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SURFACE TRANSFORMATION CLASSES OF ALGEBRAICALLY FINITE TYPE

BY

JAKOB NIELSEN

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TABLE OF CONTENTS

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INTRODUCTION

The theory of surface transformations has received a great deal of attention from many authors during the last 60 years. This is, of course, partly due to the rôle they play in the theory of analytic functions. But from the very beginning, even in the fundamental work of HENRI POINCARÉ, the theory of surfaces has had an interest of its own from a purely topological point of view. In recent developments of topology it is chiefly in the theory of surfaces that it has been possible to penetrate beyond the general line characterized by an extensive use of homology theory.

So far as surface transformations are concerned, many investigations have, for good reasons, been focussed upon periodic transformations. This may be seen from the list of works given at the end of this paper, a list to which we refer by numbers in square brackets and which does not pretend to cover all investigations on that topic. In view of generalizations transformation classes of finite order naturally present themselves. They do not, however, furnish essentially new features. This is explained by the fact, proved in [15], that every transformation class of finite order contains a periodic transformation. Hence it seemed to me that a true generalization might start from the fact that in the homology theory of periodic transformations all multipliers, i. e. roots of the characteristic polynomial, are roots of unity. It turns out that this quality belongs to a far-reaching totality of transformation classes. Imagine some part of a surface bounded by a set of simple closed curves and this set carried into a homotopic set by the members of some transformation class. One may then choose a transformation of the class such that this set is carried into itself and then look on the transformation class as carrying that part of the surface into itself.

The class may then be of finite order for that part of the surface, even if it is not so for the surface as a whole. Moreover, the entire surface may be decomposed into such parts. This is, roughly speaking, the idea of the present investigation. To carry it out requires a thorough application of the general properties of surface transformations, especially the methods of the universal covering surface and its limit points. In order not to refer the reader to investigations scattered in many different papers, I outline in part I the general foundations without proof. Part II then deals with a full investigation of the transformation classes concerned and part III with their homology theory.

In the homology theory of transformations the so-called trace formula has hitherto been one of the chief means. The trace is the sum of the roots of the characteristic polynomial. In this matter I want to show that to get full results one should not confine oneself to the trace but deal with the polynomial itself. It contains a good deal of information concerning the transformation class in question (section 22). In section 16 the general form of this polynomial is to be found. All its roots are roots of unity. Although it is actually not proved that all transformation classes with this quality are embraced,—which may well be the case for reasons which I do not intend to discuss here—I propose for the transformation classes investigated the term "classes of algebraically finite type".

The chief means of our investigation is the transformation group induced in the set of limit points of the universal covering surface by a prescribed transformation class. This group is an invariant of the transformation class, thus properties common to all transformations of the class, which concern their behaviour in a twodimensional field, are reflected by a topological transformation in a onedimensional set. It is evident that this means must be of great efficiency. If a generalization of this comprehensive invariant to manifolds of higher dimensions were discovered, a new development of the theory of their transformations might well be expected.

I am indebted to my friends S. Lauritzen and S. Bundgaard for reading the proof and suggesting many valuable improvements.

Part I.

Foundations.

1. Group F as starting point. The subject of the investigations to follow is the orientable, closed or bounded, surface of finite connectivity. Let p denote the genus and r the number of boundary curves of the surface S. These numbers are only submitted to the restriction $2p+r \geqq 3$, thus excluding the cases $p = r = 0$ (sphere), $p = 0$, $r = 1$ (circular disc), $p = 0$, $r = 2$ (circular ring) and $p = 1$, $r = 0$ (torus), that is to say all cases in which the natural metric of the surface is spherical or euclidean. Since in the case $2p+r \geqq 3$ the natural metric of S is hyperbolic, the universal covering surface of S may be mapped into the unit circular disc X of the plane of a complex variable x in such a way that the elements of the Poincaré group F of S correspond to linear hyperbolic transformations of x carrying X into itself and leaving two points of the bounding unit circle E of X invariant.

For the sake of clearness and generality, we may put the starting point in the following way: Let an arbitrary group F of (fractional) linear transformations of x be given, which is not abelian and the elements of which carry X into itself and —apart from unity—are all hyperbolic. I have shown in [13], § 4, that under the assigned conditions F is properly discontinuous in X, i. e. that each point x of X is imbedded in a neighbourhood containing no image point ($\neq x$) of x under the transformations of F. Let f be any element of F different from unity; the points U_f and V_f of E left invariant by f will be termed the *fundamental points* of f, the arc of circle joining them in X at right angles to E the *axis* of f. In terms of hyperbolic geometry based on E the axis is a straight line, and f is a translation of the hyperbolic plane along the axis in the direction from U_f towards V_f, from the *negative* towards the *positive* fundamental point of f. Two different axes have no fundamental point in common, otherwise F would contain a parabolic transformation.

The set of fundamental points of all elements of F is called the *set of fundamental points of* F and denoted by G_F. Under the assigned conditions for F the elements of F are enumerable in consequence of the discontinuity of F in X, so G_F is an enumerable set of points of E. The closure $\overline{G}_F$ of G_F containing G_F and derived from G_F by adding all limit points of G_F is called the *set of limit points of* F. The set $\overline{G}_F$ is perfect (no point of $\overline{G}_F$ is isolated). Two cases may present themselves:

a) G_F is not dense on E. In this case $E-\overline{G}_F$ is made up of intervals, which are dense on E and which will be termed *intervals of regularity*,

b) G_F is dense on E. In this case $\overline{G}_F$ coincides with the entire circumference E.

The smallest subset of $X+E$ containing $\overline{G}_F$ and convex in a non-euclidean sense is denoted by $\overline{K}_F$. In case a) $\overline{K}_F$ is obtained by removing from $X+E$ each interval of regularity together with the non-euclidean half-plane bounded by it. Inside X therefore $\overline{K}_F$ is bounded by arcs of circle at right angles to E joining the end points of some interval of regularity. In case b) $\overline{K}_F$ coincides with $X+E$. In both cases the set of points common to $\overline{K}_F$ and X is termed *convex region* of F and denoted by K_F. The set $\overline{K}_F$ is obtained from K_F by adding $\overline{G}_F$, which is precisely the set of limit points of K_F on E. In case b) the convex region of F coincides with X.

Now the group F evidently transforms G_F into itself, the fundamental points of f being transformed into the fundamental points of gfg^{-1} by an element g of F. Therefore by continuity F transforms $\overline{G}_F$ into itself and, being a group of non-euclidean displacements, transforms $\overline{K}_F$ and X and thus their common part K_F into itself. Speaking of F as a group of transformations of K_F we may in abstracto consider the whole set Fx of images of x under the transformations of F as one point $\{x\}$. The set of these points $\{x\}$ then form a manifold (surface), which we may denote by K_F mod F.

The surface K_F mod F obtained in this way from an arbitrary group F subject to the above conditions need not, however, be of finite connectivity. To obtain this, one more condition must be imposed on the group F, viz. to be generated by a finite number of its elements. As shown in [13], § 11—12 this is equivalent to the condition that there exists a non-euclidean finite region of X, a circle inside E, say, which contains at least one point of every set Fx, x being a point of K_F, in its interior. Under this condition all boundaries of K_F inside X are axes of F and they arise from a finite number r of distinct boundaries by the transformations of F. Then K_F mod F is a surface of finite connectivity with r bounding curves, which are closed geodesics in the sense of the hyperbolic metric imposed on the surface. In case b), $K_F = X$, K_F mod F is a closed surface, $r = 0$. In both cases a certain genus p arises; in case a) p may be zero, provided $r \geqq 3$.

To sum up, we start with a transformation group F of X subject to three conditions:

F is not abelian.

All elements of F other than identity are hyperbolic.

F is generated by a finite number of elements.

Then K_F mod F is our orientable surface S of finite connectivity with a certain genus p and a certain number r of boundary curves. S may be illustrated by an image in ordinary space. A hyperbolic metric derived from X is impressed upon S. In this metric every axis of F corresponds to a closed geodesic of S; especially the boundary curves are such geodesics. F is isomorphic to the Poincaré group of S. The minimum number of generators of F is $2p$ in case $r = 0$ and $2p+r-1$ in case $r > 0$. In the latter case F is a free group.

Some consequences are immediate: Let A be an axis of F. The set of its images under the transformations of F is called the *congruence class* of A and denoted by FA. All axes of FA correspond to one closed geodesic a of S. Let λ be the non-euclidean length of a. There exists an element f of F with the axis A, which displaces the points of A at the distance λ along A. The element f (and likewise its inverse f^{-1}) is called *primary element of F to the axis A*, and λ is called *primary displacement length* belonging to the axis A. All elements of F belonging to A are powers of f, and their displacement lengths are multiples of λ. The class FA does not accumulate in X. All its axes have the same primary displacement length. The number of elements of F the displacement length of which is inferior to a given positive constant is finite.

The properties of groups F and corresponding surfaces S shortly recalled in this introductory section are investigated at length in my paper [13], to which reference may be made for the proofs. In part I and II of the present paper most of the investigations are carried out in the convex region K_F of X, the universal covering surface of S with its group F, but incidentally we may draw conclusions directly on the illustrative model S.

2. Transformation functions. Let τ denote a topological (i. e. one-to-one and continuous) transformation of S into itself preserving orientation. Let $\{x_0\}$ be any point of S and $\{x_0'\}$ its image under τS. Let x_0 be a point of K_F representing $\{x_0\}$ and x_0' a point representing $\{x_0'\}$. Then by continuity we have one topological transformation of K_F and one only covering the surface transformation τS and carrying x_0 into x_0'. We denote this transformation of K_F by $x' = \varkappa(x)$. The transformation function $\varkappa(x)$ thus defined in K_F satisfies a system of functional equations: Let f be any element of F[1]. As x and $f(x)$ determine the same point $\{x\}$ of S, their images under the transformation $\varkappa$ must correspond to the same point $\{x'\}$ of S:

$$\varkappa\big(f(x)\big) = f'(x') = f'\big(\varkappa(x)\big), \quad f' \subset F.$$

For reasons of continuity the correspondence $f \to f'$ cannot depend on the choice of x, and it is easily seen that this correspondence constitutes an automorphic transformation of the group F into itself. Denoting this automorphism by the letter I and writing f_I instead of f', we may write the above functional equation in short

$$\varkappa f = f_I \varkappa. \tag{2.1}$$

In (2.1) f and accordingly f_I ranges over the group F and the argument x of the functions (not written explicitly in (2.1)) over the convex region K_F.

[1] We denote this by $f \subset F$, using $\subset$ as a symbol of inclusion, and write likewise $F \supset f$.

Since in defining $\varkappa$ we have chosen the representing points x_0 and x_0' of the points $\{x_0\}$ and $\{x_0'\}$ of S freely within their congruence classes Fx_0 and Fx_0' respectively, $\varkappa$ is not the only transformation function to represent the surface transformation τS. But it is evident that any transformation function covering τS can differ from $\varkappa$ by an element g of F only. Thus

$$g\varkappa = g\big(\varkappa(x)\big), \quad g \subset F,$$

is the totality of transformation functions covering τS, the element g ranging over the entire group F. Since (2.1) may be written

$$\varkappa f \varkappa^{-1} = f_I, \tag{2.2}$$

we have for $g\varkappa$ the functional equation

$$g\varkappa f \varkappa^{-1} g^{-1} = g f_I g^{-1}.$$

The automorphism $f \to g f_I g^{-1}$ corresponding to $g\varkappa$ is said to be *related* to I and is derived from I by applying the *inner* automorphism consisting in transforming by the element g. The totality of related automorphisms obtained by making g range over F is termed a *family of automorphisms* of F.

If the given topological transformation τ of S is made to vary continuously, even so as not to preserve its quality of being one-to-one, this will make $\varkappa$ vary accordingly, but for obvious reasons of continuity it will not alter the functional equation (2.1). Thus the automorphism I induced by $\varkappa$ will be unaltered. So we infer that *the family of automorphisms of F belonging to the surface transformation τS is an invariant of the transformation class of τ.*

In looking for invariants of surface transformation classes the chief means lies in the fact that transformation functions such as $\varkappa$, originally defined in K_F, extend continuously from K_F to $\overline{K}_F$, thus including the set $\overline{G}_F$ of limit points of F within the reach of their definition. For closed surfaces ($r = 0$) this has been proved in [14] I, § 28; in [15], § 4, another proof will be found, which is valid also for $r > 0$. The structure of $\varkappa$ in $\overline{G}_F$ is easily described by first taking into account the set G_F of fundamental points of F. Any point x of G_F is the positive fundamental point V_f of exactly one primary element f of F. Then x is the positive fundamental point of all elements f^n, $n > 0$, and of no other element of F. Now $\varkappa$ carries the positive fundamental point of any element of F into the positive fundamental point of the element corresponding to it under the automorphism I belonging to $\varkappa$. In symbols:

$$\varkappa\, V_{f^n} = V_{f_I^n}. \tag{2.3}$$

Since G_F is dense in $\overline{G}_F$, the extension of $\varkappa$ to $\overline{G}_F$ derives from continuity, and it is seen that $\varkappa$ is topological in $\overline{G}_F$ and preserves the order of $\overline{G}_F$ on E. Cf. [14] I, § 9, and [15], § 3.

From (2.3) we infer that the transformation $\varkappa\overline{G}_F$ depends on I only. So it does not vary even if $\varkappa$ varies continuously in K_F. So we have: *The transformations $g\varkappa$, $g \subset F$, of the set $\overline{G}_F$ of limit points of F are invariants of the transformation class.* One of these, say $\varkappa$, is sufficient to determine all the others.

3. The group *T*. Another means needed for thorough investigation of a transformation class is iteration of the transformations of the class. Writing the symbol 1 for the identical transformation of F, an enumeration of the elements of F may be given by

$$1,\ f_1,\ f_2,\ \cdots\cdots \text{ in inf.}$$

Writing τ^2 for the iterated transformation $\tau\tau S$, this transformation is covered by $\varkappa^2 = \varkappa\left(\varkappa(x)\right)$ and likewise by all functions of the sequence

$$\varkappa^2,\ f_1\varkappa^2,\ f_2\varkappa^2,\ \cdots\cdots$$

Supposing τ topological in order that τ^{-1} may be defined, this extends to all positive and negative powers of τ, and we get the following scheme written in full:

$$(T)\qquad \begin{array}{c|llll}
\cdots & \cdots & \cdots & \cdots & \cdots \\
[\tau^{-2}] & \varkappa^{-2} & f_1\varkappa^{-2} & f_2\varkappa^{-2} & \cdots \\
[\tau^{-1}] & \varkappa^{-1} & f_1\varkappa^{-1} & f_2\varkappa^{-1} & \cdots \\
[\tau^{0}] & 1 & f_1 & f_2 & \cdots \\
[\tau] & \varkappa & f_1\varkappa & f_2\varkappa & \cdots \\
[\tau^{2}] & \varkappa^{2} & f_1\varkappa^{2} & f_2\varkappa^{2} & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots
\end{array}$$

By applying the functional equation (2.1) it is easily seen that these transformation functions form a group; e. g. we get

$$f_\mu\varkappa^\alpha \cdot f_\nu\varkappa^\beta = f_\mu\left(f_\nu\right)_{I^\alpha}\varkappa^{\alpha+\beta},$$

$$\left(f_\mu\varkappa^\alpha\right)^{-1} = \varkappa^{-\alpha}f_\mu^{-1} = \left(f_\mu^{-1}\right)_{I^{-\alpha}}\varkappa^{-\alpha}.$$

This group will be denoted by the letter T. Moreover F is an invariant subgroup of T, as is seen by applying the functional equation in the form (2.2). The lines of the above scheme (T) stand for the elements of the corresponding factor group T/F.

The transformation functions written in the scheme (T) need not be different. Two functions in the same line are always different, since they differ by an element of F. Let $f_\mu\varkappa^\alpha$ and $f_\nu\varkappa^\beta$ be the same transformation function of K_F. Then $f_\nu^{-1}f_\mu\varkappa^{\alpha-\beta} = 1$ is the identical transformation of K_F. Putting $|\alpha-\beta| = n$, we find that $\varkappa^n$ is an element of F. So τ^n is the identical transformation of S. Since $n \neq 0$, τ is a *transformation of finite order* (a periodic transformation) of S. In this case T/F is cyclic. In case all functions of the scheme (T) are different, T/F is infinite, viz. a free group generated by one element, and τ is not periodic.

The above scheme is, however, capable of another aspect. In consequence of section 2 all functions of the scheme extend to the set $\overline{G}_F$ of limit points of F. Moreover, all functions of the scheme remain topological in $\overline{G}_F$, even if $\varkappa$ by continuous deformation does not remain so in K_F. Therefore the group T always exists as a group of topological transformations of $\overline{G}_F$, and by section 2 it is evident that *the group T defined in $\overline{G}_F$ is an invariant of the transformation class under consideration.*

With this new aspect of the group T we may ask what are the consequences of two elements of the scheme (T) being the same transformation function in $\overline{G}_F$. It means that $\varkappa^n$ for some positive n transforms $\overline{G}_F$ in the same way as a certain element f of F; n may be chosen as the smallest positive number with this property. Hence $f^{-1}\varkappa^n$, which is a function of the line $[\tau^n]$ of (T), leaves all points of $\overline{G}_F$ fixed. The automorphism induced by $f^{-1}\varkappa^n$ therefore is the identical automorphism. This means that $\tau^n S$ belongs to the transformation class of identity, and τ is said to belong to a *transformation class of finite order.* These classes therefore are characterized by the fact that the factor group T/F is cyclic of some finite order n with respect to $\overline{G}_F$ alone irrespective of the behaviour of the transformation functions in K_F.

Transformation classes of finite order have been investigated at length in [15]. The chief result of this paper is that a class of order n contains a transformation which is itself of order n. So one may always choose periodic transformations as representatives of transformation classes of finite order. We take advantage of this fact later on in this paper.

Periodic transformations of surfaces have been the subject of many investigations. Apart from their rôle in the theory of algebraic functions I quote in the present connection papers of L. E. J. Brouwer [3], [4], [5], B. v. Kerékjártó [8], [9], section 6, § 6, W. Scherrer [19], [20], [21], F. Steiger [22] and myself [16]. Later on we will have to enter more fully into the details of such periodic transformations.

4. Principal region and kernel. From what has been said it will be expected that a closer investigation of the behaviour of all transformation functions of the scheme (T) in the set $\overline{G}_F$ of limit points of F will be needed. This investigation has been carried out in [14]II, and we have merely to draw up the results as far as they are required for our present purpose. In one respect a slight addition has to be made: [14] only deals with the case of a closed surface, whereas we here have to take into account the possibility of S being bounded ($r > 0$). This does, however, not affect the validity of the analysis given in [14]. To be short, the difference can be eliminated by first mapping the circumference E continuously on another circumference E' in such a way that all intervals of regularity of E are mapped into single points of E' and the circular order of E is preserved.

With slight variation in formulating the results, the analysis of [14]II may be described in the following way:

Let t denote any element of T and J the automorphism induced by t. If t is unity, all points of $\overline{G}_F$ are left fixed by t, and J is the identical automorphism. We are concerned with the case when t is not unity. Then, in general, the points of a certain true subset M of $\overline{G}_F$ are left fixed by t. M is closed, since t is continuous on $\overline{G}_F$, and M may contain isolated points. As a special case, M may be empty; we have to deal with this case later on. Also, in general, the elements of a certain subgroup N of F are left fixed under the automorphism J. The fundamental points of the elements of N then belong to M and so does the set $\overline{G}_N$ of limit points of N. Inversely, a fundamental point of F belonging to M is a point of $\overline{G}_N$. As a special case, N may consist only of unity, and this may occur even if M is not empty. As another special case, N may be abelian and then consists of all elements of F belonging to a certain axis; in this case, the two fundamental points of this axis are the only fundamental points of F belonging to M. In general, the subgroup N is not abelian and then is of the same character as F itself; the set $\overline{G}_N$ of its limit points then is perfect and contained in M.

Every element of N carries points of M into points of M, thus reproduces M; it also reproduces the complementary set $E - M$.

Since M is a true subset of $\overline{G}_F$, some at least of the intervals forming $E - M$ contain points of $\overline{G}_F$. If i be such an interval, all points of $\overline{G}_F$ inside i are displaced by t in the same direction; this is easily seen to be true even in case $r > 0$, when $\overline{G}_F$ is nowhere dense on E; all intervals of regularity of F play the same rôle as single points. A point P of M will be termed an *isolated* point of M, if P is common end point of two intervals of $E - M$ and both contain points of $\overline{G}_F$. It will be termed *attractive* if the direction of displacement in both intervals goes towards P, *repulsive* if the direction of displacement in both intervals goes from P, and *neutral* if the direction of displacement in both intervals are in accordance on E.

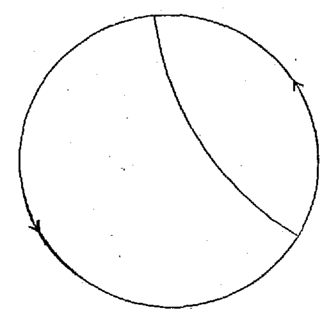

Fig. 1.

If M contains a neutral point, M is made up of exactly two points, which are end points of an axis (fig. 1). The direction of displacement is indicated by arrows.

We now define a subset M^* of M as follows. Let t belong to the line $[\tau^n]$ of the scheme (T). If $n \geq 0$, M^* is derived from M by removing all repulsive isolated points (if any). If $n < 0$, M^* is derived from M by removing all attractive isolated points (if any). (If $n = 0$, we have $t = f \neq 1$, $f \subset F$; then M consists only of the fundamental points U_f and V_f of f. If the axis of f is not a boundary axis of K_F, the former is repulsive, the latter attractive; so $M^* = V_f$.)

M^* is still a closed set. In all cases in which M^* consists of more than one point a convex region may be built on M^* in the same way as was used in deriving the convex region of F from $\overline{G}_F$ in section 1: We remove from X all non-euclidean half-planes bounded by an interval of $E - M^*$. The convex region obtained in this way will be termed the *principal region* of t. In case M^* only consists of two points, the principal region degenerates to a non-euclidean straight line. This may or may not be an axis of F. In case M^* only consists of one point, there is no principal region at all.

Moreover M^* still contains all points of $\overline{G}_N$, with the only exception of the case in which M is made up of two fundamental points (end points of an axis of F) one of which is repulsive the other attractive; in the latter case, M^* only consists of one point, and there is no principal region. So if a principal region exists, it contains the convex region of N, and this then will be termed the *kernel* of t. Two special cases should be noticed: If N only consists of identity, there is no kernel at all. If N is abelian, $\overline{G}_N$ consists of two points only (end points of an axis): if one of these is repulsive the other attractive, there is no principal region and no kernel; if not, they are either both neutral, or M^* consists of more than these two points; the kernel then degenerates to the axis of N.

The mutual situation of principal region and kernel is governed by the fact that points of M (and so of M^*) do not accumulate in the intervals of $E - \overline{G}_N$. Points of M in such an interval (if any) are therefore isolated and alternately attractive and repulsive. If there is a kernel, the end points of such an interval are end points of an axis bounding the kernel. All elements of F belonging to that axis are elements of N. Hence they reproduce M. If therefore the interval contains a point of M, it contains an infinity of points of M accumulating towards the end points of the interval.

The principal region has cuspidal points in all isolated points of M^*.

To sum up, we review the different cases which may occur. In the figures it is assumed that t belongs to a line $[\tau^n]$, $n > 0$, of the scheme (T).

A. N only consists of identity. ($\overline{G}_N$ is empty.)

A_1. M is empty. t leaves no point of $\overline{G}_F$ fixed.

A_2. M is not empty. t leaves a certain number of points of $\overline{G}_F$ fixed. This number is finite, since M does not accumulate on $E - \overline{G}_N = E$. Fig. 2 shows an example, M consisting of six points, three of which are attractive, the

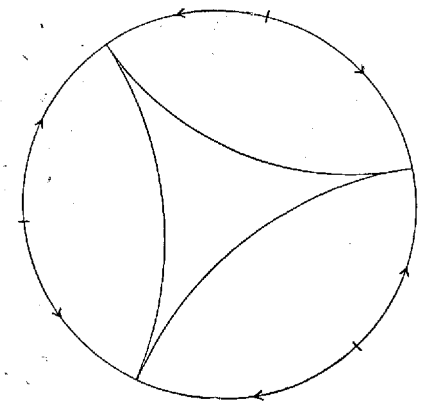

Fig. 2.

others repulsive. There is a principal region but no kernel. If there were but four points in M, the principal region would reduce to a non-euclidean straight line. If there were but two points in M, the subset M^* would reduce to one point, and the principal region would vanish.

B. N is abelian. ($\overline{G}_N$ consists of two points only.)

B_1. M consists of these two points only (the end points of the common axis of all elements of N). If these are neutral points, the axis is both principal region and kernel (fig. 1). If, on the other hand, one is attractive, the other repulsive, M^* consists of one point only, and both kernel and principal region vanish.

B_2. M contains more than these two points. In at least one of the two intervals of $E - \overline{G}_N$ there are cuspidal points of the principal region. These accumulate towards the end points of the axis (fig. 3). This situation may occur in one or in both intervals.

C. N is not abelian. ($\overline{G}_N$ is a perfect set.)

C$_1$. M coincides with $\overline{G}_N$. Both kernel and principal region are made up of the convex region of the group N. Here is embraced the case in which $N = F$, i. e. t is identity.

C$_2$. M contains more points than $\overline{G}_N$. In some of the intervals of $E - \overline{G}_N$ we have a series of cuspidal points of the same kind as in fig. 3. The kernel is the convex region of N and orms a part only of the principal region (fig. 4).

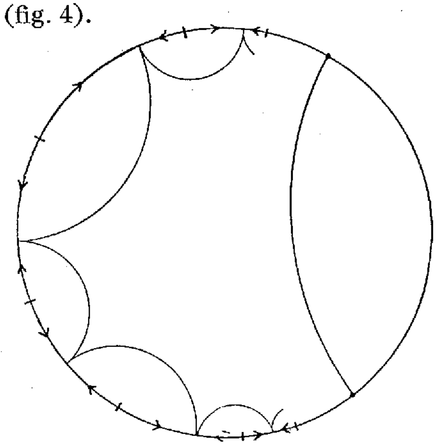

Fig. 3.

From this analysis two numbers may be derived. Let ν denote the minimum number of generators of N. In case **C** we have $\nu > 1$, and N is a free group, if for the present we leave apart the case $N = F$ and S closed. In case **B** we have a free group N with $\nu = 1$ generator. In case **A**, when N is unity, we may agree to call a group consisting of unity only "a free group with $\nu = 0$ generators".

In case **A** the number of cuspidal points of the principal region is finite, say μ. In cases **B** and **C** this number is either zero or infinite, but in the latter case all cuspidal points arise from a certain finite number of cuspidal points by applying the elements of N. So we may speak of μ as the number mod N of cuspidal points.

It is shown in [14] II, that $\nu + \mu$ is limited by some function of the number of connectivity of S. Hence only a finite number

of different types of functions t arise from this analysis. But these considerations are not necessary for our present purpose.

Any fundamental point of F is the positive fundamental point V_f of some element f of F. If V_f belongs to M, the element f is fixed under the automorphism J. So V_f belongs to G_N and hence to $\overline{G}_N$. From this we infer that all cuspidal points of the principal region, which are situated in intervals of $E - \overline{G}_N$, are limit points but not fundamental points of F.

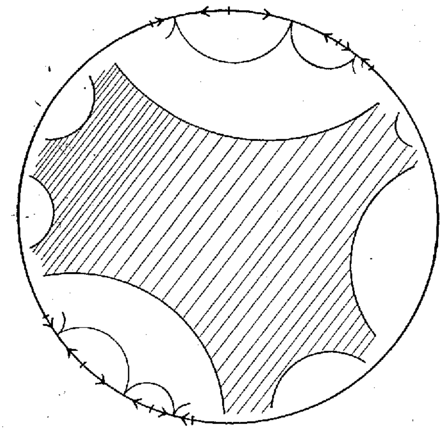

Fig. 4.

5. Classes of fixed points and index. Let us now consider the function t of the last section not only as a transformation function defined in $\overline{G}_F$ but in $\overline{K}_F = K_F + \overline{G}_F$, and let $\tau^n S$ be the corresponding transformation of the surface S. Let Q denote the set of points of K_F left fixed by t (if any). The set Q covers a certain set q of points of S left fixed by τ^n. This set q is called a *class of fixed points* of $\tau^n S$. There may be other fixed points of $\tau^n S$ than the set q, since other functions of the scheme (T) in the line $[\tau^n]$ of t, functions of the form ft, $f \subset F$, may yield a class. For the moment we are concerned with the class q only. Let x_0 be some point of Q, thus $tx_0 = x_0$. What about fx_0, $f \subset F$? By (2.1) we get immediately

$$tfx_0 = f_J t x_0 = f_J x_0.$$

So if $f = f_J$, the point fx_0 belongs to Q, and if $f \neq f_J$, it does not. Hence Q is reproduced by the subgroup N of F and by no other element of F.

Let ψ denote a fundamental region of N in K_F, i. e. a region containing exactly one point of each set Nx, $x \subset K_F$. Then the class q will be covered exactly once by the part of Q belonging to ψ. Let us assume ψ to be so chosen as to be bounded by a simple closed curve c of K_F not meeting Q. If x is a point of c, let $\varphi(x)$ be the point of E in which a non-euclidean ray from x through tx meets E. As x describes c once in a positive sense, $\varphi(x)$ will in all make a certain number $j \gtreqless 0$ of tours of E. This number j will be called the *index of the class* q in accordance with the common use of the term "index"; see for instance J. W. ALEXANDER [1], section 2, or S. LEFSCHETZ [12], p. 276: if q happens to be made up of a finite number of isolated fixed points, j is the sum of indices attributed to these single points.

This number j is computed in [14] II, its value being (with one exception assigned below)

$$j = j(t) = 1 - \nu - \mu, \tag{5.1}$$

thus only depending on the numbers ν and μ attributed to t in the preceding section. If the structure of Q be such as not to allow ψ to be so chosen as to be bounded by a simple curve avoiding Q, this may be achieved by a slight variation of t. So we fix the value (5.1) to be the index of q in all cases.

The exceptional case referred to above is a very special one well known from the homology theory of transformations: If S is closed and t belongs to the transformation class of identity, one gets

$$j = 2 - 2p, \tag{5.2}$$

while in that case $\mu = 0$ and $\nu = 2p$. This is the well known formula of BIRKHOFF [2]. The explanation of this difference is simple: In all cases which are not the BIRKHOFF case, the determination of the index of a class of fixed points takes into account an auxiliary surface (so to speak), viz. the surface corresponding to the principal region of t; and this surface in all these cases is not closed. See also [14] II, § 16.

We now examine the different possible values of j in (5.1). Since $\nu \geqq 0$ and $\mu \geqq 0$ the greatest possible value is $j = 1$ arising from $\nu = \mu = 0$. This is the case $\mathbf{A}_1$ of the previous section. The next section will be devoted to closer investigation of this case.

Then we may have $j = 0$. That requires either $\nu = 0$, $\mu = 1$, or $\nu = 1$, $\mu = 0$. If $\nu = 0$, $\mu = 1$, we are in case $\mathbf{A}_2$ with $2\mu = 2$ points in M, and no kernel nor principal region exists. If $\nu = 1$, $\mu = 0$, we are in case $\mathbf{B}_1$. If one point of M is attractive, the other repulsive, we have no kernel nor principal region. If both are neutral, we are in the case of fig. 1 with an axis of F both as kernel and principal region.

Finally we may have $j < 0$, thus $\nu + \mu > 1$. Then we are in the cases $\mathbf{A}_g$ with more than two points in M, $\mathbf{B}_2$ or $\mathbf{C}$. In all these cases there is a principal region, and if $\nu > 0$ there is a kernel.

It goes without saying that t must leave at least one point of K_F fixed, if $j \neq 0$. If $j = 0$, it is not decided whether t leaves some point of K_F fixed or not.

In consequence of the limitation of $\nu + \mu$ mentioned in section 4 there is a limit to the possible negative values of indices of classes of fixed points on a surface of given connectivity. We do not state this limit explicitly, as we do not need it for our purpose.

6. Index $j = 1$. If the set M of fixed points of $\overline{G}_F$ under the transformation t is not empty, let us examine any interval of $E - M$ containing points of $\overline{G}_F$. These are displaced by t in a definite direction common to all points of $\overline{G}_F$ in the interval, only the end points remaining fixed. By the powers $t^2, t^3, \cdots$ this displacement is increased, thus no new fixed point can arise. Hence $t^2, t^3, \cdots$ have the same set M of fixed points as t. The same is true of $t^{-1}, t^{-2}, \cdots$, in which cases the displacement goes in the opposite direction. Moreover, isolated points of M which are repulsive for t are repulsive for $t^2, t^3, \cdots$ and attractive for $t^{-1}, t^{-2}, \cdots$. In view of the definition of M^* in section 4, this set M^* therefore is common to all powers of t (except t^0, of course), and so are the principal region, the kernel, the numbers ν and μ and hence the index j. We thus infer that *an index $j \leqq 0$ of a class of fixed points is stable with respect to iteration of the transformation.*

As to an index $j = 1$, things are different. In this case M is empty, so the successive displacements of a point of $\overline{G}_F$ under the powers of t may eventually carry it back to its original position. In fact, in [14] III, § 1, it has been shown that some power t^n, $n > 0$, of t will have a set M of fixed points, which is not empty, in $\overline{G}_F$. It is true that in the proof given in [14] III the surface was assumed closed, but, as in previous cases, the extension to bounded surfaces is immediate, if one makes the intervals of regularity play the rôle of single points.

Let n be the smallest positive number for which t^n leaves some points of $\overline{G}_F$ fixed, and let M denote the set of these points. Since M is not empty, we have $j(t^n) \leqq 0$. Let P be any point of M and $P' = tP$ its image under the transformation t. Then

$$t^n P' = t^n t P = t t^n P = tP = P',$$

hence $P' \subset M$, and M is reproduced by t. Moreover, since $t \cdot t^n \cdot t^{-1} = t^n$, an isolated attractive point of M is carried into an isolated attractive point of M, a repulsive into a repulsive, a neutral into a neutral. Let N be the subgroup of fixed elements belonging to t^n. Since fundamental points are transformed by t into fundamental points, it is seen that G_N and hence $\overline{G}_N$ is reproduced by t. We now examine the different cases of section 4:

A. If t^n is of type $\mathbf{A}_2$, M consists of 2μ points, μ of which are attractive. Since the attractive points of M are interchanged by t, we must have $\mu \geqq 2$. Hence there is a principal region for t^n (in case $\mu = 2$ degenerating into one straight line) and, speaking symbolically, this principal region is "rotated" in itself by t.

B. If t^n is of type **B**, M contains two fundamental points, and these are interchanged by t, so $n = 2$. In case $\mathbf{B}_1$ these two points clearly must be neutral. So we are in the case of fig. 1 and, symbolically, the axis, which is at the same time principal region and kernel, is "reversed". In case $\mathbf{B}_2$ the two intervals are interchanged, so the two parts of the principal region separated by the axis (kernel) must have the same number mod N of cuspidal points.

C. In case **C** we merely notice that, symbolically, the principal region of t^n is "rotated" in itself by t, and so is the kernel of t^n.

Thus it is seen that t^n has in all cases a principal region, and we may say that t, though it has not itself a principal region, is *affiliated to the principal region of t^n*. This is of course the principal region of all powers of t^n too. In the same way in cases **B** and **C**, t^n has a kernel, and we may say that t is *affiliated to that kernel.*

So the index of a class of fixed points with index $j = 1$ is not stable with respect to iteration of the transformation. It is affiliated to a class of fixed points with index $j \leqq 0$ of some power of the transformation. It should be noticed that several distinct classes with index $j = 1$ may well be affiliated to one and the same class with index $j \leqq 0$ of some power of the transformation.

7. Simple axes. Equivalence classes and congruence classes.

In the first part of this section we consider T as a group of transformations of $\overline{G}_F$ only, thus abstract from the rôle of the elements of T as transformation functions of the convex region K_F of F.

Let A be any axis of F, f an element of F belonging to A, thus U_f and V_f the end points of A. Let t be any element of T and J the corresponding automorphism. Then t takes U_f and V_f into the points U_{f_J} and V_{f_J} respectively, i. e. into the end points of the axis belonging to the element f_J corresponding to f under the automorphism J. We denote this axis by tA, thus speaking *purely symbolically* of it as the image of A under the transformation t. Making t range over the whole group T we get a totality of axes, denoted by TA and termed the *equivalence class of A with respect to T.*

If TA satisfies the condition that any two axes of TA are either identical or have no point in common (thus do not intersect), A and so any axis of TA will be termed *simple with respect to T.* Examples are obvious: If the surface S is bounded, an axis bounding the convex region K_F of F cannot be crossed by any other axis of F; so it is simple with respect to T.

More generally, if A does not mean an axis of F, but merely a non-euclidean straight line joining two points of $\overline{G}_F$, the straight line joining the images of these to points under t is denoted by tA. Then the same considerations may be applied and simplicity of A with respect to T defined.

Moreover, if T' is any subgroup of T, the meaning of the denotation $T'A$ and of simplicity with respect to T' is immediate. So if F is taken as subgroup of T, simplicity of A with respect to F means that the geodesic a of S corresponding to A does not intersect; if A is an axis, a is a simple closed geodesic (without double points).

Simplicity with respect to T involves simplicity with respect to any subgroup of T, but not vice versa. So if A is simple with respect to T, a is a simple geodesic; if a is a simple geodesic, A is simple with respect to F, but A may be intersected by some tA, $t \subset T$ but not in F, and so need not be simple with respect to T.

Let t_1 and t_2 be two elements $\neq 1$ of T and let a principal region, say $\Omega(t_1)$ and $\Omega(t_2)$, exist for both of them. It is first assumed that t_1 and t_2 belong to the same line of the scheme (T), say to $[\tau^n]$. Then $n \neq 0$, since there is no principal region for an element $\neq 1$ of F. The mutual situation of $\Omega(t_1)$ and $\Omega(t_2)$ in K_F then is one of the following three cases:

1) they are identical,
2) they have no point in common,
3) they have one axis of F in common.

In case 3) both functions of course have a kernel, say $\Gamma(t_1)$ and $\Gamma(t_2)$, the axis is a bounding axis for both of these, and $\Omega(t_1)$ and $\Omega(t_2)$ and so $\Gamma(t_1)$ and $\Gamma(t_2)$ are contiguous along that axis.

The proof of this theorem is to be found in chapter 3 of [14] II with a slight modification: The paper quoted speaks of a principal region $\Omega(t)$ only if t leaves more than two points of $\overline{G}_F$ fixed, whereas we here include the case (fig. 1) of t having exactly two fixed points, these being neutral. It will, however, easily be seen that this case exactly fits into the proof too.—The situation met with later in the present paper makes our present, broader definition of the concept of principal region $\Omega(t)$ necessary. In fact $\Omega(t_2)$, say, may happen to be such an axis as in fig. 1, thus at the same time being $\Gamma(t_2)$, and may coincide with a boundary axis of $\Omega(t_1)$ (and so of $\Gamma(t_1)$).

The above assumption of t_1 and t_2 belonging to the same line of (T) is readily seen to be superfluous. In fact, since $\Omega(t_1)$ and $\Omega(t_2)$ exist, these regions are, as pointed out in section 6, principal regions of all powers of t_1 and t_2 respectively. So in choosing a line of (T) containing a power of t_1 and of t_2 simultaneously and applying the above theorem we get the same three possibilities of the mutual situation of $\Omega(t_1)$ and $\Omega(t_2)$.

So if we let t range over all elements $\neq 1$ of T for which a principal region $\Omega(t)$ exists, the totality of these regions $\Omega(t)$ has the property that any two of its members do not intersect.

Now, let t and t_1 be any two elements of T. We consider t_1 together with the conjugate element

$$t_2 = t t_1 t^{-1}.$$

It is obvious that any point of $\overline{G}_F$ left fixed by t_1 is carried by t into a point left fixed by t_2. So $M(t_2)$ is the image of $M(t_1)$ by t. As the character of being isolated, attractive etc. is preserved for the points of M, and t_1 and t_2 belong to the same line of the scheme (T) (thus $n \geqq 0$ or $n < 0$ for both in the definition of M^*) we also have $M^*(t_2) = tM^*(t_1)$. We may therefore say symbolically that $\Omega(t_2)$ is the image of $\Omega(t_1)$ by t and write

$$\Omega(t_2) = t\,\Omega(t_1).$$

The same is valid for the kernels, if any. Making t range over the whole of T we get an *equivalence class* $T\Omega(t_1)$ of principal regions and $T\Gamma(t_1)$ of kernels. All functions tt_1t^{-1} are said to form an *equivalence class of functions.*

We now infer that any non-euclidean straight line, whether axis of F or not, bounding some principal region, is simple with respect to T; this follows at once from the above theorem that two different principal regions do not intersect. Especially such a bounding line is simple with respect to F. Thus the boundaries of a region of the surface corresponding to a principal region are simple geodesics.

Since an equivalence class of principal regions is part of the totality of all such regions, any two of its members are mutually situated in one of the three possible ways indicated above. We may express this property by saying that *any principal region is simple with respect to* T.

We return once more to the comparison of two conjugate elements of T such as t_1 and $t_2 = tt_1t^{-1}$. If the element $f \subset F$ is left fixed by t_1,

$$t_1 f t_1^{-1} = f,$$

the element tft^{-1}, which also belongs to F, is left fixed by t_2. So we may write

$$N(t_2) = tN(t_1)\, t^{-1}.$$

$N(t_1)$ and $N(t_2)$ need not be conjugate subgroups of F, since t need not belong to F. But they are isomorphic, so they have the same number of generators. Hence

$$\nu(t_1) = \nu(t_2).$$

From this isomorphism and from the homeomorphism of $M^*(t_1)$ and $M^*(t_2)$, thus of $\Omega(t_1)$ and $\Omega(t_2)$, we infer that moreover

$$\mu(t_1) = \mu(t_2).$$

So we find that indices are the same:

$$j(t_1) = j(t_2).$$

We now again take T as a group of transformations of the closed convex region $\bar{K}_F$ of F.

Any two transformation functions conjugate in T such as t_1 and t_2 yield classes of fixed points with the same index j, as we have just seen. Are these classes different classes of fixed points of the surface transformation in question?

To decide this, let t be a transformation function, J the corresponding automorphism and x_0 a point of K_F left fixed by t. No other function of the line of t in the scheme (T) can leave x_0 fixed, since it differs from t by an element of F. Now let $f \neq 1$ be any element of F. The function $ftf^{-1} = ff_J^{-1}t$, which belongs to the line of t, evidently leaves fx_0 fixed; so no other function of the line of t can leave fx_0 fixed. If Q denotes the set of fixed points of t, then fQ is the set of fixed points of ftf^{-1}. So two functions t_1 and t_2 yield the same class of fixed points, if one is transformed into the other by an element of F, and in that case only.

All functions ftf^{-1}, t being a fixed element of T and f ranging over the entire group F, will be termed a *congruence class of functions* and the corresponding set $F\Omega$ of principal regions a *congruence class of principal regions*. Two functions such as t and $ftf^{-1} = ff_J^{-1}t$ are said to be *congruent*[1].

Any equivalence class is subdivided into congruence classes.

In looking for classes of fixed points of a surface transformation τS, only one representative of each congruence class of functions in the line $[\tau^1]$ has to be examined. These are still in infinite number, but only a finite number of them yield a class of fixed points which is not empty; this is shown in [14] I, § 32.

The principal regions of all functions ftf^{-1} of a congruence class are the images of $\Omega(t)$ by the transformations $f \subset F$. They all cover one and the same region of the surface S.

Part II.

Transformation classes of algebraically finite type.

8. Definition of the transformation classes concerned. The matter of part I was an outlining in brief of the general foundations of the theory of surface transformation classes needed for establishing the main invariants of such classes. We now proceed to the chief subject of this paper, a full investigation of all transformation classes for which *principal region and kernel coincide for every element of T for which a principal region exists.*

The different types of functions t are analyzed at the end of section 4. To make things clear, let us look at the summary of that section with the present assumptions. Cases $\mathbf{B}_1$ and $\mathbf{C}_1$ are clearly in accordance with the present assumptions, whereas $\mathbf{B}_2$ and $\mathbf{C}_2$ are excluded: the existence of cuspidal points makes the principal region contain more than the kernel. In case $\mathbf{A}_2$ there can only be two points in M, one attractive and one repul-

[1] In [14] the term "isogredient" has been used.

sive, since for more than two points in M there would be a principal region without a kernel. Case $\mathbf{A}_1$ is permitted, but it has to be remembered that such a function is affiliated to a principal region corresponding to a certain power of the function, and this principal region must then belong to one of the types permitted.

The characterization of the transformation classes in question may also be put thus: The number μ, defined in section 4, is zero for all elements of T except for the type $\nu = 0$, $\mu = 1$, which is permitted, as it has no principal region.

One may ask whether transformation classes satisfying this condition are to be found at all. It is readily seen that all transformation classes of finite order are embraced. For such classes T/F is a cyclic group of some order n, as pointed out in section 3. It has been shown in §§ 6, 7 of [15], that any element $\neq 1$ of such a group T is *either* of type $j = 0$ with $\nu = 1$, $\mu = 0$, one end point of the axis being attractive, the other repulsive, and so has no principal region, *or* of type $j = 1$ ($\nu = \mu = 0$). In the latter case the principal region to which the element is affiliated is the whole of K_F, as the n-th power of the element is identity. So for the entire group T the convex region K_F is the only existing principal region and obviously is kernel too. Hence the condition imposed above is fulfilled.

We are thus concerned with a rather far-reaching generalization of transformation classes of finite order, for which the term *algebraically finite classes* is proposed for reasons which will be mentioned at the end of this paper.

9. Existence of simple axes. In the rest of this paper τS means a *class of surface transformations*, the corresponding group T of which, *defined in* $\overline{G}_F$, satisfies the condition of section 8.

We first ask whether axes which are simple with respect to T, thus are not intersected by any of their equivalents under T, exist. If S is bounded, the axes forming the boundary of K_F in X are simple with respect to T. So let S be closed. We then use the following theorem: For some power of τ, at least, the algebraic sum of the indices of all fixed points of any transformation belonging to the class is $\neq 0$. This theorem has been proved in [17] by purely algebraic means of homology theory, using J. W. Alexander's theorem [1] concerning the sum of indices; see also [18]. Now since the sum of indices of all fixed points is equal to the sum of indices of all different classes of fixed points, there must exist some function $t \neq 1$ in T, belonging to τ or to some power of τ, the index of which is $j(t) \neq 0$.

First let $j(t) < 0$. Then we have $1 - \nu - \mu < 0$; now $\mu = 0$, since in the only exceptional case $\nu = 0$, $\mu = 1$, we have $j = 0$. Hence $\nu > 1$ and we are in case $\mathbf{C}$ of section 4. More precisely, t belongs to the type $\mathbf{C}_1$, since $\mu = 0$ (no cuspidal points).

Then let $j(t) = 1$. As described in section 6, t is affiliated to the principal region of some power t^n. Now t^n must be of one of the types enumerated in section 6. Of these, $\mathbf{A}_2$, $\mathbf{B}_2$ and $\mathbf{C}_2$ clearly cancel on our present assumptions. In case $\mathbf{B}_1$ we only have the situation of an axis with neutral end points (fig. 1), and this axis is kernel. In $\mathbf{C}_1$ we have a kernel too.

To sum up, under the conditions imposed upon the transformation class, the group T contains an element which has a kernel (here coinciding with the principal region). Now if a kernel is bounded, a bounding axis of the kernel (which may make up the entire kernel, case $\mathbf{B}_1$) is simple with respect to T, as shown in section 7.

So we are left with the last possibility of a kernel belonging to some element of T and coinciding with the complete convex region K_F, which for a closed surface is the whole circular disc X. In that case the element 1 belongs to some line $[\tau^n]$, $n \neq 0$, of the scheme (T), and this means that τ is a transformation class of finite order (section 3).

As transformation classes of finite order are fully investigated in [15], we may here assume that τ is not of this special nature. So we have established that an axis, simple with respect to T, always exists for transformation classes of the kind under consideration.

To be precise, even for transformation classes of finite order such simple axes do exist in general. As pointed out in [15], there is only one extremely special case of a transformation class of finite order for which axes simple with respect to T do not exist; see [15], § 23.

10. Decomposition of S by a maximum system of geodesics. Let A_1 be any axis simple with respect to T. We consider the totality TA_1 forming its equivalence class (section 7). As A_1 is simple with respect to the subgroup F of T, it corresponds to a simple closed geodesic a_1 on S, and so do all axes of TA_1. As any two of these simple closed geodesics do not intersect and S is of finite connectivity, their number is finite. For the groups F and T this has the following bearing. $\varkappa$ being chosen as a transformation function of $\overline{G}_F$ corresponding to τS, we recall, that $\varkappa A_1$ is meant symbolically to denote the axis the end points of which are the images of the end points of A_1 under the transformation $\varkappa$. Moreover, if an orientation is assigned to A_1 by taking its end points in a definite order, this is transferred as a definite orientation of $\varkappa A_1$. Then in the sequence

$$(10.1) \qquad A_1, \varkappa A_1, \varkappa^2 A_1, \cdots$$

there is a first axis, $\varkappa^{\alpha_1} A_1$ say, corresponding to A_1 by an element of F,

$$\varkappa^{\alpha_1} A_1 = fA_1, \quad f \subset F,$$

orientation included. If α_1 is an even number, it may happen that $\varkappa^{\frac{\alpha_1}{2}} A_1$ corresponds to A by an element of F with orientation reversed; in that case A_1 will be termed an *amphidrome axis.*

On the surface S we may denote symbolically by

$$(10.2) \qquad a_1, \tau a_1, \tau^2 a_1, \cdots$$

the simple closed geodesics corresponding to the axes (10.1), although the real image of a_1 by any special transformation of the class τ does not, in general, coincide with the geodesic covered by the axis $\varkappa A_1$, but is only homotopic to it.

With this denotation we get α_1 simple closed geodesics without common points

$$a_1, \tau a_1, \tau^2 a_1, \cdots, \tau^{\alpha_1 - 1} a_1$$

on S from (10.2) in case A_1 is *not amphidrome,* whilst

$$\tau^{\alpha_1} a_1 = a_1$$

with orientation preserved. These geodesics are covered by the axes

$$A_1, \varkappa A_1, \varkappa^2 A_1, \cdots, \varkappa^{\alpha_1 - 1} A_1.$$

The equivalence class TA_1 then is made up of α_1 *congruence classes*

$$FA_1, F\varkappa A_1, F\varkappa^2 A_1, \cdots, F\varkappa^{\alpha_1 - 1} A_1.$$

(Of course, α_1 may be 1.) If on the other hand A_1 is *amphidrome,* we get $\frac{\alpha_1}{2}$ geodesics

$$a_1, \tau a_1, \cdots, \tau^{\frac{\alpha_1}{2} - 1} a_1,$$

whilst $\tau^{\frac{\alpha_1}{2}} a_1$ coincides with a_1 with orientation reversed. This requires $\alpha_1 \geqq 2$ and even. The equivalence class of A_1 is made up of $\frac{\alpha_1}{2}$ congruence classes irrespective of orientation.

Let A_2 be an axis simple with respect to T, not comprised in TA_1 and not crossing any axis of TA_1, if any such A_2 exists. Then TA_1 and TA_2 have no point in common and, α_2 denoting the number analogous to α_1, we have on S in all $\alpha_1 + \alpha_2$ simple closed geodesics without common points, in case both A_1 and A_2 are not amphidrome, and otherwise a smaller number. Then we may look for a third axis A_3 and so on. This process comes to an end in a finite number of steps in view of the finite connectivity of S. Let

$$(10.3) \qquad A_1, A_2, \cdots, A_\xi$$

be the *maximum system* finishing the process. This system then has the following property:

1) In the set of axes

$$(10.4) \qquad TA_1 + TA_2 + \cdots + TA_\xi$$

any two axes do not intersect.

2) Any axis simple with respect to T and not comprised in the set (10.4) crosses at least one axis of this set.

The numbers $\frac{\alpha_i}{2}$ or α_i denoting the number of congruence classes in TA_i, according as A_i is or is not amphidrome,

$i = 1, 2, \cdots, \xi$, we have on S in all, at most, $\alpha_1 + \alpha_2 + \cdots + \alpha_\xi$ simple closed geodesics without common points. These divide S into a certain number $\geqq 1$ of parts.

In the case of a bounded surface it may be noticed that in whatever way the system (10.3) is chosen, any bounding axis of K_F is comprised in the set (10.4). Otherwise it would contradict condition 2. Moreover, a bounding axis of K_F cannot be amphidrome, since one of the intervals determined by it on E does not contain points of $\overline{G}_F$.

The set (10.4) of axes clearly is reproduced by any element $t \subset T$. Moreover, the arrangement of these axes in K_F is preserved, since t preserves the circular order of the points of $\overline{G}_F$. So the division of K_F by the set (10.4) is reproduced under t: Let B be any region of that division. The boundary of B inside X is made up of a subset of (10.4). If A, A', A'' are any three axes of this subset, none of them separates the two others. If A is an axis not in the subset, axes A' and A'' of the subset may be so chosen that A' separates A and A''. As these separating qualities are preserved by t, the subset of (10.4) into which the subset bounding B is transformed by t, is the boundary of some region of the division. This region will be denoted symbolically by tB. So we may form the concept of equivalence class TB and of congruence class FB of B, and split any equivalence class into congruence classes. In the sequence

$$B,\ \varkappa B,\ \varkappa^2 B,\ \cdots\cdot$$

let $\varkappa^\beta B$ be the first to be congruent to B:

$$\varkappa^\beta B = gB,\quad g \subset F. \qquad (\beta \geqq 1)$$

Then TB is made up of β congruence classes

$$FB,\ F\varkappa B,\ F\varkappa^2 B,\ \cdots,\ F\varkappa^{\beta-1} B. \tag{10.5}$$

If in the division of S corresponding to the division of K_F by the set (10.4) the part covered by B is denoted by b,

$$b,\ \tau b,\ \tau^2 b,\ \cdots,\ \tau^{\beta-1} b \tag{10.6}$$

will be β different parts, each covered by one of the congruence classes (10.5) of regions of K_F.

11. Transformation class of finite order in the single regions. We now consider some definite region B of the division of K_F by the set (10.4) and denote by T_B the subgroup of T and by F_B the subgroup of F carrying B into itself. F_B is the Poincaré group of the part b of S covered by B, and B is the convex region of the group F_B. If β is the least positive number such that

$$\varkappa^\beta B = gB,\quad g \subset F,$$

we put

$$g^{-1}\varkappa^\beta = \varkappa_B.$$

So $\varkappa_B B = B$. It should be noticed that g is not unique but may be replaced by gf_B, f_B being any element of F_B. Then $\varkappa_B$ is replaced by $f_B^{-1}\varkappa_B$. This replacement has no influence on the following argument.

Elements of T_B are to be found in those lines of the scheme (T) which contain powers of $\varkappa_B$. Moreover, F_B is clearly invariant in T_B, and we get

$$T_B = F_B + F_B\varkappa_B + F_B\varkappa_B^2 + \cdots$$
$$+ F_B\varkappa_B^{-1} + F_B\varkappa_B^{-2} + \cdots$$

each of these parts of T_B being contained in some line of the scheme (T). If we assume the factor group T/F to be infinite, τ not being a transformation class of finite order, even the factor group T_B/F_B will be infinite.

One may, however, regard T_B merely as a group of transformations of the set $\overline{G}_{F_B}$ of limit points of F_B; that is the set of all those boundary points of B which are situated on E. Then T_B defines a transformation class τ_b (and all its powers) of the surface $b = B \bmod F_B$ just in the same way as T, given in $\overline{G}_F$, defines a transformation class τ (and all its powers) of the surface $S = K_F \bmod F$. In this respect, an element of T_B being unity only means that it leaves fixed all points of $\overline{G}_{F_B}$ regardless of its behaviour in the rest of $\overline{G}_F$. So in this view the question as to the order of the factor group T_B/F_B arises anew.

To make this important point quite clear, we may put it in the following way in the language of group theory. Let T_{0B} denote the subset of T_B the elements of which transform $\overline{G}_{F_B}$ in the same way as some element of F_B, i. e. T_{0B} consists of all such elements t of T_B to which an element f of F_B exists, making the element ft leave all points of $\overline{G}_{F_B}$ fixed. This subset T_{0B} forms a group: Let t_1 and t_2 belong to T_{0B} and f_1 and f_2 be the corresponding elements of F_B. Then, if J denotes the automorphism induced by t_1,

$$f_1 t_1 \cdot f_2 t_2 = f_1 f_{2J}\, t_1 t_2$$

leaves all points of $\overline{G}_{F_B}$ fixed and $f_1 f_{2J} \subset F_B$; hence $t_1 t_2 \subset T_{0B}$. Moreover T_{0B} is invariant in T_B: Let $t_1 \subset T_{0B}$ with f as corresponding element, $t \subset T_B$ with J as corresponding automorphism and the point $P \subset \overline{G}_{F_B}$. What is the effect of tt_1t^{-1} upon P? Put $t^{-1}P = P_1 \subset \overline{G}_{F_B}$. Then $t_1 P = f^{-1} P_1$. Finally $tf^{-1} P_1 = f_J^{-1} tP_1 = f_J^{-1} P$. Hence the element $f_J tt_1t^{-1}$ leaves every point P of $\overline{G}_{F_B}$ fixed, and $f_J \subset F_B$. Thus $tt_1t^{-1} \subset T_{0B}$.

The invariant subgroup T_{0B} of T_B evidently contains F_B, but it may contain more. So the corresponding factor group T_B/T_{0B} may well become finite, even if T_B/F_B is not. We set out to prove that it actually does.

We first establish the fact, that *the transformation class τ_b of b satisfies the condition imposed on the class τ of S in section 8,*

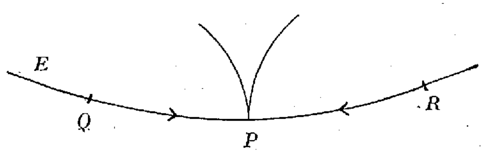

Fig. 5.

viz. that every principal region coincides with its kernel. In fact, assume t to be an element of T_B the principal region of which has a cuspidal point $P \subset \overline{G}_{F_B}$. Let P be attractive, and let Q and R be the neighbouring repulsive points of $\overline{G}_{F_B}$; see fig. 5. All points of $\overline{G}_{F_B}$ contained in the segment QPR are displaced

towards P by t except Q, P and R, which are left fixed. Moreover, Q, P and R are accumulation points of such points of $\overline{G}_{F_B}$. Now let i be any interval of $E - \overline{G}_{F_B}$ belonging to the segment QPR. The end points of i belong to $\overline{G}_{F_B}$, but are not Q, P or R, since these points are accumulation points of $\overline{G}_{F_B}$ on either side. So the end points of i are displaced into the end points of some other interval ti nearer to P. If then i contains points of $\overline{G}_F$ (not in $\overline{G}_{F_B}$) and we regard t as a transformation function in the whole of $\overline{G}_F$, these points are not left fixed by t. So even as we regard t as an element of T, the points Q and R are fixed points neigbouring P and the corresponding principal region would have P as a cuspidal point.—If P is not a cuspidal point, properly speaking, but t leaves only four points of $\overline{G}_{F_B}$ fixed, say P and P' attractive and Q and R repulsive, the principal region degenerates into a simple non-euclidean straight line; but this then is seen by the same argument to be principal region even for t as an element of T in contradiction to the assumptions made in section 8.—So the proof is complete.

Since B is bounded, F_B is a free group with a certain minimum number ν, say, of generators. *Then at least one of the transformation classes*

$$\tau_b,\ \tau_b^2,\ \cdots,\ \tau_b^\nu$$

of b yields a class of fixed points the index of which is not zero. This is seen by the argument of [17], p. 202—203[1].

[1] As the proof given in the paper quoted only takes closed surfaces into account, we shortly indicate the modification required, preserving the notations of the paper quoted: The algebraic sum of the indices in the r-th power of the transformation class is $1 - s_r$ instead of $2 - s_r$, since the surface is bounded. If this sum were to be zero for $\tau_b, \tau_b^2, \cdots, \tau_b^\nu$; we get

$$s_1 = s_2 = \cdots = s_\nu = 1.$$

Now using equations (1), (2), $\cdots$, (ν) of the paper quoted, we get in turn

$$\begin{aligned} a_1 &= -1 \text{ from (1)} \\ a_2 &= 0 \text{ from (2)} \\ &\cdots\cdots\cdots \\ a_\nu &= 0 \text{ from } (\nu). \end{aligned}$$

But a_ν is the determinant of the matrix Γ and so is $\neq 0$. This completes the proof.

Now, by section 9, the existence of an element of T_B the index of which is not zero involves the existence of a kernel. Such a kernel must be bounded, since B is bounded. Let A be an axis of F_B bounding some kernel. Then by section 7, A is simple with respect to T_B. From this we infer that A is simple with respect to T. Indeed, let t be any element of T. If t belongs to T_B, tA does not cross A, since A is simple with respect to T_B; if t does not belong to T_B, it takes B into some other region tB not intersecting B, so $A \subset B$ and $tA \subset tB$ do not intersect. Hence A is simple with respect to T. If A were interior to B, i. e. not a bounding axis of B, it would not belong to the set (10.4) and so would contradict condition 2, which characterizes (10.4) as a maximum set. So the kernel in question must coincide with B. This means that the unity element of T_B occurs in some set $F_B \varkappa_B^n$, $n \neq 0$, and so T_B/F_B is cyclic of some finite order n_B, if these groups are only considered in $\bar{G}_{F_B}$. In other words, using the above notation of the subgroup T_{0B} defined in $\bar{G}_F$, the factor group T_B/T_{0B} is cyclic of order n_B.

Hence the transformation class of $b = B \bmod F_B$ defined by the element $\varkappa_B$ is a class of finite order.

12. Screw numbers. Let A be any oriented axis of the set (10.4) and f_A the primary element of F belonging to A, so all powers of f_A forming the subgroup F_A of F with A as axis. Let T_A^* denote the subgroup of T the elements of which leave A fixed, orientation included, thus including the end points of A in their set of fixed points in $\bar{G}_F$. This group T_A^* may be found in a similar way as the group T_B of the preceding section: In the sequence analogous to (10.1)

$$A,\ \varkappa A,\ \varkappa^2 A, \cdots$$

we determine the least positive number α for which

$$\varkappa^\alpha A = fA, \quad f \subset F,$$

orientation included, and then put

$$f^{-1}\varkappa^\alpha = \varkappa_A,$$

hence $\varkappa_A A = A$. (The element f is not unique but may be replaced by ff_A^n for any n, thus replacing $\varkappa_A$ by $f_A^{-n}\varkappa_A$. Compare the corresponding remark as to $\varkappa_B$ in the preceding section.) Then we get

$$T_A^* = F_A + F_A\varkappa_A + F_A\varkappa_A^2 + \cdots$$
$$+ F_A\varkappa_A^{-1} + F_A\varkappa_A^{-2} + \cdots$$

Denoting by T_A the subgroup of T the elements of which carry A into itself irrespective of orientation, we have $T_A = T_A^*$,

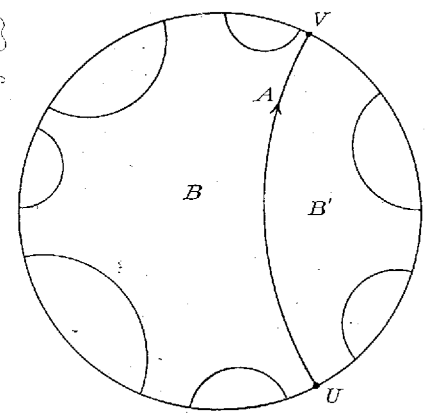

Fig. 6.

if A is not amphidrome, whereas for an amphidrome A there exist elements of T reversing A, and then T_A^* is an (invariant) subgroup of index 2 in T_A. As will be remembered, an amphidrome axis is not boundary axis of K_F.

We now assume A to be an *inner axis* (not boundary axis) of K_F.

Let B and B', (fig. 6[1]), denote the two regions contiguous to A in the division of K_F by the set (10.4), B on the left hand side of A, say. Let the numbers β and β' belong to B and B' respectively in the sense of the preceding section. Clearly T_A^* is a subgroup both of T_B and $T_{B'}$. The elements of T_A^* are contained in those lines $[\tau^n]$ of the scheme (T) for which n is a multiple of α, the elements of T_B and $T_{B'}$ in the lines for which n is a

[1] The figure is schematic. The number of bounding arcs of B and B' is of course infinite.

multiple of β and β' respectively. Hence α is a common multiple of β and β'. On the other hand, let n_B and $n_{B'}$ denote the order of the transformation class of finite order assigned in the preceding section to B and B' respectively. Then the line $[\tau^{n_B\beta}]$ of (T) contains an element leaving all boundary points of B on E fixed, thus belonging to T_A^*. Hence α divides $n_B\beta$. Likewise α divides $n_{B'}\beta'$.

Denoting by L the least common multiple of $n_B\beta$ and $n_{B'}\beta'$, the line $[\tau^L]$ contains both an element t leaving the boundary points of B on E fixed and an element t' leaving the boundary points of B' on E fixed. ($\varkappa_A^{\frac{L}{\alpha}}$ belongs to $[\tau^L]$, since $\varkappa_A$ belongs to $[\tau^\alpha]$. α divides L, since it divides both $n_B\beta$ and $n_{B'}\beta'$.) Since t and t' are in the same line of (T), they differ by an element f of F. Since both t and t' leave the end points of A fixed, f does so too, and so is a power f_A^e of f_A. Hence

$$t = f_A^e t'. \tag{12.1}$$

Now $t\overline{G}_{F_B}$ is the identical transformation of $\overline{G}_{F_B}$ and

$$t\overline{G}_{F_{B'}} = f_A^e t' \overline{G}_{F_{B'}} = f_A^e \overline{G}_{F_{B'}}.$$

The element t, which leaves all points of $\overline{G}_{F_B}$ fixed, displaces all points of $\overline{G}_{F_{B'}}$ in the same way, as does the element f_A^e of F. The rational number

$$s_A = \frac{e\alpha}{L} \tag{12.2}$$

will be termed the *screw number* of the axis A.

The screw number of an axis remains invariant under the operations of T. [To be sure, replace A by uA, $u \subset T$. Then $\varkappa_A$ is replaced by $u\varkappa_A u^{-1}$, and α remains unaltered. B and B' are replaced by uB and uB', uB being to the left of uA. The numbers $\beta, \beta', n_B, n_{B'}$ and hence L remain unaltered. f_A, t and t' are replaced by uf_Au^{-1}, utu^{-1} and $ut'u^{-1}$ respectively. So e and hence s_A remain unaltered.] Thus the screw number may be said to belong to the equivalence class TA.

Since, in case S is bounded, all boundary axes of K_F are comprised in (10.4), it should be emphasized that screw numbers are only assigned to *inner* axes.

13. Division of S into complete kernels. Now suppose the screw number s_A of A to be zero, thus $e = 0$ and by (12.1) $t = t'$. The element t then leaves both the limit point set of F_B and $F_{B'}$ fixed, thus both B and B' belong to the kernel of t. For all other axes of the equivalence class TA the situation is the same, as they all have their screw number equal to zero. So we may take A to be one of the axes of the set (10.3), A_λ say. *We then omit the subset TA_λ from the set (10.4).* Proceeding so for all values of the subscript λ for which $s_{A_\lambda} = 0$ we reduce (10.3) and so (10.4) to a smaller set, if any $s = 0$ occurs.

It may happen that the entire set (10.3) and so (10.4) or, if S is bounded, all inner axes of these sets cancel in this way. This means that the convex region K_F is no more subdivided. So the whole of K_F is the kernel of some element $t \subset T$ not in the line $[\tau^0]$ of the scheme (T). This line then is identical with the line $[\tau^0] = F$, as far as only $\overline{G}_F$ is concerned. As stated in section 3, this means that τ is a transformation class of finite order. This case is fully investigated in [15] and we have nothing to add.

In order to get a true generalization of transformation classes of finite order we thus suppose that at least one inner axis of (10.3) does not cancel. We denote anew by

$$A_1, A_2, \cdots, A_i, A_{i+1}, \cdots, A_{i+r} \quad \begin{pmatrix} i \geqq 1 \\ r \geqq 0 \end{pmatrix} \tag{13.1}$$

the remaining axes and by

$$TA_1 + TA_2 + \cdots + TA_i + TA_{i+1} + \cdots + TA_{i+r} \tag{13.2}$$

the set of their equivalence classes taken together. $A_1, A_2, \cdots, A_i$ are taken to denote inner axes, $A_{i+1}, \cdots, A_{i+r}$ are representatives of the r equivalence classes of boundary axes of K_F corresponding to the r ($\geqq 0$) bounding geodesics of S. The set (13.2) has still property 1) attributed to the set (10.4) but, in general, not property 2), as some axes of (10.4) may have been omitted.

As $i \geqq 1$ we still have a division of K_F by (13.2). We continue to use the letter B to denote some region of that division. So every region B is now the complete kernel of some function of T. Inversely, every kernel of T is someone of the regions B or someone of the axes separating them, since a kernel does not cross any other kernel.

Hence the transformation classes of algebraically finite type constituting the subject of this paper may be characterized as such transformation classes for which K_F is made up of kernels, or, if we transfer the notation of "kernel" to a region b of S covered by some B, for which S *is made up of a finite number of kernels.*

It should not be overlooked that even some or all of the inner axes $A_1, A_2, \cdots, A_i$ of (13.1) and their equivalents by T may play the rôle of independent kernels. Let us look closer at the axis A of fig. 6 together with its neighbouring regions B and B' and use the notations of section 12. As A is to be one of the axes still in (13.2), $e \neq 0$; let us assume $e < -1$. Then under the transformation t all points of $\overline{G}_{F_B}$ are fixed and all points of $\overline{G}_{F_{B'}}$ are displaced in the same way as by f_A^e, thus towards $U = U_{f_A}$. By the element $f_A t \subset T$ all points of $\overline{G}_{F_B}$ are displaced in the same way as by f_A, thus towards $V = V_{f_A}$, whereas all points of $\overline{G}_{F_{B'}}$ are displaced in the same way as by f_A^{e+1}, thus still towards U. So these two directions of displacement coincide on E, both V and U are neutral, and A is the kernel of $f_A t$, being of the type of fig. 1. The index of $f_A t$ is zero, since we have $\nu = 1$, $\mu = 0$.—The case $e > 1$ may be treated accordingly.

A case of special importance is that of A being amphidrome. Let h be an element of T reversing A. Then h clearly leaves no point of $\overline{G}_F$ fixed, so the index of h is 1. The regions B and B' contiguous to A are interchanged by h; so they are equivalent. The element h^2 leaves U and V fixed. Now suppose that h^2 leaves some point P of $\overline{G}_F$ other than U and V fixed. Since

$$h^2 hP = hh^2 P = hP,$$

h^2 also leaves hP fixed. The axis A separates P and hP. Now, since f_A belongs to the subgroup N of elements left fixed by the automorphism induced by h^2, both P and hP are carried into other fixed points by all powers of f_A. So in both intervals determined on E by U and V the points of $\overline{G}_F$ left fixed by h^2 are in infinite number. Thus there exists a principal region for the element h^2, and this region contains A as an inner axis.

This is in contradiction to the fact that the principal region is at the same time a kernel, and that A is not inner axis of the kernel of some element $\neq 1$ of T; we have $h^2 \neq 1$, since T contains no element of finite order. Thus we infer that no point of $\overline{G}_F$ other than U and V is left fixed by h^2.

If the direction of displacement by h^2 to the left of A goes from U to V, say, to the right of A it goes from V to U; this follows immediately by using the equation $h \cdot h^2 \cdot h^{-1} = h^2$. So these two directions coincide on E, and both U and V are neutral (case **B** of section 6). Hence A is the kernel of h^2, belonging to the type of fig. 1. So we see that an amphidrome axis is always an independent kernel.

As we have just seen, in contrast to a non-amphidrome axis A, for which all elements of T with A as their kernel have their index equal to zero, an amphidrome axis A gives rise to an element $h \subset T$ with $j(h) = 1$. One may ask if there are more elements of T with $j = 1$, affiliated to the same kernel A, in the transformation class to which h belongs. As such an element has its place in the line of h in the scheme (T), it has the form fh, $f \subset F$. As fh is affiliated to A and $j(fh) = 1$, fh interchanges U and V, and as h does too, f must leave U and V fixed, thus $f \subset F_A$, $f = f_A^n$ for some $n \neq 0$. Hence all elements

$$(13.3) \qquad f_A^n h, \quad n \text{ arbitrary},$$

and no other elements of the line of h interchange U and V and so are affiliated to A. Each of the elements (13.3) thus defining a class of fixed points with index 1 in the surface transformation class given by h, we have to ask how many of these classes are different, i. e. what is the number of congruence classes (section 7) into which the functions (13.3) fall.

If J denotes the automorphism of F induced by h, J carries F_A into itself, since h interchanges U and V. So J involves an automorphism of F_A, and this is not the identical one. Now, F_A consisting of all powers of the primary element f_A, there is only one non-identical automorphism of F_A, viz. the replacement of f_A by f_A^{-1}. So

$$hf_A h^{-1} = (f_A)_J = f_A^{-1}.$$

Hence we get

$$f_A^m h f_A^{-m} = f_A^m (f_A^{-m})_J h = f_A^{2m} h$$

$$f_A^m (f_A h) f_A^{-m} = f_A^m f_A (f_A^{-m})_J h = f_A^{2m+1} h.$$

Thus all elements of (13.3) with n even belong to the congruence class of h and all elements with n odd belong to the class of $f_A h$. It remains to be seen whether these classes are identical or different. Suppose they are identical. Then an element $f \subset F$ exists such that

$$f_A h = f h f^{-1} = f f_J^{-1} h,$$

hence

$$f f_J^{-1} = f_A \tag{13.4}$$

and by applying J

$$f_J f_{J^2}^{-1} = (f_A)_J = f_A^{-1}.$$

From this we get by multiplying these two equations

$$f f_{J^2}^{-1} = 1.$$

So f is left fixed by the automorphism J^2 induced by h^2. Now, as shown above, h^2 leaves only U and V fixed, so J^2 leaves all elements of F_A and no other element of F fixed. Hence

$$f = f_A^m$$

and by (13.4)

$$f_A^{2m} = f_A,$$

which is impossible. Hence the congruence classes of h and $f_A h$ are different.—So we have:

One of the inner axes of the system (13.1) which is amphidrome for some transformation class, τ^n say, and so is reversed by some element of the line $[\tau^n]$ of the scheme (T), gives rise to *exactly two classes of fixed points of the surface transformation class* $\tau^n S$ *each with index 1.*—This may be illustrated by the fact, that if a surface transformation of the class $\tau^n S$ is so chosen as to carry the closed geodesic corresponding to the axis into itself with orientation reversed, exactly two fixed points will arise on the closed curve.

14. Construction of a special transformation. The analysis of the preceding sections enables us to construct a surface transformation of a prescribed class τ of algebraically finite type such that each class of fixed points with index $\neq 0$ is "satisfied" by one single point of the surface and classes of index zero are completely avoided. In order to distinguish between the transformation class τ and the special transformation to be constructed, we denote the latter by ζ.

Let a denote the closed geodesic on S corresponding to some inner axis A of the set (13.2). We assign to a a narrow band $\tilde{a}$ of constant breadth of S enclosing a as its middle line. The part of K_F covering $\tilde{a}$ is a strip $\tilde{A}$ enclosing A and bounded by two circular arcs, all points of which are at the same non-euclidean distance from A; the end points of these arcs coincide with the end points of A. Of course, all strips of the congruence class $F\tilde{A}$ also cover $\tilde{a}$.

This construction is made for all closed geodesics of S corresponding to inner axes of the set (13.2). Since these geodesics are in finite number, the bands may be chosen so narrow as not to have common points. Then also any two of the strips arising in K_F do not interfere. For convenience we may take all bands equally wide.

We then have a division of K_F by the equivalence classes of strips

$$T\tilde{A}_1 + T\tilde{A}_2 + \cdots + T\tilde{A}_i \tag{14.1}$$

instead of by the inner axes of the set (13.2). We continue to use the letter B to denote any region of that division and to denote by b the corresponding region of S. A boundary curve of b which is not a boundary curve of S is then no more a closed geodesic, but a closed simple curve (boundary curve of some band), all points of which are at constant non-euclidean distance from a closed geodesic. This evidently makes no difference in speaking of tB as the region corresponding to B by the element $t \subset T$, of the equivalence class TB, and so on.

Now let b be any region of the division of S, B a corresponding region of K_F and the number β and an element $\varkappa_B$ defined as in section 11. It has been shown in that section that $\varkappa_B$ defines a transformation class of some finite order n_B

(or, as we may write, n_b) of the sub-surface b. By the chief theorem of [15] this class may be represented by a *periodic transformation* of b of order n_b. We construct such a periodic transformation of b by the process outlined in the paper quoted and denote it by $\zeta^{\beta} b$.

If $\beta > 1$, $\varkappa B = B_1$ (section 10) is a region symbolically equivalent to B (by T), but not congruent to B (by F); so B_1 corresponds to some region b_1 different from b in the division of S. Denoting as in section 2 by I the automorphism corresponding to $\varkappa$, the group F_B is carried into F_{B_1} by I. To this automorphism between F_B and F_{B_1} corresponds a class of transformations of b into b_1. We choose any topological transformation of b into b_1 belonging to that class and denote it by ζb. Then $\zeta\zeta^{\beta}\zeta^{-1}$ is a periodic transformation of $b_1 = \zeta b$ into itself of order $n_b = n_{b_1}$, which is denoted by $\zeta^{\beta} b_1$.

If $\beta > 2$, we proceed in the same way for $\varkappa^2 B = B_2$ and the corresponding region $\zeta^2 b = b_2$ and continue this process, till we reach $\zeta^{\beta-1} b = b_{\beta-1}$. Now, since $\varkappa^{\beta} B = gB$ for some $g \subset F$ (section 11), we have to transform $b_{\beta-1}$ into b by a topological transformation ζ, and there is no choice left; it has to be such that

$$\zeta b_{\beta-1} = \zeta\zeta^{\beta-1} b = \zeta^{\beta} b$$

is the transformation of b into itself already constructed. Now we have achieved the construction of a group of transformations consisting of all powers of ζ in the set of sub-surfaces $b, b_1, \cdots, b_{\beta-1}$. These regions are interchanged cyclically by $\zeta, \zeta^2, \cdots, \zeta^{\beta-1}$, whereas ζ^{β} is for all of them a periodic transformation of order n_b. So $\zeta^{\beta n_b}$ is the identical transformation in these β regions.

If there are more than these β regions on S, we choose another one and repeat the process for its equivalence class. After a finite number of steps ζ is defined in all regions of the division of S.

We now have to consider what this construction of the transformation ζ in the regions of S means in K_F. It will be remembered that all elements of F are defined in K_F, but all other elements of T have so far only been defined in the set $\overline{G}_F$ of limit points by way of the given transformation class τ and its powers. F carries the set (14.1) of strips into itself and so also carries the complementary set of K_F, i. e. the set of regions, into itself. So F is a transformation group of the set of regions of K_F. To extend this property from F to T, take any region B of K_F. The notation $\varkappa B = B_1$ has hitherto been meant symbolically to denote the region of K_F the limit points of which are the images of the limit points of B by $\varkappa$. Now, the region b of S covered by B is subject to a topological transformation ζ into the region b_1 covered by B_1, and this has been so constructed as to correspond to the automorphism I induced by $\varkappa$ and taking F_B into F_{B_1}. So there is one topological transformation, and one only, mapping B into B_1 and covering ζ so as to correspond to I. We may denote this transformation by the same letter $\varkappa$, so that $\varkappa B = B_1$ now literally indicates the mapping of B into B_1 by $\varkappa$. As this applies to all regions B and extends to all powers of $\varkappa$, we have extended T to denote a certain group of topological transformations of the set of regions of K_F. This group satisfies the set of functional equations (2.1) or (2.2).

We now have to extend T to the strips of K_F thus defining ζ in the bands of S.

Let A be an inner axis of the set (13.1) and α the number assigned to it in section 12. We first assume A not to be amphidrome and take $\alpha > 1$. Then $\varkappa A = A_1$[1]) means symbolically the axis the end points of which are the images of the end points of A by $\varkappa$. Now, boundary arcs of strips are boundary arcs of regions too, so the mapping function $\varkappa$ is defined on them. Hence the boundary of the strip $\tilde{A}$ imbedding A is mapped by $\varkappa$ upon the boundary of the strip $\tilde{A}_1$ imbedding A_1. We have to extend this mapping function to the interior of $\tilde{A}$.

For convenience we represent the strips $\tilde{A}$ and $\tilde{A}_1$ by the strip $0 \leq y \leq 1$ of a euclidean xy-plane and the strip $0 \leq y' \leq 1$ of a euclidean $x'y'$-plane respectively (fig. 7). (This may be achieved by some auxiliary mapping function). The axes A and A_1 are represented by the lines $y = \frac{1}{2}$ and $y' = \frac{1}{2}$ respectively. The primary translations f_A and f_{A_1} are both represented by

1) Here subscripts have no connection with the notation of (13.1), of course.

translations of length 1 of the euclidean strips. So two points of $\tilde{A}$ corresponding by an element of F_A are represented by two points having the same value of y and having values of x the difference of which is an integer.

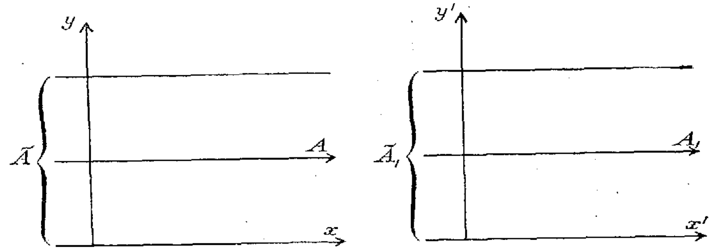

Fig. 7.

On the boundary $y = 0$ of $\tilde{A}$ we have by $\varkappa$ a transformation function $x' = \varkappa_1(x)$ taking this boundary into the boundary $y' = 0$ of $\tilde{A}_1$ and satisfying the functional equation

$$\varkappa_1(x+1) = \varkappa_1(x)+1,$$

since $\varkappa f_A = f_{A_1}\varkappa$; the element f_{A_1} corresponds to f_A by the automorphism I induced by $\varkappa$. In the same way on $y = 1$ we have a function $x' = \varkappa_2(x)$ taking this boundary into the boundary $y' = 1$ of $\tilde{A}_1$ and such that

$$\varkappa_2(x+1) = \varkappa_2(x)+1.$$

If now we carry the straight segment joining $(x, 0)$ and $(x, 1)$ into the straight segment joining $(\varkappa_1(x), 0)$ and $(\varkappa_2(x), 1)$ by an affine transformation, we get a topological mapping $\varkappa$ of the strip $\tilde{A}$ upon the strip $\tilde{A}_1$ coinciding with $\varkappa_1$ and $\varkappa_2$ respectively on the boundaries and satisfying the functional equation

$$\varkappa f_A = f_{A_1}\varkappa.$$

(If (x, y) is carried into (x', y'), then $y = y'$ and $(x+1, y)$ is carried into $(x'+1, y')$.) So this transformation $\varkappa$ of the strip $\tilde{A}$ upon $\tilde{A}_1$ covers a topological transformation of the band $\tilde{a}$ of S covered by $\tilde{A}$ upon the band $\tilde{a}_1$ covered by $\tilde{A}_1$. This transformation may be denoted by ζ, since it links up with the transformation ζ of the adjacent regions of S already defined.

If $\alpha > 2$, we define $\varkappa^2\tilde{A}$ and so $\zeta^2\tilde{a}$ in the same way and continue this process until we reach $\varkappa^{\alpha-1}\tilde{A} = \tilde{A}_{\alpha-1}$ covering the band $\tilde{a}_{\alpha-1}$. We now have to define $\varkappa^\alpha\tilde{A} = \varkappa\tilde{A}_{\alpha-1}$. Let us first suppose this to be done in the same way as before. The strip $\tilde{A}_\alpha$, upon which $\tilde{A}$ is mapped by $\varkappa^\alpha$ is congruent to $\tilde{A}$,

$$\varkappa^\alpha\tilde{A} = f\tilde{A}, \quad f \subset F,$$

according to section 12. $\zeta^\alpha\tilde{a}$ is a transformation of the band $\tilde{a}$ into itself. As the values of y are not altered by our construction, a curve of $\tilde{a}$ corresponding to constant y is carried into itself. So fixed points may arise in $\tilde{a}$ under the transformation ζ^α. In order to avoid this we first replace the point (x, y) of $\tilde{A}$ by the point (x, y^2) and then apply the above construction, i. e. the image of (x, y) under $\varkappa^\alpha$ is the image of (x, y^2) by the above construction applied to the strips $\tilde{A}$ and $\tilde{A}_\alpha$. So we get the final definition of $\zeta^\alpha\tilde{a}$ and thus the definition of all powers of ζ in $\tilde{a}$ and in $\zeta\tilde{a}, \zeta^2\tilde{a}, \cdots, \zeta^{\alpha-1}\tilde{a}$ too.—In case $\alpha = 1$, the definition of ζ^α just given is that of ζ itself.

Since $\varkappa$ is now defined in the whole equivalence class $T\tilde{A}$ by means of the functional equation (2.1), the whole group T is defined in that totality of strips $T\tilde{A}$. Let us look at the regions B and B' neighbouring $\tilde{A}$ and use all notations of section 12. The element $\varkappa_A$ transforms $\tilde{A}$ into itself. In the line of $\varkappa_A^{\frac{L}{\alpha}}$ of the scheme (T), which is the line $[\tau^L]$, there is a function t leaving all limit points of B fixed; this element t now is defined in all regions of K_F and in the strip $\tilde{A}$ too. It leaves all points of B fixed, since we have in B a periodic transformation and the limit points of B are left fixed. In the same way t' leaves all points of B' fixed. Then by (12.1) t displaces all points of B' in the same way as does f_A^e. Consider a curve of $\tilde{A}$ joining $(x, 1)$ and $(x, 0)$ in fig. 7, a segment at right angles to A, say. $(x, 1)$ is left fixed by t, since it is on the boundary of B, and $(x, 0)$ is carried into $(x+e, 0)$ by t. For the transformation ζ^L of the band $\tilde{a}$ into itself this means that ζ^L is the identical transformation on both boundaries of the band but carries a straight segment joining two opposite points into a curve which winds e times round the band. This explains the definition of the screw number given in (12.2). As $\tilde{a}$ is transformed into itself by the powers of ζ^{α} only, we divide e by $\frac{L}{\alpha}$.

M. Dehn calls a transformation such as that just defined in the band $\tilde{a}$ "Verschraubung". Such transformations are the chief means of investigation in his paper [6]. See especially § 2, p. 141—142 of [6].

Finally we have to assume A to be amphidrome. Then we have $\alpha \geqq 2$ and even. In this case the definition of $\varkappa \tilde{A}$, $\varkappa^2 \tilde{A}$, $\cdots$ described above goes without limitation, thus the auxiliary transformation replacing (x, y) by (x, y^2), applied in the former case in defining $\varkappa^{\alpha}$ in order to avoid fixed points, does not come into play. The effect is as follows: As to $\varkappa \tilde{A}$, $\varkappa^2 \tilde{A}$, $\cdots$, $\varkappa^{\frac{\alpha}{2}-1} \tilde{A}$, there is no difference. $\varkappa^{\frac{\alpha}{2}} A$ is congruent to A by an element of F, say f. Then $f^{-1} \varkappa^{\frac{\alpha}{2}}$ carries the strip $\tilde{A}$ into itself with boundaries interchanged. Then if $\varkappa^{\frac{\alpha}{2}}$ is defined as a transformation of the strip $\tilde{A}$ in the prescribed way, $f^{-1} \varkappa^{\frac{\alpha}{2}}$ carries the middle line A of $\tilde{A}$ into itself with orientation reversed. Thus $\zeta^{\frac{\alpha}{2}}$ transforms a into itself leaving exactly two points of a and no other point of the band fixed. Two fixed points cannot be avoided, since they represent two different classes of fixed points both with index 1 (section 13).

After this construction has been made for all i inner axes of the set (13.1), a topological transformation ζ of the surface S into itself has been established, and ζ belongs to the class τ prescribed. What are the fixed points of ζ? As to the bands, there are no fixed points in the interior of a band, if it is not amphidrome, or if it is amphidrome with $\alpha > 2$, since in the latter case $\frac{\alpha}{2}$ bands are interchanged cyclically by ζ. If it is amphidrome with $\alpha = 2$, there are exactly two fixed points each with index 1. As to the regions, there is no fixed point in a region, if $\beta > 1$, since in that case β regions are interchanged cyclically by ζ. If $\beta = 1$, the region b is subject to a periodic transformation into itself. If $n_b > 1$, there may be single invariant

points, each of index 1 (section 8), or there may be none. If $n_b = 1$, all points of the region are fixed by ζ, and their totality forms a class of fixed points of negative index; the index is $1 - \nu$, if ν is the minimum number of generators of the Poincaré group of the region. In this case it is easily seen that ζ may be slightly deformed in b so as to leave only one point of b fixed, the index of which then is $1 - \nu$; see [14] I, p. 314.

ζS thus satisfies the conditions asked for in the beginning of this section.

15. The equivalence problem. Without going into details we shortly indicate a problem which may be solved by the preceding analysis. Let τ denote a transformation class of S into itself and γ a topological transformation of S upon a surface S^*, which may coincide with S or not. Then $\gamma \tau \gamma^{-1}$ will be a transformation class of S^* into itself. This will be called *equivalent to* τ. The *equivalence problem* then consists in establishing a set of invariants of a transformation class such that it is necessary and sufficient for two classes being equivalent that they agree in this set of invariants. For classes of finite order such a set of invariants has been given in [16], § 11. For classes of algebraically finite type a set of invariants may be derived from the considerations of this paper. Indeed, the division of S into complete kernels, the numbers β and n_b of a kernel, the number α of an axis of the set (13.1), its screw number and its character of being amphidrome or not, are readily seen to be invariants of τ not altered by γ. Moreover, the transformation class of finite order assigned to a region b of S must be equivalent to that assigned to the region γb of S^*, the conditions for which are known from [16]. On the other hand, if two classes of transformations of two homeomorphic surfaces S and S^* agree in these invariants, and we construct special transformations ζ and ζ^* of S and S^* respectively as described in the preceding section, then ζ and ζ^* become equivalent by a suitable transformation γ of S into S^*, and so do the classes to which they belong.

Part III.

Homology theory.

16. Enouncement of the main theorem. Hitherto the Poincaré group F of S has been the chief means of our investigation. In the following sections we fix our attention upon the *homology group* H of S, i. e. the factor group of the commutator group in F. The homology group H is abelian, and we may speak of the elements of F as elements of H provided we make them interchange freely; thus for instance a set of conjugate elements of F yields one element of H. As stated in section 1, the minimum number δ of generators of F is $\delta = 2p$, if S is closed, and $\delta = 2p + r - 1$, if F is bounded; in both cases H is a free abelian group with δ generators (for S closed all relations in F are identically satisfied in H). A set of δ free generators of H is called a *homology base*.

Any automorphism of F carries with it an automorphism of H, especially an inner automorphism of F the identical automorphism of H. So a complete family of automorphisms of F, which corresponds to a transformation class of S (section 2), yields one automorphism of H. While I continues to denote the automorphism induced in F by an element $\varkappa$ corresponding to a transformation class τ, let J denote the corresponding automorphism of H. Choosing a homology base of δ elements for H and using the sign of addition to denote the combination of elements of H, we describe J by a linear homogeneous transformation of the δ basic elements. Let Δ denote the matrix of that linear transformation and E_δ the unity matrix of δ rows and columns. Then if we put

$$(16.1) \qquad P(x) = (-1)^\delta \left| \Delta - xE_\delta \right| = x^\delta + \cdots + (-1)^\delta,$$

it is known that $P(x)$ only depends on J irrespective of the choice of homology base. $P(x)$ is a polynomial of degree δ in x and is called the *characteristic polynomial of* J. The roots of the equation $P(x) = 0$ are called the *characteristic roots* (or multipliers) of J. *It is our aim to establish the general form of $P(x)$, if τ is a class of algebraically finite type.*

A special case of a class of algebraically finite type is a class of finite order. Moreover, by the theorem of [15] already used, such a class contains a periodic transformation, and the general form of $P(x)$ for a periodic transformation has been shown in [16] to be

$$(16.2) \qquad P(x) = \frac{(x-1)^{1+\omega}(x^n-1)^{2q+s-2}}{(x^{m_1}-1)(x^{m_2}-1)\cdots(x^{m_u}-1)},$$

n denoting the order of the periodic transformation and ω being 0 or 1, according as the surface is bounded or closed. As to the numbers $q, s, u, m_1, \cdots, m_u$ a more detailed explanation is needed: In [4] Brouwer has shown that a certain auxiliary surface M, termed *modular surface*, may be assigned to any periodic transformation of order n of a surface S in such a way that S may be looked upon as a regular Riemann surface consisting of n sheets over M, and that the transformation consists in interchanging the sheets of S over M. Then M is closed or bounded according as S is closed or bounded. S may or may not ramify over M. Then q denotes the genus and u the number of ramification points of M; s is the sum of u and the number of boundary curves of M. While S has n distinct points over every ordinary point of M, it has a certain number m of distinct points over a ramification point; this number m is less than n and divides n. The set $m_1, m_2, \cdots, m_u$ denotes these numbers m for all ramification points.

In sections 17—21 we are concerned with the proof of the following generalization of this result concerning classes of finite order:

Theorem: *The characteristic polynomial of a transformation class of algebraically finite type takes the form*

$$(16.3) \quad P(x) = (x-1)^{1+\omega} \prod_l \frac{\left(x^{\beta_l n_l} - 1\right)^{2q_l + s_l - 2}}{\left(x^{\beta_l m_{l1}} - 1\right)\left(x^{\beta_l m_{l2}} - 1\right)\cdots\left(x^{\beta_l m_{lu_l}} - 1\right)}$$

Here again ω is 0 or 1 according as S is bounded or closed. l ranges over all equivalence classes of regions and bands, of which S consists. We now discuss the notations in the factor corresponding to the equivalence class of number l. In all cases

β_l is the number of congruence classes into which the equivalence class falls.

First, let this class be made up of regions. Then β_l is the number of regions of S in the class. So if b is one of these regions and B a region of K_F covering b, β_l means the number β assigned to B in section 11. The element $\varkappa_B$ of section 11 defines a transformation class of finite order for b, and n_l means the order of that transformation class. In section 14 this class is represented by a periodic transformation $\zeta^{\beta_l}b$. This periodic transformation gives rise to a modular surface M_l, and the numbers $q_l, u_l, s_l, m_{l1}, \cdots, m_{lu_l}$ are defined as above.

If $\overline{M}_l$ denotes the surface derived from M_l by removing u_l small elements, each containing one ramification point, s_l is the number of boundaries of $\overline{M}_l$. If δ_l denotes the minimum number of generators of the Poincaré group of $\overline{M}_l$, we get for the exponent in the numerator of $P(x)$, provided $s_l > 0$,

$$2q_l + s_l - 2 = \delta_l - 1. \tag{16.4}$$

Then let the equivalence class consist of bands belonging to amphidrome axes. β_l then denotes the number of bands in the class, hence $\beta_l = \frac{\alpha}{2}$, the number α being defined as in section 12. Then by section 14, if $\tilde{a}$ is one of the bands, $\zeta^{\beta_l}\tilde{a} = \zeta^{\frac{\alpha}{2}}\tilde{a}$ is a transformation class of order 2 for $\tilde{a}$. Thus we take n_l to be 2. This class contains a periodic transformation of order 2 interchanging the boundaries of $\tilde{a}$. So the modular surface M_l has only one boundary; M_l is clearly seen to be an element with 2 ramification points. Hence we get

$$q_l = 0, \quad u_l = 2, \quad s_l = 3, \quad m_{l1} = m_{l2} = 1.$$

The factor of $P(x)$ corresponding to such an equivalence class of amphidrome bands thus reduces to

$$\frac{x^{2\beta_l} - 1}{(x^{\beta_l} - 1)^2}. \tag{16.5}$$

Finally, let the equivalence class consist of bands belonging to non-amphidrome axes. β_l denotes the number of bands in the class, thus $\beta_l = \alpha$, the number of section 12. Now, if $\tilde{a}$ is one of the bands, the transformation $\zeta^{\alpha}\tilde{a}$ of section 14 belongs to the class of identity. The corresponding periodic transformation is the identical transformation, so M_l coincides with $\tilde{a}$ and we have

$$n_l = 1, \quad u_l = 0, \quad q_l = 0, \quad s_l = 2,$$

and no factor arises at all in the numerator or denominator of $P(x)$. So this value of l only yields the factor 1 in $P(x)$.

We may thus restrict l to range over the equivalence classes consisting of regions or of amphidrome bands.

17. Preparations for the proof. The proof of the theorem expressed by (16.3) will be given by induction using the number i of equivalence classes of inner axes in (13.1,2) as number of induction. Since for $i = 0$ no inner axis exists, K_F is not divided and so forms one single kernel. Then l only takes the value $l = 1$. Since K_F is the only region, we have $\beta = 1$. The transformation class considered is a class of finite order, and (16.3) clearly reduces to (16.2). So the theorem is true for $i = 0$. We have to show that it is true for any $i > 0$, if it is true for all smaller values of i.

We pick out one of the inner axes of the set (13.1), denote it by A and fix our attention upon the division of K_F by the equivalence class TA. This division is reproduced by every element of T. If C is any region of that division, we denote by T_C the subgroup of T and by F_C the subgroup of F reproducing C. In the sequence

$$C, \varkappa C, \varkappa^2 C, \cdots$$

let γ be the least positive number such that

$$\varkappa^{\gamma} C = fC, \quad f \subset F,$$

and put

$$f^{-1}\varkappa^{\gamma} = \varkappa_C.$$

Then we have

$$T_C = F_C + F_C\varkappa_C + F_C\varkappa_C^2 + \cdots$$
$$+ F_C\varkappa_C^{-1} + F_C\varkappa_C^{-2} + \cdots.$$

If c denotes the region of S covered by C, a transformation class of c is defined by $\varkappa_C$. This transformation class is of algebraically finite type. To be sure we might repeat the argument of section 11, showing that if the principal region of some element $t \subset T_C$ had a cuspidal point in $\overline{G}_{F_C}$, this would be a cuspidal point in $\overline{G}_F$ too. We may also proceed as follows: If an element $t \subset T_C$, regarded as a transformation function of $\overline{G}_{F_C}$ only, has a kernel, this is at the same time the complete kernel of t regarded as a transformation function of $\overline{G}_F$, since every boundary of C inside K_F belongs to TA and so is boundary of a kernel of T. Inversely, if $t \subset T$ has its kernel inside C, then $t \subset T_C$ with the same kernel. So C is made up of complete kernels of transformation functions belonging to T_C.

If C consists of more than one kernel, the axes dividing C into kernels belong to the set (13.2). If any two of these axes are equivalent with respect to T,

$$A' = tA'', \quad t \subset T,$$

then $t \subset T_C$, since t carries an inner axis of C into another inner axis of C; so A' and A'' are equivalent with respect to T_C; the inverse is obvious. So the distribution of inner axes of C belonging to (13.2) into equivalence classes with respect to T_C is the same as with respect to T. Now the class TA only yields boundary axes of C. *So the number of equivalence classes of inner axes for C is less than the number i of (13.1,2).*

Thus by the assumption of our proof of induction the polynomial $P_c(x)$ belonging to the transformation class of c given by $\varkappa_C$ takes the form (16.3), moreover with $\omega = 0$, since c is bounded. (Even if S happens to be closed, it has been cut along one closed geodesic at least.)

Now suppose γ to be > 1. We then have on S an equivalence class of regions

$$(17.1) \qquad c,\ \tau c = c_1,\ \tau^2 c = c_2, \cdots,\ \tau^{\gamma-1} c = c_{\gamma-1}$$

covered by the regions of K_F

$$(17.2) \qquad C,\ \varkappa C,\ \varkappa^2 C, \cdots,\ \varkappa^{\gamma-1} C$$

respectively. There may or may not be more regions than (17.1) in the division of S by the geodesics covered by the axes of the class TA. For each of the regions (17.1) we have a transformation class of algebraically finite type defined by

$$\varkappa_C,\ \varkappa\varkappa_C\varkappa^{-1},\ \varkappa^2\varkappa_C\varkappa^{-2}, \cdots,\ \varkappa^{\gamma-1}\varkappa_C\varkappa^{-(\gamma-1)}$$

respectively, and they all have the same characteristic polynomial $P_c(x)$.

Now we may look upon the subsurfaces (17.1) of S as γ distinct surfaces irrespective of their connection on S. The homology group of this set of surfaces then is defined as the direct sum of the γ isomorphic homology groups belonging to the single subsurfaces. If there are δ generators in the group belonging to c, then the combined homology group is the free abelian group with $\gamma\delta$ generators.

A transformation class of this set of surfaces is given by $\varkappa$. In fact $\varkappa$ defines a transformation class of c upon c_1, of c_1 upon $c_2, \cdots$, of $c_{\gamma-2}$ upon $c_{\gamma-1}$. In applying $\varkappa$ to $\varkappa^{\gamma-1}C$ we get $\varkappa^\gamma C = fC$, which covers c and defines the same transformation class of c upon itself as $f^{-1}\varkappa^\gamma C = \varkappa_C C$. Since $\varkappa$ represents the transformation class τ, we may prefer to say that a transformation class of the set (17.1) is given by the prescribed class τ; this is expressed in the notation (17.1).

We intend to find the characteristic polynomial of the transformation class of the set (17.1) given by τ. Let δ elements be chosen as a base of the homology group $H(c)$ of c; let Δ_c be the transformation matrix of this base corresponding to the transformation class τ^γ and $P_c(x)$ the corresponding characteristic polynomial (16.3). By τ (i. e. under the isomorphism between $H(c)$ and $H(c_1)$ induced by τ) these δ elements correspond to certain δ elements of $H(c_1)$ forming a homology base of c_1. By τ^2 they correspond to δ elements of $H(c_2)$ forming a homology base of c_2, and so on till we reach $c_{\gamma-1}$. The $\gamma\delta$ elements obtained in this way form a homology base for the set (17.1). The δ elements of $H(c_{\gamma-1})$ correspond by τ to δ elements of $H(c)$, which are the transforms of the elements chosen as homology base for c by the matrix Δ_c. So the matrix of the automorphism of the homology group $H(c + c_1 + \cdots + c_{\gamma-1})$

of the set (17.1) is easily formed. To find the characteristic polynomial we subtract $xE_{\gamma\delta}$ and take the determinant. In short this determinant may be written

$$\begin{vmatrix} -xE_\delta & E_\delta & 0 & \cdots & 0 & 0 \\ 0 & -xE_\delta & E_\delta & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & -xE_\delta & E_\delta \\ \Delta_c & 0 & 0 & \cdots & 0 & -xE_\delta \end{vmatrix}$$

Here every symbol stands for a matrix with δ rows and colums and there are γ symbols in each row and column. To compute the determinant we multiply the first $\gamma-1$ rows by $x^{\gamma-1}$, $x^{\gamma-2}, \cdots, x$ respectively and add all to the last row; so the determinant reduces to

$$(-1)^{\gamma\delta} \left| \Delta_c - x^\gamma E_\delta \right|.$$

Thus we get:

The characteristic polynomial of the surface set (17.1) belonging to τ is $P_c(x^\gamma)$, if $P_c(x)$ is the polynomial belonging to τ^γ for each separate surface.

It should be noted, that the polynomial of the surface set (17.1) belonging to the transformation class τ^γ is $[P_c(x)]^\gamma$. This is evident.

If there are more equivalence classes of regions in the division of K_F by TA than that of C, all other classes may be treated in the same way.

18. First part of the proof. In this section we assume that the axis A of section 17 is not amphidrome, and that S is not decomposed by the geodesics corresponding to the equivalence class TA. According to the notations of section 12 the number of these geodesics is α, and they are represented by the axes

(18.1) $$A,\ \varkappa A,\ \varkappa^2 A,\ \cdots,\ \varkappa^{\alpha-1} A$$

of K_F. The corresponding geodesics may be denoted by

(18.2) $$a,\ \tau a = a_1,\ \tau^2 a = a_2,\ \cdots,\ \tau^{\alpha-1} a = a_{\alpha-1}.$$

We then have $\tau^\alpha a = a$[1].

As S is not decomposed by being cut along these α geodesics, only one region c arises. So there is but one equivalence class and moreover the number γ of (17.1) is 1. It should be noted that in cutting S along the α geodesics the genus p of S decreases by α and the number r of boundary curves increases by 2α. So if S is bounded, the number of generators of the homology group remains unaltered, whereas if S is closed, it decreases by 1.

Now by the assumption of our proof of induction, the polynomial $P_c(x)$ of the transformation class of c given by τ^γ takes the form (16.3), and $\omega = 0$, since c is bounded. The degree of the polynomial is δ, equal to the number of generators of $H(c)$. We intend to find the polynomial $P(x)$ of the transformation class of S given by τ. Its degree is δ, if S is bounded, and $\delta+1$, if S is closed.

If we orient A and transfer its orientation to all curves (18.1) and (18.2), we may speak of α boundary curves

(18.3) $$a',\ a_1',\ a_2',\ \cdots,\ a_{\alpha-1}'$$

of c as left hand borders and of another α boundary curves

(18.4) $$a'',\ a_1'',\ a_2'',\ \cdots,\ a_{\alpha-1}''$$

of c as right hand borders of $a, a_1, a_2, \cdots, a_{\alpha-1}$ respectively.

Now we first assume S to be bounded. Then c has more than these 2α boundary curves. So we may allow both (18.3) and (18.4) to be members of a homology base for c. Let such a homology base be chosen, and let it be arranged so as first to put $\delta-2\alpha$ elements not in (18.3,4) then the α elements (18.3) and finally the α elements (18.4). Then the matrix Δ_c describing the automorphism of $H(c)$ corresponding to the transformation class given by $\varkappa_C$ by a linear transformation of the base chosen may be composed of 9 blocks,

[1] To avoid misunderstanding, we recall that the notation $a_1 = \tau a$ etc. is symbolic and means that any closed curve on S homotopic to a is transformed by any transformation of the class τ into a curve homotopic to a_1 (section 7).

$$\Delta_c = \left\{ \begin{array}{c|c|c} I & IV & V \\ \hline VI & II & VII \\ \hline VIII & IX & III \end{array} \right\}, \tag{18.5}$$

I being a square matrix of $\delta - 2\alpha$ rows and both II and III square matrices of α rows each. Now Δ_c is known to interchange the elements (18.3) cyclically and equally for (18.4). So we have

$$II = III = \begin{Bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{Bmatrix}, \tag{18.6}$$

and all elements of VI, VII, $VIII$, IX are zero. Hence we get irrespective of IV, V

$$(-1)^\delta P_c(x) = \left|\Delta_c - xE_\delta\right|$$
$$= \left|I - xE_{\delta-2\alpha}\right| \left|II - xE_\alpha\right| \left|III - xE_\alpha\right|.$$

The last two factors are easily computed in the way used in the preceding section, and we get

$$\left|II - xE_\alpha\right| = \left|III - xE_\alpha\right| = (-1)^\alpha(x^\alpha - 1). \tag{18.7}$$

Then let S be closed. So c has exactly 2α boundaries given by (18.3) and (18.4). Thus we have the homology relation

$$a' + a_1' + \cdots + a'_{\alpha-1} - a'' - a_1'' - \cdots - a''_{\alpha-1} = 0 \tag{18.8}$$

taking the orientation of the curves (18.3) and (18.4) into account. We choose a homology base of c in the same way and in the same order as before, only omitting $a''_{\alpha-1}$. So in (18.5) III now is a square matrix of $\alpha - 1$ rows. II remains equal to (18.6) and all elements of VI and VII are zero. Under the transformation Δ_c the element a'' of (18.4) goes into a_1'', this into a_2'', and so on until $a''_{\alpha-2}$. Now $a''_{\alpha-2}$ goes into $a''_{\alpha-1}$, but as this element is not in the base, it has to be replaced by

$$a' + a_1' + \cdots + a'_{\alpha-1} - a'' - a_1'' - \cdots - a''_{\alpha-2}$$

in consequence of (18.8). So all elements of $VIII$ are zero, all elements of IX are zero except for the last row of IX, which is made up of numbers 1, and for III we get the following square matrix of $\alpha - 1$ rows

$$III = \begin{Bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ -1 & -1 & -1 & \cdots & -1 \end{Bmatrix}. \tag{18.9}$$

Since matrices VI and $VIII$ and moreover VII vanish, we get

$$(-1)^\delta P_c(x) = \left|\Delta_c - xE_\delta\right|$$
$$= \left|I - xE_{\delta-2\alpha+1}\right| \left|II - xE_\alpha\right| \left|III - xE_{\alpha-1}\right|.$$

The last factor is easily computed from (18.9) and we get

$$\left|III - xE_{\alpha-1}\right| = (-1)^{\alpha-1}\frac{x^\alpha - 1}{x - 1}. \tag{18.10}$$

To sum up, we remember that the letter ω means 0 or 1 according as S is bounded or closed. Then we may take both cases together in saying that $\alpha - \omega$ elements of (18.4) enter into the base chosen and that these elements alone yield the factor (from (18.7) and (18.10), irrespective of sign)

$$\frac{x^\alpha - 1}{(x-1)^\omega} \tag{18.11}$$

in the polynomial $P_c(x)$ of c.

In order to get S from c we let the boundaries (18.4) of c coincide in turn with the boundaries (18.3). We get a homology base for S by taking the homology base for c, arranged in the same way as before, then cancelling the last $\alpha - \omega$ elements, since these become identical with elements of (18.3) already in the base, and replacing α new homology elements arising from the α new connections. So let

$$(18.12) \qquad b,\, b_1,\, b_2,\, \cdots,\, b_{\alpha-1}$$

denote homology elements corresponding to cycles of S, of which the first, b, crosses a in one point from left to right and does not intersect $a_1, a_2, \cdots, a_{\alpha-1}$, and the same for b_1 and a_1, b_2 and a_2 and so on. These cycles may be taken to have one point of c in common. Evidently there is no homology relation between the elements (18.12), and they make the homology base of S complete.

We may call (18.12) a *homology base of connection*. We now divide the matrix Δ of S into 9 blocks in the same way as in (18.5):

$$\Delta = \left\{\begin{array}{c|c|c} I^* & IV^* & V^* \\ \hline VI^* & II^* & VII^* \\ \hline VIII^* & IX^* & III^* \end{array}\right\}.$$

Here III^* is a square matrix with α rows corresponding to the elements (18.12). All other rows of Δ correspond to the same basic elements as in Δ_c.

As the elements (18.3) are interchanged cyclically, we have $II^* = II$, and both VI^* and VII^* vanish.—We now look at the basic elements belonging to the rows of I or I^*. Their transformation by Δ_c depends on the matrices I, IV and V. Now the basic elements (18.4), belonging to the columns of V have been replaced by the corresponding elements of (18.3). This means that $I^* = I$, but IV^* may differ from IV. Moreover these elements of the rows of I have their intersection number with curves (18.2) equal to zero, and this is not changed by the transformation; so their transform by Δ does not contain any element of (18.12) with a coefficient $\neq 0$; thus V^* vanishes.—Finally we look at the basic elements belonging to the rows of III^*. As b has its intersection number with a equal to 1 and with $a_1, a_2, \cdots, a_{\alpha-1}$ equal to zero, and as (18.2) are interchanged cyclically, we infer that the element corresponding to b by Δ has its intersection number with a_1 equal to 1 (the transformation class preserves the orientation of S) and with $a_2, \cdots, a_{\alpha-1}$, a equal to zero. So b_1 is the only element of (18.12) to appear in the transform of b, and b_1 has 1 as its coefficient. As to the transforms of $b_1, b_2, \cdots, b_{\alpha-1}$, things are analogous. So III^* is equal to (18.6). As to $VIII^*$ and IX^* nothing is known, but that does not matter, since V^* and VII^* are known to vanish. Using the fact that VI^* vanishes too, we get

$$(-1)^{\delta+\omega} P(x) = \left|\Delta - xE_{\delta+\omega}\right|$$
$$= \left|I^* - xE_{\delta-2\alpha+\omega}\right| \; \left|II^* - xE_\alpha\right| \; \left|III^* - xE_\alpha\right|.$$

Since $I^* = I$ and $II^* = II$, the two first factors are the same as before, and since III^* is equal to the matrix (18.6), the last factor is

$$(-1)^\alpha (x^\alpha - 1)$$

as in (18.7).

Hence we have the following result: The cancelling of $\alpha-\omega$ elements of (18.4) means multiplication of $P_c(x)$ by $\dfrac{(x-1)^\omega}{(x^\alpha-1)}$ (from 18.11) and the replacement by α new basic elements means multiplication by $(x^\alpha-1)$. So the total effect is multiplication by $(x-1)^\omega$.

Thus if S is bounded, we have $P(x) = P_c(x)$, and if S is closed, we have $P(x) = (x-1)\,P_c(x)$.

So we have to ask if this is actually the polynomial $P(x)$ we have to look for according to the description following (16.3). The initial factor, which was $x-1$ in $P_c(x)$, is now $(x-1)^{1+\omega}$. As to the factors of the product $\prod_l$ we recall that such a factor with all its numbers $\beta_l, n_l, q_l, s_l, u_l, m_{l1}, \cdots, m_{lu_l}$ arises from an equivalence class of regions or of amphidrome axes in c. Now every such equivalence class of c is a class of S too with all numbers unchanged. And no new class arises on S. It is true that there is a class of inner axes on S which is not a class of inner axes in c, viz. the class (18.1,2) used for cutting S. But as these axes are not amphidrome, they do not yield a factor in $P(x)$. — So our theorem is proved in the case considered.

19. Second part of the proof. In this section we assume that the geodesic a is not amphidrome, and that S is decomposed by the geodesics (18.2). Let c denote the region of this decomposition to the left of a; so c has the left hand border a' (18.3) of a as one of its boundaries. c may or may not have some of the right hand borders (18.4) as a boundary. In this section we make the further assumption that it has not. Since τ interchanges the geodesics (18.2), all regions to the left of these geodesics are equivalent by T, and so are all regions to the right of these geodesics. On our present assumption these two equivalence classes are different. As every region of the decomposition has at least one of the geodesics (18.2) as boundary, we have exactly these two equivalence classes of regions of the decomposition of S. If c' denotes the region to the right of a, the two classes are represented by c and c'.

Let γ and γ' denote the number of regions in these equivalence classes (section 17). Then

$$(19.1) \qquad c, \quad \tau c = c_1, \quad \tau^2 c = c_2, \ \cdots, \quad \tau^{\gamma-1} c = c_{\gamma-1}$$

are all regions of the equivalence class of c and

$$(19.2) \qquad c', \quad \tau c' = c_1', \quad \tau^2 c' = c_2', \ \cdots, \quad \tau^{\gamma'-1} c' = c'_{\gamma'-1}$$

are all regions of the equivalence class of c'. We then have $\tau^{\gamma} c = c$ and $\tau^{\gamma'} c' = c'$. From $\tau^{\alpha} a = a$ we infer that $\tau^{\alpha} c = c$; hence γ divides α and so does γ'.

Let $g = (\gamma, \gamma')$ be the greatest common measure of γ and γ', and put $\gamma = g\gamma_1$, $\gamma' = g\gamma_1'$. Then, if we suppose $g > 1$,

$$(19.3) \qquad c, \quad c_g, \quad c_{2g}, \ \cdots, \quad c_{(\gamma_1-1)g}$$

is a subset of (19.1) and

$$(19.4) \qquad c', \quad c_g', \quad c_{2g}', \ \cdots, \quad c'_{(\gamma_1'-1)g}$$

is a subset of (19.2). All geodesics

$$(19.5) \qquad a, \quad a_g, \quad a_{2g}, \ \cdots, \quad a_{\alpha-g}$$

and no other geodesics of the set (18.2) are boundaries of (19.3) and (19.4). So by joining the regions (19.3) and (19.4) along the geodesics (19.5) we get a subsurface S^* of S, and this subsurface has no boundary in common with the subsurfaces τS^*, $\tau^2 S^*$, $\cdots$, $\tau^{g-1} S^*$. So S would consist of g distinct surfaces, while throughout this paper we suppose S to be one coherent surface. So we have

$$(\gamma, \gamma') = 1,$$

γ and γ' are relatively prime.

The geodesics (18.2) or, more precisely, their borders (18.3) and (18.4) are so distributed on the subsurfaces (19.1) and (19.2) that

$$(19.6) \qquad a_\nu', \ a_{\nu+\gamma}', \ a_{\nu+2\gamma}', \ \cdots, \ a_{\nu+\alpha-\gamma}'$$

are boundaries of c_ν $(\nu = 0, 1, 2, \cdots, \gamma-1)$ and

$$(19.7) \qquad a_\nu'', \ a_{\nu+\gamma'}'', \ a_{\nu+2\gamma'}'', \ \cdots, \ a_{\nu+\alpha-\gamma'}''$$

are boundaries of c_ν' $(\nu = 0, 1, 2, \cdots, \gamma'-1)$.

τ^{γ} is a transformation class of algebraically finite type for c (section 17) and its characteristic polynomial $P_c(x)$ takes the form (16.3) with $\omega = 0$:

$$P_c(x) = (x-1) \prod_l f_l(x),$$

if we agree to denote the factor corresponding to l by

$$f_l(x) = \frac{(x^{\beta_l n_l} - 1)^{2q_l + s_l - 2}}{(x^{\beta_l m_{l1}} - 1) \cdots (x^{\beta_l m_{lu_l}} - 1)}.$$

Accordingly for $\tau^{\gamma'}$ as a transformation class of c' we have

$$P_{c'}(x) = (x-1) \prod_{l'} f_{l'}(x).$$

τ is a transformation class of the set (19.1) of γ distinct subsurfaces, and the corresponding polynomial is by section 17:

$$P_c(x^{\gamma}) = (x^{\gamma} - 1) \prod_l f_l(x^{\gamma}).$$

τ also is a transformation class of the set (19.2), and the corresponding polynomial is

$$P_{c'}(x^{\gamma'}) = (x^{\gamma'}-1)\prod_{l'} f_{l'}(x^{\gamma'}).$$

We may thus speak of τ as a transformation class of the *decomposed surface* S' (being a set of $\gamma+\gamma'$ distinct subsurfaces); the corresponding polynomial $P_{S'}(x)$ then is the product

$$P_{S'}(x) = P_c(x^{\gamma})\,P_{c'}(x^{\gamma'}) = \tag{19.8}$$

$$= (x^{\gamma}-1)(x^{\gamma'}-1)\prod_{l} f_l(x^{\gamma})\prod_{l'} f_{l'}(x^{\gamma'}).$$

We now ask what is the effect on this polynomial of passing from S' to S by joining the boundaries (18.4) in turn to the boundaries (18.3). This effect is found in two steps as in the preceding section, a) by cancelling some elements of the homology base of S' and b) by introducing new ones by a homology base of connection.

a) First let S be bounded. Then at least one of the subsurfaces c and c' has a boundary not belonging to (18.3,4). Let c have such a boundary; then all subsurfaces (19.1) have. So all elements of (18.3) may be allowed to enter into a homology base of S'. By joining (18.3) to (18.4) these elements (18.3) become equal to the elements (18.4) of the homology group of $c'+c'_1+\cdots+c'_{\gamma'-1}$; these elements may or may not be in the homology base of S' and may even be zero; in all cases they are independent of (18.3). So the elements (18.3) cancel, and as they are interchanged cyclically by τ, this evidently means multiplication of (19.8) by $\frac{1}{x^{\alpha}-1}$; see (18.6,7).

Then let S be closed. Thus (18.3) and (18.4) are the only boundaries of S'. As a preliminary case let us assume $\gamma' = \alpha$; thus (19.2) is a set of the maximum number α of subsurfaces, each bounded by one of the curves (18.4), and these curves are thus all homologous to zero. Then we have $\gamma = 1$, since γ divides α and is relatively prime to γ'. So (19.1) consists only of one surface c bounded by all curves (18.3). So we may take $a', a'_1, \cdots, a'_{\alpha-2}$ into the homology base of S'. The corresponding part of the transformation matrix then is (18.9). Since these $\alpha-1$ basic elements vanish by being identified with (18.4), it is seen from (18.10) that (19.8) is multiplied by $\frac{x-1}{x^{\alpha}-1}$. Of course it is the same for $\gamma=\alpha$, $\gamma'=1$.

We now consider the general case, both $\gamma<\alpha$ and $\gamma'<\alpha$. The set of subsurfaces (19.1) is made up of γ surfaces, each with $\frac{\alpha}{\gamma}$ boundaries from (18.3), and (19.2) is made up of γ' surfaces, each with $\frac{\alpha}{\gamma'}$ boundaries from (18.4). For each surface the number of boundaries is greater than 1, and their sum is homologous to zero. Thus the sum of the elements of (19.6) is zero for every value of ν and so is the sum of (19.7).

A homology base of (19.1) is now so chosen as to include the *first* $\alpha-\gamma$ elements of (18.3), thus excluding the last γ, as they can be expressed by the first $\alpha-\gamma$. By τ every element is replaced by the following except the last, $a'_{\alpha-\gamma-1}$, which is replaced by the element $a'_{\alpha-\gamma}$ not in the base. So we express it from (19.6) with $\nu=0$:

$$a'_{\alpha-\gamma} = -a' - a'_{\gamma} - a'_{2\gamma} - \cdots - a'_{\alpha-2\gamma}.$$

We then get the part of the transformation matrix of τ which belongs to the $\alpha-\gamma$ basic elements in question:

$$\begin{Bmatrix} 0 & 1 & 0 & \cdots & . & . & . & . & \cdots & . & . & \cdots \\ 0 & 0 & 1 & \cdots & . & . & . & . & \cdots & . & . & \cdots \\ . & . & . & \cdots & . & . & . & . & \cdots & . & . & \cdots \\ -1 & 0 & 0 & \cdots & 0 & -1 & 0 & 0 & \cdots & 0 & -1 & \cdots \end{Bmatrix}$$

the last row being alternately -1 and a group of $\gamma-1$ zeros. From this it is easily computed that the corresponding part of the polynomial, denoted above by $P_c(x^{\gamma})$, is $\frac{x^{\alpha}-1}{x^{\gamma}-1}$. As the same is true for the set (19.2) with γ' instead of γ, we may say that in (19.8) a factor

$$\frac{(x^{\alpha}-1)^2}{(x^{\gamma}-1)(x^{\gamma'}-1)} \tag{19.9}$$

is due to the boundaries of S'.

In choosing a homology base for (19.2) we take the $\alpha-\gamma'$ *last* elements of (18.4) into the base. So the part of the basic elements of S' corresponding to boundary curves of S' may be written

$$\begin{array}{l} a'\ a_1' \cdots a_{\gamma'-1}'\ a_{\gamma'}' \cdots a_{\alpha-\gamma-1}' \\ \qquad\qquad a_{\gamma'}'' \cdots a_{\alpha-\gamma-1}''\ a_{\alpha-\gamma}'' \cdots a_{\alpha-1}''. \end{array}$$

These two lines have the subscripts from γ' to $\alpha-\gamma-1$ in common and that is at least one subscript. Since

$$(\alpha-\gamma-1)-(\gamma'-1) = \alpha-(\gamma+\gamma')$$

and both γ and γ' are less than α, divide α and are relatively prime, this is positive. In the empty places of the first line we may substitute linear combinations of the elements written, and likewise in the second line. Then passing from S' to S means equating corresponding members of the two lines and reducing the system. From this it may be directly computed that the elements

$$a', a_{\gamma'}', \cdots, a_{\alpha-\gamma-1}'$$

may be taken as basic elements of the resulting system, and that they yield the factor

$$\frac{(x^{\alpha}-1)(x-1)}{(x^{\gamma}-1)(x^{\gamma'}-1)} \tag{19.10}$$

in the polynomium of S.

Instead of carrying out this direct computation we may obtain the result more easily in the following way, if we allow the ring of coefficients of the homology group to be complex. We first consider the α boundary curves (18.3) of the surface set (19.1). Putting

$$\varepsilon = e^{\frac{2\pi i}{\alpha}}$$

we introduce the linear transformation

$$\left\{\begin{array}{l} d_0 = a'+a_1'+a_2'+\cdots+a_{\alpha-1}' \\ d_1 = a'+\varepsilon^{-1}a_1'+\varepsilon^{-2}a_2'+\cdots+\varepsilon^{-(\alpha-1)}a_{\alpha-1}' \\ d_2 = a'+\varepsilon^{-2}a_1'+\varepsilon^{-4}a_2'+\cdots+\varepsilon^{-2(\alpha-1)}a_{\alpha-1}' \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ d_{\alpha-1} = a'+\varepsilon^{-(\alpha-1)}a_1'+\varepsilon^{-2(\alpha-1)}a_2'+\cdots+\varepsilon^{-(\alpha-1)^2}a_{\alpha-1}' \end{array}\right. \tag{19.11}$$

the determinant of which is $\neq 0$, since it is the product of all differences of $1, \varepsilon, \varepsilon^2, \cdots, \varepsilon^{\alpha-1}$. Then τ replaces d_0 by d_0, d_1 by εd_1, d_2 by $\varepsilon^2 d_2, \cdots, d_{\alpha-1}$ by $\varepsilon^{\alpha-1}d_{\alpha-1}$. So the multipliers of the set (19.11) are $1, \varepsilon, \varepsilon^2, \cdots, \varepsilon^{\alpha-1}$ respectively, i. e. all roots of the polynomial $x^{\alpha}-1$. If (18.3) were an independent set, the set (19.11) would be so too. But from (19.6) we have

$$\begin{array}{c} a_{\nu}'+a_{\nu+\gamma}'+\cdots+a_{\nu+\alpha-\gamma}' = 0, \\ \nu = 0, 1, \cdots, \gamma-1. \end{array} \tag{19.12}$$

From this we get $d_0 = 0$; moreover, since $\varepsilon^{\alpha} = 1$:

$$\begin{aligned} d_{\frac{\alpha}{\gamma}} &= a'+\varepsilon^{-\frac{\alpha}{\gamma}}a_1'+\varepsilon^{-\frac{2\alpha}{\gamma}}a_2'+\cdots+\varepsilon^{-\frac{(\alpha-1)\alpha}{\gamma}}a_{\alpha-1}' \\ &= a'+a_{\gamma}'+a_{2\gamma}'+\cdots+a_{\alpha-\gamma}' \\ &+\varepsilon^{-\frac{\alpha}{\gamma}}\left[a_1'+a_{1+\gamma}'+\cdots+a_{\alpha-\gamma+1}'\right] \\ &+\varepsilon^{-\frac{2\alpha}{\gamma}}\left[a_2'+a_{2+\gamma}'+\cdots+a_{\alpha-\gamma+2}'\right] \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &+\varepsilon^{-\frac{(\gamma-1)\alpha}{\gamma}}\left[a_{\gamma-1}'+a_{2\gamma-1}'+\cdots+a_{\alpha-1}'\right], \end{aligned}$$

hence $d_{\frac{\alpha}{\gamma}} = 0$ by (19.12). In the same way we get $d_{\frac{2\alpha}{\gamma}} = 0$, $d_{\frac{3\alpha}{\gamma}} = 0, \cdots, d_{\frac{(\gamma-1)\alpha}{\gamma}} = 0$. So the multipliers

$$1, \varepsilon^{\frac{\alpha}{\gamma}}, \varepsilon^{\frac{2\alpha}{\gamma}}, \cdots, \varepsilon^{\frac{(\gamma-1)\alpha}{\gamma}} \tag{19.13}$$

cancel; they are the roots of $x^{\gamma}-1$. Hence

$$\frac{x^{\alpha}-1}{x^{\gamma}-1}$$

is the corresponding part of $P_c(x^{\gamma})$, as stated before (19.9).

To pass from S' to S we put

$$a'_{\mu} = a''_{\mu}, \quad \mu = 0, 1, \cdots, \alpha-1.$$

Since we have relations from (19.7)

$$a''_{\nu} + a''_{\nu+\gamma'} + \cdots + a''_{\nu+\alpha-\gamma'} = 0, \quad \nu = 0, 1, \cdots, \gamma'-1,$$

we get the corresponding relations

$$(19.12\text{a}) \qquad \begin{aligned} a'_{\nu} + a'_{\nu+\gamma'} + \cdots + a'_{\nu+\alpha-\gamma'} = 0, \\ \nu = 0, 1, \cdots, \gamma'-1, \end{aligned}$$

besides (19.12). From this we infer in the same way that the multipliers

$$(19.14) \qquad 1, \varepsilon^{\frac{\alpha}{\gamma'}}, \varepsilon^{\frac{2\alpha}{\gamma'}}, \cdots, \varepsilon^{\frac{(\gamma'-1)\alpha}{\gamma'}}$$

cancel; they are the roots of $x^{\gamma'}-1$. Now (19.13) and (19.14) have only the multiplier 1 in common; for since γ and γ' are relatively prime, $\frac{\alpha}{\gamma}$ and $\frac{\alpha}{\gamma'}$ have α as least common multiple. Hence (19.10) is in fact the factor in the polynomial of S derived from the system (18.2) of geodesics used in decomposing S; the factor $x-1$ in the numerator of (19.10) is due to the fact that the multiplier 1 has been omitted twice.

Comparing (19.9) and (19.10) we see that multiplication by

$$\frac{x-1}{x^{\alpha}-1}$$

is the effect of passing from S' to the closed surface S so far as the bounding geodesics of S' are concerned. For S bounded,

the corresponding factor was $\frac{1}{x^{\alpha}-1}$. So we may sum up the effect of the step a) in the following way:

The effect of cancelling some elements of the homology base of S' by joining the boundary curves is multiplication of the polynomial $P_{S'}(x)$ (19.8) by the factor

$$(19.15) \qquad \frac{(x-1)^{\omega}}{x^{\alpha}-1}.$$

b) We now have to introduce some new elements in the homology base of S according to the new connections established by joining 2α boundary curves (18.3) and (18.4) of S'. This may be done in the following way. We choose a point in the interior of each region of the decomposition of S and denote the point chosen in the region c by $\{c\}$. Then we join the points $\{c\}$ and $\{c'\}$ by a segment b, crossing a in one point from the left hand side to the right hand side. In this way we join $\{\tau^{\mu} c\} = \{c_{\mu}\}$ and $\{\tau^{\mu} c'\} = \{c'_{\mu}\}$ by a segment b_{μ} crossing $\tau^{\mu} a = a_{\mu}$ in one point and not meeting any other of the geodesics (18.2); here $\mu = 0, 1, \cdots, \alpha-1$, and $b_0 = b$. As the equivalence class of c consists of γ regions, we have

$$\{c_{\nu}\} = \{c_{\nu+\gamma}\} = \{c_{\nu+2\gamma}\} = \cdots = \{c_{\nu+\alpha-\gamma}\}$$

and $\frac{\alpha}{\gamma}$ segments eradiate from this point. Subscripts of c and c' only count modulo γ and modulo γ' respectively.

The totality of these segments form a coherent complex, since S is coherent. This complex consists of $\gamma+\gamma'$ points and α segments. Hence it contains

$$(19.16) \qquad p_1 = 1 + \alpha - (\gamma+\gamma')$$

independent cycles. Then any p_1 independent cycles of this complex may be taken as a homology base of connection to complete the homology base of S.

$p_1 = 0$ arises only in the case $\gamma = \alpha$ and hence $\gamma' = 1$ (or inversely). In this case we have one subsurface c' and α subsurfaces $c, c_1, \cdots, c_{\alpha-1}$, each of which is adjacent to c' along one of the α geodesics; so evidently no new element has to be introduced in the base.

So we assume $\gamma < \alpha$, $\gamma' < \alpha$ and hence

$$\gamma \leqq \frac{\alpha}{2}, \quad \gamma' \leqq \frac{\alpha}{2} \quad \text{and} \quad p_1 > 0.$$

Any region c_i of the equivalence class of c neighbours any region c'_j of the equivalence class of c' along at least one geodesic $\tau^n a$. For since γ and γ' are relatively prime, if i is one of the numbers $0, 1, \cdots, \gamma - 1$ and j is one of the numbers $0, 1, \cdots, \gamma' - 1$, then n may be so chosen among the numbers $0, 1, \cdots, \gamma\gamma' - 1$, that $n \equiv i \pmod{\gamma}$ and $n \equiv j \pmod{\gamma'}$, and this in exactly one way. Then $\tau^n c = c_i$ neighbours $\tau^n c' = c'_j$ along $\tau^n a$. Since both γ and γ' divide α and are relatively prime, we have

$$\alpha = \lambda\gamma\gamma' \tag{19.17}$$

and infer that c_i and c'_j have exactly λ of the geodesics (18.2) as common boundary. So λ of the segments b_μ join the points $\{c_i\}$ and $\{c'_j\}$.

We first consider the simplest case $\lambda = 1$. So there goes exactly one segment b_μ from any point $\{c_i\}$ to any point $\{c'_j\}$. Any cycle of the complex consists of an even number of segments as the two equivalence classes of regions or, as we may say, of points, alternate. Thus the simplest cycle consists of four segments. Such a cycle is

$$k_0 = b_0 - b_{\gamma'} + b_{\gamma'+\gamma} - b_\gamma;$$

for b_0 goes from $\{c\}$ to $\{c'\}$, $-b_{\gamma'}$ from $\{c'_{\gamma'}\} = \{c'\}$ to $\{c_{\gamma'}\}$, $b_{\gamma'+\gamma}$ from $\{c_{\gamma'+\gamma}\} = \{c_{\gamma'}\}$ to $\{c'_{\gamma'+\gamma}\} = \{c'_\gamma\}$ and $-b_\gamma$ from $\{c'_\gamma\}$ to $\{c_\gamma\}$, which is the starting point $\{c\}$. From this we get α cycles

$$\begin{aligned} k_\mu &= b_\mu - b_{\mu+\gamma'} + b_{\mu+\gamma'+\gamma} - b_{\mu+\gamma}, \\ \mu &= 0, 1, \cdots, \alpha - 1, \end{aligned} \tag{19.18}$$

remembering that subscripts of b only count modulo α. These cycles k_μ are not independent, since their number is $> p_1$.

(19.18) contains a homology base of connection, i. e. p_1 cycles completing the homology base of S. To see this we have to

show that any cycle of the complex, which without restriction may be taken without double points, is a linear combination of the k_μ.

Any cycle with more than four segments is the sum of cycles with four segments each. In fact, let

$$Z = b_x - b_y + b_z - b_u + \cdots$$

be a cycle of n segments. If t is so determined as to satisfy

$$t \equiv z \pmod{\gamma'}, \quad t \equiv x \pmod{\gamma},$$

we may write

$$Z = b_x - b_y + b_z - b_t + b_t - b_u + \cdots.$$

Here the first four segments form a cycle, since $\{c'_z\} = \{c'_t\}$ and $\{c_x\} = \{c_t\}$. Omitting this cycle we reduce Z to $n-2$ segments and continue the same process. ($t \equiv u \pmod{\gamma'}$, since $t \equiv z$ and $z \equiv u \pmod{\gamma'}$.)

So we have to show that any cycle of four segments may be obtained by linear combination of the cycles (19.18). Since the last two segments in k_μ are equal to the first two segments of $k_{\mu+\gamma}$ with opposite sign, we get for any n

$$\begin{aligned} q_\mu &= k_\mu + k_{\mu+\gamma} + k_{\mu+2\gamma} + \cdots + k_{\mu+(n-1)\gamma} \\ &= b_\mu - b_{\mu+\gamma'} + b_{\mu+\gamma'+n\gamma} - b_{\mu+n\gamma}, \end{aligned}$$

from this for any y

$$\begin{aligned} & q_\mu + q_{\mu+\gamma'} + q_{\mu+2\gamma'} + \cdots + q_{\mu+(y-1)\gamma'} \\ &= b_\mu - b_{\mu+y\gamma'} + b_{\mu+y\gamma'+n\gamma} - b_{\mu+n\gamma} \end{aligned}$$

and finally, replacing μ by $\mu + x\gamma$ for any x and putting $x + n = z$,

$$Z = b_{\mu+x\gamma} - b_{\mu+x\gamma+y\gamma'} + b_{\mu+z\gamma+y\gamma'} - b_{\mu+z\gamma}.$$

This cycle Z goes from the point

$$\{c_\mu\} = \{c_{\mu+x\gamma}\} \text{ to } \{c'_{\mu+x\gamma}\} = \{c'_{\mu+x\gamma+y\gamma'}\},$$

from there to $\{c_{\mu+x\gamma+y\gamma'}\} = \{c_{\mu+y\gamma'}\} = \{c_{\mu+z\gamma+y\gamma'}\}$,

from there to $\{c'_{\mu+z\gamma+y\gamma'}\} = \{c'_{\mu+z\gamma}\}$,

from there to $\{c_{\mu+z\gamma}\} = \{c_\mu\}$.

Now, since the four numbers μ, x, y, z, may be chosen arbitrarily and γ and γ' are relatively prime, the points $\{c_\mu\}$, $\{c_{\mu+y\gamma'}\}$ and $\{c'_{\mu+x\gamma}\}$, $\{c'_{\mu+z\gamma}\}$ may be any given points in the equivalence classes of $\{c\}$ and $\{c'\}$ respectively. So Z becomes any cycle composed of four segments. This completes the proof.

Relations between the generating cycles (19.18) are readily found. Taking $n = \frac{\alpha}{\gamma}$ in q_μ and remembering that subscripts of b only count modulo α, we find $q_\mu = 0$ for this particular n, thus

$$k_\mu + k_{\mu+\gamma} + k_{\mu+2\gamma} + \cdots + k_{\mu+\alpha-\gamma} = 0, \quad (\mu = 0, 1, \cdots, \gamma-1), \tag{19.19}$$

and in the same way

$$k_\mu + k_{\mu+\gamma'} + k_{\mu+2\gamma'} + \cdots + k_{\mu+\alpha-\gamma'} = 0 \quad (\mu = 0, 1, \cdots, \gamma'-1). \tag{19.19 a}$$

These $\gamma+\gamma'$ relations between the α generators are, however, not independent, since the sum of all γ left hand members of (19.19) is equal to the sum of all γ' left hand members of (19.20). So one relation is abundant. Hence we may cancel $\gamma+\gamma'-1$ generators properly chosen and so are left with p_1 generators in accordance with (19.16).

We now complete the homology base of S by adding these p_1 new generators derived from the connections. The matrix Δ corresponding to the transformation class τ of S then takes the form

$$\Delta = \left\{ \begin{array}{c|c} I & II \\ \hline III & IV \end{array} \right\}$$

IV being a matrix with p_1 rows and columns. If there are δ elements in all in the base of S, the transforms of the first $\delta - p_1$ elements (derived from S' by the considerations under a)) do not contain the p_1 new generators. So all elements of II are zero, and we have

$$\left| \Delta - xE_\delta \right| = \left| I - xE_{\delta-p_1} \right| \left| IV - xE_{p_1} \right|.$$

Now take any of the new generators, k_μ say. The intersection numbers of k_μ with $a_\mu, a_{\mu+\gamma'}, a_{\mu+\gamma'+\gamma}, a_{\mu+\gamma}$ are in turn $1, -1, 1, -1$, and with all other geodesics (18.2) they are zero. The transform of k_μ then must have the same intersection numbers $1, -1, 1, -1$ with $a_{\mu+1}, a_{\mu+\gamma'+1}, a_{\mu+\gamma'+\gamma+1}, a_{\mu+\gamma+1}$, and zero with the rest. From this it follows that it contains $k_{\mu+1}$ with coefficient 1 and all other elements of (19.18) with coefficient zero; if $k_{\mu+1}$ is not in the base, it has of course to be replaced by its expression by the p_1 elements chosen. From this the matrix IV may be derived. III does not matter, since II vanishes.

To compute the polynomial

$$(-1)^{p_1} \left| IV - xE_{p_1} \right| \tag{19.20}$$

we follow the same way as in proving (19.10) by means of the transformation (19.11). In fact, the deduction is literally the same. Instead of the elements $a', a'_1, \cdots, a'_{\alpha-1}$ of (19.11) we have the α elements (19.18). The relations (19.12) and (19.12 a) correspond exactly to the relations (19.19) and (19.19 a). So we find (19.20) equal to (19.10).

We now have the following result: (19.8) was the polynomial $P_{S'}(x)$ belonging to the transformation class τ of the decomposed surface S', and we had to consider the effect on it from joining the boundary curves in order to get the surface S. This means first cancelling some elements of the homology base of S' with the effect of introducing the factor (19.15) into the polynomial, and then introducing new elements into the homology base of S with the effect of introducing the factor (19.10). Hence

$$(19.21)\qquad P(x) = (x-1)^{1+\omega}\prod_{l} f_l(x^{\gamma})\prod_{l'} f_{l'}(x^{\gamma'})$$

belongs to the transformation class τ of S.

This is in accordance with the theorem stated by (16.3). For since a is not amphidrome, no factor arises from the geodesics (18.2); and all regions of S' and all other geodesics of S' (if any) evidently play the same rôle in S.

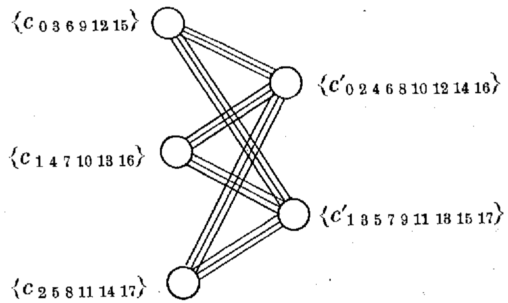

Fig. 8.

Finally an addition has to be made, as we have assumed the factor λ in (19.17) to be 1.

Let λ be greater than 1. Each point $\{c_i\}$ is connected with each point $\{c'_j\}$ by λ segments. In fig. 8 the case $\alpha = 18$, $\gamma = 3$, $\gamma' = 2$, thus $\lambda = 3$, is illustrated in a schematic way; each of the five points carries all subscripts belonging to it; so e. g. $\{c_0\} = \{c_3\} = \{c_6\} = \{c_9\} = \{c_{12}\} = \{c_{15}\}$. Any segment b_μ is one of the segments leading from the c-point with subscript μ to the c'-point with subscript μ. All segments connecting these two points then are

$$(19.22)\qquad b_\mu,\ b_{\mu+\gamma\gamma'},\ b_{\mu+2\gamma\gamma'},\ \cdots,\ b_{\mu+(\lambda-1)\gamma\gamma'}.$$

From these we may form λ cycles

$$(19.23)\quad \begin{cases} l_\mu = b_\mu - b_{\mu+\gamma\gamma'} \\ l_{\mu+\gamma\gamma'} = b_{\mu+\gamma\gamma'} - b_{\mu+2\gamma\gamma'} \qquad (\mu = 0, 1, \cdots, \gamma\gamma'-1) \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ l_{\mu+(\lambda-1)\gamma\gamma'} = b_{\mu+(\lambda-1)\gamma\gamma'} - b_\mu. \end{cases}$$

The sum of these λ cycles is zero for every μ. Letting μ range from 0 to $\gamma\gamma'-1$, we get α cycles in all, which for short we call the l-cycles.

The l-cycles (19.23) together with the the k-cycles (19.18) contain a homology base of connection consisting of p_1 generating cycles of the b-complex: Any segment of a k-cycle may be replaced by another segment connecting the same two points by adding or subtracting some suitable l-cycles. So modulo the subgroup generated by the l-cycles we may look upon the k-cycles as if we join all λ segments (19.22) to one string. The building up of the complex from k-cycles of these strings then is the same as in the case $\lambda = 1$.

If we take $n' = \gamma'$ in the cycle q_μ used previously, we get

$$q_\mu = b_\mu - b_{\mu+\gamma'} + b_{\mu+\gamma'+\gamma'\gamma} - b_{\mu+\gamma'\gamma}.$$

Replacing μ in turn by $\mu+\gamma'$, $\mu+2\gamma'$, $\cdots$, $\mu+(\gamma-1)\gamma'$ and adding we get the cycle

$$\begin{aligned} u_\mu &= b_\mu - b_{\mu+\gamma\gamma'} + b_{\mu+2\gamma\gamma'} - b_{\mu+\gamma\gamma'} \\ &= l_\mu - l_{\mu+\gamma\gamma'}. \end{aligned}$$

Hence by replacing μ we get the cycles

$$\begin{aligned} u_\mu &= l_\mu - l_{\mu+\gamma\gamma'} \\ u_{\mu+\gamma\gamma'} &= l_{\mu+\gamma\gamma'} - l_{\mu+2\gamma\gamma'} \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ u_{\mu+(\lambda-1)\gamma\gamma'} &= l_{\mu+(\lambda-1)\gamma\gamma'} - l_\mu, \end{aligned}$$

since $l_{\mu+\alpha} = l_\mu$. Multiplying in turn by λ, $\lambda-1$, $\lambda-2$, $\cdots$, 1 and adding, we get

$$\lambda u_\mu + (\lambda-1)\,u_{\mu+\gamma\gamma'} + \cdots + u_{\mu+(\lambda-1)\gamma\gamma'} = \lambda\, l_\mu,$$

since the sum of the cycles (19.23) is zero for μ fixed.

So λl_μ is seen to be a linear combination of the k-cycles (19.18) for $\mu = 0, 1, \cdots, \alpha-1$. Now if any cycle Z belongs to a certain multiplier (i. e. root of the characteristic polynomial), so does λZ, and λZ is expressible by the k-cycles. In other words, if we allow the ring of coefficients of the homology group of connections to consist of all rational numbers, then p_1 of the k-cycles (19.18), properly chosen, may be taken as generators. We thus get the same part IV of the matrix $\varDelta$ and the same characteristic polynomial (19.21) as in the case $\lambda = 1$.

20. Third part of the proof. In this section we assume that the geodesic a is not amphidrome, that S is decomposed by the geodesics $a, a_1, \cdots, a_{\alpha-1}$ of (18.2), and that the region c to the left of a (thus with the left hand border a' of (18.3) on its boundary) has also one of the right hand borders (18.4) on its boundary. This is transformed into a by a certain power of τ, and so the region to the right of a is equivalent to c. Hence we get only one equivalence class of regions

$$c,\ \tau c = c_1,\ \tau^2 c = c_2,\ \cdots,\ \tau^{\gamma-1} c = c_{\gamma-1} \tag{20.1}$$

instead of (19.1) and (19.2). Moreover $\gamma > 1$, since S is decomposed.

Let $\tau^h c = c_h$ be the region of the set (20.1) which is adjacent to c along a. Then, c_μ being the region to the left of a_μ, $c_{\mu+h}$ is the region to the right of a_μ, and $c_{\mu+h} = \tau^h c_\mu$. So we may pass from any region to any other region by a power of τ^h. From this we infer that h and γ are relatively prime. h and α may or may not be relatively prime.

Subscripts of c only counting modulo γ, we may write the set (20.1) in another way:

$$c,\ c_h,\ c_{2h},\ \cdots,\ c_{(\gamma-1)h}. \tag{20.2}$$

In this arrangement they form a ring, each region c_μ neighbouring $c_{\mu-h}$ and $c_{\mu+h}$ only.

The α geodesics may be arranged in the following way, if we put

$$\alpha = \gamma \xi,$$

and count their subscripts modulo α:

$$\left\{\begin{array}{lllll} a & a_\gamma & a_{2\gamma} & \cdots & a_{(\xi-1)\gamma} \\ a_h & a_{h+\gamma} & a_{h+2\gamma} & \cdots & a_{h+(\xi-1)\gamma} \\ a_{2h} & a_{2h+\gamma} & a_{2h+2\gamma} & \cdots & a_{2h+(\xi-1)\gamma} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{(\gamma-1)h} & a_{(\gamma-1)h+\gamma} & & \cdots & a_{(\gamma-1)h+(\xi-1)\gamma}. \end{array}\right. \tag{20.3}$$

The geodesics of the rows beginning with $a_{\mu h}$ and $a_{(\mu-1)h}$ are on the boundary of $c_{\mu h}$, and these two rows bound $c_{\mu h}$ in

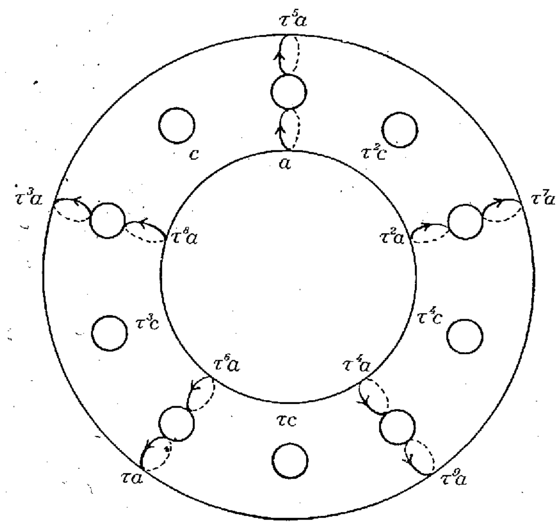

Fig. 9.

opposite senses. Using the notation (18.3) and (18.4) we may say that

$$a'_{\mu h} + a'_{\mu h+\gamma} + \cdots + a'_{\mu h+(\xi-1)\gamma} - (a''_{(\mu-1)h} + \cdots + a''_{(\mu-1)h+(\xi-1)\gamma}) \tag{20.4}$$

is part of the boundary of c_μ.

To illustrate these facts, fig. 9 shows the case $\alpha = 10$, $\gamma = 5$, $h = 2$. One may replace $h = 2$ by $h = 3$ and so obtain a case in which h and α are relatively prime.

The construction of the characteristic polynomial in the case of this section goes along similar lines as in the preceding sections. Let

$$P_c(x) = (x-1)\prod_l \frac{\left(x^{\beta_l n_l} - 1\right)^{2q_l + s_l - 2}}{\left(x^{\beta_l m_{l1}} - 1\right)\cdots}$$

be the polynomial (16.3) belonging to c under the transformation class τ^{γ}. Then by section 17 $P_c(x^{\gamma})$ is the polynomial of the decomposed surface S' under the transformation class τ. We now have to take the same two steps as in the preceding section, a) passing from S' to S by joining boundary curves and b) introducing a homology base of connection.

a) If S is bounded, all elements a' and a'' of (18.3) and (18.4) may be included in a homology base of S'. The effect of joining the boundary curves then simply means cancelling the a''. As in section 19, this means multiplication of $P_c(x^{\gamma})$ by $\frac{1}{x^{\alpha}-1}$.

So let S be closed. Then (20.4) is the complete boundary of c_{μ} and we have γ relations, which are obviously independent,

$$(20.5) \qquad (a'_{\mu h} + a'_{\mu h+\gamma} + \cdots) - (a''_{(\mu-1)h} + a''_{(\mu-1)h+\gamma} + \cdots) = 0 \qquad (\mu = 0, 1, \cdots, \gamma-1)$$

for the a' and a'' in the homology base of S'. One of these γ relations may be replaced by the sum of all:

$$(20.6) \qquad a' + a'_1 + \cdots + a'_{\alpha-1} = a'' + a''_1 + \cdots + a''_{\alpha-1}.$$

Both the left hand member and the right hand member of this equation are homology elements with the multiplier 1. The effect of putting them equal to one another is to cancel the factor $x-1$. So if (20.6) were not used, $P_c(x^{\gamma})$ would have to be multiplied by $(x-1)$.

Now passing from S' to S means replacing every a'' by the corresponding a'. All a'' may be included in the homology base of S', as the relations (20.5) may be used for eliminating some of the a'. The cancelling of the a'' means multiplication by $(x^{\alpha}-1)^{-1}$. As to the a' the only effect on the polynomial arises from the fact that (20.6) is satisfied identically; thus the factor $x-1$ has to be restored. In all, we get the factor $\frac{x-1}{x^{\alpha}-1}$, if S is closed.

To sum up for S bounded or closed, the effect of the operation a) is to multiply $P_c(x^{\gamma})$ by

$$(20.7) \qquad \frac{(x-1)^{\omega}}{x^{\alpha}-1}.$$

b) To set up a homology base of connection we again introduce the segments b_{μ}, $\mu = 0, 1, \cdots, \alpha-1$, crossing a_{μ} from the left hand side to the right hand side; here they connect the points $\{c_{\mu}\}$ and $\{c_{\mu+h}\}$. As we have a coherent complex of γ points and α segments, we get

$$p_1 = 1 + \alpha - \gamma$$

independent cycles in the complex; cf. (19.16). From the b_{μ} we form the cycles

$$l_{\mu} = b_{\mu} - b_{\mu+\gamma}, \quad \mu = 0, 1, \cdots, \alpha-1,$$

and

$$k = b_0 + b_h + b_{2h} + \cdots + b_{(\gamma-1)h}.$$

l_{μ} runs from $\{c_{\mu}\}$ to $\{c_{\mu+h}\}$ and back to $\{c_{\mu}\}$, since both c_{μ} and $c_{\mu+h}$ are left invariant by τ^{γ}. The cycle k runs once through the ring formed by the γ regions (20.2). It is easily seen that the $\alpha+1$ cycles l_{μ} and k contain a homology base of connection: Any cycle formed of the b_{μ} which on its way runs from the point $\{c_{\mu}\}$ of some region c_{μ} to the point $\{c_{\mu\pm h}\}$ of one of the two neighbouring regions $c_{\mu\pm h}$ and then returns to $\{c_{\mu}\}$ may be reduced to a cycle of fewer segments by adding or subtracting some suitable cycles l_{μ}. Any cycle which runs once through the ring may be reduced to k by adding a suitable combination of the l_{μ}.

The l_{μ} fulfill γ relations

$$(20.8) \qquad l_{\mu} + l_{\mu+\gamma} + l_{\mu+2\gamma} + \cdots l_{\mu+(\xi-1)\gamma} = 0, \qquad \mu = 0, 1, \cdots, \gamma-1,$$

by means of which γ of the l_{μ} may be eliminated. The remaining $\alpha-\gamma$ cycles l_{μ} together with k then are independent and form a homology base of connection of p_1 elements.

The setting up of the part *IV* of the transformation matrix due to these p_1 basic elements is easily performed as in the preceding section and the factor corresponding to (19.20) computed: From the invariance of intersection numbers it is seen that the transform of l_μ contains $l_{\mu+1}$ with coefficient 1 and no other l-cycles nor k. Likewise the transform of k contains k with coefficient 1. So from k the factor $x-1$ is derived, and from the l-cycles we get $\frac{x^\alpha-1}{x^\gamma-1}$ in taking account of the relations (20.8); this follows directly from the computation attached to (19.11) and (19.12).

So in all we have to multiply by

$$\frac{(x^\alpha-1)(x-1)}{x^\gamma-1} \tag{20.9}$$

as a result of operations b).

As the final result of both a) and b) we have: In passing from S' to S the polynomial $P_c(x^\gamma)$ of S' has to be multiplied by the factors (20.7) and (20.9), thus giving the polynomial $P(x)$ of S for the transformation class τ:

$$P(x) = (x-1)^{1+\omega}\prod_l \frac{(x^{\gamma\beta_l n_l}-1)^{2q_l+s_l-2}}{(x^{\gamma\beta_l m_{l1}}-1)\cdots}.$$

This is in fact the polynomial set up in (16.3), since a is not amphidrome and any kernel of c passes into kernels of c_1, $c_2, \cdots, c_{\gamma-1}$ before returning to c.

In sections 18, 19, 20 we have given the proof of the theorem stated in (16.3), in case the geodesics of the equivalence class used in dividing S are not amphidrome. In the following section we complete the proof by dealing with an amphidrome a.

21. Fourth part of the proof. In this section we assume a to be amphidrome. If again α is the number assigned to a in section 10 or 12, α is even, and there are $\frac{\alpha}{2}$ different geodesics

$$a,\ \tau a = a_1,\ \tau^2 a = a_2,\ \cdots,\ \tau^{\frac{\alpha}{2}-1} a = a_{\frac{\alpha}{2}-1} \tag{21.1}$$

in the equivalence class of a. Let the left hand and right hand borders of $a, a_1, \cdots$ as before be denoted by $a', a_1', \cdots$ and $a'', a_1'', \cdots$ respectively. They are oriented in the same way as the geodesics (21.1).

We first assume that S is not decomposed by the system (21.1). So S' consists of one region c only. Let

$$P_c(x) = (x-1)\prod_l \frac{(x^{\beta_l n}-1)^{2q+s-2}}{(x^{\beta m_{l1}}-1)\cdots}$$

be the characteristic polynomial of c for the transformation class τ. The $\frac{\alpha}{2}$ geodesics (21.1) yield α boundary curves of c.

If S is bounded, we may take the curves

$$a', a_1', \cdots, a'_{\frac{\alpha}{2}-1}, -a'', -a_1'', \cdots, -a''_{\frac{\alpha}{2}-1} \tag{21.2}$$

as members of a homology base of S'. As they are interchanged cyclically by τ, we get $x^\alpha-1$ as the factor in $P_c(x)$ derived from these α boundary curves. If S is closed, the sum of the elements (21.2) is zero. We then get $\frac{x^\alpha-1}{x-1}$ as the corresponding factor.

Taking both cases together, we get

$$\frac{x^\alpha-1}{(x-1)^\omega}$$

as the factor of $P_c(x)$ derived from the boundaries (21.1). Compare the corresponding more elaborate proof in section 18.

Now joining the boundary curves in turn, the a''-elements cancel and we retain the a'-elements, which we may denote as a-elements. Then, by τ, a is replaced by $a_1, \cdots, a_{\frac{\alpha}{2}-2}$ by $a_{\frac{\alpha}{2}-1}$ and this by $-a$. No homology relation exists between them. The corresponding polynomial is easily computed to be

$$x^{\frac{\alpha}{2}}+1.$$

So the effect of joining the boundary curves of S' is to multiply $P_c(x)$ by

$$\frac{(x-1)^{\omega}}{x^{\frac{\alpha}{2}}-1}. \tag{21.3}$$

To introduce a homology base of connection we choose a point $\{c\}$ in c and denote by b_μ $(\mu = 0, 1, \cdots, \alpha-1)$ a cycle which starts from $\{c\}$, crosses $\tau^\mu a$ once from left to right, and returns to $\{c\}$. Since

$$\tau^{\mu+\frac{\alpha}{2}} a = -\tau^\mu a,$$

we may choose the cycles b_μ so that

$$b_{\mu+\frac{\alpha}{2}} = -b_\mu.$$

It is seen from intersection numbers that b_μ has the coefficient zero in the transforms of all basic elements derived from S' and in the transforms of all b-cycles except $b_{\mu-1}$; in the transform of $b_{\mu-1}$ the coefficient of b_μ is 1. So the contribution of the b-elements to the polynomial is that of α elements which are interchanged cyclically by τ and which satisfy the condition

$$b_\mu + b_{\mu+\frac{\alpha}{2}} = 0.$$

It then follows (e. g. by a transformation analogous to (19.11)), that the corresponding factor in the polynomial is $\dfrac{x^\alpha-1}{x^{\frac{\alpha}{2}}-1}$.

So the polynomial $P(x)$ of S is found by multiplying $P_c(x)$ by this factor and (21.3):

$$P(x) = \frac{(x-1)^{\omega}}{x^{\frac{\alpha}{2}}-1} \cdot \frac{x^\alpha-1}{x^{\frac{\alpha}{2}}-1} \cdot (x-1) \prod_l .$$

The part $\dfrac{x^\alpha-1}{\left(x^{\frac{\alpha}{2}}-1\right)^2}$ has to be taken into the sign $\prod_l$, since it is the factor corresponding to the amphidrome bands arising from the axes (21.1); cf. (16.5) with the notation $\beta_l = \dfrac{\alpha}{2}$. The result

then obviously is in accordance with (16.3) and the explanation attached to that formula.

Secondly we assume that S is decomposed by the system (21.1). Let c be the region to the left and c_1 the region to the right of a. These two regions are not identical, otherwise any two adjacent regions would be identical and S would not be decomposed. Under the transformation class $\tau^{\frac{\alpha}{2}}$ all geodesics (21.1) are inverted, hence $\tau^{\frac{\alpha}{2}} c = c_1$. Thus all those geodesics of the set (21.1) which bound c must bound c_1 too, and vice versa. If these were not all geodesics (21.1), S would not be coherent. So we infer that there are exactly two regions; c and c_1; the number γ of regions in the equivalence class of c is 2, and we have $\tau c = c_1$, $\tau^2 c = c$. Thus $\dfrac{\alpha}{2}$ must be an odd number.

Let the above polynomial $P_c(x)$ belong to c. Then, as pointed out in section 17, the polynomial

$$P_{S'}(x) = P_c(x^2) = (x^2-1) \prod \frac{\left(x^{2\beta_l n_l}-1\right)^{2q_l+s_l-2}}{\left(x^{2\beta_l m_{l1}}-1\right)\cdots}$$

belongs to the decomposed surface S'.

The curves (21.2) are on the boundary of S' and are interchanged cyclically by τ. If S is bounded, both c and c_1 have boundaries not belonging to (21.2). So these α curves may be taken as members of a homology base of S' and yield the factor $x^\alpha-1$ as before. If S is closed, we remember that all even powers of τ transform c into itself and c_1 into itself. So

$$a' + a_2' + a_4' + \cdots + a'_{\frac{\alpha}{2}-1} - a_1'' - a_3'' - \cdots - a''_{\frac{\alpha}{2}-2} = 0, \tag{21.4}$$

since these curves constitute the complete boundary of c, and

$$a_1' + a_3' + a_5' + \cdots + a'_{\frac{\alpha}{2}-2} - a'' - a_2'' - \cdots - a''_{\frac{\alpha}{2}-1} = 0, \tag{21.5}$$

since these curves constitute the complete boundary of c_1. Then using a transformation analogous to (19.11) we see that $\dfrac{x^\alpha-1}{x^2-1}$ is the factor in $P_{S'}(x)$ derived from the boundaries. Taking both cases together

$$\frac{x^{\alpha}-1}{(x^2-1)^{\omega}} \tag{21.6}$$

is the factor of $P_{S'}(x)$ derived from the boundaries (21.2).

We now join the boundaries by putting

$$a''_{\mu} = a'_{\mu}, \quad \mu = 0, 1, \cdots, \frac{\alpha}{2}-1,$$

and denoting it simply by a_{μ}. In the sequence (21.1) each element is by τ replaced by the following and the last element, $a_{\frac{\alpha}{2}-1}$, is replaced by $-a$. If S is bounded, no homology relation has to be observed, and the corresponding factor in the polynomial is as in the first case equal to

$$x^{\frac{\alpha}{2}}+1 = \frac{x^{\alpha}-1}{x^{\frac{\alpha}{2}}-1}.$$

If S is closed, we get from (21.4)

$$a - a_1 + a_2 - a_3 + \cdots + a_{\frac{\alpha}{2}-1} = 0,$$

and the same relation is derived from (21.5). From this we may express $a_{\frac{\alpha}{2}-1}$. The matrix corresponding to the transformation of the set $a, a_1, \cdots, a_{\frac{\alpha}{2}-2}$ then becomes

$$\left\{\begin{array}{cccccc} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & 1 \\ -1 & 1 & -1 & 1 & \cdots & 1 \end{array}\right\}$$

with $\frac{\alpha}{2}-1$ rows and columns. Subtracting $xE_{\frac{\alpha}{2}-1}$ and then adding the second, third, etc. columns with coefficients x, x^2, etc. to the first, one finds the corresponding polynomial equal to

$$-\left[(1-x)\,x^{\frac{\alpha}{2}-2} - x^{\frac{\alpha}{2}-3} + x^{\frac{\alpha}{2}-4} \cdots + x - 1\right]$$

$$= x^{\frac{\alpha}{2}-1} - x^{\frac{\alpha}{2}-2} + x^{\frac{\alpha}{2}-3} - \cdots - x + 1$$

$$= \frac{x^{\frac{\alpha}{2}}+1}{x+1} = \frac{(x^{\alpha}-1)(x-1)}{\left(x^{\frac{\alpha}{2}}-1\right)(x^2-1)}.$$

Taking both cases together, we find that

$$\frac{(x^{\alpha}-1)(x-1)^{\omega}}{\left(x^{\frac{\alpha}{2}}-1\right)(x^2-1)^{\omega}} \tag{21.7}$$

is the factor corresponding to the $\frac{\alpha}{2}-\omega$ basic geodesics after joining the boundary curves.

From a comparison of (21.6) and (21.7) we find that multiplication of $P_{S'}(x)$ by

$$\frac{(x-1)^{\omega}}{x^{\frac{\alpha}{2}}-1} \tag{21.8}$$

is the effect of joining the boundary curves (21.2) of S'.

Finally we have to introduce a homology base of connection. We choose a point $\{c\}$ in c and a point $\{c_1\}$ in c_1 and connect them by α segments, the segment b_{μ} crossing $\tau^{\mu}a$ from left to right. Since $a_{\mu+\frac{\alpha}{2}} = -a_{\mu}$, we take $b_{\mu+\frac{\alpha}{2}} = -b_{\mu}$. So we have two points and $\frac{\alpha}{2}$ segments in a coherent complex, thus $\frac{\alpha}{2}-1$ independent cycles. (For $\alpha = 2$ there is only one common boundary of c and c_1 and no new cycle has to be introduced.) We form the cycles

$$l_{\mu} = b_{\mu} + b_{\mu+1}, \quad \mu = 0, 1, \cdots, \alpha-1.$$

This system of α cycles evidently contains a homology base of connection. Relations are obvious: We have

$$l_\mu + l_{\mu+\frac{\alpha}{2}} = 0 \tag{21.9}$$

and moreover

$$l_\mu - l_{\mu+1} + l_{\mu+2} - \cdots + l_{\mu+\frac{\alpha}{2}-1} = 0. \tag{21.10}$$

$\frac{\alpha}{2}$ of the relations (21.9) together with one of the relations (21.10) form an independent set of relations, and we are left with $\frac{\alpha}{2}-1$ independent cycles l_μ.

If we introduce

$$\varepsilon = e^{\frac{2\pi i}{\alpha}}$$

and set up the transformation analogous to (19.11), we would get all α powers of ε as multipliers, if the α cycles l_μ were independent. Because of (21.9) all even powers of ε cancel, and because of (21.10) $\varepsilon^{\frac{\alpha}{2}} = -1$ cancels too. So we get

$$\frac{x^\alpha - 1}{\left(x^{\frac{\alpha}{2}} - 1\right)(x+1)} = \frac{(x^\alpha - 1)(x-1)}{\left(x^{\frac{\alpha}{2}} - 1\right)(x^2-1)} \tag{21.11}$$

as the contribution of the homology base of connections. (For $\alpha = 2$ this factor is equal to 1.)

We thus get the following result: In passing from S' to S we get the polynomial $P(x)$ corresponding to the transformation class τ of S by multiplying $P_{S'}(x)$ by (21.8) on account of joining the boundary curves and by (21.11) on account of the new connections thereby established. So we get

$$P(x) = (x-1)^{1+\omega} \cdot \frac{x^\alpha - 1}{\left(x^{\frac{\alpha}{2}} - 1\right)^2} \cdot \prod_l .$$

The second factor has to be taken under the sign $\prod_l$ as the contribution of the amphidrome bands corresponding to the amphidrome geodesics (21.1); cf. (16.5). So the result is clearly seen to be in accordance with (16.3) and the explanation attached to it.

This completes the proof of the main homology theorem stated by formula (16.3).

22. Final remarks. In section 14 we have constructed a transformation ζ of S, belonging to a prescribed transformation class τ of algebraically finite type, such that classes of fixed points with index $j = 0$ are completely avoided. We may therefore term such classes *unessential.* Classes with index $j \neq 0$, which we term *essential classes,* cannot, of course, be avoided, but are by ζ "satisfied" by one point each.

The function (16.1)

$$P(x) = (-1)^\delta \left| \Delta - x E_\delta \right|$$

is a polynomial in x. So if we write it in the fractional form (16.3), the denominator divides the numerator. It will now be pointed out that one of the advantages of writing $P(x)$ in the fractional form (16.3) is to put all essential classes of fixed points, together with the indices of these classes, into evidence. For this purpose we examine the factors corresponding to the different values of l, which, it will be remembered, ranges over all equivalence classes of kernels, i. e. regions or amphidrome geodesics, of S. — The statement as to the fixed points of ζ at the end of section 14 should be compared.

If $\beta_l > 1$, the β_l kernels of the equivalence class in question are interchanged cyclically by τ, so they do not give rise to any fixed point of the special transformation ζ of section 14 and hence not to any essential class.

If $\beta_l = 1$ and the kernel is an amphidrome geodesic, it is seen from (16.5) that the corresponding factor is $\frac{x^2-1}{(x-1)^2}$. We then associate the two factors $x-1$ of the denominator with the two essential classes of fixed points, each of index $j = 1$, which are known to arise in the amphidrome band in the construction of ζ, each class being represented by one point.

If $\beta_l = 1$ and the kernel is a region, we may first have $n_l > 1$. The region is then mapped into itself by ζ in such a way that ζ^{n_l} is the identical transformation, thus ζ is periodic in the region with n_l as its order. (The region is then kernel of τ^{n_l}.) Fixed points of ζ are such ramification points of the region over its modular surface M_l for which all n sheets hang together, thus for which the corresponding number m in the set

$$m_{l1}, m_{l2}, \cdots, m_{lu_l} \tag{22.1}$$

has the value 1. Hence we get as many essential classes as the number 1 occurs in the set (22.1), and the index of each class is $j = 1$.

If $\beta_l = 1$ and $n_l = 1$, the region is a kernel of τ itself and so is identical with its modular surface $M_l = \overline{M}_l$. The minimum number δ_l of generators of its Poincaré group is given by (16.4). Putting $\mu = 0$ and δ_l instead of ν in (5.1), we get the index

$$j_l = 2 - s_l - 2\,q_l < 0.$$

Since there are no ramification points, the set (22.1) is empty, and the factor in (16.3) corresponding to a region with $\beta_l n_l = 1$ is

$$(x-1)^{-j_l}.$$

To sum up, we have the following theorem:

The number of classes of fixed points with index $j = 1$ is equal to the number of factors with $\beta_l m_{li} = 1$ in the denominator of $P(x)$ written in the fractional form (16.3). *The number of classes with negative index is equal to the number of values of l with $\beta_l n_l = 1$, and the corresponding indices are the exponents with the opposite sign $2 - s_l - 2q_l$.*

It is seen that this result is due to a close combination of methods of homotopy with methods of homology. It is easy to deduce from our theorem a well known theorem of pure homology theory concerning the algebraic sum Ξ of all indices. This theorem makes use of the trace of the transformation matrix Δ. This trace does not depend on the choice of the homology base used and is equal to the sum of the roots of the characteristic polynomial $P(x)$. So we have to determine this sum of roots from (16.3). The initial factor $(x-1)^{1+\omega}$ yields $1+\omega$ as its contribution. In $\prod_l$ a factor of the numerator with $\beta_l n_l > 1$ or of the denominator with $\beta_l m_{li} > 1$ has the sum of its roots equal to zero. A factor of the numerator with $\beta_l n_l = 1$ yields

$$2q_l + s_l - 2 = -j_l$$

as its contribution, a factor of the denominator with $\beta_l m_{li} = 1$ yields 1 and counts for -1, since it is placed in the denominator. So if we denote by Ξ^- the sum of all negative indices and by Ξ^+ the sum of all positive indices, we get

$$\text{trace } \Delta = 1 + \omega - \Xi^- - \Xi^+,$$

hence

$$\Xi = \Xi^- + \Xi^+ = 1 + \omega - \text{trace } \Delta.$$

This formula is due to J. W. Alexander [1] in its first form concerning surface transformations. It has received a far-reaching generalization by the investigations of S. Lefschetz [10, 11] and these have been treated in a modified form by H. Hopf [7]. It is seen from the present paper how it is possible for transformation classes of algebraically finite type to split the algebraic sum Ξ given by the trace formula into its different terms, positive or negative, due to the single essential classes of fixed points. To do this requires not only taking into consideration the sum of roots of the characteristic polynomial of the transformation class, but this polynomial itself.

It is seen from (16.3) that all roots of $P(x)$ are roots of unity. Now the roots of the polynomial belonging to τ^n are the n-th powers of the roots of $P(x)$. Owing to this the polynomial of τ^n is easily deduced from the polynomial of τ. Hence trace Δ^n is limited for all values of n, and so is $\Xi(\tau^n)$. This justifies the notation "algebraically finite type" for the transformation classes in question. But the full justification lies in a conjecture, which I have so far not been able to prove: It will be remembered that in section 8 classes of algebraically finite type were defined by the character that no principal region possesses cuspidal points. In all cases known to the author the existence of cuspidal points involves the existence of multipliers the numerical value of which is greater than 1, and then Ξ is not limited for the powers of τ. If this were proved to be true in general, then the transformation classes of algebraically finite type would be capable of a purely algebraic definition: They would be the only transformation classes for which all multipliers are roots of unity.

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