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ON THE INTERNAL CONSTITUTION OF RELATIVISTICALLY DEGENERATE STARS

BY

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It is well known that in a first approximation white dwarf stars are such equilibrium configurations, which masses of matter with completely degenerate electrons take up under the influence of their own internal gravitational fields. The gravitational forces act on the electrons mainly through a small radial displacement of the heavy particles relative to the electrons. CHANDRASEKHAR [1] has in his theory of white dwarf stars taken account of the relativistic relation between energy and momentum of a particle in finding the equation of state of a relativistically degenerate electron gas.

We shall here investigate the influence of another relativistic effect, namely the "spin-orbit interaction", which is well known from the theory of the fine structure of the hydrogen spectrum. The star will be considered as a kind of THOMAS-FERMI atom, and we are thus using an approximation, which is well suited for the problem in question, even if it is not very good in the case of ordinary atoms. In the stellar interior we may namely deal with volume elements having linear dimensions that are small by a factor of about 10^9 in comparison with the dimensions of the star as a whole and still large by a similar factor in comparison with electronic wave lengths.

We place the origin of our co-ordinate system at the center of the spherically symmetric star and are then going to use DIRAC's equations for electrons in a central field.

The angle-dependent part of the solution is well known and is, independent of the form of the potential as a function of the distance from the center, leading to the following two simultaneous differential equations for two radial functions R_1 and R_2: (*cf.* A. SOMMERFELD: Wellenmechanik, Ch. IV, § 7. [2])

$$\left.\begin{aligned}\left(\frac{d}{dr}+\frac{1-k}{r}\right)R_1 &= \frac{1}{\hbar c}(E-V+E_0)\,R_2\\ \left(\frac{d}{dr}+\frac{1+k}{r}\right)R_2 &= \frac{1}{\hbar c}(-E+V+E_0)R_1\,.\end{aligned}\right\} \quad (1)$$

The notations used here and in the following have their usual meaning. The quantum number k is restricted to positive and negative integers.

Introducing the functions $P_1 = rR_1$ and $P_2 = rR_2$ we get

$$\left.\begin{aligned}\left(\frac{d}{dr}-\frac{k}{r}\right)P_1 &= \frac{1}{\hbar c}(E-V+E_0)\,P_2\\ \left(\frac{d}{dr}+\frac{k}{r}\right)P_2 &= \frac{1}{\hbar c}(-E+V+E_0)\,P_1\,.\end{aligned}\right\} \quad (2)$$

In order to deduce a wave equation that enables us to apply the principles of quantum statistics we proceed as follows: We differentiate the first of equations (2) and substitute for $\frac{dP_2}{dr}$ the expression from the second equation and get

$$\left.\begin{aligned}&\frac{d^2P_1}{dr^2}-\frac{k}{r}\frac{dP_1}{dr}+\frac{k}{r^2}P_1 =\\ &\frac{1}{\hbar c}(E-V+E_0)\left[\frac{1}{\hbar c}(-E+V+E_0)\,P_1-\frac{k}{r}P_2\right]-\frac{1}{\hbar c}\frac{dV}{dr}P_2\,.\end{aligned}\right\} \quad (3)$$

Then P_2 is eliminated from the bracket by the aid of the first of equations (2), so that we get

$$\frac{d^2P_1}{dr^2}+\left[\frac{(E-V)^2-E_0^2}{\hbar^2c^2}-\frac{k^2-k}{r^2}\right]P_1 = -\frac{1}{\hbar c}\frac{dV}{dr}P_2\,. \quad (4)$$

By an exactly similar procedure we find

$$\frac{d^2P_2}{dr^2}+\left[\frac{(E-V)^2-E_0^2}{\hbar^2c^2}-\frac{k^2+k}{r^2}\right]P_2 = +\frac{1}{\hbar c}\frac{dV}{dr}P_1\,. \quad (5)$$

In the case of a vanishing potential gradient these two differential equations are two wave equations. They have identical

eigenfunctions and eigenvalues for values of k that differ by one, and they are changed into each other by the interchange of $+k$ and $-k$.

In order to treat the general case of a non-vanishing potential gradient, when equations (4) and (5) describe coupled oscillations, we introduce a linear combination

$$Q = a_1P_1 + a_2P_2\,, \quad (6)$$

where a_1 and a_2 are as yet undetermined constants. Multiplying (4) and (5) by a_1 and a_2 respectively and adding, we get

$$\left.\begin{aligned}&\frac{d^2Q}{dr^2}+\left[\frac{(E-V)^2-E_0^2}{\hbar^2c^2}-\frac{k^2}{r^2}\right]Q+\\ &\frac{k}{r^2}a_1P_1+\frac{1}{\hbar c}\frac{dV}{dr}a_1P_2-\frac{k}{r^2}a_2P_2-\frac{1}{\hbar c}\frac{dV}{dr}a_2P_1 = 0\,.\end{aligned}\right\} \quad (7)$$

We can now determine the ratio of the a's and a new constant, g, and arrive at a wave equation for Q of the following form:

$$\frac{d^2Q}{dr^2}+\left[\frac{(E-V)^2-E_0^2}{\hbar^2c^2}-\frac{k^2+g}{r^2}\right]Q = 0, \quad (8)$$

provided $r^2\frac{dV}{dr}$ can be treated as a constant in that region, characterized by a small interval of r, which we will consider. Equating the coefficients of P_1 and P_2 in (7) and (8) we get the following two equations

$$\left.\begin{aligned}\frac{k}{r^2}a_1-\frac{1}{\hbar c}\frac{dV}{dr}a_2 &= -\frac{g}{r^2}a_1\\ \frac{1}{\hbar c}\frac{dV}{dr}a_1-\frac{k}{r^2}a_2 &= -\frac{g}{r^2}a_2\,.\end{aligned}\right\} \quad (9)$$

Writing k_0 for the constant $\frac{r^2}{\hbar c}\frac{dV}{dr}$ we have

$$\left.\begin{aligned}ka_1-k_0a_2+ga_1 &= 0\\ k_0a_1-ka_2+ga_2 &= 0.\end{aligned}\right\} \quad (10)$$

In order that the above equations have finite solutions for the a's, the determinant

$$\begin{vmatrix} k+g & -k_0 \\ k_0 & -k+g \end{vmatrix}$$

must vanish.

This leads to the (secular) equation

$$g^2 = k^2 - k_0^2. \tag{11}$$

The constant g may then take one of the two values

$$g = \pm \sqrt{k^2 - k_0^2}. \tag{12}$$

If we choose the upper sign we have the differential equation

$$\frac{d^2Q_{\mathrm{I}}}{dr^2} + \left[\frac{(E-V)^2 - E_0^2}{\hbar^2 c^2} - \frac{k^2 + \sqrt{k^2 - k_0^2}}{r^2}\right] Q_{\mathrm{I}} = 0, \tag{13}$$

and with the lower sign

$$\frac{d^2Q_{\mathrm{II}}}{dr^2} + \left[\frac{(E-V)^2 - E_0^2}{\hbar^2 c^2} - \frac{k^2 - \sqrt{k^2 - k_0^2}}{r^2}\right] Q_{\mathrm{II}} = 0. \tag{14}$$

The functions Q_{I} and Q_{II} then are two different linear combinations of P_1 and P_2. The request that P_1 and P_2 both fulfill the boundary conditions leads to a similar request for the Q's.

The equations (13) and (14) may both, independent of the sign of g, be written as

$$\frac{d^2Q}{dr^2} + \left[\frac{(E-V)^2 - E_0^2 - \left(r\frac{dV}{dr}\right)^2}{\hbar^2 c^2} - \frac{g(g+1)}{r^2}\right] Q = 0. \tag{15}$$

The above equation is in the case of a hydrogen atom identical with the iterated DIRAC equation given by TEMPLE [3]. Our Q is equal to his Ψ multiplied by r. In that special case the terms V^2 and $-\left(r\frac{dV}{dr}\right)^2$ cancel each other, and k_0 is equal to the fine-structure constant.

For every energy value determined by this equation there is a $2|k|$-fold degeneracy due to the angular parts of the functions. The quantum number j is namely equal to $|k| - \frac{1}{2}$ (*cf.* SOMMERFELD, loc. cit. Ch. IV, § 8). For the hydrogen atom *e.g.* the ${}^2S_{\frac{1}{2}}$- and ${}^2P_{\frac{1}{2}}$-states both correspond to $k^2 = 1$ and are both double. Similarly the ${}^2P_{\frac{3}{2}}$- and ${}^2D_{\frac{3}{2}}$-states, which correspond to $k = \pm 2$, are both quadruple etc.

For each sign and numerical value of g the number of states with energy constants lower than a maximal value E_m, characteristic of the star, in a volume element in the form of a shell concentric with the star, is equal to $2|k|$ times the number of half oscillations of the radial function Q for the maximum energy value, because each state has one node less than that lying immediately above it.

The minimum value of k^2 is k_0^2, because a smaller value would cause E to be complex. (For the hydrogen atom this is no problem, since the fine-structure constant is much smaller than one, the lowest allowed value of k).

The minimum radial wave length $\lambda_{\min}$ is determined as a function of E_m and g by

$$\left(\frac{2\pi}{\lambda_{\min}}\right)^2 = \frac{(E_m - V)^2 - E_0^2 - \left(r\frac{dV}{dr}\right)^2}{\hbar^2 c^2} - \frac{g(g+1)}{r^2}. \tag{16}$$

For a thickness of the shell of one cm the total number of states with a certain numerical value of g then is equal to

$$2 \cdot \frac{2}{\lambda_{\min}} \cdot 2|k| = \frac{4|k|}{\pi r}\left[r^2 \frac{(E_m - V)^2 - E_0^2 - \left(r\frac{dV}{dr}\right)^2}{\hbar^2 c^2} - g^2\right]^{\frac{1}{2}}, \tag{17}$$

because we need not here distinguish between g and $g+1$. The first factor two to the left is due to the double sign of g.

We find the total number of states by integrating over $|k|$ from $g^2 = 0$ to its maximal value, which value makes the integrand vanish. We use the relation

$$2|k|\, d|k| = d(g^2). \tag{18}$$

The total number N of states with energy less than E_m is then determined by

$$N = \frac{2}{\pi r}\int_{g=0}^{g\,\max}\left[r^2\frac{(E_m - V)^2 - E_0^2 - \left(r\frac{dV}{dr}\right)^2}{\hbar^2 c^2} - g^2\right]^{\frac{1}{2}} d(g^2), \qquad (19)$$

and we find

$$N = \frac{4}{3\pi r}\left[r^2\frac{(E_m - V)^2 - E_0^2 - \left(r\frac{dV}{dr}\right)^2}{\hbar^2 c^2}\right]^{\frac{3}{2}}. \qquad (20)$$

The number of states per cubic cm is found by dividing N by $4\pi r^2$. When all states with energy constants less than E_m are occupied by electrons, the material density ϱ is found by multiplying the density of states by $\mu_e m_H$, the mass per electron, where the mass of a hydrogen atom is denoted by m_H. For pure hydrogen the molecular weight μ_e is equal to one. We get

$$\varrho = \frac{\mu_e m_H}{3\pi^2\hbar^3 c^3}\left[(E_m - V)^2 - E_0^2 - \left(r\frac{dV}{dr}\right)^2\right]^{\frac{3}{2}}. \qquad (21)$$

If the gravitational potential per unit mass is called U, we have POISSON's equation:

$$\frac{d^2U}{dr^2} + \frac{2}{r}\frac{dU}{dr} = 4\pi G\varrho. \qquad (22)$$

The potential function V is equal to $\mu_e m_H U$, so that we arrive at the following differential equation for V (writing h for $2\pi\hbar$):

$$\frac{d^2V}{dr^2} + \frac{2}{r}\frac{dV}{dr} = \frac{32\pi^2\mu_e^2 m_H^2 G}{3h^3c^3}\left[(E_m - V)^2 - E_0^2 - \left(r\frac{dV}{dr}\right)^2\right]^{\frac{3}{2}}. \qquad (23)$$

In deriving (21) and (23) we have neglected such non-uniformity in the distribution of the heavy particles as has been taken into account by SCHATZMAN [4].

Introducing as a new variable

$$y = \frac{E_m - V}{E_0} \qquad (24)$$

we get

$$\frac{d^2y}{dr^2} + \frac{2}{r}\frac{dy}{dr} = -\frac{32\pi^2\mu_e^2 m_H^2 m_0^2 Gc}{3h^3}\left[y^2 - \left(r\frac{dy}{dr}\right)^2 - 1\right]^{\frac{3}{2}}, \qquad (25)$$

where we have made use of the relation

$$E_0 = m_0 c^2. \qquad (26)$$

If the term $-\left(r\frac{dy}{dr}\right)^2$ in the bracket in equation (25) is neglected, we find CHANDRASEKHAR's equation [5]. Our aim is, however, to find the effect of this term on the mass-radius relation for white dwarf stars. If we introduce CHANDRASEKHAR's variables

$$\left.\begin{aligned} r &= \alpha\eta,\quad y = y_0\varphi,\\ \alpha &= \frac{1}{4\pi\mu_e m_H m_0 y_0}\left(\frac{3h^3}{2Gc}\right)^{\frac{1}{2}} = l_1 y_0^{-1} = \frac{7{,}71\cdot 10^8\ \mathrm{cm}}{\mu_e y_0}, \end{aligned}\right\} \qquad (27)$$

we can write the differential equation as

$$\frac{1}{\eta^2}\frac{d}{d\eta}\left(\eta^2\frac{d\varphi}{d\eta}\right) = -\left(\varphi^2 - \left(\eta\frac{d\varphi}{d\eta}\right)^2 - \frac{1}{y_0^2}\right)^{\frac{3}{2}}, \qquad (28)$$

where φ has to take the value one at the center. The boundary condition is $\frac{d\varphi}{d\eta} = 0$ at the center. The surface is found where the density vanishes (at $\eta = \eta_1$).

In the limiting case when $\frac{1}{y_0^2}$ is very near one, the limiting solution is, just as is that of CHANDRASEKHAR's equation, that of an EMDEN equation of index $\frac{3}{2}$. (Then all relativistic effects are negligible).

Following CHANDRASEKHAR we deduce the following expression for the mass of the whole configuration:

$$M = -\frac{\sqrt{3}}{4\sqrt{2}\,\pi\mu_e^2 m_H^2}\left(\frac{hc}{G}\right)^{\frac{3}{2}}\left(\eta^2\frac{d\varphi}{d\eta}\right)_{\eta=\eta_1}, \qquad (29)$$

where the numerical factor is 2,85 μ_e^{-2} solar masses. (For comparison with CHANDRASEKHAR's results we have used his adopted values of the natural constants throughout this paper).

The differential equation (28) has been integrated numerically for three values of the parameter $\frac{1}{y_0^2}$. The results for the radii and masses of the corresponding stellar configurations are given below. For comparison, CHANDRASEKHAR's values are also given. Our central densities are the same as in his models for the same parameter values.

TABLE I.

$\frac{1}{y_0^2}$	$\frac{R}{l_1}$	$-\left(\eta^2\frac{d\varphi}{d\eta}\right)_{\eta=\eta_1}$	Chandrasekhar's values	
			$\frac{R}{l_1}$	$-\left(\eta^2\frac{d\varphi}{d\eta}\right)_{\eta=\eta_1}$
0,5	2,58	0,597	2,50	0,707
0,2	1,84	0,926	1,67	1,243
0,1	1,55	1,091	1,29	1,519

The density distributions are given in *Table II*. The unit of density is

$$B = \frac{8\pi\mu_e m_H m_0^3 c^3}{3h^3} = 9{,}82\cdot 10^5 \mu_e \,\mathrm{g\,cm^{-3}}.$$

The unit of radius is l_1.

A comparison of the results with those of CHANDRASEKHAR shows that the radii are larger and the masses smaller than his for the same values of $\frac{1}{y_0^2}$. For the same value of the mass the radius is smaller than CHANDRASEKHAR's.

The limiting case of vanishing $\frac{1}{y_0^2}$ has also been treated by numerical integration. *Table III* gives the variable φ together with $-\eta^2\varphi'$ and $\frac{\varrho}{\varrho_c}$ as functions of η. In this case there is no definite radius measured in units of $l_1 y_0^{-1}$. For any value of the parameter we have namely the following limiting form of φ as a function of η:

$$\varphi \sim c_1 + \frac{c_2}{\eta}. \tag{30}$$

TABLE II.

$\frac{r}{l_1}$ \ $\frac{1}{y_0^2}$ =	0,5	0,2	0,1
0,0	1,00	8,00	27,00
0,1	0,99	7,81	25,71
0,2	0,97	7,29	22,01
0,3	0,94	6,44	16,59
0,4	0,89	5,37	11,03
0,5	0,83	4,21	6,68
0,6	0,76	3,13	3,86
0,7	0,69	2,22	2,20
0,8	0,61	1,53	1,26
0,9	0,53	1,03	0,73
1,0	0,45	0,68	0,42
1,1	0,38	0,45	0,24
1,2	0,32	0,29	0,13
1,3	0,26	0,19	0,07
1,4	0,20	0,12	0,03
1,5	0,16	0,07	0,01
1,6	0,13	0,04	..
1,7	0,10	0,02	..
1,8	0,07	0,00	..
1,9	0,05	..	..
2,0	0,04	..	..
2,1	0,03	..	..
2,2	0,02	..	..
2,3	0,01	..	..
2,4	0,00	..	..
2,5	0,00	..	..

In the case of $\frac{1}{y_0^2} = 0$, however, the constant c_1 vanishes, because we have here

$$|\varphi| \to \left|\eta\frac{d\varphi}{d\eta}\right| \tag{31}$$

as we approach the surface. The radius might therefore be finite in units of l_1.

The limiting mass can be estimated from the data in the table to be some 85 per cent of CHANDRASEKHAR's limiting mass. This, which is that of an EMDEN polytrope of index 3, corresponds to a value of $-\eta^2\frac{d\varphi}{d\eta}$ equal to 2,018 at the surface.

A more detailed investigation of the model considered would not be of very much interest, because we have here still neglected the influence of exchange effects. EDDINGTON's criticisms of the

"current" theory of white dwarf stars have therefore not yet been properly answered, also because we have still preserved dividing walls (here spherical) inside the star for determining energy states instead of determining them for the star as a whole. The present method of approach to the problem might serve as a starting point for investigations as to the effect due to the introduction of such refinements into the theory.

TABLE III.

η	φ	$-\eta^2\varphi'$	ϱ/ϱ_c	η	φ	$-\eta^2\varphi'$	ϱ/ϱ_c
0,0.....	1,0000	0,0000	1,000	3,9	0,3747	1,2453	0,008
0,1.....	0,9984	0,0003	0,995	4,0	0,3667	1,2563	0,007
0,2.....	0,9934	0,0026	0,980	4,1	0,3590	1,2669	0,006
0,3.....	0,9853	0,0088	0,955	4,2	0,3516	1,2768	0,006
0,4.....	0,9740	0,0203	0,920	4,3	0,3445	1,2862	0,005
0,5.....	0,9600	0,0386	0,876	4,4	0,3377	1,2949	0,005
0,6.....	0,9433	0,0643	0,823	4,5	0,3311	1,3038	0,004
0,7.....	0,9244	0,0978	0,763	4,6	0,3248	1,3120	0,004
0,8.....	0,9035	0,1389	0,697	4,7	0,3187	1,3194	0,003
0,9.....	0,8811	0,1868	0,628	4,8	0,3129	1,3270	0,003
1,0.....	0,8576	0,2402	0,558	4,9	0,3072	1,3338	0,003
1,1.....	0,8332	0,2978	0,489	5,0	0,3017	1,3407	0,003
1,2.....	0,8084	0,3580	0,424				
1,3.....	0,7836	0,4195	0,364	5,5	0,2771	1,371	0,0018
1,4.....	0,7589	0,4806	0,310	6,0	0,2561	1,395	0,0012
1,5.....	0,7346	0,5406	0,262	6,5	0,2381	1,416	0,0009
1,6.....	0,7109	0,5984	0,221	7,0	0,2225	1,433	0,0007
1,7.....	0,6879	0,6536	0,186	7,5	0,2087	1,448	0,0005
1,8.....	0,6657	0,7058	0,156	8,0	0,1966	1,461	0,0004
1,9.....	0,6443	0,7544	0,131	8,5	0,1858	1,473	0,0003
2,0.....	0,6239	0,8000	0,110	9,0	0,1762	1,484	0,0002
2,1.....	0,6043	0,8424	0,092	9,5	0,1675	1,493	0,0002
2,2.....	0,5857	0,8815	0,078	10,0	0,1596	1,501	0,0002
2,3.....	0,5679	0,9179	0,066				
2,4.....	0,5510	0,9515	0,056	15	0,1087	1,56	0,000034
2,5.....	0,5349	0,9826	0,048	20	0,0825	1,59	0,000012
2,6.....	0,5195	1,0112	0,041	25	0,0665	1,61	0,000005
2,7.....	0,5049	1,0377	0,035	30	0,0558	1,62	0,000003
2,8.....	0,4910	1,0624	0,030	35	0,0481	1,63	0,000002
2,9.....	0,4778	1,0855	0,026	40	0,0422	1,64	0,000001
3,0.....	0,4652	1,1067	0,023	45	0,0376	1,65	0,000001
3,1.....	0,4532	1,1266	0,020	50	0,0340	1,65	..
3,2.....	0,4418	1,1448	0,017	55	0,0310	1,66	..
3,3.....	0,4308	1,1620	0,015	60	0,0284	1,66	..
3,4.....	0,4204	1,1781	0,014	65	0,0263	1,67	..
3,5.....	0,4104	1,1933	0,012	70	0,0245	1,67	..
3,6.....	0,4009	1,2075	0,011	75	0,0229	1,68	..
3,7.....	0,3918	1,2210	0,009	80	0,0215	1,68	..
3,8.....	0,3831	1,2333	0,008	85	0,0202	1,68	..
				90	0,0191	1,69	..

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