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ON THE INTRODUCTION OF MEASURES IN INFINITE PRODUCT SETS

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1. Introduction. It is shown that an important theorem on the introduction of measures in a real space of an infinite number of dimensions cannot be extended to abstract sets. From the theory of product measures in abstract product sets follows that the extension is valid in the case, where, in the terminology of the theory of probability, the coordinates are independent. It will be shown by an example that the extension need not be possible when the coordinates are dependent.

2. Formulation of the result. Let I denote an infinite set of indices i, and let there for every $i \varepsilon I$ be given a non-empty set E_i and a Borel field $\mathfrak{F}_i$ of sub-sets of E_i such that $E_i \varepsilon \mathfrak{F}_i$. By $E = (E_i)$ we denote the product of the sets E_i, consisting of all symbols $x = (x_i)$ where $x_i \varepsilon E_i$ for every $i \varepsilon I$. The elements x_i are called the coordinates of x. The smallest Borel field in E, which for every i and every $A_i \varepsilon \mathfrak{F}_i$ contains the set of all x for which $x_i \varepsilon A_i$, will be denoted by $\mathfrak{F} = (\mathfrak{F}_i)$.

For an arbitrary finite sub-set $I' = \{i_1, \cdots, i_n\}$ of I we may consider the corresponding partial product $E' = (E_{i_1}, \cdots, E_{i_n})$ with elements $x' = (x_{i_1}, \cdots, x_{i_n})$ and in E' the smallest Borel field $\mathfrak{F}' = (\mathfrak{F}_{i_1}, \cdots, \mathfrak{F}_{i_n})$ which for every $i \varepsilon I'$ and every $A_i \varepsilon \mathfrak{F}_i$ contains the set of all x' for which $x_i \varepsilon A_i$.

For an arbitrary $x \varepsilon E$ the element $x' \varepsilon E'$ for which the coordinates are equal to the corresponding coordinates of x is called the projection of x on E'. The set of all x for which the projection x' belongs to a set A' in E' will be denoted by $\{x' \varepsilon A'\}$; it is called the cylinder with base A' in E'. It belongs to $\mathfrak{F}$ if and only if A' belongs to $\mathfrak{F}'$.

Our problem is now the following:

Let there for every finite partial product E' of E be given a measure μ' defined on the system of all cylinders of $\mathfrak{F}$ with base in E' and with $\mu'(E') = 1$. Suppose that any two of these measures coincide in the common part of their domains, so that a set function λ is defined on the system of all cylinders of $\mathfrak{F}$ with base in some finite partial product by placing $\lambda(A) = \mu'(A)$ when A is a cylinder with base in E'. Is it then possible to extend λ to a measure defined on $\mathfrak{F}$?

It is clear that the domain of λ is a field and that λ is additive. Since $\mathfrak{F}$ is the smallest Borel field containing the domain of λ, a necessary and sufficient condition that λ may be extended to a measure defined on $\mathfrak{F}$ is that λ is completely additive. The extension is then unique.

3. Two important cases are known in which the answer is affirmative.

The one is the case where each E_i is the real axis $-\infty < x_i < +\infty$ and $\mathfrak{F}_i$ is the system of Borel sets on E_i. Then E is the real space whose dimension is the cardinal number of I, and $\mathfrak{F}$ is (by definition) the system of Borel sets in E. The explicit formulation of the possibility of the extension under these conditions is due to Kolmogoroff [9, pp. 24—30]; essentially, the result goes back to Daniell [3], [4].

The other is the case where there is given a measure μ_i in every E_i with domain $\mathfrak{F}_i$ such that $\mu_i(E_i) = 1$ and where $\mu'(A)$ for a set of the type $A = \{x_{i_1} \varepsilon A_{i_1}\} \cdots \{x_{i_n} \varepsilon A_{i_n}\}$, where $A_{i_1} \varepsilon \mathfrak{F}_{i_1}$, $\cdots$, $A_{i_n} \varepsilon \mathfrak{F}_{i_n}$, is equal to the product $\mu_{i_1}(A_{i_1}) \cdots \mu_{i_n}(A_{i_n})$. By this condition the measures μ' are uniquely determined. The extension of λ to $\mathfrak{F}$ is the product measure $\mu = (\mu_i)$ generated by the measures μ_i. Proofs of this result, which has first been formulated by Łomnicki and Ulam [10], have been given by von Neumann [11, pp. 105—129] and Jessen [7], cf. Sparre Andersen and Jessen [2, pp. 18—22]. The latter proof has also been found by Kakutani [8].

These examples make it natural to expect that the answer is always affirmative. Attempts to prove that it is so have been made by Doob [6, pp. 90—93], who considers the case with equal E_i and equal $\mathfrak{F}_i$, and, independently, by Sparre Andersen [1]. That these proofs are incomplete has been pointed out in Sparre Andersen and Jessen [2, p. 22]. We shall now prove by the construction of an example that actually the theorem fails. More explicitly we shall prove:

For an arbitrary infinite set I there exists a case with equal E_i and equal $\mathfrak{F}_i$ for which the answer to the problem is negative.

In the terminology of the theory of probability this means, that the case of dependent variables cannot be treated for abstract variables in the same manner as for unrestricted real variables. Professor Doob has kindly pointed out, what was also known to us, that this case may be dealt with along similar lines as the case of independent variables (product measures) when conditional probability measures are supposed to exist. This question will be treated in a forthcoming paper by Doob and Jessen. That conditional probability measures need not always exist, has, as we have been informed, already been shown by Dieudonné in a paper which is about to appear [5, p. 42]. This is also seen from our example.

4. Construction of the example. Let C denote a circle of length 1 and let m^* denote the exterior (linear) Lebesgue measure on C. Let C be divided in Hausdorff's manner into disjoint sets $\cdots$, C_{-1}, C_0, C_1, $\cdots$ which are congruent by rotation. For each $i \varepsilon I$ we take $E_i = C - C_j$ for some $j = j(i)$ and choose as $\mathfrak{F}_i$ the system of all sets which are the common part of E_i and a Borel set on C. We suppose, as we may since I is infinite, that every integer j occurs as value of $j(i)$ for at least one i. With this choice the sets E_i are not actually equal, but they are congruent, which amounts to the same; if by rotation we make the E_i equal, the $\mathfrak{F}_i$ will also be equal.

The product $E = (E_i)$ is a sub-set of the torus-space $Q = (Q_i)$, where all $Q_i = C$. Similarly, for an arbitrary finite sub-set $I' = \{i_1, \cdots, i_n\}$ of I the partial product E' is a sub-set of the torus-space $Q' = (Q_{i_1}, \cdots, Q_{i_n})$.

A point $x = (t, \cdots, t)$ on the 'diagonal' of Q' will belong to E' if and only if t belongs to the set $D' = (C - C_{j(i_1)}) \cdots$

$(C - C_{j(i_n)}) = C - (C_{j(i_1)} + \cdots + C_{j(i_n)})$ on C. Since C contains an infinite number of disjoint sets congruent to $C_{j(i_1)} + \cdots + C_{j(i_n)}$ we have $m^*(D') = 1$.

We now consider a cylinder $A = \{x' \varepsilon A'\}$ with base $A' \varepsilon \mathfrak{F}'$ in E'. The base A' is the common part of E' and a Borel set in Q'. Hence the set $S_{A'}$ of points t on C for which the corresponding point $x' = (t, \cdots, t)$ on the diagonal of Q' belongs to A' is the common part of D' and a Borel set on C. We may therefore define a measure μ' on the system of these cylinders A by placing $\mu'(A) = m^*(S_{A'})$. Plainly $\mu'(E) = 1$.

We shall now prove that any two of these measures coincide in the common part of their domains. It will be sufficient to prove that if μ_0' is the measure corresponding to a sub-set $I_0' = \{i_1, \cdots, i_n, i_{n+1}, \cdots, i_{n_0}\} \supset I'$ then μ' is a contraction of μ_0'. To see this it is sufficient to notice that when a cylinder $\{x' \varepsilon A'\}$ with base in E' is considered as a cylinder $\{x_0' \varepsilon A_0'\}$ with base in the partial product E_0' corresponding to I_0' then the set $S_{A'}$ is replaced by a set $S_{A_0'} \subseteq S_{A'}$. Hence $\mu_0'(A) \leq \mu'(A)$. If this inequality is applied to the complementary set $E - A$ we obtain, since $\mu_0'(E) = \mu'(E)$, the opposite inequality $\mu_0'(A) \geq \mu'(A)$. Hence $\mu_0'(A) = \mu'(A)$.

Finally we shall prove that the set-function λ defined by the measures μ' cannot be extended to a measure defined on $\mathfrak{F}$. To see this we consider for every finite partial product E' the cylinder $A = \{x' \varepsilon A'\}$ whose base A' consists of all $x' \varepsilon E'$ which do not lie on the diagonal of Q' (this set A' belongs to $\mathfrak{F}'$ since it is the common part of E' and a Borel set in Q'). Then $S_{A'}$ is empty so that $\lambda(A) = \mu'(A) = 0$. Let $i_1, i_2, \cdots$ be a sequence of elements of I such that the numbers $j(i_1), j(i_2), \cdots$ are *all* integers, and let A_n be the set A formed, in the manner just explained, for $I' = \{i_1, \cdots, i_n\}$. Then the sets $A_1, A_2, \cdots$ will cover E, since for any element $x = (x_i)$ of E the coordinates $x_{i_1}, x_{i_2}, \cdots$ cannot all be equal. Consequently λ is not completely additive.

5. The idea of the above example may briefly be described as follows: We choose $E = (E_i)$ as a sub-set of the torus-space $Q = (Q_i)$ choosing the sets E_i in such a manner that any finite partial product E' of E contains a set of exterior linear Lebesgue measure 1 on the diagonal of the corresponding partial product Q' of Q whereas there exists an enumerable partial product E'' of E (corresponding to the indices $i_1, i_2, \cdots$) which contains no points on the diagonal of the corresponding partial product Q'' of Q. The first property makes it possible to define the measures μ' by 'projection' on E' of an equidistribution of unit mass on the diagonal of Q, and the second property prevents the complete additivity of the corresponding set-function λ, since by projection on Q'' the whole mass falls outside E''.