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ON THE CONVERGENCE PROBLEM FOR DIRICHLET SERIES

BY

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Introduction.

Let $\sum_{1}^{\infty} a_n n^{-s}$ be a Dirichlet series in the complex variable $s = \sigma + it$. With such a series are connected several values of the abscissa σ which are characteristic for the series in question.

Firstly, as $|a_n n^{-s}| = |a_n| n^{-\sigma}$, it is evident that there exists an "abscissa of absolute convergence" σ_A $(-\infty \leqq \sigma_A \leqq \infty)$ such that the series is absolutely convergent for $\sigma > \sigma_A$ but not for $\sigma < \sigma_A$.

Secondly, as first proved by Jensen, the series also possesses an "abscissa of convergence" σ_C, i.e. there exists a number σ_C such that the series is convergent for $\sigma > \sigma_C$, divergent for $\sigma < \sigma_C$. Between σ_A and σ_C we have the relation $0 \leqq \sigma_A - \sigma_C \leqq 1$. For any $\varepsilon > 0$, $K > 0$ the series is uniformly convergent in $\sigma > \sigma_C + \varepsilon$, $|s| < K$; hence the series represents in its half-plane of convergence $\sigma > \sigma_C$ an analytic function $f(s)$.

Thirdly we have the "abscissa of uniform convergence" σ_U, introduced by the author as the g.l.b. of the abscissae σ_0 for which the series is uniformly convergent in the whole half-plane $\sigma > \sigma_0$ (and not merely in any limited part of it). Obviously $\sigma_C \leqq \sigma_U \leqq \sigma_A$.

Finally it is often convenient to introduce a fourth abscissa, which we shall denote by σ^*, defined as the g.l.b. of the abscissae σ_0 for which the terms of the series $\sum a_n n^{-(\sigma_0 + it)}$ remain bounded (i.e. $|a_n| n^{-\sigma_0} < K$ for all n). Except for the special cases where the series is convergent everywhere or nowhere (where all the abscissae introduced are either $-\infty$ or $+\infty$ respectively) the abscissa σ^* may also be defined as the—certainly existing—smallest number σ for which $a_n n^{-\sigma} = O(n^{\delta})$ for every $\delta > 0$. Evidently $\sigma_C \geqq \sigma^*$ and $\sigma_A \leqq \sigma^* + 1$.

There is no real difficulty in giving explicit expressions fo the four abscissae, introduced above, in terms of the coefficients a_n of the series. As regards σ^*, it follows immediately from the definition of this number that

$$\sigma^* = \overline{\lim_{n\to\infty}} \frac{\log|a_n|}{\log n},$$

and for the three abscissae of convergence one finds by partial summation, in case the series is divergent for $s = 0$, the well known expressions

$$\sigma_C = \overline{\lim} \frac{\log|S_n|}{\log n}, \qquad \sigma_A = \overline{\lim} \frac{\log A_n}{\log n}, \qquad \sigma_U = \overline{\lim} \frac{\log U_n}{\log n}$$

where

$$S_n = \sum_{\nu=1}^{n} a_\nu, \qquad A_n = \sum_{\nu=1}^{n} |a_\nu|, \qquad U_n = \underset{-\infty<t<\infty}{\text{l. u. b.}} \left| \sum_{\nu=1}^{n} a_\nu \nu^{-it} \right|.$$

In the following we shall, however, use only a simple consequence of the expression for σ_C, viz., if for some $\alpha > 0$ the partial sum $S_n = a_1 + \cdots + a_n$ is *not* $o(n^\alpha)$ for $n \to \infty$, then the abscissa of convergence σ_C is certainly $\geq \alpha$.

While the four numbers σ_A, σ_C, σ_U, σ^* are defined directly from the series itself, there are some other abscissae, important for the theory of Dirichlet series, which are determined in a more indirect manner, namely from the behaviour of the analytic function $f(s)$ given by the series in its half-plane of convergence. We shall mention three such numbers, viz.

ω = g. l. b. of those σ_0 such that $f(s)$ is regular and bounde in $\sigma > \sigma_0$,

ω_1 = g. l. b. of those σ_0 such that $f(s)$ is regular and $O(|t|$ in $\sigma > \sigma_0$ for any $\varepsilon > 0$,

Ω = g. l. b. of those σ_0 such that $f(s)$ is regular and of finit order in $\sigma > \sigma_0$, i. e. $= O(|t|^K)$ for some value of $K = K(\sigma_0$

Obviously $\omega \geqq \omega_1 \geqq \Omega$.

By the "convergence-problem" for Dirichlet series is unde stood the problem of finding relations between the properties of the series $\sum a_n n^{-s}$ as such—especially the values of the different abscissae of convergence—and the properties of th analytic function $f(s)$ defined by the series.

Only in the case of the abscissa of uniform convergence do we have a simple and general solution of the problem; in fact, as proved by the author, we have for every Dirichlet series the relation

$$\sigma_U = \omega,$$

i. e. the series is uniformly convergent just so far to the left as the function $f(s)$ represented by the series remains regular and bounded.

In the case of the abscissa of absolute convergence the problem of characterizing the abscissa σ_A has turned out to be in the main a problem of discussing the possible values of $\sigma_A - \sigma_U$, i. e. of $\sigma_A - \omega$, and this latter problem has in turn, as shown by the author, an intimate relation with problems from the theory of power series in infinitely many variables. In this way it was found that the difference $\sigma_A - \sigma_U$ (which evidently is ≤ 1) is always $\leqq \frac{1}{2}$ and, as proved by Hille and Bohnenblust, that this upper bound $\frac{1}{2}$ can not be diminished.

The problem of determining, or rather estimating, the abscissa of convergence σ_C itself by means of analytical properties of the function $f(s)$ was first attacked by Landau who by his researches opened the whole field of investigations in question. Starting from the well-known fact that $f(s) = O(|t|)$ for $\sigma > \sigma_C + \varepsilon$, which so to speak limits the extension of the half-plane of convergence, he proved the other way round that if the order of magnitude of $f(s)$ is known in a certain half-plane $\sigma > \eta$ one may conclude that the half-plane of convergence cannot be too small. Landau's estimation of σ_C was sharpened and generalized first by Schnee, who proved among other things that σ_C is always $\leq \omega_1$, and later on by Landau himself. The best result, obtained by these successive improvements, is stated in the following theorem[1], the "Landau-Schnee theorem":

L.-S. Theorem. *For the Dirichlet series $\sum a_n n^{-s}$ let the abscissa σ^* be ≤ 0, i. e. let $a_n = O(n^\delta)$ for every $\delta > 0$, so that the series is certainly absolutely convergent for $\sigma > 1$. Further we assume*

[1] See e. g. E. Landau, Handbuch der Lehre von der Verteilung der Primzahlen, Leipzig und Berlin 1909, p. 853 and 857.

the existence of two constants $\eta < 1$ and $k \geqq 0$ such that the function $f(s)$ defined by the series is regular for $\sigma > \eta$ and

$$f(s) = O(|t|^{k+\varepsilon}) \quad \textit{for} \quad \sigma > \eta + \varepsilon.$$

Then the series is convergent beyond the line $\sigma = 1$, namely at any rate for

$$\sigma > \operatorname{Min}\left\{\frac{\eta+k}{1+k}, \quad \eta+k\right\}.$$

In other words, under the above conditions the abscissa of convergence satisfies the relation

$$\sigma_C \leqq \frac{\eta+k}{1+k} \quad \text{if} \quad \eta+k \geqq 0$$

and

$$\sigma_C \leqq \eta+k \quad \text{if} \quad \eta+k \leqq 0.$$

When applying the L.-S. theorem to the classical Dirichlet series occurring in the analytical theory of numbers, such as the zeta-series with alternating signs $\sum(-1)^{n-1} n^{-s}$, the results obtained come out to be rather far from the real truth, i.e. the values obtained as an upper bound for σ_C are far too big. This might indicate that the theorem could be improved, i.e. that the number $\operatorname{Min}\left\{\frac{\eta+k}{1+k}, \eta+k\right\}$ could be replaced by a smaller one. From an earlier investigation by the author, carried out with a somewhat different purpose, it can, however, be concluded that this is not always the case; in fact, from the study of a certain artificially constructed Dirichlet series of the "gap type", it may be shown that in the special cases $0 \leqq k \leqq 1$, $\eta = -k$, the L.-S. theorem cannot be improved. Recently, I have taken the whole question up for a renewed investigation and by generalizing my earlier construction I have proved that the L.-S. theorem is in the very strictest sense, i.e. for every pair of values (η, k) with $\eta < 1$, $k \geqq 0$, the best possible one. In fact the following theorem holds.

Theorem 1. *For any given pair of values (η, k) with $\eta < 1$, $k \geqq 0$, there exists a Dirichlet series $\sum a_n n^{-s}$ with $\sigma^* \leqq 0$ for which the function $f(s)$ represented by the series is regular in $\sigma > \eta$ and, for any $\varepsilon > 0$, is $O(|t|^{k+\varepsilon})$ in $\sigma > \eta + \varepsilon$ and such that the value of the abscissa of convergence σ_C is given exactly by the equation*

$$\sigma_C = \operatorname{Min}\left\{\frac{\eta+k}{1+k}, \ \eta+k\right\}.$$

By means of trivial transformations (s being replaced by $s+s_0$) it is easily seen that we may confine ourselves to consider the case $\eta+k \geqq 0$, $\sigma^* = 0$, i.e. to prove the

Theorem 1a. *Let (η, k) be an arbitrary point in the domain determined by $\eta < 1$, $k \geqq 0$, $\eta + k \geqq 0$. Then there exists a Dirichlet series $f(s) = \sum a_n n^{-s}$ with $\sigma^* = 0$, with $f(s)$ regular in $\sigma > \eta$ and $O(|t|^{k+\varepsilon})$ in $\sigma > \eta + \varepsilon$, such that its abscissa of convergence σ_C is just the point $\frac{\eta+k}{1+k}$ ($\geqq 0$ and < 1), i.e. the point in which the line connecting the two points (η, k) and $(1, -1)$ cuts the axis of abscissae.*

For a Dirichlet series we introduce in the usual way the Lindelöf μ-function $\mu = \mu(\sigma)$ ($\Omega < \sigma < \infty$) defined for each $\sigma_0 > \Omega$ as the g.l.b. of the values of l such that

$$f(\sigma_0 + it) = O(|t|^l)$$

or, what amounts to the same thing, that

$$f(\sigma + it) = O(|t|^l) \quad \text{for} \quad \sigma \geqq \sigma_0 \quad (\text{or } \sigma > \sigma_0).$$

The μ-function is a continuous convex function in $\Omega < \sigma < \infty$ and is equal to zero for sufficiently large values of σ, at any rate for $\sigma > \sigma_A$. By help of the μ-function the L.-S. theorem, in the case $\eta + k \geqq 0$, $\sigma^* = 0$, may easily be shown[1] to be equivalent with the following theorem.

Other form of the L.-S. theorem. *Let α be an arbitrary number in the interval $0 \leqq \alpha < 1$, and let $f(s) = \sum a_n n^{-s}$ be a Dirichlet series with $\sigma^* = 0$ and $\sigma_C = \alpha$. Then the μ-function*

[1] Compare K. Grandjot, Ueber das absolute Konvergenzproblem der Dirichletschen Reihen, Dissertation, Göttingen 1922.

attached to $f(s)$ will in the whole interval $\sigma > \Omega$ satisfy the condition

$$\mu(\sigma) \geqq M_\alpha(\sigma)$$

where $M_\alpha(\sigma)$ is the simple convex function characterizing the broken line which consists of two half-lines meeting in the point $(\alpha, 0)$, namely of the part of the axis of abscissae to the right of $\sigma = \alpha$ and of the prolongation of the segment from the point $(1, -1)$ to the point $(\alpha, 0)$ beyond this latter point, i. e.

$$M_\alpha(\sigma) = \begin{cases} 0 & \text{for} \quad \sigma \geqq \alpha \\ \dfrac{\sigma - \alpha}{\alpha - 1} & \text{for} \quad \Omega \leqq \sigma \leqq \alpha. \end{cases}$$

Furthermore, using this form of the L.-S. theorem it is readily seen that our theorem 1a (i. e. the theorem 1) is certainly true if the following theorem holds good.

Theorem 2. *For any α in the interval $0 \leqq \alpha < 1$ there exists a Dirichlet series $f(s) = \sum a_n n^{-s}$ with $\sigma^* = 0$ and $\sigma_C = \alpha$, such that $\Omega = -\infty$ and the μ-function attached to $f(s)$ is given by*

$$\mu(\sigma) = M_\alpha(\sigma) = \begin{cases} 0 & \text{for} \quad \sigma \geqq \alpha \\ \dfrac{\sigma - \alpha}{\alpha - 1} & \text{for} \quad \sigma \leqq \alpha. \end{cases}$$

The main object of the present paper is to prove this theorem 2. I may add, however, that the types of Dirichlet series constructed below may also be used to illustrate various points in the theory of CESARO-summability of the Dirichlet series $\sum a_n n^{-s}$ as developed independently by M. RIESZ and the author. I may return to these summability problems in a later paper; here I shall only recall one of the most striking results of this theory, viz. the relation

$$\sigma_S = \Omega$$

where σ_S denotes the g. l. b. of those σ_0 for which the series is C-summable of some order in the half-plane $\sigma > \sigma_0$. Thus σ_S like σ_U (but in contrast to σ_C and σ_A) can be fully determined by means of simple characteristic properties of the function $f(s)$ represented by the series.

The paper is divided into two sections. In § 1 we prove our theorem 2 for the special case $\alpha = 0$, and in § 2 we treat the general and somewhat more difficult case $0 < \alpha < 1$.

Added remark. After having completed this paper, containing, for an arbitrarily given α in the interval $0 \leqq \alpha < 1$, the construction of a Dirichlet series $\sum a_n n^{-s}$ satisfying the conditions of theorem 2, I became aware that quite a similar problem had already been treated and solved in an interesting paper by NEDER[1] who used a method of construction which shows characteristic relationship with that applied by the author. The results of NEDER, however, do not cover the results of this paper. In order to explain the connection between the results I have for a moment to consider not only the "ordinary" Dirichlet series $\sum a_n n^{-s} = \sum a_n e^{-s \log n}$, with which we exclusively deal in the present paper, but also the "general" Dirichlet series $\sum a_n e^{-\lambda_n s}$ where $0 < \lambda_1 < \lambda_2 < \cdots$ is an arbitrary increasing sequence tending to ∞. For a certain class of such general Dirichlet series we have, as well known, a theorem quite similar to (and including) the L.-S. theorem mentioned above. In fact, if the λ_n-sequence satisfies, for a positive finite l and any $\delta > 0$, the condition

$$(*) \qquad \frac{1}{\lambda_{n+1} - \lambda_n} = O\left(e^{\lambda_n (l+\delta)}\right),$$

introduced by the author (and shown to be the widest class for which theorems of the type in question hold), then the L.-S. theorem in the second formulation is still valid when only α runs through the interval $0 \leqq \alpha < l$ (instead of the interval $0 \leqq \alpha < 1$) and the dominator $1 - \alpha$ in the expression for $M_\alpha(\sigma)$ is replaced by $l - \alpha$. Now, the problem treated and solved by NEDER was just to prove the converse of this last theorem concerning the general class of Dirichlet series $\sum a_n e^{-\lambda_n s}$ satisfying the condition (*), say with $l = 1$, namely to each $0 \leqq \alpha < 1$ to construct a series $\sum a_n e^{-\lambda_n s}$ of the type in question with $\sigma^* = 0$, $\sigma_C = \alpha$ and its μ-function given by $\mu(\sigma) = M_\alpha(\sigma)$. If

[1] L. NEDER, Ueber Umkehrungen der Konvergenzsätze für Dirichletsche Reihen, Berichte der Akademie der Wissenschaften zu Leipzig, Math.-Phys. Klasse, Bd. LXXIV, 1922.

this had been done for an *arbitrarily given* sequence of exponents satisfying (*), for $l = 1$, NEDER's results would obviously have contained our theorem 2. But this was not the case; in fact, NEDER only proved that for any α in $0 \leqq \alpha < 1$ there exists *some* sequence $\lambda_1, \lambda_2, \cdots$ satisfying (*) for $l = 1$ and a corresponding Dirichlet series $\sum a_n e^{-\lambda_n s}$ with $\sigma^* = 0$, $\sigma_C = \alpha$ and $\mu(\sigma) = M_\alpha(\sigma)$; indeed, he just faciliated his construction by choosing an "artificial" sequence λ_n especially suited for his purpose, namely a sequence which—in contrast to the sequence $\lambda_n = \log n$—is composed by parts of different arithmetical progressions.

In this connection it may be emphasized that many problems concerning Dirichlet series essentially depend on the special character of the sequence λ_n of exponents. Thus, to mention an interesting example, nearly related to the problem in question, it has been proved by NEDER in his paper quoted above that there exists a general Dirichlet series $\sum a_n e^{-\lambda_n s}$, with exponents satisfying (*) for $l = 1$, such that $\sigma_C = 0$, $\sigma_A = 1$ and $\mu(\sigma) = \frac{1}{2} - \sigma$ for $\sigma \leqq \frac{1}{2}$, while it is an open problem whether there exists a Dirichlet series of the ordinary type $\sum a_n n^{-s}$ with the same properties. [Only under the assumption of the so-called LINDELÖF hypothesis—which certainly holds good if the RIEMANN hypothesis does—we possess, as emphasized by GRANDJOT, such an example, namely the series $\zeta(s)(1-2^{1-s}) = \sum(-1)^{n+1} n^{-s}$].

§ 1. The case $\alpha = 0$.

In this section we shall prove theorem 2 in the special case $\alpha = 0$, i. e. we shall construct a Dirichlet series $f(s) = \sum a_n n^{-s}$ with $\sigma^* = 0$, $\sigma_C = 0$ and $\Omega = -\infty$ such that the μ-function is given by

$$\mu(\sigma) = \begin{cases} 0 & \text{for } \sigma \geqq 0 \\ -\sigma & \text{for } \sigma \leqq 0. \end{cases}$$

The idea of the construction, namely to build a gap-series containing differences of higher and higher order, has been used previously by the author[1].

[1] H. BOHR, Bidrag til de Dirichlet'ske Rækkers Theori, Habilitationsskrift, København 1910.

Let $p_1 < p_2 < p_3 \cdots$ be a sequence of positive integers which increases so rapidly that $p_{m+1} > p_m + m$ and that the infinite series

$$\sum_{m=1}^{\infty} 2^m p_m^{-\varepsilon}$$

converges for every $\varepsilon > 0$.

We consider the Dirichlet series

$$\sum_{n=1}^{\infty} a_n n^{-s} = p_1^{-s} - (p_1+1)^{-s} + p_2^{-s} - 2(p_2+1)^{-s} + (p_2+2)^{-s} + \cdots$$

the terms of which consist of groups of 2, 3, $\cdots$ elements such that the m^{th} group is given by

$$p_m^{-s} - \binom{m}{1}(p_m+1)^{-s} + \binom{m}{2}(p_m+2)^{-s} - \cdots + (-1)^m (p_m+m)^{-s}$$

and where, on account of the inequality $p_{m+1} > p_m + m$, the groups do not overlap. Applying the usual terminology for differences of 1st and higher order, i. e.

$$\Delta u_p = u_p - u_{p+1}, \quad \Delta^2 u_p = u_p - 2u_{p+1} + u_{p+2}, \cdots$$

$$\Delta^m u_p = u_p - \binom{m}{1} u_{p+1} + \binom{m}{2} u_{p+2} - \cdots + (-1)^m u_{p+m}, \cdots$$

we may write our Dirichlet series, with its terms in groups, in the form

$$f(s) = \sum_{n=1}^{\infty} a_n n^{-s} = \sum_{m=1}^{\infty} \Delta^m (p_m^{-s}).$$

For these differences we shall have to use, besides the general trivial estimations

$$|\Delta^m u_p| \leqq |u_p| + \binom{m}{1}|u_{p+1}| + \cdots + |u_{p+m}| \leqq 2^m \cdot \underset{p \leqq \nu \leqq p+m}{\operatorname{Max}} |u_\nu|$$

and, for $1 \leqq h < m$,

$$|\Delta^m u_p| = |\Delta^{m-h} \Delta^h u_p| \leqq 2^{m-h} \cdot \underset{p \leqq \nu \leqq p+m-h}{\operatorname{Max}} |\Delta^h u_\nu|,$$

the special, well-known, integral-representation for $u_p = p^{-s}$ ($p \geqq 1$, s arbitrary complex)

$$\Delta^h(p^{-s}) = (-1)^h \int_p^{p+1} dx_1 \int_{x_1}^{x_1+1} dx_2 \cdots \int_{x_{h-1}}^{x_{h-1}+1} \frac{d^h}{dx_h^h}(x_h^{-s})\, dx_h =$$

$$s(s+1)\cdots(s+h-1)\int_p^{p+1} dx_1 \cdots \int_{x_{h-1}}^{x_{h-1}+1} x_h^{-s-h} dx_h$$

which, in case $\sigma + h > 0$, immediately gives the inequality

$$|\Delta^h(p^{-s})| \leqq |s|\cdot|s+1|\cdots|s+h-1|\cdot p^{-\sigma-h}.$$

We shall now show that our series $f(s) = \sum a_n n^{-s} = \sum \Delta^m(p_m^{-s})$ has all the properties stated.

Firstly we have $\sigma^* = 0$. On the one hand σ^* is certainly $\geqq 0$, as a_n does not tend to zero for $n \to \infty$ (e. g. $a_n = 1$ for $n = p_m$). On the other hand $a_n = O(n^\varepsilon)$ for each $\varepsilon > 0$; in fact since $2^m p_m^{-\varepsilon} \to 0$, we have for each n for which $a_n \neq 0$, i. e. for $n = p_m + \nu$ $(0 \leqq \nu \leqq m)$ and m sufficiently large

$$|a_n| = \left|(-1)^\nu \binom{m}{\nu}\right| \leqq 2^m = (2^m p_m^{-\varepsilon})\, p_m^\varepsilon < p_m^\varepsilon \leqq n^\varepsilon.$$

Secondly we have $\sigma_C = 0$. As $\sigma_C \geqq \sigma^* = 0$ we need only show that the series is convergent for $\sigma > 0$. We shall see that it is even absolutely convergent for $\sigma > 0$, and thus not only $\sigma_C = 0$ but also $\sigma_A = 0$. In fact for each $\sigma > 0$ the sum of the numerical values of the terms in the group $\Delta^m(p_m^{-s})$ is equal to

$$\sum_{\nu=0}^{m} \binom{m}{\nu} (p_m + \nu)^{-\sigma} \leqq p_m^{-\sigma} \sum_{\nu=0}^{m} \binom{m}{\nu} = 2^m p_m^{-\sigma},$$

and the series $\sum 2^m p_m^{-\sigma}$ is convergent for $\sigma > 0$.

Thirdly we have $\Omega = -\infty$, i. e. the function $f(s)$ defined by the series in the half-plane of convergence $\sigma > 0$ is an integral function and of finite order in each half-plane $\sigma > \sigma_0$. To this purpose we consider, for an arbitrary fixed positive integer h, the half-plane $\sigma > -h$ and in this half-plane estimate

the group of terms $\Delta^m(p_m^{-s})$ for each $m > h$. By help of the estimations indicated above we get, for $\sigma > -h$, $m > h$

$$|\Delta^m(p_m^{-s})| \leqq 2^{m-h} \cdot \underset{p_m \leqq \nu \leqq p_m+m-h}{\operatorname{Max}} |\Delta^h(\nu^{-s})|$$

$$\leqq 2^{m-h}|s||s+1|\cdots|s+h-1| \cdot \underset{p_m \leqq \nu \leqq p_m+m-h}{\operatorname{Max}} \nu^{-\sigma-h} = 2^{m-h}|s|\cdots|s+h-1|\, p_m^{-\sigma-h}.$$

Writing

$$f(s) = \sum_{m=1}^{\infty} \Delta^m(p_m^{-s}) = \sum_{m=1}^{h} \Delta^m(p_m^{-s}) + \sum_{m=h+1}^{\infty} \Delta^m(p_m^{-s}) = f_1(s) + f_2(s),$$

the finite sum $f_1(s) = \sum_{m=1}^{h}$ is evidently an integral function bounded in every half-plane $\sigma > \sigma_0$, while the series $f_2(s) = \sum_{m=h+1}^{\infty}$ is majorized in the half-plane $\sigma > -h + \varepsilon$ by the convergent series

$$2^{-h}|s|\cdots|s+h-1| \sum_{m=h+1}^{\infty} 2^m p_m^{-\varepsilon}.$$

Hence $f_2(s)$ is analytic in $\sigma > -h + \varepsilon$ (the series being uniformly convergent in $\sigma > -h + \varepsilon$, $|s| < K$) and $= O(|t|^h)$ for $|t| \to \infty$, uniformly in $-h + \varepsilon < \sigma < 2$ and hence also in $-h + \varepsilon < \sigma < \infty$. Therefore $f(s) = f_1(s) + f_2(s)$ is regular and $= O(|t|^h)$ in $\sigma > -h + \varepsilon$. Thus $\Omega \leqq -h + \varepsilon$ for each h, i. e. $\Omega = -\infty$.

Finally we have to consider the μ-function $\mu(\sigma)$, defined for all σ. As $\sigma_A = 0$ we have immediately $\mu(\sigma) = 0$ for $\sigma > 0$ and hence, as $\mu(\sigma)$ is a continuous function,

$$\mu(\sigma) = 0 \quad \text{for} \quad \sigma \geqq 0.$$

We shall show that $\mu(\sigma) = -\sigma$ for $\sigma \leqq 0$. We apply the L.-S. theorem in its second form. Since for our series $f(s) = \sum a_n n^{-s}$ we have $\sigma^* = 0$, $\sigma_C = 0$ and $\Omega = -\infty$, the theorem (for $\alpha = 0$) states that $\mu(\sigma)$ is certainly $\geqq M_0(\sigma) = -\sigma$ for $\sigma \leqq 0$; thus, in order to prove that $\mu(\sigma) = -\sigma$ for $\sigma \leqq 0$, we have only to show that

$$\mu(\sigma) \leqq -\sigma \quad \text{for} \quad \sigma \leqq 0.$$

As $\mu(\sigma)$ is convex, and $\mu(0) = 0$, it suffices to prove that

$$\mu(-h) \leqq h \quad \text{for} \quad h = 1, 2, 3, \cdots.$$

This, however, follows immediately from the above relation, holding for an arbitrary positive integer h,

$$f(s) = O(|t|^h) \quad \text{in } \sigma > -h + \varepsilon$$

which shows that $\mu(\sigma) \leqq h$ for $\sigma > -h$ and hence also, by reason of continuity, $\mu(-h) \leqq h$.

§ 2. The case $0 < \alpha < 1$.

In this last section we shall prove theorem 2 in the general case, i. e. for an arbitrary α in the interval $0 < \alpha < 1$. We have to prove the existence of a Dirichlet series $f(s) = \sum a_n n^{-s}$ with $\sigma^* = 0$, $\sigma_C = \alpha$, $\Omega = -\infty$ and the μ-function

$$\mu(\sigma) = \begin{cases} 0 & \text{for} \quad \sigma \geqq \alpha \\ \dfrac{\sigma - \alpha}{\alpha - 1} & \text{for} \quad \sigma \leqq \alpha. \end{cases}$$

As in the case $\alpha = 0$ we start from a rapidly increasing sequence of positive integers $p_1 < p_2 \cdots$. But now—in order that $\sum a_n n^{-s}$ shall converge only for $\sigma > \alpha$, and not for $\sigma > 0$—we must take care that the partial sums $S_n = a_1 + \cdots + a_n$ are not for all values of n too small compared with n but, roughly speaking, for some large n's are of the order of magnitude n^α. To this end we introduce, besides the numbers $p_1, p_2, \cdots$, other numbers $q_1, q_2, \cdots$ such that q_m is of the order p_m^α, and choose $a_n = 1$ for the values of n between p_m and $p_m + q_m$. Specifically, we choose the positive integers $(1 <) p_1 < p_2 \cdots$; $q_1, q_2, \cdots$ such that

$$p_m^\alpha < q_m < 2 p_m^\alpha, \qquad p_{m+1} > p_m + (m+1) q_m - 1$$

and (as in § 1) such that

$$\sum_{m=1}^{\infty} 2^m p_m^{-\varepsilon}$$

converges for every $\varepsilon > 0$.

The Dirichlet series to be constructed is based on the same idea as that of § 1 but is of a somewhat more complicated structure. Instead of the differences $\Delta^m(p^{-s})$ we have now to use differences $\Delta_q^m(p^{-s})$ with a "span" q (where q tends to infinity together with p), defined in the usual manner by

$$\Delta_q u_p = u_p - u_{p+q}, \quad \Delta_q^2 u_p = u_p - 2 u_{p+q} + u_{p+2q}, \cdots$$

$$\Delta_q^m u_p = u_p - \binom{m}{1} u_{p+q} + \binom{m}{2} u_{p+2q} - \cdots + (-1)^m u_{p+mq}, \cdots.$$

For these differences we have for $1 \leqq h < m$ the trivial estimation

$$|\Delta_q^m u_p| = |\Delta_q^{m-h} \Delta_q^h u_p| = \left|\Delta_q^h u_p - \binom{m-h}{1} \Delta_q^h u_{p+q} + \cdots + (-1)^{m-h} \Delta_q^h u_{p+(m-h)q}\right|$$

$$\leqq 2^{m-h} \cdot \operatorname*{Max}_{p \leqq \nu \leqq p+(m-h)q} |\Delta_q^h u_\nu|,$$

and for $p > 0$, $s = \sigma + it$ and $\sigma + h > 0$

$$|\Delta_q^h(p^{-s})| = \left| \int_p^{p+q} dx_1 \int_{x_1}^{x_1+q} dx_2 \cdots \int_{x_{h-1}}^{x_{h-1}+q} \frac{d^h}{dx_h^h}(x_h^{-s})\, dx_h \right| \leqq$$

$$|s||s+1| \cdots |s+h-1| \, q^h p^{-\sigma-h}.$$

In our series $\sum a_n n^{-s}$ the first group of terms is no longer (as in § 1) of the simple form $p_1^{-s} - (p_1+1)^{-s} = \Delta(p_1^{-s})$ but of the form

$$+ (p_1+1)^{-s} + \cdots + (p_1+q_1-1)^{-s} - (p_1+q_1)^{-s} - (p_1+q_1+1)^{-s} - \cdots -$$

$$(p_1 + 2q_1 - 1)^{-s} = \Delta_{q_1}(p_1^{-s}) + \Delta_{q_1}(p_1+1)^{-s} + \cdots + \Delta_{q_1}(p_1+q_1-1)^{-s}$$

and analogously for the following groups, i. e. the m^{th} group consists of the $(m+1) q_m$ terms (here not indicated in their natural order)

$$\sum_{\mu=0}^{q_m-1} \Delta_{q_m}^m (p_m + \mu)^{-s}.$$

Our Dirichlet series is then defined by

$$f(s) = \sum a_n n^{-s} = \sum_{m=1}^{\infty} \sum_{\mu=0}^{q_m-1} \Delta_{q_m}^m (p_m+\mu)^{-s}$$

where the different groups, on account of $p_{m+1} > p_m + (m+1)q_m - 1$, do not overlap.

Firstly we have to show that $\sigma^* = 0$. As in § 1 we see immediately that σ^* is $\geqq 0$, as for instance $a_n = 1$ for $n = p_m$. On the other hand $a_n = O(n^\varepsilon)$ for every $\varepsilon > 0$, since for each n with $a_n \neq 0$, i. e. $n = p_m + \mu \; (0 \leqq \mu < (m+1)\,q_m)$ we have for sufficiently large m (on account of $2^m p_m^{-\varepsilon} \to 0$)

$$|a_n| \leqq 2^m = 2^m p_m^{-\varepsilon} p_m^\varepsilon < p_m^\varepsilon \leqq n^\varepsilon.$$

Secondly we have $\sigma_C = \alpha$, and also $\sigma_A = \alpha$. To prove this, we first show that $\sum a_n n^{-s}$ is absolutely convergent for $\sigma > \alpha$, i. e. that $\sigma_A \leqq \alpha$, and secondly that $\sigma_C \geqq \alpha$. In order to prove the convergence of $\sum |a_n| n^{-\sigma}$ for $\sigma > \alpha$, we simply observe that for each $\sigma > \alpha$ the sum of the numerical values of the $(m+1)\,q_m$ terms in the group

$$\sum_{\mu=0}^{q_m-1} \Delta_{q_m}^m (p_m+\mu)^{-s}$$

is equal to

$$\sum_{\mu=0}^{q_m-1} \sum_{\nu=0}^{m} \binom{m}{\nu} (p_m+\mu+\nu q_m)^{-\sigma} < p_m^{-\sigma} 2^m q_m < 2 \cdot 2^m p_m^{\alpha-\sigma}$$

and that the series $\sum 2^m p_m^{\alpha-\sigma}$ is convergent for $\sigma > \alpha$. Next, in order to show that $\sigma_C \geqq \alpha$, it suffices (compare a remark in the introduction) to show that $S_n = a_1 + \cdots + a_n$ is not $o(n^\alpha)$. But this is certainly the case, as the assumption $S_n = o(n^\alpha)$ would imply that

$$S_{p_m-1} = o(p_m-1)^\alpha = o(p_m^\alpha), \quad S_{p_m+q_m-1} = o(p_m+q_m-1)^\alpha = o(p_m^\alpha)$$

and hence

$$S_{p_m+q_m-1} - S_{p_m-1} = o(p_m^\alpha)$$

which contradicts the fact that $a_n = 1$ for $p_m \leqq n \leqq p_m + q_m - 1$ and thus

$$S_{p_m+q_m-1} - S_{p_m-1} = q_m > p_m^\alpha.$$

Thirdly, in order to prove that $\Omega = -\infty$, it suffices to show that the analytic function $f(s)$ defined by the series for $\sigma > \alpha$ for some decreasing sequence of real numbers $\sigma_h \; (h = 1, 2, \cdots)$ tending to $-\infty$ is regular and of finite order in $\sigma > \sigma_h + \varepsilon$. As these numbers σ_h we choose the equidistant numbers

$$\sigma_h = \alpha - h(1-\alpha) \qquad (h = 1, 2, \cdots),$$

i. e. the numbers σ_h for which the linear function $M_\alpha(\sigma) = \frac{\sigma-\alpha}{\alpha-1} = h$. We proceed in a way analogous to that in § 1, i. e. we divide, for a fixed h, the function $f(s)$ into two parts, viz.

$$f(s) = f_1(s) + f_2(s) = \sum_{m=1}^{h} \sum_{\mu=0}^{q_m-1} \Delta_{q_m}^m (p_m+\mu)^{-s} + \sum_{m=h+1}^{\infty} \sum_{\mu=0}^{q_m-1} \Delta_{q_m}^m (p_m+\mu)^{-s}$$

where the finite sum $f_1(s) = \sum_{m=1}^{h}$ is obviously an integral function bounded in every half-plane $\sigma > \sigma_0$. In the second sum $f_2(s) = \sum_{m=h+1}^{\infty}$ we estimate each term for $\sigma > \sigma_h + \varepsilon$ and find for an arbitrary one of its q_m components

$$\left|\Delta_{q_m}^m (p_m+\mu)^{-s}\right| \leqq 2^{m-h} \cdot \underset{p_m+\mu \leqq \nu \leqq p_m+\mu+(m-h)q_m}{\operatorname{Max}} \left|\Delta_{q_m}^h \nu^{-s}\right|$$

where the right-hand side, on account of $\sigma + h > \alpha(1+h) + \varepsilon$ for $\sigma > \sigma_h + \varepsilon$, is again

$$\leqq 2^{m-h} |s|\,|s+1| \cdots |s+h-1|\, q_m^h \cdot \underset{p_m+\mu \leqq \nu \leqq p_m+\mu+(m-h)q_m}{\operatorname{Max}} \nu^{-\alpha(1+h)-\varepsilon}$$

$$\leqq 2^{m-h} |s| \cdots |s+h-1|\, q_m^h\, p_m^{-\alpha(1+h)-\varepsilon}.$$

Thus for $\sigma > \sigma_h + \varepsilon$ we find for the whole m^{th} group consisting of q_m components the estimation

$$\left| \sum_{\mu=0}^{q_m-1} \Delta_{q_m}^m (p_m + \mu)^{-s} \right| \leqq q_m \cdot (2^{m-h} |s| \cdots |s+h-1| q_m^h p_m^{-\alpha(1+h)-\varepsilon})$$

which, on account of $q_m < 2 p_m^\alpha$, is again

$$< 2^{m+1} |s| \cdots |s+h-1| p_m^{\alpha(1+h)-\alpha(1+h)-\varepsilon} = 2|s| \cdots |s+h-1| \cdot 2^m p_m^{-\varepsilon}.$$

Now, as $\sum 2^m p_m^{-\varepsilon}$ is convergent, we conclude, just as in § 1, that $f_2(s)$ and hence also $f(s) = f_1(s) + f_2(s)$, is regular and $O(|t|^h)$ in $\sigma > \sigma_h + \varepsilon$, and thus we have proved that $\Omega = -\infty$.

Finally we shall prove that the μ-function, defined for all σ, is given by

$$\mu(\sigma) = \begin{cases} 0 & \text{for } \sigma \geqq \alpha \\ M_\alpha(\sigma) = \dfrac{\sigma - \alpha}{\alpha - 1} & \text{for } \sigma \leqq \alpha. \end{cases}$$

That $\mu(\sigma) = 0$ for $\sigma \geqq \alpha$ follows immediately from $\sigma_A = \alpha$. Since, according to the L.-S. theorem in the formulation of pag. 7, we have certainly

$$\mu(\sigma) \geqq M_\alpha(\sigma) = \frac{\sigma - \alpha}{\alpha - 1} \quad \text{for } \sigma \leqq \alpha$$

we need only show that

$$\mu(\sigma) \leqq M_\alpha(\sigma) \quad \text{for} \quad \sigma \leqq \alpha,$$

and as $\mu(\sigma)$ is convex, and $\mu(\alpha) = 0$, it suffices to show that

$$\mu(\sigma_h) \leqq M_\alpha(\sigma_h) = h \quad \text{for} \quad h = 1, 2, \cdots.$$

This, however, is an immediate consequence of the relation

$$f(s) = O(|t|^h) \quad \text{for} \quad \sigma > \sigma_h + \varepsilon,$$

which shows that $\mu(\sigma) \leqq h$ for $\sigma > \sigma_h$ and hence also for $\sigma = \sigma_h$.