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SOME LIMIT THEOREMS ON INTEGRALS IN AN ABSTRACT SET

BY

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1. Introduction. The present paper deals with two limit theorems on integrals in an abstract set. The first limit theorem is a generalization of the well-known theorem on differentiation on a net, the net being replaced by an increasing sequence of σ-fields. The second limit theorem is a sort of counterpart of the first, the sequence of σ-fields being now decreasing. The proofs follow the lines of the proof of the theorem on differentiation on a net.

In case of integrals in an infinite product set the theorems lead to known results, when for the n^{th} σ-field of the sequence we take either the system of measurable sets depending on the n first coordinates only, or the system of measurable sets depending on all except the n first coordinates.

If the abstract theory of integration is interpreted as probability theory, our theorems lead to two theorems concerning conditional mean values.

For the convenience of the reader the main definitions and theorems used are stated at the beginning of the paper. For references and proofs we may refer to the book by Saks [1] or to a series of articles by Jessen [2], which we follow closely.

2. Sets and functions. Let E be a set containing at least one element. Elements of E will be denoted by $x, y, \cdots$ and sub-sets of E by $A, B, \cdots$. The set E itself and the empty set 0 will be reckoned among the sub-sets of E. The notation $x \varepsilon A$ means that the element x belongs to the set A, while $x \bar{\varepsilon} A$ means that x does not belong to A. If A is a subset of B we write $A \subseteqq B$ or $B \supseteqq A$, while $A \subset B$ or $B \supset A$ means that A is a proper sub-set of B. By $\underset{n}{\mathfrak{S}} A_n$ or $A_1 \dotplus A_2 \dotplus \cdots$ we denote the sum of the sets $A_1, A_2, \cdots$. If no two of the sets have elements in common the signs $\mathfrak{S}$ and $\dot{+}$ may be replaced by Σ and $+$. By $\underset{n}{\mathfrak{D}} A_n$ or $A_1 A_2 \cdots$ we denote the common part of the sets $A_1, A_2, \cdots$. The notation $A - B$ will be used only when $A \supseteqq B$, and denotes the difference between A and B.

A real function f in E is given, when to every element x of a set A there corresponds a value $f(x), -\infty \leqq f(x) \leqq +\infty$. The set A is called the domain of f. Functions in E will be denoted by $f, g, \cdots$. By $[\cdots]$, where $\cdots$ indicates a number of expressions or relations involving functions in E, we denote the set of elements x of E for which these expressions are defined and the relations are valid. For example $[f]$ denotes the domain of f.

3. Systems of sets and set-functions. Let $\mathfrak{M}$ denote the set of all sub-sets of E. Sub-sets of $\mathfrak{M}$ will be denoted by $\mathfrak{F}, \mathfrak{G}, \cdots$, and will be called systems of sets, the notation 'set' being reserved for sub-sets of E.

A system $\mathfrak{F}$ is called *additive*, if $A_1 \dot{+} \cdots \dot{+} A_n \varepsilon \mathfrak{F}$ when all $A_i \varepsilon \mathfrak{F}$, and *multiplicative*, if $A_1 \cdots A_n \varepsilon \mathfrak{F}$ when all $A_i \varepsilon \mathfrak{F}$. It is called *subtractive*, if $A - B \varepsilon \mathfrak{F}$ when $A \varepsilon \mathfrak{F}$, $B \varepsilon \mathfrak{F}$, and $A \supseteqq B$. It is called *completely additive*, if $A_1 \dot{+} A_2 \dot{+} \cdots \varepsilon \mathfrak{F}$ when all $A_i \varepsilon \mathfrak{F}$, and *completely multiplicative*, if $A_1 A_2 \cdots \varepsilon \mathfrak{F}$ when all $A_i \varepsilon \mathfrak{F}$.

A system of sets is called a *field*, if it contains at least one set and is additive and subtractive (and hence also multiplicative).

A system of sets is called a *σ-field*, if it contains at least one set and is completely additive and subtractive (and hence also completely multiplicative).

Functions in $\mathfrak{M}$, i.e. functions for which the domain is a system of sets, are called *set-functions* and will be denoted by $\mu, \nu, \cdots$.

A set-function μ with domain $\mathfrak{F}$ is called *additive*, if $\mu(A_1 + \cdots + A_n) = \mu(A_1) + \cdots + \mu(A_n)$ when all $A_i \varepsilon \mathfrak{F}$ and $A_1 + \cdots + A_n \varepsilon \mathfrak{F}$. It is called *completely additive*, if $\mu(A_1 + A_2 + \cdots) = \mu(A_1) + \mu(A_2) + \cdots$ when all $A_i \varepsilon \mathfrak{F}$ and $A_1 + A_2 + \cdots \varepsilon \mathfrak{F}$.

If two set-functions μ and ν have the domains $\mathfrak{F}$ and $\mathfrak{G}$, we call ν an *extension* of μ or μ a *contraction* of ν and write $\nu \supseteq \mu$ or $\mu \subseteq \nu$ when $\mathfrak{F} \subset \mathfrak{G}$ and $\mu(A) = \nu(A)$ for all $A \varepsilon \mathfrak{F}$.

4. Content and measure. A set-function μ is called a *content*, if

(i) its domain $\mathfrak{F}$ is a field,
(ii) $0 \leqq \mu(A) \leqq +\infty$ for any $A \varepsilon \mathfrak{F}$,
(iii) μ is additive,
(iv) to every $A \varepsilon \mathfrak{F}$ there corresponds a set $\underset{n}{\mathfrak{S}} A_n$ where all $A_n \varepsilon \mathfrak{F}$, such that $A \subseteqq \underset{n}{\mathfrak{S}} A_n$ and $\mu(A_n) < +\infty$ for all n.[1]

A set-function μ is called a *measure*, if

(i) its domain $\mathfrak{F}$ is a σ-field,
(ii) $0 \leqq \mu(A) \leqq +\infty$ for any $A \varepsilon \mathfrak{F}$,
(iii) μ is completely additive,
(iv) to every $A \varepsilon \mathfrak{F}$ there corresponds a set $\underset{n}{\mathfrak{S}} A_n$ where all $A_n \varepsilon \mathfrak{F}$, such that $A \subseteqq \underset{n}{\mathfrak{S}} A_n$ and $\mu(A_n) < +\infty$ for all n.[1]

Of fundamental importance is the following

Extension Theorem. If μ is a content, then there exists a measure $\omega \supseteqq \mu$ if, and only if, μ is completely additive. If so, there exists a unique measure $\nu \supseteqq \mu$, such that $\omega \supseteqq \nu$ for any measure $\omega \supseteqq \mu$. The domain of ν is the smallest σ-field containing the domain of μ.

The measure ν is called the *narrowest extension of μ to a measure.*

For the complete additivity of a content we have the following criterion:

A finite content μ with domain $\mathfrak{F}$ is completely additive if, and only if, $\lim\limits_{n} \mu(A_n) = 0$ for any sequence of sets $A_n \varepsilon \mathfrak{F}$, for which $A_1 \supseteqq A_2 \supseteqq \cdots$ and $\underset{n}{\mathfrak{D}} A_n = 0$.

5. Integration with respect to a measure. By the *system of functions over a σ-field* $\mathfrak{F}$ we mean the system of all functions f for which $[f \geqq a] \varepsilon \mathfrak{F}$ for any a, $-\infty \leqq a \leqq +\infty$. If $\mathfrak{F}$ is the domain of a measure μ, the functions of this system are called *μ-measurable.*

The theory of *integration* of μ-measurable functions with respect to the measure μ may be developed in the usual way.

[1] This is the definition adopted in Jessen [2]. In the sequel we shall in the main only consider contents and measures for which $E \varepsilon \mathfrak{F}$ and $\mu(E) < +\infty$.

For the integral of the function f over the μ-measurable set $A \subseteq [f]$ we shall use the notation

$$\int_A f(x)\,\mu(dE).$$

If the integral of f over its domain $[f]$ exists and is finite, the function f is called *μ-integrable.* The set-function

$$\varphi(A) = \int_A f(x)\,\mu(dE)$$

is then defined for all sub-sets $A \varepsilon \mathfrak{F}$ of $[f]$ and is called the *indefinite integral* of f.

Two μ-integrable functions f and g with $[f] = [g]$ have the same indefinite integral if, and only if, $\mu([f \mp g]) = 0$.

6. Completely additive set-functions. Let φ denote a *bounded* set-function with domain $\mathfrak{F}$. We define two other set-functions φ^+ and φ^- with the same domain by placing

$$\varphi^+(A) = \text{upper bound } \varphi(B) \quad \text{and} \quad \varphi^-(A) = \text{lower bound } \varphi(B),$$

where the upper and lower bounds are taken with respect to all sub-sets $B \varepsilon \mathfrak{F}$ of A. We then have the following

Decomposition Theorem. If $\mathfrak{F}$ is a field, and φ is completely additive, then φ^+ and $-\varphi^-$ are completely additive contents, and $\varphi = \varphi^+ + \varphi^-$. If moreover $\mathfrak{F}$ is a σ-field, then φ^+ and $-\varphi^-$ are measures.

The set-functions φ^+ and φ^- are called the *positive* and *negative parts of* φ.

Moreover we have the following

Extension Theorem. If $\mathfrak{F}$ is a field, and φ is completely additive, then there exists a unique set-function $\psi \supseteq \varphi$, which is bounded and completely additive and is defined on the smallest σ-field containing $\mathfrak{F}$. Moreover $\psi^+ \supseteq \varphi^+$ and $\psi^- \supseteq \varphi^-$, i.e. ψ^+ and $-\psi^-$ are the narrowest extensions of the contents φ^+ and $-\varphi^-$ to measures.

This theorem implies that if $\mathfrak{F}$ is a field, and $\mathfrak{G}$ is the smallest σ-field containing $\mathfrak{F}$, then a bounded, completely additive set-function defined on $\mathfrak{G}$, for which the contraction to $\mathfrak{F}$ is non-negative, will itself be non-negative.

7. Continuous and singular set-functions. Let μ denote a measure in E with domain $\mathfrak{F}$, and suppose that $E \varepsilon \mathfrak{F}$.

A bounded, completely additive set-function φ defined on $\mathfrak{F}$ is called *μ-continuous*, if $\varphi(M) = 0$ for any $M \varepsilon \mathfrak{F}$ with $\mu(M) = 0$. It is called *μ-singular*, if there exists a set $N \varepsilon \mathfrak{F}$ with $\mu(N) = 0$, such that $\varphi(A) = 0$ for every sub-set $A \varepsilon \mathfrak{F}$ of $E - N$.

We have the following

Decomposition Theorem. Any bounded, completely additive set-function φ defined on $\mathfrak{F}$ admits of a unique representation $\varphi = \varphi_c + \varphi_s$, where φ_c and φ_s are bounded, completely additive set-functions defined on $\mathfrak{F}$, of which φ_c is μ-continuous, while φ_s is μ-singular.

The set-functions φ_c and φ_s are called the *μ-continuous* and *μ-singular parts of φ.*

A set-function φ defined on $\mathfrak{F}$ is the indefinite integral of a μ-integrable function f with $[f] = E$ if, and only if, it is bounded, completely additive, and μ-continuous.

8. First limit theorem. Let E be a set containing at least one element, and μ a measure in E with domain $\mathfrak{F}$, such that $E \varepsilon \mathfrak{F}$ and $\mu(E) = 1$. Let $\mathfrak{F}_1 \subseteq \mathfrak{F}_2 \subseteq \cdots$ be an increasing sequence of σ-fields contained in $\mathfrak{F}$, such that $E \varepsilon \mathfrak{F}_1$. The system $\mathfrak{G} = \underset{n}{\mathfrak{S}} \mathfrak{F}_n$ is a field. The smallest σ-field containing $\mathfrak{G}$ will be denoted be $\mathfrak{F}'$.

The contraction of μ to $\mathfrak{F}_n$ is a measure, which will be denoted by μ_n. Similarly the contraction of μ to $\mathfrak{F}'$ is a measure, which will be denoted by μ'.

Let φ be a bounded, completely additive set-function defined on $\mathfrak{F}$, whose contraction φ_n to $\mathfrak{F}_n$ is μ_n-continuous for any n. By § 7 this means, that φ_n is the indefinite integral with respect to μ_n of a μ_n-integrable function f_n, i. e. there exists for every n a μ_n-integrable function f_n, such that

$$\int_A f_n(x)\,\mu(dE) = \varphi(A) \quad \text{for every } A \varepsilon \mathfrak{F}_n.$$

This function f_n need not be uniquely determined. In the sequel f_n denotes for every n some such function. We shall consider the functions

$$\underline{f} = \liminf_n f_n \quad \text{and} \quad \bar{f} = \limsup_n f_n.$$

Evidently these functions are μ'-measurable. The contraction of φ to $\mathfrak{F}'$ will be denoted by φ'.

The first limit theorem now states:

With the above notations we have the relation

$$\mu([\underline{f}<\bar{f}]) = 0,$$

and $\varphi(C) = 0$ *for any sub-set* $C \varepsilon \mathfrak{F}'$ *of* $[\underline{f}<\bar{f}]$.

Moreover $\underline{f}$ *and* $\bar{f}$ *are* μ'*-integrable, and their indefinite integrals with respect to* μ' *are the* μ'*-continuous part* φ'_c *of* φ', *i. e. for any* $A \varepsilon \mathfrak{F}'$ *we have*

$$\varphi'_c(A) = \int_A \underline{f}(x)\,\mu(dE) = \int_A \bar{f}(x)\,\mu(dE).$$

Finally, the positive and negative parts of the μ'*-singular part* φ'_s *of* φ' *satisfy for any* $A \varepsilon \mathfrak{F}'$ *the relations*

$$\varphi'^{+}_s(A) = \varphi\left(A\,[\underline{f} = +\infty]\right) \quad \textit{and} \quad \varphi'^{-}_s(A) = \varphi\left(A\,[\bar{f} = -\infty]\right).$$[1]

9. The proof will be based on the following lemma:

Placing $H = [\underline{f} \leqq h]$ *and* $K = [\bar{f} \geqq k]$ *for arbitrary numbers* h *and* k *we have for any* $C \varepsilon \mathfrak{F}'$ *the inequalities*

$$\varphi(HC) \leqq h\,\mu(HC) \quad \textit{and} \quad \varphi(KC) \geqq k\,\mu(KC).$$

In order to prove the first inequality we put

$$H_n = [\inf_p f_{n+p} < h_n]$$

and

$$H_{np} = \begin{cases} [f_{n+1} < h_n] \text{ for } p = 1 \\ [f_{n+1} \geqq h_n, \cdots, f_{n+p-1} \geqq h_n, f_{n+p} < h_n] \text{ for } p > 1, \end{cases}$$

where $h_1, h_2, \cdots$ denotes a decreasing sequence of numbers converging towards h. Then $H_{np} \varepsilon \mathfrak{F}_{n+p}$ and $H_{np} \subseteqq [f_{n+p} < h_n]$. Clearly (for a given n) no two of the sets H_{np} have elements in common, and $H_n = \sum_p H_{np}$. Further $H_1 \supseteqq H_2 \supseteqq \cdots$ and $H = \mathfrak{D}_n H_n$.

[1] The assumption $\mu(E) = 1$ has been introduced for the sake of convenience. The theorem may easily be extended to the case where $E \varepsilon \mathfrak{F}$ and $\mu(E)$ is arbitrary (finite or infinite).

Now, if $C \varepsilon \mathfrak{G} = \underset{n}{\mathfrak{S}} \mathfrak{F}_n$, we shall have $C \varepsilon \mathfrak{F}_n$ for all $n >$ (some) n_0; hence $H_{np} C \varepsilon \mathfrak{F}_{n+p}$ for $n \geqq n_0$ and all p. We therefore have

$$\varphi(H_n C) = \varphi\left(\sum_p H_{np} C\right) = \sum_p \varphi(H_{np} C) = \sum_p \int_{H_{np}C} f_{n+p}(x)\,\mu(dE)$$
$$\leqq \sum_p h_n \mu(H_{np} C) = h_n \mu\left(\sum_p H_{np} C\right) = h_n \mu(H_n C).$$

Since $H_1 C \supseteqq H_2 C \supseteqq \cdots$ and $HC = \underset{n}{\mathfrak{D}} H_n C$, we have $\mu(HC) = \lim_n \mu(H_n C)$ and $\varphi(HC) = \lim_n \varphi(H_n C)$. We therefore obtain $\varphi(HC) \leqq h\mu(HC)$.

We now define a set-function $\varkappa$ on $\mathfrak{F}'$ by placing

$$\varkappa(C) = h\mu(HC) - \varphi(HC).$$

Clearly $\varkappa$ is bounded and completely additive. Moreover, since $\varphi(HC) \leqq h\mu(HC)$ for $C \varepsilon \mathfrak{G}$, the contraction of $\varkappa$ to $\mathfrak{G}$ is non-negative. Hence, by § 6, the set-function $\varkappa$ is itself non-negative, i.e. the inequality $\varphi(HC) \leqq h\mu(HC)$ is valid for all $C \varepsilon \mathfrak{F}'$.

The inequality $\varphi(KC) \geqq k\mu(KC)$ is proved analogously.

10. By means of the lemma we shall now first prove that $\mu([\underline{f} < \overline{f}]) = 0$, and that $\varphi(C) = 0$ for any sub-set $C \varepsilon \mathfrak{F}'$ of $[\underline{f} < \overline{f}]$.

Since $[\underline{f} < \overline{f}] = \mathfrak{S}([\underline{f} < h, \overline{f} > k])$, where the summation is with respect to all pairs of rational numbers h and k, for which $h < k$, it is sufficient to prove that $\mu(C) = 0$ and $\varphi(C) = 0$ for any sub-set $C \varepsilon \mathfrak{F}'$ of $[\underline{f} < h, \overline{f} > k]$, when $h < k$.

This follows from the lemma. For, when $C \varepsilon \mathfrak{F}'$ is a sub-set of $[\underline{f} < h, \overline{f} > k]$, we have $C \subseteqq H$ and $C \subseteqq K$, and hence

$$h\mu(C) \geqq \varphi(C) \geqq k\mu(C).$$

Since $h < k$, this shows that $\mu(C) = 0$, and hence also $\varphi(C) = 0$.

11. Next we prove that the functions $\underline{f}$ and $\overline{f}$ are μ'-integrable and that their indefinite integrals with respect to μ' are the μ'-continuous part φ'_c of φ'.

Placing $N^+ = [\underline{f} = +\infty]$ and $N^- = [\overline{f} = -\infty]$ we shall first prove that $\mu(N^+) = 0$ and $\mu(N^-) = 0$.

For every k we have $N^+ \subseteq [\overline{f} = +\infty] \subseteq [\overline{f} \geqq k]$. From the lemma it follows that when $k > 0$ we have

$$\mu(N^+) \leqq \mu([\overline{f} \geqq k]) \leqq \frac{1}{k} \varphi([\overline{f} \geqq k]) \leqq \frac{1}{k} \varphi^+(E).$$

Making $k \to +\infty$ we obtain $\mu(N^+) = 0$. Analogously it is proved that $\mu(N^-) = 0$.

For an arbitrary finite $d > 0$ we now put

$$f_d(x) = \begin{cases} nd \text{ for } x \,\varepsilon\, [nd \leqq \underline{f} = \overline{f} < (n+1)\,d], -\infty < n < +\infty, \\ 0 \text{ for } x \,\overline{\varepsilon}\, [-\infty < \underline{f} = \overline{f} < +\infty] \end{cases}$$

and apply for an arbitrary set $A \,\varepsilon\, \mathfrak{F}'$ the lemma on the set $C_n = A\,[nd \leqq \underline{f} = \overline{f} < (n+1)\,d]$ together with $H_n = [\underline{f} \leqq (n+1)\,d]$ and $K_n = [\overline{f} \geqq nd]$. This yields

$$nd\,\mu(C_n) \leqq \varphi(C_n) \leqq (n+1)\,d\,\mu(C_n).$$

If we choose $A = E$, these inequalities show that f_d is μ'-integrable. For an arbitrary $A \,\varepsilon\, \mathfrak{F}'$ they show, together with the relations $\mu(A[\underline{f} < \overline{f}]) = 0$ and $\varphi(A[\underline{f} < \overline{f}]) = 0$, that

$$\int_{AD} f_d(x)\,\mu(dE) \leqq \varphi(AD) \leqq \int_{AD} f_d(x)\,\mu(dE) + d,$$

where $D = E - (N^+ + N^-)$.

Since in the set $D - [\underline{f} < \overline{f}]$ we have $f_d \leqq \underline{f} = \overline{f} < f_d + d$, it is plain that $\underline{f}$ and $\overline{f}$ are μ'-integrable, and that for an arbitrary $A \,\varepsilon\, \mathfrak{F}'$

$$\int_{AD} f_d(x)\,\mu(dE) \leqq \left\{ \begin{matrix} \int_{AD} \underline{f}(x)\,\mu(dE) \\ \int_{AD} \overline{f}(x)\,\mu(dE) \end{matrix} \right\} \leqq \int_{AD} f_d(x)\,\mu(dE) + d.$$

Since d may be chosen arbitrarily small, the preceding inequalities show that

$$\varphi(AD) = \int_{AD} \underline{f}(x)\,\mu(dE) = \int_{AD} \overline{f}(x)\,\mu(dE).$$

The set-function $\psi(A) = \varphi(AD)$ defined on $\mathfrak{F}'$ is therefore μ'-continuous, and since $\varphi'(A) = \varphi(AD) + \varphi(A(E-D))$, where $\mu(E-D) = 0$, so that $\chi(A) = \varphi(A(E-D))$ is μ'-singular, it follows from the decomposition theorem of § 7 that ψ and χ must be the μ'-continuous and μ'-singular parts of φ'. Since the integrals of $\underline{f}$ and $\overline{f}$ over AD are equal to the integrals over A, the last relations may therefore also be written

$$\varphi'_c(A) = \int_{A} \underline{f}(x)\,\mu(dE) = \int_{A} \overline{f}(x)\,\mu(dE).$$

12. The set-function χ being the μ'-singular part of φ', it is plain that for any $A \varepsilon \mathfrak{F}'$

$$\varphi'_s(A) = \varphi(A(E-D)) = \varphi(AN^+) + \varphi(AN^-).$$

Since $AN^+ \subseteqq [\overline{f} \geqq 0]$, it follows from the lemma that $\varphi(AN^+) \geqq 0$ for any $A \varepsilon \mathfrak{F}'$. Similarly it is proved that $\varphi(AN^-) \leqq 0$.

For any sub-set $B \varepsilon \mathfrak{F}'$ of A we therefore have $0 \leqq \varphi(BN^+) \leqq \varphi(AN^+)$ and $\varphi(AN^-) \leqq \varphi(BN^-) \leqq 0$. Since $\varphi'_s(B) = \varphi(BN^+) + \varphi(BN^-)$, this shows that $\varphi(AN^-) \leqq \varphi'_s(B) \leqq \varphi(AN^+)$.

Hence by the definition of § 6 we have for any $A \varepsilon \mathfrak{F}'$

$$\varphi_s'^{+}(A) = \varphi(AN^+) \text{ and } \varphi_s'^{-}(A) = \varphi(AN^-).$$

This completes the proof of the theorem.

13. Corollaries of the first limit theorem. If in particular $\mathfrak{F}' = \mathfrak{F}$, we have $\mu' = \mu$ and $\varphi' = \varphi$, so that the first limit theorem contains statements about the set-function φ itself.

Even if $\mathfrak{F}' \subset \mathfrak{F}$, we may, however, under certain additional assumptions, deduce less precise results regarding the set-function φ.

Let us first assume that *to any set $A \varepsilon \mathfrak{F}$ there exists a set $B \varepsilon \mathfrak{F}'$, such that $B \supseteqq A$ and $\mu(B-A) = 0$.* We shall then prove two results:

(i) *If φ is non-negative, then the indefinite integrals of $\underline{f}$ and $\overline{f}$ with respect to μ are the μ-continuous part φ_c of φ, and the μ-singular part of φ satisfies for any $A \varepsilon \mathfrak{F}$ the relation $\varphi_s(A) = \varphi(A[\overline{f} = +\infty])$.*

Proof. We have the decomposition $\varphi(A) = \varphi(AD) + \varphi(AN^+)$. The set-function $\varphi(AN^+)$ is μ-singular, since $\mu(N^+) = 0$. The set-function $\varphi(AD)$ is μ-continuous. For if $A \varepsilon \mathfrak{F}$ and $\mu(A) = 0$, there exists a set $B \varepsilon \mathfrak{F}'$, such that $B \supseteqq A$ and $\mu(B-A) = 0$, i.e. $\mu(B) = 0$. Hence, since φ is non-negative, $0 \leqq \varphi(AD) \leqq \varphi(BD) = 0$, and therefore $\varphi(AD) = 0$. Finally $\varphi(AD)$ is the indefinite integral of $\underline{f}$ and $\overline{f}$ with respect to μ. For to an arbitrary $A \varepsilon \mathfrak{F}$ there exists a $B \varepsilon \mathfrak{F}'$, such that $B \supseteqq A$ and $\mu(B-A) = 0$, and we then have

$$\varphi(AD) = \varphi(BD) = \begin{cases} \int\limits_{BD} \underline{f}(x)\,\mu(dE) = \int\limits_{AD} \underline{f}(x)\,\mu(dE) \\ \int\limits_{BD} \overline{f}(x)\,\mu(dE) = \int\limits_{AD} \overline{f}(x)\,\mu(dE). \end{cases}$$

(ii) *Without restriction on the sign of φ the indefinite integrals of $\underline{f}$ and $\overline{f}$ with respect to μ are the μ-continuous part φ_c of φ.*

Proof. The statement follows from the decomposition $\varphi = \varphi^+ + \varphi^-$, when we apply the previous result on each of the set-functions φ^+ and $-\varphi^-$.

We mention that not only the relations $\varphi_s^+(A) = \varphi(AN^+)$ and $\varphi_s^-(A) = \varphi(AN^-)$, but even the relation $\varphi_s(A) = \varphi(A(N^+ + N^-))$, need not hold generally. This is shown by the following example:

Let $\mathfrak{F}$ consist of all sub-sets of a set E of three elements a, b, and c, and let $\mu(\{a\}) = 1$, $\mu(\{b\}) = \mu(\{c\}) = 0$, and $\varphi(\{a\}) = 0$, $\varphi(\{b\}) = 1$, $\varphi(\{c\}) = -1$. Let each of the σ-fields $\mathfrak{F}_1, \mathfrak{F}_2, \cdots$ consist of all sets containing either both or none of the elements b and c. Then the above condition is satisfied, but φ is singular, and there exists no set $N \varepsilon \mathfrak{F}'$ for which $\varphi(A) = \varphi(AN)$ for all $A \varepsilon \mathfrak{F}$.

14. Next, let us assume that *to any set $A \varepsilon \mathfrak{F}$ there exists a set $B \varepsilon \mathfrak{F}'$, such that $\mu(A + B - AB) = 0$.* Evidently this condition is weaker than the preceding one. We shall then prove:

If φ is μ-continuous, then φ is the indefinite integral of $\underline{f}$ or $\overline{f}$ with respect to μ.

Proof. Let $A \varepsilon \mathfrak{F}$ be arbitrary, and let $B \varepsilon \mathfrak{F}'$ be chosen such that $\mu(A + B - AB) = 0$. Since $\mu(A - AB) = 0$ and $\mu(B - AB) = 0$, we have

$$\varphi(A) = \varphi(B) + \varphi(A - AB) - \varphi(B - AB) = \varphi(B),$$

and, denoting by f any of the functions $\underline{f}$ and $\overline{f}$,

$$\int_A f(x)\,\mu(dE) = \int_B + \int_{A-AB} - \int_{B-AB} f(x)\,\mu(dE) = \int_B f(x)\,\mu(dE),$$

from which the result follows.

15. Differentiation on a net. Let E be a set containing at least one element, and μ a measure in E with domain $\mathfrak{F}$, such that $E \varepsilon \mathfrak{F}$ and $\mu(E) = 1$.

By a *partition of E with respect to μ* we shall mean a partition $E = \sum_p D_p$ of E into sets $D_p \varepsilon \mathfrak{F}$, for which $\mu(D_p) > 0$. These sets D_p are called the meshes of the partition. By a *net in E with respect to μ* we shall mean a sequence of partitions $E = \sum_p D_p^1$, $E = \sum_p D_p^2, \cdots$ with respect to μ, each of which is a sub-partition of the preceding one.

If we denote by $\mathfrak{F}_n$ the σ-field consisting of all sums of meshes from the n^{th} partition $E = \sum_p D_p^n$, it is plain that the conditions $\mathfrak{F}_1 \subseteqq \mathfrak{F}_2 \subseteqq \cdots \subseteqq \mathfrak{F}$ and $E \varepsilon \mathfrak{F}_1$ of § 8 are satisfied. Moreover, since $\mu(D_p^n) > 0$ for all meshes, it is plain that for any bounded, completely additive set-function φ defined on $\mathfrak{F}$, the contraction φ_n to $\mathfrak{F}_n$ is μ_n-continuous. Thus the first limit theorem is applicable. The μ_n-integrable function f_n in this case is uniquely determined by

$$f_n(x) = \frac{\varphi(D_p^n)}{\mu(D_p^n)} \text{ for } x \varepsilon D_p^n.$$

The two functions $\underline{f}$ and $\bar{f}$ are called the *lower and upper derivatives of φ with respect to μ on the net.*

16. Density of a set with respect to a σ-field. Let μ be a fixed measure in E with domain $\mathfrak{F}$, such that $E \varepsilon \mathfrak{F}$ and $\mu(E) = 1$, and let $\mathfrak{H}$ be a σ-field contained in $\mathfrak{F}$. Let ν denote the contraction of μ to $\mathfrak{H}$.

For an arbitrary set $A \varepsilon \mathfrak{F}$ the set-function φ on $\mathfrak{F}$ defined by $\varphi(B) = \mu(AB)$ is bounded, completely additive, and μ-continuous. Its contraction ψ to $\mathfrak{H}$ is therefore bounded, completely additive, and ν-continuous, and is therefore the indefinite integral with respect to ν of a ν-integrable function f. By the *density* of A with respect to $\mathfrak{H}$ (and the measure μ) we mean any such function f, i.e. any ν-integrable function f with $[f] = E$, such that

$$\mu(AB) = \int_B f(x)\,\mu(dE) \quad \text{for any } B \varepsilon \mathfrak{H}.$$

Suppose now, as in § 8, that a sequence of σ-fields $\mathfrak{F}_1 \subseteqq \mathfrak{F}_2 \subseteqq \cdots$ contained in $\mathfrak{F}$ is given, such that $E \varepsilon \mathfrak{F}_1$, and let $\mathfrak{F}'$ denote the smallest σ-field containing $\mathfrak{G} = \underset{n}{\mathfrak{S}} \mathfrak{F}_n$. Let further $A \varepsilon \mathfrak{F}$ be arbitrary. From the first limit theorem then follows:

Denoting by f_n the density of A with respect to $\mathfrak{F}_n$ and by f' the density of A with respect to $\mathfrak{F}'$ we have $\mu([\lim_n f_n = f']) = 1$.

If to the set A there exists a set $C \varepsilon \mathfrak{F}'$, such that $\mu(A + C - AC) = 0$, we have the more precise result, that $\mu([\lim_n f_n = f]) = 1$, where f denotes the characteristic function of A.

This implies the following *nought-or-one-theorem*:

If for every n the density f_n of the set A with respect to $\mathfrak{F}_n$ satisfies the relation $\mu([f_n = k_n]) = 1$ for some number k_n, and there exists a set $C \varepsilon \mathfrak{F}'$ such that $\mu(A + C - AC) = 0$, then the measure of the set A is either 0 *or* 1.

We shall also give an independent proof of this theorem: From the relation $\mu(AB) = k_n \mu(B)$ for any $B \varepsilon \mathfrak{F}_n$ we obtain, by placing $B = E$, the equality $\mu(A) = k_n$. Hence $\mu(AB) = \mu(A)\mu(B)$ for any $B \varepsilon \mathfrak{G}$. By the extension theorem of § 4 the relation $\mu(AB) = \mu(A)\mu(B)$ is then valid for any $B \varepsilon \mathfrak{F}'$. Choosing B such that $\mu(A + B - AB) = 0$, we have $\mu(AB) = \mu(A)$ and $\mu(B) = \mu(A)$. The relation therefore becomes $\mu(A) = \mu(A)^2$, which shows that $\mu(A)$ is either 0 or 1.

17. Second limit theorem. Let E be a set containing at least one element, and μ a measure in E with domain $\mathfrak{F}$, such that $E \varepsilon \mathfrak{F}$ and $\mu(E) = 1$. Let now $\mathfrak{F}_1 \supseteqq \mathfrak{F}_2 \supseteqq \cdots$ be a decreasing sequence of σ-fields contained in $\mathfrak{F}$, such that $E \varepsilon \mathfrak{F}_n$ for every n. The system $\mathfrak{F}' = \underset{n}{\mathfrak{D}} \mathfrak{F}_n$ is a σ-field, and $E \varepsilon \mathfrak{F}'$.

The contraction of μ to $\mathfrak{F}_n$ is a measure, which will be denoted by μ_n. Similarly the contraction of μ to $\mathfrak{F}'$ is a measure which will be denoted by μ'.

We shall consider a μ-integrable function f with $[f] = E$. Its indefinite integral

$$\varphi(A) = \int_A f(x)\, \mu(dE)$$

with respect to μ is, by § 7, a bounded, completely additive and μ-continuous set-function in $\mathfrak{F}$. Since the contraction of φ to $\mathfrak{F}_n$ is for every n a μ_n-continuous set-function, there exists, by § 7, a μ_n-integrable function f_n, such that

$$\int_A f_n(x)\, \mu(dE) = \varphi(A) \quad \text{for any } A \varepsilon \mathfrak{F}_n.$$

Similarly there exists a μ'-integrable function f', such that

$$\int_A f'(x)\, \mu(dE) = \varphi(A) \quad \text{for any } A \varepsilon \mathfrak{F}'.$$

The functions f_n and f' need not be uniquely determined. In the sequel f_n and f' will denote some such functions.

The second limit theorem now states:

With the above notations we have the relation

$$\mu([\lim_n f_n = f']) = 1.$$

To prove this it is sufficient to prove that $\mu([\underline{f} < f']) = 0$ and $\mu([f' < \overline{f}]) = 0$, where

$$\underline{f} = \liminf_n f_n \quad \text{and} \quad \overline{f} = \limsup_n f_n.$$

For $E - [\lim_n f_n = f'] \subseteqq [\underline{f} < f'] + [f' < \overline{f}]$.

18. The proof will be based on the following lemma:

Placing $H = [\inf_n f_n < h]$ *and* $K = [\sup_n f_n > k]$ *for arbitrary numbers* h *and* k, *we have for any* $C \varepsilon \mathfrak{F}'$ *the inequalities*

$$\varphi(HC) \leqq h\mu(HC) \quad \textit{and} \quad \varphi(KC) \geqq k\mu(KC).$$

In order to prove the first inequality it is sufficient to prove, that if for an arbitrary n we put

$$H_n = [\min_{p \leqq n} f_p < h],$$

we have $\varphi(H_n C) \leqq h\mu(H_n C)$ for any $C \varepsilon \mathfrak{F}'$. For $H_1 \subseteqq H_2 \subseteqq \cdots$ and $H = \mathfrak{S}_n H_n$. Hence $\mu(HC) = \lim_n \mu(H_n C)$ and $\varphi(HC) = \lim_n \varphi(H_n C)$.

To prove the inequality $\varphi(H_n C) \leqq h\mu(H_n C)$ we put

$$H_{np} = \begin{cases} [f_p < h,\ f_{p+1} \geqq h, \cdots, f_n \geqq h] & \text{for } p < n \\ [f_n < h] & \text{for } p = n. \end{cases}$$

Then $H_{np} \varepsilon \mathfrak{F}_p$ and $H_{np} \subseteqq [f_p < h]$. Moreover $H_n = \sum_{p \leqq n} H_{np}$. Since $C \varepsilon \mathfrak{F}_p$ for any p, this implies

$$\varphi(H_n C) = \sum_{p \leqq n} \varphi(H_{np} C) = \sum_{p \leqq n} \int_{H_{np}C} f_p(x)\,\mu(dE) \leqq \sum_{p \leqq n} h\mu(H_{np}C) = h\mu(H_n C).$$

The inequality $\varphi(KC) \geqq k\mu(KC)$ is proved analogously.

19. The proof of the theorem now runs as follows:

In order to prove that $\mu([\underline{f} < f']) = 0$ it is sufficient to prove that $\mu([\underline{f} < h, f' > k]) = 0$ for any pair of rational numbers h

and k for which $h < k$. For $[f < f'] = \mathfrak{S}[f < h, f' > k]$, where the summation is with respect to all such pairs.

We now apply the inequality $\varphi(HC) \leqq h\mu(HC)$ to the set $C = [f < h, f' > k]$. Since f is μ_n-measurable for all n, it is μ'-measurable; hence $C \varepsilon \mathfrak{F}'$. As $C \subseteqq H$, we obtain

$$\varphi(C) \leqq h\mu(C).$$

On the other hand, since $C \subseteqq [f' > k]$, we have

$$\varphi(C) \geqq k\mu(C).$$

Since $h < k$, these two inequalities show that $\mu(C) = 0$.

The relation $\mu([f' < \bar{f}]) = 0$ is proved analogously.

20. Corollary of the second limit theorem. If in particular μ' only attains the values 0 and 1, i. e. if for any set $A \varepsilon \mathfrak{F}'$ either $\mu(A) = 0$ or $\mu(A) = 1$, we must have

$$\mu([f' = \int_E f(x)\mu(dE)]) = 1,$$

since otherwise one of the sets

$$[f > \int_E f(x)\mu(dE)] \quad \text{or} \quad [f' < \int_E f(x)\mu(dE)]$$

would have the measure 1, which is impossible, as

$$\int_E f'(x)\mu(dE) = \int_E f(x)\mu(dE).$$

By the second limit theorem we therefore in this case have

$$\mu([\lim_n f_n = \int_E f(x)\mu(dE)]) = 1.$$

21. Approximation of a Lebesgue integral by Riemann sums. Let E be the real axis $-\infty < x < +\infty$, $\mathfrak{F}$ the system of Lebesgue measurable sets A on E with period 1, and $\mu(A)$ the measure of a period of A. Let $\mathfrak{F}_n$ denote the system of Lebesgue measurable sets of period $\frac{1}{2^n}$. Then $\mathfrak{F} \supseteqq \mathfrak{F}_1 \supseteqq \mathfrak{F}_2 \supseteqq \cdots$, and

$E \varepsilon \mathfrak{F}_n$ for all n. The systems $\mathfrak{F}, \mathfrak{F}_1, \mathfrak{F}_2, \cdots$ are σ-fields. The σ-field $\mathfrak{F}' = \underset{n}{\mathfrak{D}} \mathfrak{F}_n$ consists of all Lebesgue measurable sets having the period $\frac{1}{2^n}$ for any n. Hence[1] for any $A \varepsilon \mathfrak{F}'$ we have either $\mu(A) = 0$ or $\mu(A) = 1$. The corollary of the second limit theorem is therefore applicable and leads to the following theorem:[2]

If $f(x)$ is a Lebesgue integrable function of period 1, *then the sequence of functions*

$$f_n(x) = \frac{1}{2^n} \sum_{k=0}^{2^n-1} f\left(x + \frac{k}{2^n}\right)$$

converges for almost all x towards the integral

$$\int_0^1 f(x)\, dx .$$

22. Product sets. Let $E_1, E_2, \cdots$ denote a finite or infinite sequence of sets. By the finite or infinite *product*

$$E = (E_1, E_2, \cdots)$$

we shall mean the set of all symbols

$$x = (x_1, x_2, \cdots)$$

where $x_n \varepsilon E_n$ for every n. The elements x_n are called the *coordinates* of x.

For every n except the last in case of a finite product we shall write

$$E'_n = (E_1, \cdots, E_n) \quad \text{and} \quad E''_n = (E_{n+1}, E_{n+2}, \cdots).$$

For an arbitrary element $x = (x_1, x_2, \cdots)$ of E, the corresponding elements

[1] By the well-known theorem, that a Lebesgue measurable set with arbitrarily small periods is either a null-set or the complement of a null-set. We mention that in case of the periods $\frac{1}{2^n}$ this theorem is an easy consequence of the nought-or-one-theorem of § 16.

[2] Jessen [1].

$$x'_n = (x_1, \cdots, x_n) \quad \text{and} \quad x''_n = (x_{n+1}, x_{n+2}, \cdots)$$

of E'_n and E''_n are called the *projections* of x on E'_n and E''_n. We may write

$$E = (E'_n, E''_n) \quad \text{and} \quad x = (x'_n, x''_n).$$

If A'_n is a set in E'_n, the set (A'_n, E''_n) in E is called a *cylinder in E with base A'_n in E'_n*; it consists of all elements $x \varepsilon E$ for which the projection x'_n on E'_n belongs to A'_n. Similarly, if A''_n is a set in E''_n, the set (E'_n, A''_n) is called a *cylinder in E with base A''_n in E''_n*.

Suppose now that every E_n contains at least one element, and let $\mathfrak{F}_n$ for every n be a field in E_n such that $E_n \varepsilon \mathfrak{F}_n$. A set $A = (A_1, A_2, \cdots)$ in E, where $A_n \varepsilon \mathfrak{F}_n$ for every n, and $A_n = E_n$ for all n from a certain stage in case of an infinite product, will be called a *simple set* in E with respect to the fields $\mathfrak{F}_n$. We notice that in case of an infinite product any simple set A is a cylinder with base in some E'_n.

The smallest field containing all simple sets will be denoted by

$$\mathfrak{F} = [\mathfrak{F}_1, \mathfrak{F}_2, \cdots].$$

This field $\mathfrak{F}$ consists of all sets in E which are a sum of a finite number of simple sets no two of which have elements in common. Hence, in case of an infinite product, any set in $\mathfrak{F}$ is a cylinder with base in some E'_n.

The smallest σ-field containing all simple sets will be denoted by

$$\mathfrak{G} = (\mathfrak{F}_1, \mathfrak{F}_2, \cdots).$$

On placing

$$\mathfrak{F}'_n = [\mathfrak{F}_1, \cdots, \mathfrak{F}_n], \quad \mathfrak{F}''_n = [\mathfrak{F}_{n+1}, \mathfrak{F}_{n+2}, \cdots],$$
$$\mathfrak{G}'_n = (\mathfrak{F}_1, \cdots, \mathfrak{F}_n), \quad \mathfrak{G}''_n = (\mathfrak{F}_{n+1}, \mathfrak{F}_{n+2}, \cdots),$$

it is easily seen that

$$\mathfrak{F} = [\mathfrak{F}'_n, \mathfrak{F}''_n] \text{ and } \mathfrak{G} = (\mathfrak{F}'_n, \mathfrak{F}''_n) = (\mathfrak{G}'_n, \mathfrak{G}''_n).$$

23. Measure and integration in product sets. Let μ_n for every n be a content in E_n with domain $\mathfrak{F}_n$, such that $E_n \varepsilon \mathfrak{F}_n$ and

$\mu_n(E_n) = 1$. Then it is easily seen that there exists a unique content μ in $E = (E_1, E_2, \cdots)$ with domain $\mathfrak{F} = [\mathfrak{F}_1, \mathfrak{F}_2, \cdots]$, such that

$$\mu(A) = \mu_1(A_1)\mu_2(A_2)\cdots$$

for any simple set $A = (A_1, A_2, \cdots)$. We notice that the factors of the product $\mu_1(A_1)\mu_2(A_2)\cdots$ are 1 from a certain stage in case of an infinite product.

This content μ will be denoted by

$$\mu = [\mu_1, \mu_2, \cdots].$$

On placing

$$\mu'_n = [\mu_1, \cdots, \mu_n] \quad \text{and} \quad \mu''_n = [\mu_{n+1}, \mu_{n+2}, \cdots],$$

it is easily seen that

$$\mu = [\mu'_n, \mu''_n].$$

We shall now prove the following theorem:

If the contents μ_n are all completely additive, then the content $\mu = [\mu_1, \mu_2, \cdots]$ is also completely additive.

By the criterion of § 4 it is sufficient to prove that for any sequence $A_1 \supseteq A_2 \supseteq \cdots$ of sets $A_m \varepsilon \mathfrak{F}$ for which $\mu(A_m) \geq$ (some) $k > 0$ for all m, there exists an element $x^* = (x^*_1, x^*_2, \cdots)$ of E which belongs to all A_m.

In the proof we shall use the relations

$$E''_n = (E_{n+1}, E''_{n+1}) \quad \text{and} \quad \mu''_n = [\mu_{n+1}, \mu''_{n+1}].$$

For an arbitrary set A in E and an arbitrary element $x'_n = (x_1, \cdots, x_n) \varepsilon E'_n$ we shall denote by $A(x'_n) = A(x_1, \cdots, x_n)$ the set of all elements $x''_n \varepsilon E''_n$ for which $x = (x'_n, x''_n)$ belongs to A.

We choose an arbitrary sequence of numbers

$$k > k_1 > k_2 > \cdots > 0.$$

Corresponding to the relations $E = (E_1, E''_1)$ and $\mu = [\mu_1, \mu''_1]$ we begin by considering for every m the set B_m of all $x_1 \varepsilon E_1$ for which

$$\mu''_1(A_m(x_1)) > k_1.$$

A simple consideration shows that

$$\mu_1(B_m) + k_1(1 - \mu_1(B_m)) \geqq \mu(A_m) > k,$$

whence

$$\mu_1(B_m) > \frac{k - k_1}{1 - k_1}.$$

Since $B_1 \supseteqq B_2 \supseteqq \cdots$, and μ_1 is completely additive, this implies the existence of an element $x_1^* \varepsilon E_1$, which belongs to all B_m. Thus for this x_1^* we have for all m

$$\mu_1''(A_m(x_1^*)) > k_1.$$

Corresponding to the relations $E_1'' = (E_2, E_2'')$ and $\mu_1'' = [\mu_2, \mu_2'']$ we may now repeat the argument to the sets $A_1(x_1^*) \supseteqq A_2(x_1^*) \supseteqq \cdots$ in E_1''. This proves the existence of an element $x_2^* \varepsilon E_2$, such that

$$\mu_2''(A_m(x_1^*, x_2^*)) > k_2$$

for all m. Continuing in this manner we arrive at a sequence $x_1^*, x_2^*, \cdots$, where $x_n^* \varepsilon E_n$, such that

$$\mu_n''(A_m(x_1^*, \cdots, x_n^*)) > k_n$$

for every n and all m.

If the product $E = (E_1, E_2, \cdots)$ is infinite, the element $x^* = (x_1^*, x_2^*, \cdots)$ of E must belong to all A_m. For every A_m is a cylinder with base in some E_n', and the set $A_m(x_1^*, \cdots, x_n^*)$ is not empty.

If the product $E = (E_1, E_2, \cdots)$ is finite, say $E = (E_1, \cdots, E_p)$, the above procedure breaks off for $n = p - 1$, and the last relation becomes

$$\mu_p(A_m(x_1^*, \cdots, x_{p-1}^*)) > k_{p-1}.$$

Since $A_1(x_1^*, \cdots, x_{p-1}^*) \supseteqq A_2(x_1^*, \cdots, x_{p-1}^*) \supseteqq \cdots$, and μ_p is completely additive, this implies the existence of an element $x_p^* \varepsilon E_p$, which belongs to all $A_m(x_1^*, \cdots, x_{p-1}^*)$. The element $x^* = (x_1^*, \cdots, x_p^*)$ of E then belongs to all A_m.

24. The conditions of the previous theorem are in particular satisfied if the contents μ_n are measures. Applying the extension theorem of § 4 we therefore obtain the following theorem:[1]

If $E = (E_1, E_2, \cdots)$, and μ_n is for every n a measure in E_n with domain $\mathfrak{F}_n$, for which $E_n \varepsilon \mathfrak{F}_n$ and $\mu_n(E_n) = 1$, then there exists a unique measure ν in E with domain $\mathfrak{G} = (\mathfrak{F}_1, \mathfrak{F}_2, \cdots)$, such that

$$\nu(A) = \mu_1(A_1)\mu_2(A_2)\cdots$$

for any simple set $A = (A_1, A_2, \cdots)$.

This measure ν will be denoted by

$$\nu = (\mu_1, \mu_2, \cdots).$$

On placing

$$\nu_n' = (\mu_1, \cdots, \mu_n) \quad \text{and} \quad \nu_n'' = (\mu_{n+1}, \mu_{n+2}, \cdots)$$

it is easily seen that

$$\nu = (\nu_n', \nu_n'').$$

25. Regarding integration in a product of two sets the usual theorem on repeated integration is valid. Applying this theorem to $E = (E_n', E_n'')$ and $\nu = (\nu_n', \nu_n'')$ we obtain the following results:

If f is a ν-integrable function defined in E, then on placing

$$f_n'(x_n') = \int_{E_n''} f(x_n', x_n'')\, \nu_n''(dE_n'')$$

when the integral exists, we have $[f_n'] \varepsilon \mathfrak{G}_n'$ and $\mu_n'(E_n' - [f_n']) = 0$, and for every set $A_n' \varepsilon \mathfrak{G}_n'$

$$\int_{(A_n', E_n'')} f(x)\, \nu(dE) = \int_{[f_n']A_n'} f_n'(x_n')\, \nu_n'(dE_n').$$

[1] Łomnicki and Ulam [1] have given an incomplete proof of this theorem (in the proof of lemma 4 the number N is chosen twice). The proof given here is taken from Jessen [2, article 4]. An analogous theorem on arbitrary measures in product sets has been given by Doob [1], but his proof seems incomplete (it is not seen how the sets $\tilde{A}_n$ on p. 92 are chosen). The proof by Sparre Andersen of a more general theorem is incomplete (the relation $\sup_n f_n^{(2)}(x_2) \leq 1$ on p. 21 needs not be valid).

Similarly, on placing

$$f''_n(x''_n) = \int_{E'_n} f(x'_n, x''_n)\, \nu'_n(dE'_n)$$

when the integral exists, we have $[f''_n] \varepsilon \mathfrak{S}''_n$ and $\mu''_n(E''_n - [f''_n]) = 0$, and for every set $A''_n \varepsilon \mathfrak{S}''_n$

$$\int_{(E'_n, A''_n)} f(x)\, \nu(dE) = \int_{[f''_n] A''_n} f''_n(x''_n)\, \nu''_n(dE''_n).$$

26. Let $\mathfrak{S}^{n'}$ denote the system of all cylinders in E with a base in E'_n belonging to $\mathfrak{S}'_n$, i.e. the system of all sets (A'_n, E''_n), where $A'_n \varepsilon \mathfrak{S}'_n$. Evidently $\mathfrak{S}^{n'}$ is a σ-field, $\mathfrak{S}^{1'} \subseteq \mathfrak{S}^{2'} \subseteq \cdots$, and $\mathfrak{S}$ is the smallest σ-field containing all $\mathfrak{S}^{n'}$. Finally $E \varepsilon \mathfrak{S}^{1'}$.

Let f be a ν-integrable function defined in E, and let

$$\varphi(A) = \int_A f(x)\, \nu(dE)$$

be its indefinite integral. Let $f^{n'}$ denote the function f'_n introduced in § 25, considered as a function in E which is independent of x''_n. Then $[f^{n'}] \varepsilon \mathfrak{S}^{n'}$ and $\nu(E - [f^{n'}]) = 0$, and for every set $A \varepsilon \mathfrak{S}^{n'}$

$$\varphi(A) = \int_{[f^{n'}] A} f^{n'}(x)\, \nu(dE).$$

By the first limit theorem we therefore obtain the following result:[1]

If $f(x) = f(x_1, x_2, \cdots)$ is a ν-integrable function defined in E, then the sequence of integrals

$$f^{n'}(x) = \int_{E''_n} f(x'_n, x''_n)\, \nu''_n(dE''_n)$$

converges towards $f(x)$ for all x outside a set $N \varepsilon \mathfrak{S}$ with $\nu(N) = 0$.

27. Let in particular f be the characteristic function of a set $S \varepsilon \mathfrak{S}$ having the property, that any two elements $x = (x_1, x_2, \cdots)$

[1] This theorem, and the two which follow, have been stated without proof in Jessen and Wintner [1], where some applications are given. Proofs were given in Jessen [2, article 4].

and $y = (y_1, y_2, \cdots)$, for which $x_n = y_n$ for all n from a certain stage, either both belong to S or both do not belong to S. Then f is for every n actually independent of x'_n. The integral $f^{n'}(x)$ is therefore in this case defined for all x and is a constant k_n.

The nought-or-one-theorem in § 16 therefore gives the following result:

If $S \varepsilon \mathfrak{S}$ is a set in E, such that any two elements $x = (x_1, x_2, \cdots)$ and $y = (y_1, y_2, \cdots)$, for which $x_n = y_n$ for all n from a certain stage, either both belong to S or both do not belong to S, then $\nu(S)$ is either 0 or 1.

28. Let $\mathfrak{S}^{n''}$ denote the system of all cylinders in E with a base in E''_n belonging to $\mathfrak{S}''_n$, i. e. the system of all sets (E'_n, A''_n), where $A''_n \varepsilon \mathfrak{S}''_n$. Then $\mathfrak{S}^{1''} \supseteqq \mathfrak{S}^{2''} \supseteqq \cdots$ is a decreasing sequence of σ-fields contained in $\mathfrak{S}$, and $E \varepsilon \mathfrak{S}^{n''}$ for all n. The system $\mathfrak{H} = \underset{n}{\mathfrak{D}} \mathfrak{S}^{n''}$ is the system of sets $S \varepsilon \mathfrak{S}$, which for every n is a cylinder with a base in E''_n, i. e. satisfying the condition of § 27. Thus $\nu(S)$ is either 0 or 1 for any $S \varepsilon \mathfrak{H}$.

Let f be a ν-integrable function defined in E, and let

$$\varphi(A) = \int_A f(x)\, \nu(dE)$$

be its indefinite integral. Let $f^{n''}$ denote the function f''_n introduced in § 25, considered as a function in E which is independent of x'_n. Then $[f^{n''}] \varepsilon \mathfrak{S}^{n''}$ and $\nu(E - [f^{n''}]) = 0$, and for every set $A \varepsilon \mathfrak{S}^{n''}$

$$\varphi(A) = \int_{[f^{n''}]A} f^{n''}(x)\, \nu(dE).$$

By the corollary of the second limit theorem we therefore obtain the following result:

If $f(x) = f(x_1, x_2, \cdots)$ is a ν-integrable function defined in E, then the sequence of integrals

$$f^{n''}(x) = \int_{E'_n} f(x'_n, x''_n)\, \nu'_n(dE'_n)$$

converges towards

$$\int_E f(x)\,\nu(dE)$$

for all x outside a set $N \varepsilon \mathfrak{G}$ with $\nu(N) = 0$.

29. Applications to the theory of probability. Let E be a set containing at least one element. A measure μ in E with domain $\mathfrak{F}$, for which $E \varepsilon \mathfrak{F}$ and $\mu(E) = 1$, may also be called a *probability distribution* in E. The measure $\mu(A)$ of a set $A \varepsilon \mathfrak{F}$ is then called the *probability* of the *event* A. A μ-measurable function f with $[f] = E$ is called a *random variable*, and the integral

$$\mathfrak{M}(f) = \int_E f(x)\,\mu(dE),$$

when it exists, is called the *mean value* of f.

Besides E we shall now consider another set E^*. We suppose that to every $x \varepsilon E$ is assigned a definite element $x^* \varepsilon E^*$. Let $\mathfrak{G}^*$ be a σ-field in E^* such that $E^* \varepsilon \mathfrak{G}^*$. For every set $A^* \varepsilon \mathfrak{G}^*$ we consider the set A of all elements $x \varepsilon E$ for which x^* belongs to A^*. The system of all sets $A \varepsilon \mathfrak{F}$ of this particular type will be denoted by $\mathfrak{G}$. As is easily proved, $\mathfrak{G}$ is a σ-field, and $E \varepsilon \mathfrak{G}$. The contraction of μ to $\mathfrak{G}$ will be denoted by ν.

Let now f be a random variable for which the mean value $\mathfrak{M}(f)$ exists. Let

$$\varphi(A) = \int_A f(x)\,\mu(dE)$$

be its indefinite integral, and let g be some ν-integrable function with $[g] = E$, for which

$$\varphi(A) = \int_A g(x)\,\mu(dE) \quad \text{for any } A \varepsilon \mathfrak{G}.$$

The function g evidently depends on x^* only, i. e. it has the same value for any two elements $x \varepsilon E$ with the same corresponding element $x^* \varepsilon E^*$. We call $g(x)$ the *conditional mean value of f by known x^**, taken with respect to the σ-field $\mathfrak{G}^*$, and shall use the notation

$$g(x) = \mathfrak{M}_{x^*}(f).^{1}$$

[1] If for $\mathfrak{G}^*$ we take the system of all sets in E^*, the conditional mean value is that defined by Kolmogoroff [1, chap. V]. The modification here adopted is necessary for the results of §§ 33—34.

From the definition of g it follows, that

$$\mathfrak{M}(g) = \mathfrak{M}(\mathfrak{M}_{x^*}(f)) = \mathfrak{M}(f).$$

When f is the characteristic function of a set $A \varepsilon \mathfrak{F}$, the conditional mean value $\mathfrak{M}_{x^*}(f)$ is also called the *conditional probability* of A by known x^*, taken with respect to $\mathfrak{G}^*$.

30. Let $\mathfrak{G}_1 \subseteqq \mathfrak{G}_2^* \subseteqq \cdots$ be an arbitrary increasing sequence of σ-fields in E^*, such that $E^* \varepsilon \mathfrak{G}_1^*$, and let $\mathfrak{G}^*$ be the smallest σ-field containing $\underset{n}{\mathfrak{S}} \mathfrak{G}_n^*$. Let $\mathfrak{G}_1, \mathfrak{G}_2, \cdots$, and $\mathfrak{G}$, be the corresponding σ-fields in E. Clearly $\mathfrak{G}_1 \subseteqq \mathfrak{G}_2 \subseteqq \cdots \subseteqq \mathfrak{G}$. Assuming that for every set $A^* \varepsilon \mathfrak{G}^*$ the corresponding set A in E belongs to $\mathfrak{F}$ we shall now prove that $\mathfrak{G}$ is the smallest σ-field containing $\underset{n}{\mathfrak{S}} \mathfrak{G}_n$.

Let for the moment this smallest σ-field be denoted by $\mathfrak{G}'$, and let $\mathfrak{H}^*$ be the system of sets A^* in E^*, for which the corresponding set A in E belongs to $\mathfrak{G}'$. From the mere fact that $\mathfrak{G}'$ is a σ-field, follows easily that $\mathfrak{H}^*$ is a σ-field. Moreover, by our assumption every $\mathfrak{G}_n^* \subseteqq \mathfrak{H}^*$. Hence $\mathfrak{G}^* \subseteqq \mathfrak{H}^*$. Thus for every set $A^* \varepsilon \mathfrak{G}^*$ the corresponding set A in E belongs to $\mathfrak{G}'$, i. e. $\mathfrak{G} \subseteqq \mathfrak{G}'$, and hence $\mathfrak{G} = \mathfrak{G}'$.

31. Next, let $\mathfrak{G}_1^* \supseteqq \mathfrak{G}_2^* \supseteqq \cdots$ be an arbitrary decreasing sequence of σ-fields in E^*, such that $E^* \varepsilon \mathfrak{G}_n^*$ for all n. Then $\mathfrak{G}^* = \underset{n}{\mathfrak{D}} \mathfrak{G}_n^*$ is also a σ-field, and $E^* \varepsilon \mathfrak{G}^*$. Let $\mathfrak{G}_1$, $\mathfrak{G}_2, \cdots$, and $\mathfrak{G}$, be the corresponding σ-fields in E. Clearly $\mathfrak{G}_1 \supseteqq \mathfrak{G}_2 \supseteqq \cdots \supseteqq \mathfrak{G}$. We shall now prove that $\mathfrak{G} = \underset{n}{\mathfrak{D}} \mathfrak{G}_n$.

To see this, we have to prove that any set $A \varepsilon \underset{n}{\mathfrak{D}} \mathfrak{G}_n$ corresponds to some $A^* \varepsilon \mathfrak{G}^*$. Since $A \varepsilon \mathfrak{G}_n$, it corresponds to a set $A_n^* \varepsilon \mathfrak{G}_n^*$. These sets A_n^* will differ only by elements x^* which do not correspond to any x. Hence any set A^* in E^* for which $\underset{n}{\mathfrak{D}} A_n^* \subseteqq A^* \subseteqq \underset{n}{\mathfrak{S}} A_n^*$ will have A as its corresponding set. Let us take $A^* = \limsup\limits_{n} A_n^* = \underset{n}{\mathfrak{D}} \underset{p}{\mathfrak{S}} A_{n+p}^*$. For every m we have $A_n^* \varepsilon \mathfrak{G}_m^*$ for $n \geqq m$. Hence $A^* \varepsilon \mathfrak{G}_m^*$ for all m, i. e. $A^* \varepsilon \mathfrak{G}^*$.

32. Let us now consider a finite or infinite sequence of random variables $g_1, g_2, \cdots$, and let E^* be the space $(R_1, R_2, \cdots)$, where each R_n is the real axis $-\infty \leqq x_n \leqq +\infty$. To every $x \varepsilon E$, let us assign the point

$$x^* = (g_1(x), g_2(x), \cdots)$$

of E^*. Let $\mathfrak{G}^*$ denote the system of Borel sets in E^*, i. e. the smallest σ-field containing all sets $x_n \leqq a_n$. The corresponding σ-field $\mathfrak{G}$ in E will then be denoted by $\mathfrak{G}_{g_1, g_2, \cdots}$, and the corresponding conditional mean value $\mathfrak{M}_{x^*}(f)$ will be called the *conditional mean value of f by known values of $g_1, g_2, \cdots$*, and will be denoted by

$$\mathfrak{M}_{g_1, g_2, \cdots}(f).$$

If f is the characteristic function of a set $A \varepsilon \mathfrak{F}$, the conditional mean value $\mathfrak{M}_{g_1, g_2, \cdots}(f)$ is also called the conditional probability of A by known values of $g_1, g_2, \cdots$.

33. Let us now suppose that the sequence $g_1, g_2, \cdots$ is infinite. Let $\mathfrak{G}_n^*$ for every n denote the system of Borel sets in E^*, which are cylinders with base in $(R_1, \cdots, R_n)$. The bases of these sets being just all Borel sets in $(R_1, \cdots, R_n)$, the σ-field in E corresponding to $\mathfrak{G}_n^*$ will be $\mathfrak{F}_{g_1, \cdots, g_n}$.

Now $\mathfrak{G}_1^* \subseteqq \mathfrak{G}_2^* \subseteqq \cdots$, and $\mathfrak{G}^*$ is the smallest σ-field containing $\underset{n}{\mathfrak{S}} \mathfrak{G}_n^*$; moreover, it is easily seen that for every set $A^* \varepsilon \mathfrak{G}^*$ the corresponding set A in E belongs to $\mathfrak{F}$. Hence by § 30 we have $\mathfrak{F}_{g_1} \subseteqq \mathfrak{F}_{g_1, g_2} \subseteqq \cdots$, and $\mathfrak{F}_{g_1, g_2, \cdots}$ is the smallest σ-field containing $\underset{n}{\mathfrak{S}} \mathfrak{F}_{g_1, \cdots, g_n}$. The first limit theorem is therefore applicable and yields the following result:

The conditional mean value $\mathfrak{M}_{g_1, \cdots, g_n}(f)$ of f by known values of $g_1, \cdots, g_n$ converges for $n \to \infty$ with the probability 1 towards the conditional mean value $\mathfrak{M}_{g_1, g_2, \cdots}(f)$ of f by known values of $g_1, g_2, \cdots$.

When f is the characteristic function of a set $A \varepsilon \mathfrak{F}$, the theorem becomes a theorem on conditional probabilities. If in particular $A \varepsilon \mathfrak{F}_{g_1, g_2, \cdots}$, the theorem shows that the conditional probability of A by given values of $g_1, \cdots, g_n$ converges for $n \to \infty$ with the probability 1 towards 1 in A and 0 in $E - A$.[1]

[1] Lévy [1, pp. 128—130].

34. Still assuming the sequence $g_1, g_2, \cdots$ to be infinite, let us now by $\mathfrak{G}_n^*$ denote the system of Borel sets in E^*, which are cylinders with base in $(R_{n+1}, R_{n+2}, \cdots)$. The bases of these sets being just all Borel sets in $(R_{n+1}, R_{n+2}, \cdots)$, the σ-field in E corresponding to $\mathfrak{G}_n^*$ will be $\mathfrak{F}_{g_{n+1}, g_{n+2}, \cdots}$.

We have $\mathfrak{G}_1^* \supseteqq \mathfrak{G}_2^* \supseteqq \cdots$. Let us put $\mathfrak{G}^* = \underset{n}{\mathfrak{D}} \mathfrak{G}_n^*$. Then $\mathfrak{G}^*$ is the system of Borel sets A^* in E^* with the property that two points $x^* = (x_1, x_2, \cdots)$ and $y^* = (y_1, y_2, \cdots)$, for which $x_n = y_n$ for all n from a certain stage, either both belong to A^* or both do not belong to A^*.

The class of all sequences obtained from a given sequence $x_1, x_2, \cdots$ by changing only a finite number of elements will briefly be called an *end*, and will be denoted by $\{x_1, x_2, \cdots\}$. The σ-field $\mathfrak{G}$ in E corresponding to the above σ-field $\mathfrak{G}^* = \underset{n}{\mathfrak{D}} \mathfrak{G}_n^*$ will be denoted by $\mathfrak{F}_{\{g_1, g_2, \cdots\}}$, and the corresponding conditional mean value $\mathfrak{M}_{x^*}(f)$ will be called the *conditional mean value of f by known end of the sequence* $g_1, g_2, \cdots$, and will be denoted by

$$\mathfrak{M}_{\{g_1, g_2, \cdots\}}(f).$$

From § 31 it follows that $\mathfrak{F}_{g_1, g_2, \cdots} \supseteqq \mathfrak{F}_{g_2, g_3, \cdots} \supseteqq \cdots$, and that $\mathfrak{F}_{\{g_1, g_2, \cdots\}} = \underset{n}{\mathfrak{D}} \mathfrak{F}_{g_{n+1}, g_{n+2}, \cdots}$. The second limit theorem is therefore applicable and yields the following result:

The conditional mean value $\mathfrak{M}_{g_{n+1}, g_{n+2}, \cdots}(f)$ of f by known values of $g_{n+1}, g_{n+2}, \cdots$ converges for $n \to \infty$ with the probability 1 towards the conditional mean value $\mathfrak{M}_{\{g_1, g_2, \cdots\}}(f)$ of f by known end of the sequence $g_1, g_2, \cdots$.

The corollary of the second limit theorem shows that if the probability of any event $A \varepsilon \mathfrak{F}_{\{g_1, g_2, \cdots\}}$ is either 0 or 1, then the conditional mean value $\mathfrak{M}_{g_{n+1}, g_{n+2}, \cdots}(f)$ converges for $n \to \infty$ with the probability 1 towards the mean value $\mathfrak{M}(f)$. In particular, this will be so when the probability of any event A whatsoever with the property, that two elements x and y for which $g_n(x) = g_n(y)$ for all n from a certain stage, either both belong to A or both do not belong to A, is either 0 or 1.

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