume element in the functional space. The weighting function $\varrho(C)$ must be normalized according to

$$\int dC\,\varrho(C) = 1,$$

and such that (8,6) is invariant under all displacements.

The introduction of exponential factors into the interaction term can be pictured as describing the effect of the electric charge jumping between the points x' and x'''. The path C can be considered as the path followed by the electric charge between the two points and $\varrho(C)$ as a sort of probability distribution of all possible paths. The function $\varrho(C)$, however, need not be positive everywhere.

The interpretation of the extra interaction term as describing the motion of the charge between x' and x''', that is over distances of the order of λ, shows that its effects will, presumably, be small. It was, however, important to show that no contradiction with the requirement of gauge invariance arises from the introduction of form functions. It is remarkable that the electromagnetic properties of charged particles are modified by the interaction with a neutral field when the interaction is of a non-localized type.

b) *Transition probabilities.*

The transition probability between a state a and a state b can, in principle, be computed as the average value in state a of the projection operator on state b, or conversely. The projection operators can be computed by the methods developed above, and the convergence proof holds. Practically, however, it is simpler to guess the lowest order terms of the S-matrix from the equations (4,9) (7,8). In connection with the difficulty of solving the latter equations, it can be asked whether another method of quantization would not give the S-matrix more directly. In fact, the Lagrangian formulation of quantum mechanics developed by FEYNMAN[18] can be applied to the present problem. The result is quite simple: the only modification to the Feynman rules is the introduction of form functions at every vertex of the graphs. This solution, however, cannot be accepted since the corresponding matrix is not unitary. A calculation, for instance, of the second order unitarity condition yields after some manipulations an irreducible sum of terms involving factors such as

$$F(x_1', x_1'', x_1''') S_+(x_1''' - x_2') S_-(x_1' - x_2''') F(x_2', x_2'', x_2'''),$$

which clearly vanish in the limit of a local interaction, due to the properties of the retarded and advanced Green's functions, but do not vanish in the more general case.

In conclusion, I should like to express my gratitude to Professor C. MØLLER for his advice and encouragement, and to Professor N. BOHR for the hospitality of the Institute of Theoretical Physics during my stay. I am indebted to Dr. R. GLAUBER for many helpful comments on the manuscript. The foregoing work was supported by the Direction des Mines et de la Sidérurgie in Paris.

Appendix I.

Definition of some singular functions.

The singular functions used in field theory are conveniently defined by the Fourier integral

$$D(x) = -(2\pi)^{-4}\int dk\, e^{ikx}/(k^2 + m^2).$$

Since the integrand is singular for $k^2 + m^2 = 0$, it is convenient to perform the integration with respect to k^4 in the complex plane

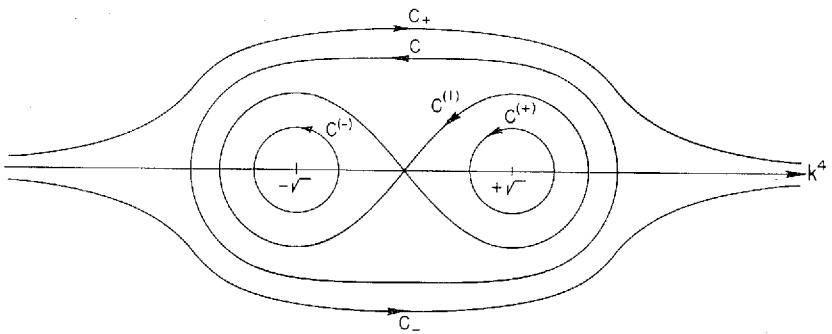

Fig. 3.

along a contour avoiding the singularities. The integrand has two poles at $k^4 = \sqrt{k^2 + m^2}$, and $k^4 = -\sqrt{k^2 + m^2}$. The contours C_+ and C_- (Fig. 3) yield the retarded and advanced Green's functions D_+ and D_-. The contour $C = C_- - C_+$ yields the function D occurring in the commutation relations. The decomposition $C = C^{(+)} + C^{(-)}$ corresponds to the decomposition of D into the positive and negative frequency parts $D^{(+)}$ and $D^{(-)}$. Finally, the contour $C^{(1)}$ yields the function $D^{(1)}$.

If we call $\Delta(x)$ the functions similar to the D functions where the mass m is replaced by M, the functions S are given by

$$S(x) = \left(-\gamma^\mu \frac{\partial}{\partial x^\mu} + M\right)\Delta(x).$$

Appendix II.

Commutation relations of the outgoing fields.

The outgoing fields are solutions of the free field equations. Assuming that the fields are enclosed in a large cube of volume V, we can expand them in Fourier series as

$$\left.\begin{aligned} \psi^{\text{out}}(x) &= V^{-1/2}\sum_{\boldsymbol{K},r} a^r_{\boldsymbol{K}}\psi^r_+(K)\,e^{iKx} + b^{*r}_{\boldsymbol{K}}\psi^r_-(K)\,e^{-iKx}, \\ u^{\text{out}}(x) &= V^{-1/2}\sum_{\boldsymbol{k}} (2k^4)^{-1/2}(v_{\boldsymbol{k}}e^{ikx} + v^*_{\boldsymbol{k}}e^{-ikx}), \end{aligned}\right\} \quad \text{(II,1)}$$

where $\boldsymbol{K}$ is the space part of the vector K ($K^4 = +\sqrt{\boldsymbol{K}^2 + M^2}$), $\boldsymbol{k}$ is the space part of k^μ ($k^4 = +\sqrt{\boldsymbol{k}^2 + m^2}$), and $\psi^r_\pm(K)$ ($r = 1,2$) are the positive and negative energy solutions, respectively, of the equations $(\pm iK_\mu\gamma^\mu + M)\psi^r_\pm(K) = 0$, orthonormalized according to

$$\psi^{*r}_+(K)\psi^s_+(K) = \delta^{rs}, \qquad \psi^{*r}_-(K)\psi^s_-(K) = \delta^{rs}.$$

Substituting these developments into the expressions of the constants of collision we get

$$\left.\begin{aligned} G^\mu &= \sum_{\boldsymbol{K},r} K^\mu(a^{*r}_{\boldsymbol{K}}a^r_{\boldsymbol{K}} - b^r_{\boldsymbol{K}}b^{*r}_{\boldsymbol{K}}) + (1/2)\sum_{\boldsymbol{k}} k^\mu(v^*_{\boldsymbol{k}}v_{\boldsymbol{k}} + v_{\boldsymbol{k}}v^*_{\boldsymbol{k}}), \\ Q &= \varepsilon\sum_{\boldsymbol{K},r} a^{*r}_{\boldsymbol{K}}a^r_{\boldsymbol{K}} + b^r_{\boldsymbol{K}}b^{*r}_{\boldsymbol{K}}, \end{aligned}\right\} \quad \text{(II,2)}$$

and a similar expression for the angular momentum. Substitution of the developments (II,1) and (II,2) into the relations (4,8) gives

$$\left.\begin{aligned} [a^r_{\boldsymbol{K}}, H^{+r}_{\boldsymbol{K}}] &= a^r_{\boldsymbol{K}}, & [a^{*r}_{\boldsymbol{K}}, H^{+r}_{\boldsymbol{K}}] &= -a^{*r}_{\boldsymbol{K}}, \\ [b^{*r}_{\boldsymbol{K}}, H^{-r}_{\boldsymbol{K}}] &= b^{*r}_{\boldsymbol{K}}, & [b^r_{\boldsymbol{K}}, H^{-r}_{\boldsymbol{K}}] &= -b^r_{\boldsymbol{K}}, \\ [v_{\boldsymbol{k}}, H_{\boldsymbol{k}}] &= v_{\boldsymbol{k}}, & [v^*_{\boldsymbol{k}}, H_{\boldsymbol{k}}] &= -v^*_{\boldsymbol{k}}, \end{aligned}\right\} \quad \text{(II,3)}$$

where $H^{+r}_{\boldsymbol{K}} = a^{*r}_{\boldsymbol{K}}a^r_{\boldsymbol{K}}$, $H^{-r}_{\boldsymbol{K}} = b^r_{\boldsymbol{K}}b^{*r}_{\boldsymbol{K}}$ and $H_{\boldsymbol{k}} = (1/2)(v^*_{\boldsymbol{k}}v_{\boldsymbol{k}} + v_{\boldsymbol{k}}v^*_{\boldsymbol{k}})$: every other commutator of a quantity a, b or v with an H vanishes.

The incoming fields can be developed in the same way as the outgoing fields. The coefficients in the Fourier series will be operators a^{in}, b^{in} and v^{in} which not only satisfy commutation relations similar to (II,3), but also the following ones which result from (4,4)

$$[a_{\boldsymbol{K}}^{\mathrm{in},r},\ a_{\boldsymbol{K}}^{*\mathrm{in},r}]_+ = 1, \quad [b_{\boldsymbol{K}}^{\mathrm{in},r},\ b_{\boldsymbol{K}}^{*\mathrm{in},r}]_+ = 1, \quad [v_{\boldsymbol{k}}^{\mathrm{in}},\ v_{\boldsymbol{k}}^{*\mathrm{in}}] = 1; \text{ (II,4)}$$

all other commutators (or anticommutators if two operators a or b are involved) vanish.

The field equations give expressions of the outgoing operators as power series of the coupling constants. Each term of these series is a polynomial of the incoming operators. Thus the outgoing operators are continuous functions of the coupling constants, and have the incoming operators as limit if the coupling vanishes.

The problem now is to show that, as a consequence of (II,3), the outgoing operators satisfy commutation relations identical with (II,4).

WIGNER[19] has given the general form of the operators satisfying the relations

$$[v, H] = v, \quad [v^*, H] = -v^*, \quad H = (1/2)(v^*v + vv^*). \qquad \text{(II,5)}$$

His method can be applied with very little change (although the result is very different) to the case of operators satisfying the relations

$$[a, H] = a, \quad [a^*, H] = -a^*, \quad H = a^*a. \qquad \text{(II,6)}$$

One finds in this case an infinite number of irreducible representations with 0, 1, 2, 3, $\cdots$ or ∞ dimensions. Only in the case of a two-dimensional representation do the operators satisfy the relations

$$[a, a]_+ = 0, \quad [a^*, a^*]_+ = 0, \quad [a, a^*]_+ = 1. \qquad \text{(II,7)}$$

However, as there is only a discrete set of possibilities, and as the outgoing operators go continuously over to the incoming operators as the coupling goes to zero, it follows that the outgoing operators satisfy the relations (II,7). The same applies to any operator a or b.

Next, we take two operators a_1 and a_2. They satisfy besides the relations (II,7) the relations

$$[a_1, H_2] = [a_1^*, H_2] = [a_2, H_1] = [a_2^*, H_1] = 0. \qquad \text{(II,8)}$$

It is easily seen that these relations have as consequences the relations

$$a_1a_2 = ca_2a_1, \quad a_1a_2^* = c^*a_2^*a_1, \quad a_1^*a_2 = c^*a_2a_1^*, \quad a_1^*a_2^* = ca_2^*a_1^*, \qquad \text{(II,9)}$$

where c is any number such that $|c|^2 = 1$. The relations (II,9) are not symmetrical in a_1 and a_2. In order to make it more obvious we can write, for instance,

$$a_1 a_2 = c_{12} a_2 a_1, \qquad a_2 a_1 = c_{21} a_1 a_2.$$

We have then $c_{21} = c_{12}^*$. The operators a_1 and a_2 appear in fact as Fourier coefficients of two plane waves with propagation vectors $\boldsymbol{K}_1$ and $\boldsymbol{K}_2$ and spin orientations r_1 and r_2, and c must then be an invariant function of these quantities. As all invariant functions of two propagation vectors and two spin orientations are symmetrical we must have $c_{12} = c_{21}$. It follows that $c = +1$ or -1, and by continuity we see that $c = -1$. The same applies to any couple of operators a or b. Thus, we have shown that the fields $\psi^{+\,\mathrm{out}}$ and ψ^{out} satisfy the same commutation relations as the incoming fields.

The case of the operators v can be treated by very similar considerations. A slight complication comes from the fact shown by WIGNER that the representations of the operators satisfying the relations (II,5) depend on a continuous parameter which fixes in particular the zero-point energy. So the continuity argument does not apply here. It is, however, easily seen that the zero—point energy must be the same for the outgoing fields as for the incoming fields as a consequence of the conservation equation

$$G^4(\mathrm{out}) = G^4(\mathrm{in}).$$

The proof is then easily completed.

Appendix III.

On the solution of the field equations.

1. *Localized interaction.*

We assume that the formulas (5,6) hold up to the order $n-1$. The terms of order n can then be computed from the field equations (4,3). For ψ, for instance, one finds

$$\psi^{(n)}(x_0) = i \int dx_1 S_+(x_0 - x_1) \{gA(x_1) - \Delta M \psi^{(n-1)}(x_1)\}, \qquad \text{(III,1)}$$

where

$$A(x_1) = \sum_{p=0}^{n-1} u^{(p)}(x_1)\psi^{(n-p-1)}(x_1).$$

Explicitly $A(x_1)$ reads

$$A(x_1) = \sum_{p=0}^{n-1} \int_{\substack{x_1 > x'_1 > x'_2 > \cdots > x'_p \\ x_1 > x''_1 > x''_2 > \cdots > x''_{n-p-1}}} dx'_1 dx'_2 \cdots dx'_p dx''_1 dx''_2 \cdots dx''_{n-p-1}$$

$$[\cdots[u^{\text{in}}(x_1), H'_1], \cdots H'_p][\cdots[\psi^{\text{in}}(x_1), H''_1], \cdots H''_{n-p-1}].$$

If we call $x_2, x_3, \cdots x_n$ the points $x'_1, \cdots x'_p, x''_1, \cdots x''_{n-p-1}$ chronologically reordered, we can write for $A(x_1)$

$$A(x_1) = \int_{x_1 > x_2 > \cdots > x_n} dx_2 \cdots dx_n \sum \Big[\cdots\Big[u^{\text{in}}(x_1), H_{j_1}\Big], \cdots H_{j_p}\Big] \Big[\cdots\Big[\psi^{\text{in}}(x_1), H_{j_{p+1}}\Big], \cdots H_{j_{n-1}}\Big]$$

where the summation $\sum$ is extended to all permutations $j_1, j_2, \cdots j_{n-1}$ of $2, 3, \cdots n$ such that $j_1 < j_2 < \cdots < j_p$ and $j_{p+1} < j_{p+2} < \cdots < j_{n-1}$. It is now easily seen that

$$A(x_1) = \int_{x_1 > x_2 > \cdots > x_n} dx_2 dx_3 \cdots dx_n [\cdots[u^{\text{in}}(x_1)\psi^{\text{in}}(x_1), H_2], \cdots H_n]. \qquad \text{(III,2)}$$

After substitution in (III,1) of (III,2) and of the expression of $\psi^{(n-1)}$, we get

$$\psi^{(n)}(x_0) = i\int_{x_1 > x_2 > \cdots > x_n} dx_1 dx_2 \cdots dx_n S_+(x_0 - x_1)\left[\cdots\left[\frac{\delta L_{\text{I}}^{\text{in}}}{\delta\psi^+(x_1)}, H_2\right]\cdots H_n\right]$$

$$= \int_{x_1 > x_2 > \cdots > x_n} dx_1 dx_2 \cdots dx_n [\cdots[\psi^{\text{in}}(x_0), H_1], H_2], \cdots H_n]$$

where use has been made of the relation

$$\left.\begin{aligned} iS_+(x_0 - x_1)\frac{\delta L_{\text{I}}^{\text{in}}}{\delta\psi^+(x_1)} &= [\psi^{\text{in}}(x_0), H_1] \quad \text{if} \quad x_0 > x_1, \\ &= 0 \qquad\qquad\quad \text{if} \quad x_0 < x_1. \end{aligned}\right\}$$

A similar treatment applies to the other field functions. Thus, (5,6) holds to all orders.

As for the outgoing fields, the term of order n of ψ, for instance, is given by

$$\psi^{\text{out}\,(n)}(x) = i\int dx_1 S(x-x_1)\{gA(x_1) - \Delta M\psi^{(n-1)}(x_1)\},$$

and on using (III,2) and the relation

$$iS(x-x_1)\frac{\delta L_{\text{I}}^{\text{in}}}{\delta\psi^{+}(x_1)} = [\psi^{\text{in}}(x),\ H_1],$$

we get

$$\psi^{\text{out}\,(n)}(x) = \int_{x_1>x_2>\cdots>x_n} dx_1 dx_0 \cdots dx_n [\cdots [\psi^{\text{in}}(x),\ H_1],\cdots H_n].$$

2. *Non-localized interaction.*

The preceding proof can be extended immediately to the case of a non-localized interaction. The only delicate point is the transformation of the expression called $A(x_1)$. Presently, we have to show the identity

$$\left.\begin{aligned}\sum_{p=0}^{n-1}\int_t d\xi_1'\cdots d\xi_n'[\cdots[u^{\text{in}}(x_0''),\ H_1'],\cdots H_p']\int_t d\xi_1''\cdots d\xi_{n-p-1}''\\ [\cdots[\psi^{\text{in}}(x_0'''),\ H_1''],\cdots H_{n-p-1}''] =\\ = \int_t d\xi_1\cdots d\xi_n[\cdots[u^{\text{in}}(x_0'')\psi^{\text{in}}(x_0'''),\ H_1],\ H_2],\cdots H_n].\end{aligned}\right\} \quad \text{(III,3)}$$

In the left-hand side the variables ξ' on the one hand, and the variables ξ'' on the other hand, are ordered in time independently. We have to develop both sides of (III,3) and to compare the results. The integrand in the right-hand side can first be expanded as

$$\sum\left[\cdots\left[u^{\text{in}}(x_0''),\ H_{j_1'}\right],\cdots H_{j_p'}\right]\left[\cdots\left[\psi^{\text{in}}(x_0'''),\ H_{j_1''}\right],\cdots H_{j_{n-p-1}''}\right], \quad \text{(III,4)}$$

where the summation is extended to all permutations of 1, 2, 3, $\cdots n$ such that $j_1' < j_2' < \cdots < j_p'$, and $j_1'' < j_2'' < \cdots < j_{n-p-1}''$. Then, the term in u^{in} and the term in ψ^{in} have to be developed. Thus a term T of (III,4) is the product of a term T_u coming from the first factor, and a term T_ψ coming from the second factor. The term T_u will also appear in the development of the first factor in the left-hand side of (III,3), and T_ψ will appear in the development of the second factor. The associated domains are clearly the same in both sides. Finally, the rule that each term T should be multiplied by $1/N(T)$ is conveniently replaced here by the

rule that only one term in each family of equivalent terms should be taken into account. It follows that each term appears the same number of times in both sides of (III,3), and this completes the proof.

The formulas for the outgoing fields and the products of outgoing fields are merely generalizations of (III,3).

References.

(1) R. E. Peierls and H. MacManus, Proc. Roy. Soc. A **195**, 323, (1948).
(2) H. Yukawa, Phys. Rev. **76**, 300, 1731, (1949), **77**, 219, (1950).
(3) C. Bloch, Dan. Mat. Fys. Medd. **26**, no. 1, (1950).
(4) C. N. Yang and D. Feldman, Phys. Rev. **79**, 972, (1950).
(5) G. Källen, Arkiv för Fysik, **2**, 371, (1950).
(6) C. Bloch, Prog. theor. Phys. **5**, 606, (1950.)
(7) J. Rayski, Phil. Mag. **42**, 1289, (1951).
(8) C. Møller, Communication at the Conference on Nuclear Physics, Copenhagen, (1951).
(9) A. Pais and G. E. Uhlenbeck, Phys. Rev. **79**, 145, (1950).
(10) R. P. Feynman, Phys. Rev. **76**, 749, 769, (1949).
(11) J. Schwinger, Phys. Rev. **75**, 651, (1949).
(12) W. Pauli and F. Villars, Rev. Mod. Phys. **21**, 434, (1949).
(13) G. N. Watson, Theory of Bessel functions, (Cambridge).
(14) C. Møller, Dan. Mat. Fys. Medd. **23**, no. 1, (1945).
(15) J. Schwinger, Phys. Rev. **74**, 1439, (1948).
(16) A. Houriet and A. Kind, Helv. Phys. Acta, **22**, 319; (1949) G. C. Wick, Phys. Rev. **80**, 268, (1950).
(17) F. J. Dyson, Phys. Rev. **82**, 428, (1951).
(18) R. P. Feynman, Phys. Rev. **80**, 440, (1950).
(19) E. P. Wigner, Phys. Rev. **77**, 711, (1950).

Indleveret til selskabet den 26. juni 1952.
Færdig fra trykkeriet den 6. november 1952.