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# ON A GENERALIZATION OF KRONECKER'S THEOREM

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## INTRODUCTION

The following well-known theorem by KRONECKER

**Theorem 1.** *If m real numbers $\lambda_1, \cdots, \lambda_m$ do not satisfy any relation*

$$r_1\lambda_1 + \cdots + r_m\lambda_m = 0,$$

*where $r_1, \cdots, r_m$ are rational numbers and at least one $r_\nu$ is $\neq 0$, then there exists to any given real numbers $v_1, \cdots, v_m$ and any positive $\varepsilon$ a number $t$ satisfying*

$$|\lambda_\nu t - v_\nu| \leqq \varepsilon \ (\text{mod. } 2\pi), \quad \nu = 1, \cdots, m,$$

is equivalent to the following

**Theorem 2.** *If $h_\nu(t) = e^{i\lambda_\nu t}$; $\nu = 1, \cdots, m$ are pure oscillations, whose frequencies $\frac{\lambda_\nu}{2\pi}$; $\nu = 1, \cdots, m$ do not satisfy any relation*

$$r_1\frac{\lambda_1}{2\pi} + \cdots + r_m\frac{\lambda_m}{2\pi} = 0,$$

*where $r_1, \cdots, r_m$ are rational numbers and at least one $r_\nu$ is $\neq 0$, then there exists to any given real numbers $v_1, \cdots, v_m$ and any positive $\varepsilon$ a number $t$ satisfying*

$$(1) \qquad |h_\nu(t) - e^{iv_\nu}| \leqq \varepsilon; \quad \nu = 1, \cdots, m.$$

If the numbers $\lambda_1, \cdots, \lambda_m$ satisfy the condition of Theorem 1, they are called *rationally independent.* Theorem 1 states that the straight line $x_\nu = \lambda_\nu t$; $\nu = 1, \cdots, m$, where $\lambda_1, \cdots, \lambda_m$ are rationally independent, is mod. $2\pi$ everywhere dense in the $m$-dimensional space.

In this paper we shall consider *phase-modulated oscillations,* i.e. functions

$$H(t) = e^{i(ct+g(t))},$$

where $c$ is a real constant and $g(t)$ a real-valued function, almost periodic in the sense of BOHR. Its frequency (in mean) is determined by the constant $c$, which is called the *mean motion* of $H(t)$. We shall prove in this paper that Theorem 2 is valid also for phase-modulated oscillations, i. e. we shall prove the following

**Theorem 3.** *Let $H_1(t), \cdots, H_m(t)$ be phase-modulated oscillations with rationally independent mean motions. To any given real numbers $v_1, \cdots, v_m$ and any positive $\varepsilon$ there exists a number $t$, satisfying*

$$|H_\nu(t) - e^{iv_\nu}| \leqq \varepsilon; \ \nu = 1, \cdots, m.$$

Apparently we lose nothing by this generalization although Theorem 3 is evidently much more far-reaching than Theorem 2. However, it is well-known that BOHL has proved that the set of numbers $t$ satisfying (1) is relatively dense and WEYL has proved that the set of numbers $t$ satisfying (1) has the relative measure $\left(\frac{2 \arc\sin\frac{\varepsilon}{2}}{\pi}\right)^m$ on the $t$-axis. It will be proved that BOHL's result is valid also in the general case, but we lose WEYL's result. This is, in fact, not valid for the single oscillation $H(t) = e^{i(t+\sin t)}$, if we take $v = 0$ and $\varepsilon < 2$.

Theorem 3 states that the curve $x_\nu = c_\nu t + g_\nu(t)$; $\nu = 1, \cdots, m$, where $c_1, \cdots, c_m$ are rationally independent, is mod $2\pi$ everywhere dense in the $m$-dimensional space.

The result is brought in closer connection with the theory of almost periodic functions by the following theorem by H. BOHR[1].

**Theorem 4.** *A complex-valued almost periodic function $f(t)$, $-\infty < t < \infty$, satisfying $|f(t)| \geqq k > 0$, can be written*

$$f(t) = r(t) \cdot H(t),$$

*where $r(t)$ is a positive almost periodic function and $H(t)$ is a phase-modulated oscillation.*

The mean motion of $H(t)$ is also called the *mean motion* of $f(t)$. If $f(t)$ is almost periodic and $a$ and $b$ are complex constants such that $f(t) - a$ and $f(t) - b$ satisfy the condition of Theorem 4, the mean motions of these functions have a rational ratio. This was first proved by JESSEN[1] and later JESSEN and FENCHEL[2] found a more general theorem concerning almost periodic movements on closed or plane surfaces. In this paper we shall deduce some generalizations of JESSEN's original theorem in another direction. For the present we observe that JESSEN's theorem is a corollary of Theorem 3. In fact, JESSEN's theorem may be expressed as the following

[1] H. BOHR: Kleinere Beiträge zur Theorie der fastperiodischen Funktionen, I. Det Kgl. Danske Videnskabernes Selskab. Math.-fys. Meddelelser, X (1930). Über fastperiodische ebene Bewegungen. Comment. math. helv. 4 (19..).



**Theorem 5.** *If two almost periodic functions $f_1(t)$ and $f_2(t)$, which do not come arbitrarily near to zero, satisfy a linear relation*

$$\alpha_1 f_1(t) + \alpha_2 f_2(t) = \alpha_3, \tag{2}$$

*where $\alpha_1$, $\alpha_2$ and $\alpha_3$ are complex constants $\neq 0$, the mean motions $c_1$ and $c_2$ of $f_1(t)$ and $f_2(t)$ are rationally dependent.*

In fact, if $c_1$ and $c_2$ were rationally independent, there would according to Theorem 3 exist a number $t$ such that the complex numbers $\alpha_1 f_1(t)$, $\alpha_2 f_2(t)$ and $-\alpha_3$ would have nearly equal arguments and that would render the relation (2) impossible.

We shall prove that Theorem 5 is valid for an arbitrary number of almost periodic functions satisfying a similar condition. If, on the other hand, we restrict the number of functions to three, we may replace the linear relation (2) by a homogeneous quadratic equation, and the theorem is still true.

## § 1. Some Preliminary Remarks.

For the convenience of the reader we shall first mention some results concerning periodic and limit periodic functions of a denumerable set of variables, which we shall use in the sequel. We shall permanently use the vector notations $\boldsymbol{x} = (x_1, x_2, \cdots)$. For two vectors $\boldsymbol{x} = (x_1, x_2, \cdots)$ and $\boldsymbol{y} = (y_1, y_2, \cdots)$ and two numbers $k$ and $l$ we define the linear combination

$$k\boldsymbol{x} + l\boldsymbol{y} = (kx_1 + ly_1, kx_2 + ly_2, \cdots),$$

[1] B. JESSEN: Über die Säkularkonstanten einer fastperiodischen Funktion. Math. Ann. 111 (1935).

[2] W. FENCHEL und B. JESSEN: Über fastperiodische Bewegungen in ebenen Bereichen und auf Flächen. Kgl. Danske Videnskabernes Selskab. Math.-fys. Meddelelser XIII, 6 (1935).

and if one of the vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ has only a finite number of coordinates $\neq 0$, we have the inner product

$$\boldsymbol{x}\boldsymbol{y} = x_1 y_1 + x_2 y_2 + \cdots.$$

A sequence $\boldsymbol{x}_\nu = (x_{\nu 1}, x_{\nu 2}, \cdots)$; $\nu = 1, 2, \cdots$ of vectors is said to converge towards a vector $\boldsymbol{x}_0 = (x_{01}, x_{02}, \cdots)$ if $x_{\nu\mu} \to x_{0\mu}$ for $\mu = 1, 2, \cdots$ and $\nu \to \infty$. A function $F(\boldsymbol{x})$ is called continuous for $\boldsymbol{x} = \boldsymbol{x}_0$ if $F(\boldsymbol{x}_\nu) \to F(\boldsymbol{x}_0)$, when $\boldsymbol{x}_\nu$ runs through a sequence of vectors belonging to the domain where $F(\boldsymbol{x})$ is defined, and converging towards $\boldsymbol{x}_0$. A function is continuous in a domain (i. e. continuous in every point of this domain) if it can be approximated uniformly with any given accuracy by a continuous function depending only on a finite number of variables. In what follows the domain in question is the real infinite-dimensional space. A function $F(\boldsymbol{x}) = F(x_1, x_2, \cdots)$ is called limit periodic with the limit period $2\pi$, if it can be approximated uniformly in the whole space by a continuous function depending only on a finite number of variables and periodic in each of these with a period that is an integral multiple of $2\pi$. Hence a limit periodic function is continuous. The function $F(\boldsymbol{x})$ can be approximated uniformly in the whole space with any given accuracy by an exponential polynomial

$$P(\boldsymbol{x}) = \sum{}^{*} a_{\boldsymbol{r}} e^{i \boldsymbol{r}\boldsymbol{x}}$$

where $\boldsymbol{r}$ runs through a finite set of vectors with rational coordinates, among which only a finite number are $\neq 0$[1].

The numbers $\lambda_1, \cdots, \lambda_m$ are called *rationally independent* if a relation

$$r_1 \lambda_1 + \cdots + r_m \lambda_m = 0$$

with rational $r_1, \cdots, r_m$ is possible only when $r_1 = \cdots = r_m = 0$. The numbers $\lambda_1, \lambda_2, \cdots$ are rationally independent if $\lambda_1, \cdots, \lambda_m$ are rationally independent for all values of $m$.

In the sequel we shall give a very brief account of some principal theorems concerning almost periodic functions[2]. We shall start with two preliminary definitions:

[1] A detailed discussion of the properties of limit periodic functions is given by H. BOHR: Zur Theorie der fastperiodischen Funktionen II. Acta math. 46 (1925).

[2] For detailed proofs, cf. H. BOHR: Fastperiodische Funktionen, Berlin 1932.



A set of real numbers is called *relatively dense* if there exists a number $l$ such that any interval of length $l$ contains at least one number of the set.

A number $\tau$ is called a *translation number* of a function $f(t)$, $-\infty < t < \infty$ corresponding to $\varepsilon > 0$, if

$$|f(t+\tau) - f(t)| \leq \varepsilon$$

for $-\infty < t < \infty$.

A function $f(t)$ is called *almost periodic* if it has the following property

(i) The set of translation numbers of $f(t)$ corresponding to any $\varepsilon > 0$ is relatively dense.

It is a main result of the theory of almost periodic functions that any of the following two properties is equivalent to the preceding one:

(ii) To any $\varepsilon > 0$ exists an exponential polynomial

$$\sum{}^{*} a_\lambda e^{i\lambda t},$$

where $\lambda$ runs through a finite set of real numbers, approximating $f(t)$ everywhere with the accuracy $\varepsilon$.

(iii) There exist a series of linearly independent real numbers $\beta_1, \beta_2, \cdots$ and a function $F(\boldsymbol{x})$ with the limit period $2\pi$ such that

$$f(t) = F(\boldsymbol{\beta} t) = F(\beta_1 t, \beta_2 t, \cdots).$$

The function $F(\boldsymbol{x})$ is called a spatial extension (or *the* spatial extension, although $F(\boldsymbol{x})$ is not uniquely determined) of $f(t)$.

The equivalence of the two latter properties is rather easily proved and it is also rather simple to prove that they imply the property (i), but it is much more difficult to prove that (i) implies (ii) or (iii). This is the main theorem in the theory of almost periodic functions. In the sequel we shall almost exclusively use the property (iii). From the theory of almost periodic functions we also have

**Theorem 6.** *The set of values assumed by the spatial extension of an almost periodic function $f(t)$ is a subset of the closure of the set of values assumed by $f(t)$.*

**Theorem 7.** *The set of common translation numbers of a fini number of almost periodic functions corresponding to an arbitrar $\varepsilon > 0$ is relatively dense.*

Sum and product of a finite number of almost periodic func tions are almost periodic, and if $g(t)$ is almost periodic th function $e^{ig(t)}$ is also almost periodic.

It is important that the numbers $\beta_1, \beta_2, \cdots$ in (iii) can b chosen in a great variety of manners. E. g. any set $\gamma_1, \gamma_2, \cdot$ of linearly independent numbers such that any $\beta_\nu$ can be writte as a linear combination with rational coefficient of a finit number of the $\gamma$ 's. From this follows further

**Theorem 8.** *To a sequence $f_1(t), f_2(t), \cdots$ of almost periodi functions exist a sequence $\beta_1, \beta_2, \cdots$ of rationally independent rea numbers and a sequence $F_1(\boldsymbol{x}), F_2(\boldsymbol{x}), \cdots$ of limit periodic func tions such that*

$$f_\nu(t) = F_\nu(\boldsymbol{\beta} t); \; \nu = 1, 2, \cdots,$$

*and the sequence $\beta_1, \beta_2, \cdots$ can be chosen such that it contai any given sequence of rationally independent numbers as a sub sequence.*

From the theory of almost periodic functions follows furthe

**Theorem 9.** *If a denumerable set $f_1(t), f_2(t), \cdots$ of almo periodic functions satisfy an equation*

$$\Phi(f_1(t), f_2(t), \cdots) = 0,$$

*where $\Phi(u_1, u_2, \cdots)$ is continuous when $u_\nu$ for $\nu = 1, 2, \cdots$ be longs to the closure of the set of values assumed by $f_\nu(t)$, th spatial extensions $F_1(\boldsymbol{x}), F_2(\boldsymbol{x}), \cdots$ satisfy the equation*

$$\Phi(F_1(\boldsymbol{x}), F_2(\boldsymbol{x}), \cdots) = 0.$$

**Theorem 10.** *If $G(\boldsymbol{x})$ has the limit period $2\pi$, and $(r_1, r_2,$ is a vector with rational coordinates, of which only a finite nu ber are $\neq 0$, the function*

$$e^{i(r_1 x_1 + r_2 x_2 + \cdots + G(\boldsymbol{x}))}$$

*has the limit period $2\pi$.*



*If $g(t)$ is almost periodic and $c$ is an arbitrary real number, the function*

$$e^{i(ct+g(t))}$$

*is almost periodic.*

## § 2. The Mean Motions of Limit Periodic and Almost Periodic Functions.

A continuous argument of a continuous function $P(\boldsymbol{x}) = P(x_1, \cdots, x_m)$ with the period $2\pi$ in each variable and not assuming the value 0 can evidently be written

$$\arg P(\boldsymbol{x}) = \boldsymbol{p}\boldsymbol{x} + Q(\boldsymbol{x}) = p_1 x_1 + \cdots + p_m x_m + Q(\boldsymbol{x}),$$

where $p_1, \cdots, p_m$ are integers and $Q(\boldsymbol{x})$ is continuous and has the period $2\pi$. For a limit periodic function we have

**Theorem 11.** *If $F(\boldsymbol{x}) = F(x_1, x_2, \cdots)$ has the limit period $2\pi$ and satisfies $|F(\boldsymbol{x})| \geqq k > 0$, a continuous argument of $F(\boldsymbol{x})$ can be written*

$$\arg F(\boldsymbol{x}) = r_1 x_1 + r_2 x_2 + \cdots + G(\boldsymbol{x}) = \boldsymbol{r}\boldsymbol{x} + G(\boldsymbol{x}),$$

*where $r_1, r_2, \cdots$ are rational numbers, of which only a finite number are $\neq 0$, and $G(\boldsymbol{x})$ has the limit period $2\pi$.*

*The vector $\boldsymbol{r}$ is called the mean motion vector of $F(\boldsymbol{x})$.*

For the proof[1] we consider a sequence $P_1(\boldsymbol{x}), P_2(\boldsymbol{x}), \cdots$ of continuous functions with the following properties: (i) Each function depends only on a finite number of variables and has a period that is an integral multiple of $2\pi$. (ii) The functions $P_\nu(\boldsymbol{x})$ converge uniformly towards $F(\boldsymbol{x})$ and satisfy

$$|F(\boldsymbol{x}) - P_\nu(\boldsymbol{x})| \leqq \frac{k}{2}; \; \nu = 1, 2, \cdots.$$

We can choose continuous arguments such that

$$(3) \qquad |\arg F(\boldsymbol{x}) - \arg P_\nu(\boldsymbol{x})| < \frac{\pi}{2}; \; \nu = 1, 2, \cdots$$

and we have for any integer $\nu$

[1] The following proof is very similar to a proof of Theorem 4 given by JESSEN loc. cit.

$$\arg P_\nu(\boldsymbol{x}) = r_1 x_1 + r_2 x_2 + \cdots + Q_\nu(\boldsymbol{x}),$$

where $r_1, r_2, \cdots$ are rational numbers, of which only a finite number are $\mp 0$, and $Q_\nu(\boldsymbol{x})$ is a continuous function depending on a finite number of variables and having a period that is an integral multiple of $2\pi$. From this and (3) follows

$$\arg F(\boldsymbol{x}) = r_1 x_1 + r_2 x_2 + \cdots + G(\boldsymbol{x}),$$

where $G(\boldsymbol{x})$ is continuous and bounded. But it also follows that the numbers $r_1, r_2, \cdots$ do not depend on $\nu$. Hence it follows that $Q_\nu(\boldsymbol{x})$ converges towards $G(\boldsymbol{x})$, which implies that $G(\boldsymbol{x})$ has the limit period $2\pi$.

Concerning almost periodic functions we have

**Theorem 12.** *Let $f(t)$ denote an almost periodic function and $F(\boldsymbol{x})$ its spatial extension such that*

$$f(t) = F(\beta_1 t, \beta_2 t, \cdots),$$

*where $\beta_1, \beta_2, \cdots$ are rationally independent real numbers. If $f(t)$ satisfies the condition $|f(t)| \geq k > 0$, continuous arguments of $F(\boldsymbol{x})$ and $f(t)$ can be written*

$$\arg F(\boldsymbol{x}) = r_1 x_1 + r_2 x_2 + \cdots + G(\boldsymbol{x})$$

*and*

$$\arg f(t) = ct + g(t),$$

*where*

$$c = r_1 \beta_1 + r_2 \beta_2 + \cdots$$

*and*

$$g(t) = G(\beta_1 t, \beta_2 t, \cdots),$$

i. e. *$g(t)$ is almost periodic.*

*The constant $c$ is called the mean motion of $f(t)$.*

The theorem is an immediate consequence of Theorems 6 and 11. It contains Theorem 4 as a special case.

## § 3. An Auxiliary Theorem on Convergence in an Infinite-Dimensional Space.

A denumerable set $\boldsymbol{a}_\mu = (a_{\mu 1}, a_{\mu 2}, \cdots)$; $\mu = 1, 2, \cdots$ of infinite-dimensional vectors, each with only a finite number of



its coordinates $\mp 0$, is called a complete set of linearly independent vectors if every vector with only a finite number of its coordinates $\mp 0$ in one and only one way can be written as a linear combination of a finite number of the vectors $\boldsymbol{a}_\mu$. This will be the case if and only if any finite number of the vectors $\boldsymbol{a}_\mu$ are linearly independent and each of the unit vectors $\boldsymbol{e}_1 = (1, 0, 0, \cdots)$, $\boldsymbol{e}_2 = (0, 1, 0, \cdots)$, $\cdots$ can be written as a linear combination of a finite number of the vectors $\boldsymbol{a}_\mu$.

For the proof of a generalization of Theorem 3 to a denumerable set of phase-modulated oscillations we shall need the following theorem.

**Theorem 13.** *Let $\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots$ be a complete set of linearly independent vectors, each with only a finite number of its coordinates $\mp 0$, and let $K_1, K_2, \cdots$ be a sequence of positive numbers. Any sequence $\boldsymbol{x}_\nu = (x_{\nu 1}, x_{\nu 2}, \cdots)$; $\nu = 1, 2, \cdots$ of vectors satisfying*

$$(4)\quad \begin{cases} |\boldsymbol{a}_\mu \boldsymbol{x}_\nu| = |a_{\mu 1} x_{\nu 1} + a_{\mu 2} x_{\nu 2} + \cdots| \leq K_\mu; \; \mu = 1, 2, \cdots, \nu; \\ \qquad \nu = 1, 2, \cdots \end{cases}$$

*possesses a convergent subsequence.*

In fact, for each $n$ we have a representation

$$\boldsymbol{e}_n = \alpha_{n1} \boldsymbol{a}_1 + \cdots + \alpha_{nN_n} \boldsymbol{a}_{N_n},$$

which implies that

$$(5)\quad x_{\nu n} = \boldsymbol{e}_n \boldsymbol{x}_\nu = \alpha_{n1} \boldsymbol{a}_1 \boldsymbol{x}_\nu + \cdots + \alpha_{nN_n} \boldsymbol{a}_{N_n} \boldsymbol{x}_\nu$$

and from (4) and (5) follows that

$$|x_{\nu n}| \leq |\alpha_{n1}| \, |\boldsymbol{a}_1 \boldsymbol{x}_\nu| + \cdots + |\alpha_{nN_n}| \, |\boldsymbol{a}_{N_n} \boldsymbol{x}_\nu| \leq |\alpha_{n1}| K_1 + \cdots + \\ + |\alpha_{nN_n}| K_{N_n}; \; \nu = n, n+1, \cdots$$

or

$$|x_{\nu n}| \leq K_n^*; \; n = 1, 2, \cdots; \; \nu = 1, 2, \cdots,$$

and it follows by the usual diagonal method that the sequence has a convergent subsequence.

It is easily proved that any sequence of vectors $\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots$, each with only a finite number of its coordinates $\mp 0$, and of which any finite number are linearly independent, can be enlarged to a complete set of linearly independent vectors, e. g. by adding to the sequence a conveniently chosen subsequence of the sequence of unit vectors.

## § 4. An Auxiliary Theorem from the Topology.

The greater part of our theory will be founded on the following topological theorem, which is an immediate consequence of the theory of the Kronecker index of a surface[1].

**Theorem 14.** *Let* $\boldsymbol{y} = \boldsymbol{f}(\boldsymbol{x})$ *or* $y_1 = f_1(\boldsymbol{x}), \cdots, y_m = f_m(\boldsymbol{x})$ *be a continuous vector function in m-dimensional space. If there exists a constant K such that* $|f_\nu(\boldsymbol{x})| \leqq K$ *for* $\nu = 1, \cdots, m$ *and all vectors* $\boldsymbol{x}$, *the vector function* $\boldsymbol{y} = \boldsymbol{x} + f(\boldsymbol{x})$ *maps the total m-dimensional space on itself.*

Generally, of course, the mapping is not a one-to-one correspondence. For the infinite-dimensional space we shall prove the somewhat more general theorem:

**Theorem 15.** *Let* $\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots$ *denote a complete set of linearly independent real vectors,* $K_1, K_2, \cdots$ *positive numbers and* $Q_1(\boldsymbol{x})$, $Q_2(\boldsymbol{x}), \cdots$ *continuous real functions satisfying* $|Q_\nu(\boldsymbol{x})| \leqq K_\nu$, $\nu = 1, 2, \cdots$. *To arbitrary real numbers* $y_1, y_2, \cdots$ *exists a corresponding vector* $\boldsymbol{x}$ *satisfying*

$$\boldsymbol{a}_\nu \boldsymbol{x} + Q_\nu(\boldsymbol{x}) = y_\nu; \quad \nu = 1, 2, \cdots. \tag{6}$$

At first we shall restrict our considerations to the equations

$$\boldsymbol{a}_\nu \boldsymbol{x} + Q_\nu(\boldsymbol{x}) = y_\nu; \quad \nu = 1, \cdots, m.$$

It is easily proved that these equations possess a solution. In fact we can enlarge the system so that we obtain a new system

$$\boldsymbol{b}_\nu \boldsymbol{x} + R_\nu(\boldsymbol{x}) = y_\nu; \quad \nu = 1, \cdots, N,$$

where $\boldsymbol{b}_1, \cdots, \boldsymbol{b}_N$ are linearly independent $N$-dimensional vectors such that for $\nu = 1, \cdots, m$ the coordinates of $\boldsymbol{b}_\nu$ are identical

[1] Cf. e. g. J. Tannery: Introduction à la théorie des fonctions d'une variable 2. éd., vol. 2, Paris 1910. Note de M. J. Hadamard.



with the $N$ first coordinates of $\boldsymbol{a}_\nu$ including all that are different from zero. If we introduce $z_\nu = \boldsymbol{b}_\nu \boldsymbol{x}$; $\nu = 1, \cdots, N$ as new variables, we obtain a new system of equations satisfying the conditions of Theorem 14 and the assertion follows. We choose zero for the coordinates $x_{N+1}, x_{N+2}, \cdots$.

The solution $\boldsymbol{x}_m$ found in this manner satisfies

$$|\boldsymbol{a}_\nu \boldsymbol{x}_m| = |y_\nu - Q_\nu(\boldsymbol{x})| \leqq |y_\nu| + K_\nu$$

and from Theorem 13 follows that some subsequence of the sequence of vectors $\boldsymbol{x}_m$ converges towards a limit vector $\boldsymbol{x}$, which evidently satisfies all the equations (6).

## § 5. The Main Theorems.

From Theorems 11 and 15 follows immediately

**Theorem 16.** *Let* $F_\nu(\boldsymbol{x}) = F_\nu(x_1, x_2, \cdots)$; $\nu = 1, 2, \cdots$ *be a sequence of functions with the limit period* $2\pi$ *and satisfying* $|F_\nu(\boldsymbol{x})| \geqq k_\nu > 0$ *for all vectors* $\boldsymbol{x}$. *If the mean motion vectors corresponding to any finite number of the functions* $F_\nu(\boldsymbol{x})$ *are linearly independent and* $v_1, v_2, \cdots$ *are arbitrary real numbers, there exists a vector* $\boldsymbol{x}$ *such that*

$$\arg F_\nu(\boldsymbol{x}) = v_\nu; \quad \nu = 1, 2, \cdots.$$

In fact, if the set of mean motion vectors is not a complete set of linearly independent real vectors, we can make it complete by enlarging the system of functions $F_\nu(\boldsymbol{x})$.

For a finite set of almost periodic functions we have

**Theorem 17.** *If* $c_1, \cdots, c_n$ *are rationally independent real numbers and* $g_1(t), \cdots, g_n(t)$ *are arbitrary almost periodic functions, there exists to any given real numbers* $v_1, \cdots, v_n$ *and any positive* $\varepsilon$ *a number* $t$ *satisfying*

$$|c_\nu t + g_\nu(t) - v_\nu| \leqq \varepsilon \ (\mathrm{mod.}\ 2\pi); \quad \nu = 1, \cdots, n, \tag{7}$$

*i. e. the curve* $x_\nu = c_\nu t + g_\nu(t)$; $\nu = 1, \cdots, n$ *is* mod. $2\pi$ *everywhere dense in the n-dimensional space.*

According to Theorem 8 we choose a sequence $\beta_1, \beta_2, \cdots$ of rationally independent real numbers, where $\beta_1 = c_1, \cdots, \beta_n = c_n$, such that

$$c_\nu t + g_\nu(t) = \beta_\nu t + G_\nu(\boldsymbol{\beta} t); \; \nu = 1, \cdots, n,$$

where the functions $G_\nu(\boldsymbol{x})$; $\nu = 1, \cdots, n$ have the limit period $2\pi$. It follows from Theorem 14 that there exists a vector $\boldsymbol{x}^* = (x_1^*, \cdots, x_n^*, 0, \cdots)$, satisfying

$$x_\nu^* + G_\nu(\boldsymbol{x}^*) = v_\nu.$$

We can further choose continuous periodic functions $Q_1(\boldsymbol{x}), \cdots, Q_n(\boldsymbol{x})$ each depending only on a finite number of variables, such that

$$| G_\nu(\boldsymbol{x}) - Q_\nu(\boldsymbol{x}) | \leq \frac{\varepsilon}{3}; \; \nu = 1, \cdots, n$$

for all vectors $\boldsymbol{x}$. From Kronecker's theorem follows the existence of a number $t$ satisfying

$$| Q_\nu(\boldsymbol{x}^*) - Q_\nu(\boldsymbol{\beta} t) | \leq \frac{\varepsilon}{6}; \; \nu = 1, \cdots, n$$

and

$$| x_\nu^* - \beta_\nu t | \leq \frac{\varepsilon}{6} \; (\text{mod. } 2\pi); \; \nu = 1, \cdots, n.$$

Hence

$$\begin{aligned} | c_\nu t + g_\nu(t) - v_\nu | &= | \beta_\nu t + G_\nu(\boldsymbol{\beta} t) - v_\nu | \leq | G_\nu(\boldsymbol{\beta} t) - Q_\nu(\boldsymbol{\beta} t) | + \\ &+ | \beta_\nu t - x_\nu^* | + | Q_\nu(\boldsymbol{\beta} t) - Q_\nu(\boldsymbol{x}^*) | + | Q_\nu(\boldsymbol{x}^*) - G_\nu(\boldsymbol{x}^*) | + \\ &+ | x_\nu^* + G_\nu(\boldsymbol{x}^*) - v_\nu | \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{6} + \frac{\varepsilon}{6} + \frac{\varepsilon}{3} + 0 = \varepsilon \; (\text{mod. } 2\pi), \end{aligned}$$

which proves the theorem.

Theorem 3, announced in the introduction, is evidently equivalent with Theorem 17. We observe that the theorem is not valid for a denumerable set of almost periodic functions. E. g. the pure oscillations $e^{i\beta_1 t}, e^{i\beta_2 t}, \cdots$, where the numbers $\beta_1, \beta_2, \cdots$ are rationally independent and converge towards zero, will never simultaneously obtain values near $-1$.

We shall further prove the following generalization of the Kronecker-Bohl theorem.



**Theorem 18.** *The set of numbers satisfying* (7) *is relatively dense.*

In fact, if we choose the real number $t_0$ satisfying

$$| c_\nu t_0 + g_\nu(t_0) - v_\nu | \leq \frac{\varepsilon}{2} \; (\text{mod. } 2\pi); \; \nu = 1, \cdots, n,$$

any number $t_0 + \tau$, where $\tau$ is a common translation number of the almost periodic functions

$$e^{i(c_\nu t + g_\nu(t) - v_\nu)}$$

corresponding to $2 \sin \frac{\varepsilon}{4}$, satisfies (7). Hence the theorem follows from Theorem 7.

## § 6. On Limit Periodic and Almost Periodic Functions Satisfying Linear Relations.

From Theorem 15 follows

**Theorem 19.** *If a sequence of functions $F_\nu(\boldsymbol{x}) = F_\nu(x_1, x_2, \cdots)$ with the limit period $2\pi$ and with the property $| F_\nu(\boldsymbol{x}) | \geq k_\nu > 0$; $\nu = 1, 2, \cdots$ for all $\boldsymbol{x}$ satisfies a linear relation*

$$\alpha_1 F_1(\boldsymbol{x}) + \alpha_2 F_2(\boldsymbol{x}) + \cdots = 0, \tag{8}$$

*where $\alpha_1, \alpha_2, \cdots$ are complex constants that are $\neq 0$, and the series on the left converges for all real vectors $\boldsymbol{x}$, the mean motion vectors $\boldsymbol{r}_1, \boldsymbol{r}_2, \cdots$ are linearly dependent,* i. e. *the set of mean motion vectors has a finite subset of vectors that are linearly dependent. This is still true even if the condition $| F_\nu(\boldsymbol{x}) | \geq k_\nu > 0$ is satisfied for only one value of $\nu$, if for every $\nu$ we have a representation*

$$F_\nu(\boldsymbol{x}) = | F_\nu(\boldsymbol{x}) | \, e^{i(\boldsymbol{r}_\nu \boldsymbol{x} + G_\nu(\boldsymbol{x}))}$$

*where $G_\nu(\boldsymbol{x})$ is continuous and bounded and $\boldsymbol{r}_\nu$ is a vector with rational coordinates, of which only a finite number are $\neq 0$.*

In fact, if the vectors $\boldsymbol{r}_\nu$ were linearly independent, according to Theorem 15 there would exist a vector $\boldsymbol{x}^*$ satisfying

$$\boldsymbol{r}_\nu \boldsymbol{x}^* + G_\nu(\boldsymbol{x}^*) + \arg \alpha_\nu = 0; \; \nu = 1, 2, \cdots,$$

and for $\varkappa = \varkappa^*$ the left side of (8) would be positive and the relation would not be satisfied.

**Theorem 20.** *Let $f_1(t), f_2(t), \cdots$ be a sequence of almost periodic functions satisfying $|f_\nu(t)| \geqq k_\nu > 0$; $\nu = 1, 2, \cdots$ and let $K_\nu$ denote the upper bound of $|f_\nu(t)|$; $\nu = 1, 2, \cdots$. If $\alpha_1, \alpha_2, \cdots$ are complex numbers different from zero such that $\sum_{\nu=0}^{\infty} |\alpha_\nu| K_\nu$ is convergent and if we have*

$$\alpha_1 f_1(t) + \alpha_2 f_2(t) + \cdots = 0, \tag{9}$$

*the mean motions $c_1, c_2, \cdots$ of $f_1(t), f_2(t), \cdots$ are rationally dependent.*

From Theorem 9 follows that the relation (9) implies an analogous relation between the spatial extensions and the theorem therefore follows immediately from Theorems 19 and 12.

We shall prove some stronger theorems concerning finite sets of almost periodic functions satisfying linear relations. For the sake of brevity a function $r(t)e^{i(ct+g(t))}$, where $g(t)$ is a real almost periodic function and $r(t)$ is a real, non-negative function, will be called a *modulated oscillation*. It will further be called *normal* if the set of zeros of $r(t)$ contains no intervals.

**Theorem 21.** *If a finite set of modulated oscillations $f_\nu(t) = r_\nu(t)e^{i(c_\nu t + g_\nu(t))}$; $\nu = 1, \cdots, n$ satisfy a linear relation*

$$\alpha_1 f_1(t) + \cdots + \alpha_n f_n(t) + \beta = 0, \tag{10}$$

*where $\alpha_1, \cdots, \alpha_n$, and $\beta$ are arbitrary complex numbers $\neq 0$, the mean motions $c_1, \cdots, c_n$ are rationally dependent.*

If the numbers $c_1, \cdots, c_n$ were rationally independent, it would follow from Theorem 17 that the inequalities

$$|c_\nu t + g_\nu(t) + \arg\alpha_\nu - \arg\beta| \leqq \frac{\pi}{2} \pmod{2\pi}; \quad \nu = 1, \cdots, n$$

were satisfied for some value of $t$ in contradiction to the relation (10).

We notice that Theorem 21 is a generalization of theorem 5 mentioned in the introduction. In the case where $\beta = 0$, we have



**Theorem 22.** *If a finite set of modulated oscillations $f_\nu(t) = r_\nu(t)e^{i(c_\nu t + g_\nu(t))}$; $\nu = 1, \cdots, n$, of which at least one is normal, satisfy a relation*

$$\alpha_1 f_1(t) + \cdots + \alpha_n f_n(t) = 0, \tag{11}$$

*where $\alpha_1, \cdots, \alpha_n$ are arbitrary complex numbers that are $\neq 0$, the mean motions $c_1, \cdots, c_n$ satisfy a linear relation*

$$r_1 c_1 + \cdots + r_n c_n = 0,$$

*where $r_1, \cdots, r_n$ are rational numbers satisfying*

$$r_1 + \cdots + r_n = 0,$$

*and not all zeros.*

We choose a real number $c$ that cannot be written $c = r_1 c_1 + \cdots + r_n c_n$ with rational $r_1, \cdots, r_n$. If the real numbers $c + c_1, \cdots, c + c_n$ were rationally independent, it would follow from Theorem 17 and the continuity of $g_1(t), \cdots, g_n(t)$ that the inequalities

$$|(c + c_\nu)t + g_\nu(t) + \arg\alpha_\nu| \leqq \frac{\pi}{4} \pmod{2\pi}; \quad \nu = 1, \cdots, n$$

were satisfied for all values of $t$ belonging to some interval $t_1 \leqq t \leqq t_2$. One of the oscillations, say $f_\mu(t)$, is normal, and hence the interval $t_1 \leqq t \leqq t_2$ contains a point $t^*$, where $r_\mu(t^*) > 0$. From this would follow that the term $\alpha_\mu f_\mu(t^*)e^{ict}$ had a positive real part and none of the terms $\alpha_\nu f_\nu(t^*)e^{ict}$ had negative real parts, and the relation (11) could not be satisfied for $t = t^*$. We have thus proved the existence of a relation

$$r_1(c + c_1) + \cdots + r_n(c + c_n) = 0, \tag{12}$$

where $r_1, \cdots, r_n$ are rational and not all zero. As $c$ cannot be written as a linear combination of $c_1, \cdots, c_n$ with rational coefficients the relation (12) implies the following two relations

$$r_1 + \cdots + r_n = 0$$

$$r_1 c_1 + \cdots + r_n c_n = 0,$$

which proves the theorem.

As a very special case of Theorem 21 we have the following theorem, which is a slight generalization of Jessen's original theorem mentioned in the introduction.

**Theorem 23.** *Let $f(t)$ be an almost periodic function. We consider all complex numbers $a$, for which we have a representation*

$$f(t)-a = |f(t)-a|\, e^{i(ct+g(t))},$$

*where $g(t)$ is an almost periodic function. The values $c$ corresponding to all possible values of $a$ are rational multiples of one real number.*

## § 7. On Almost Periodic Functions Satisfying Quadratic Relations.

**Theorem 24.** *Let $f_1(t)$, $f_2(t)$, and $f_3(t)$ be three almost periodic functions satisfying $|f_\nu(t)| \geqq k > 0$; $\nu = 1, 2, 3$, and let $c_1, c_2$, and $c_3$ denote their mean motions. If we have a relation*

$$a_1 f_1(t)^2 + a_2 f_2(t)^2 + a_3 f_3(t)^2 + b_1 f_2(t) f_3(t) + b_2 f_3(t) f_1(t) + \\ + b_3 f_1(t) f_2(t) + k = 0,$$

*where at least one of the complex numbers $a_1, a_2, a_3, b_1, b_2, b_3$ and $k$ is $\neq 0$, we have a relation*

$$r_1 c_1 + r_2 c_2 + r_3 c_3 = 0,$$

*where $r_1, r_2$, and $r_3$ are rational numbers of which at least one is $\neq 0$.*

From Theorem 9 follows that the spatial extensions $F_1(\boldsymbol{x})$, $F_2(\boldsymbol{x})$, and $F_3(\boldsymbol{x})$ satisfy the equation

$$a_1 z_1^2 + a_2 z_2^2 + a_3 z_3^2 + b_1 z_2 z_3 + b_2 z_3 z_1 + b_3 z_1 z_2 + k = 0 \tag{13}$$

and if $c_1, c_2, c_3$ were rationally independent it would follow from Theorems 12 and 16 that the equation (13) would possess solutions $z_1, z_2, z_3$ with any given set of arguments $\varphi_1, \varphi_2, \varphi_3$. Hence it is sufficient to prove the existence of a set of numbers $\varphi_1, \varphi_2, \varphi_3$ such that the equation



$$a_1 r_1^2 e^{2i\varphi_1} + a_2 r_2^2 e^{2i\varphi_2} + a_3 r_3^2 e^{2i\varphi_3} + b_1 r_2 r_3 e^{i(\varphi_2+\varphi_3)} + b_2 r_3 r_1 e^{i(\varphi_3+\varphi_1)} + \\ + b_3 r_1 r_2 e^{i(\varphi_1+\varphi_2)} + k = 0$$

is not satisfied for any positive values of $r_1, r_2, r_3$. Without restricting the generality we may suppose that $k \geqq 0$ (if not, we multiply the equation by a convenient factor $e^{i\varphi}$).

Let us first suppose that at most one of the numbers $b_1, b_2, b_3$ is zero. In this case it is sufficient to determine $\varphi_1, \varphi_2, \varphi_3$ such that

$$|2\varphi_\nu + \operatorname{Arg} a_\nu| \leqq \frac{\pi}{2} \quad (\mathrm{mod.}\ 2\pi);\ \nu = 1, 2, 3$$

$$|\varphi_2 + \varphi_3 + \operatorname{Arg} b_1| \leqq \frac{\pi}{2} \quad (\mathrm{mod.}\ 2\pi)$$

$$|\varphi_3 + \varphi_1 + \operatorname{Arg} b_2| \leqq \frac{\pi}{2} \quad (\mathrm{mod.}\ 2\pi)$$

$$|\varphi_1 + \varphi_2 + \operatorname{Arg} b_3| \leqq \frac{\pi}{2} \quad (\mathrm{mod.}\ 2\pi),$$

where the sign $<$ holds in at least two of the three last inequalities (If a coefficient is zero, its argument is in this connection defined as zero). If we put

$$\psi_\nu = \varphi_\nu + \frac{1}{2}\operatorname{Arg} a_\nu;\ \nu = 1, 2, 3, \tag{14}$$

we obtain a new set of equations

$$|\psi_\nu| \leqq \frac{\pi}{4} \quad (\mathrm{mod.}\ \pi);\ \nu = 1, 2, 3 \tag{15}$$

$$\left.\begin{aligned} &|\psi_2 + \psi_3 - \alpha_1| \leqq \frac{\pi}{2} \quad (\mathrm{mod.}\ 2\pi) \\ &|\psi_3 + \psi_1 - \alpha_2| \leqq \frac{\pi}{2} \quad (\mathrm{mod.}\ 2\pi) \\ &|\psi_1 + \psi_2 - \alpha_3| \leqq \frac{\pi}{2} \quad (\mathrm{mod.}\ 2\pi), \end{aligned}\right\} \tag{16}$$

and to prove our theorem we shall find solutions to this system such that the sign $<$ holds in two of the last three inequalities.

If at least two of the numbers $\alpha_1, \alpha_2, \alpha_3$, say $\alpha_1$ and $\alpha_2$, are neither 0 nor $\pi$ (mod. $2\pi$), we have

$$\left|\varepsilon_1 \frac{\pi}{2} - \alpha_1\right| < \frac{\pi}{2} \quad (\text{mod. } 2\pi)$$

$$\left|\varepsilon_2 \frac{\pi}{2} - \alpha_2\right| < \frac{\pi}{2} \quad (\text{mod. } 2\pi)$$

$$\left|\varepsilon_3 \frac{\pi}{2} - \alpha_3\right| \leqq \frac{\pi}{2} \quad (\text{mod. } 2\pi)$$

when $\varepsilon_1, \varepsilon_2, \varepsilon_3$ are chosen conveniently as 1 or $-1$. If we choose $\psi_1, \psi_2, \psi_3$ such that

$$(17) \quad \left\{ \begin{aligned} \psi_2 + \psi_3 &= \varepsilon_1 \frac{\pi}{2} \\ \psi_3 + \psi_1 &= \varepsilon_2 \frac{\pi}{2} \\ \psi_1 + \psi_2 &= \varepsilon_3 \frac{\pi}{2}, \end{aligned} \right.$$

the inequalities (16) are satisfied and the sign $<$ holds in at least two of them. But from the equations (17) results

$$\psi_1 + \psi_2 + \psi_3 = \pm \frac{\pi}{4} \text{ or } \pm \frac{3\pi}{4},$$

and it follows that $\psi_1, \psi_2$, and $\psi_3$ have also values $\pm \frac{\pi}{4}$ or $\pm \frac{3\pi}{4}$ which proves that the inequalities (15) are satisfied.

If at least two of the numbers $\alpha_1, \alpha_2, \alpha_3$, say $\alpha_1$ and $\alpha_2$, are 0 or $\pi$ (mod. $2\pi$), we choose $\varepsilon_3 = \pm 1$ such that $\left|\varepsilon_3 \frac{\pi}{2} - \alpha_3\right| \leqq \frac{\pi}{2}$ (mod. $2\pi$), and it is sufficient to choose $\psi_1, \psi_2, \psi_3$ as solutions of the equations

$$\begin{aligned} \psi_2 + \psi_3 &= \alpha_1 \\ \psi_3 + \psi_1 &= \alpha_2 \\ \psi_1 + \psi_2 &= \varepsilon_3 \frac{\pi}{2}. \end{aligned}$$

Finally we consider the case, where at least two of the numbers $b_1, b_2$, and $b_3$, say $b_1$ and $b_2$, are zero. It is then sufficient to determine $\psi_1, \psi_2$, and $\psi_3$ such that



$$|2\varphi_1 + \operatorname{Arg} a_1| < \frac{\pi}{2} \quad (\text{mod. } 2\pi)$$

$$|2\varphi_2 + \operatorname{Arg} a_2| < \frac{\pi}{2} \quad (\text{mod. } 2\pi)$$

$$|2\varphi_3 + \operatorname{Arg} a_3| < \frac{\pi}{2} \quad (\text{mod. } 2\pi)$$

$$|\varphi_1 + \varphi_2 \operatorname{Arg} b_3| < \frac{\pi}{2} \quad (\text{mod. } 2\pi).$$

As $\varphi_3$ occurs in only one inequality, it is sufficient to consider $\varphi_1$ and $\varphi_2$. If we use (14) once more, we obtain the inequalities

$$|\psi_3| < \frac{\pi}{4} \quad (\text{mod. } \pi)$$

$$|\psi_2| < \frac{\pi}{4} \quad (\text{mod. } \pi)$$

$$|\psi_1 + \psi_2 + \alpha| < \frac{\pi}{2} \quad (\text{mod. } 2\pi),$$

which have solutions. In fact, any angle between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ can be written as $\psi_1 + \psi_2$, where $|\psi_1| < \frac{\pi}{4}$ and $|\psi_2| < \frac{\pi}{4}$, and any angle between $\frac{\pi}{2}$ and $\frac{3\pi}{2}$ can be written $\psi_1 + \psi_2$ with $|\psi_1| < \frac{\pi}{4}$, and $\frac{3\pi}{4} < \psi_2 < \frac{5\pi}{4}$. This proves the theorem.

**Theorem 25.** *If the constant k vanishes, the numbers $r_1, r_2$, and $r_3$ can be chosen such that*

$$r_1 + r_2 + r_3 = 0.$$

We choose a real number $c$ that cannot be written $r_1 c_1 + r_2 c_2 + r_3 c_3$, and Theorem 25 follows, when Theorem 24 is applied to the functions $f_1(t)\, e^{ict}, f_2(t)\, e^{ict}, f_3(t)\, e^{ict}$.

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