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ON THE STRUCTURE OF GENERALIZED ALMOST PERIODIC FUNCTIONS

BY

ERLING FØLNER

KØBENHAVN
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§ 1. Introduction.

Throughout the paper we operate with complex Lebesgue measurable functions of a real variable defined on the whole axis. For every $p \geqq 1$ we consider the well-known norms[1] (which may be ∞)

$$\| f(x) \|_{O} = \underset{-\infty < x < \infty}{u.\,b.} | f(x) |,$$

$$\| f(x) \|_{S_L^p} = \underset{-\infty < x < \infty}{u.\,b.} \sqrt[p]{\frac{1}{L} \int_x^{x+L} | f(\xi) |^p \, d\xi} \quad (L > 0),$$

$$\| f(x) \|_{W^p} = \lim_{L \to \infty} \| f(x) \|_{S_L^p}$$

(the limit always exists),

$$\| f(x) \|_{B^p} = \overline{\lim_{T \to \infty}} \sqrt[p]{\frac{1}{2T} \int_{-T}^{T} | f(x) |^p dx}$$

as well as the corresponding limit notions, O-convergence (uniform convergence on the whole axis), S^p-convergence (the norms $\| f \|_{S_L^p}$ correspond for different values of L to the same limit notion), W^p-convergence, and B^p-convergence. Understanding by a trigonometric polynomial a finite sum of the form

$$\sum_{n=1}^{N} a_n e^{i \lambda_n x}$$

[1] See e.g. Besicovitch and Bohr [I] or Besicovitch [II]. The norms $\| f \|_{S_L^p}$, $\| f \|_{W^p}$, and $\| f \|_{B^p}$ are named after Stepanoff [I], Weyl [I], and Besicovitch [I] respectively.

where the coefficients a_n are complex numbers and the "exponens" λ_n are real numbers, the different types of almost periodic functions may be defined in the following manner:

The ordinary almost periodic functions (the O-a. p. functions) may be defined as the functions which can be O-approximated by trigonometric polynomials.

The S^p-a. p. functions may be defined as the functions which can be S^p-approximated by trigonometric polynomials.

The W^p-a. p. functions may be defined as the functions which can be W^p-approximated by trigonometric polynomials.

The B^p-a. p. functions may be defined as the functions which can be B^p-approximated by trigonometric polynomials.

The proper main theorem in the theory of O-a. p. functions is the following generalization of Weierstrass's theorem concerning continuous periodic functions:

An O-a. p. function $f(x)$ may be characterized as a continuous function possessing to every $\varepsilon > 0$ a relatively dense set of O-translation numbers τ. By a relatively dense set of numbers we understand a set to which can be found a length L such that every interval $\alpha < x < \alpha + L$ of this length contains a number of the set. By an O-translation number τ of $f(x)$ belonging to ε we understand a number satisfying

$$\| f(x+\tau) - f(x) \|_O \leqq \varepsilon, \text{ i. e. } | f(x+\tau) - f(x) | \leqq \varepsilon$$

for every x.

Quite an analogous main theorem holds good of the S^p-a. p. functions:

An S^p-a. p. function $f(x)$ may be characterized as a p-integrable[1] *function possessing for fixed $L > 0$ to every $\varepsilon > 0$ a relatively dense set of S_L^p-translation numbers τ.* By an S_L^p-translation number τ belonging to ε we understand, of course, a number satisfying the inequality

$$\| f(x+\tau) - f(x) \|_{S_L^p} \leqq \varepsilon .$$

For the W^p-a. p. and the B^p-a. p. functions the analogous

1) A function $f(x)$ is called p-integrable if $\int_a^b | f(x) |^p dx < \infty$ for every finite interval (a, b).

theorems are wrong. This may be seen from considering the example

$$F(x) = \begin{cases} 1 \text{ for } x \geqq 0 \\ -1 \text{ for } x < 0. \end{cases}$$

The limits

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T F(x)\,dx \quad \text{and} \quad \lim_{T\to\infty} \frac{1}{T}\int_{-T}^0 F(x)\,dx$$

being different the function $F(x)$ is neither W^p-a. p. nor B^p-a. p.. On the other hand for an arbitrary real number τ we have

$$|F(x+\tau)-F(x)| = \begin{cases} 2 \text{ for } x \text{ lying between } 0 \text{ and } -\tau \text{ (inclusive the smaller, exclusive the larger number)} \\ 0 \text{ elsewhere} \end{cases}$$

so that

$$\|F(x+\tau)-F(x)\|_{W^p} = \|F(x+\tau)-F(x)\|_{B^p} = 0$$

for every τ.

Still the W^p-a. p. functions may be characterized in a simple way by structural properties which are very similar to those in the O-a. p. and the S^p-a. p. cases. In fact the following main theorem is valid:

A W^p-a.p. function may be characterized as a p-integrable function possessing to every $\varepsilon > 0$ for L sufficiently large, i. e. for $L \geqq L_0(\varepsilon)$ a relatively dense set of S_L^p-translation numbers.

It is of course obvious that a W^p-translation number belonging to ε is an S_L^p-translation number belonging to—say—2ε for L sufficiently large, $L \geqq L_0'(\tau)$, but the difference between the last theorem and the (wrong) theorem analogous to the above mentioned main theorems is that in the first case the same L_0 must be applicable to all translation numbers belonging to the given ε. It is also immediately seen that the function $F(x)$ from the example above does not possess this property; the larger the modulus of τ, the larger $L_0'(\tau)$ is to be used.

In the case of B^p-a.p. functions the characterization by means of structural properties known is not nearly as simple as in the O-a. p., S^p-a. p., and W^p-a. p. cases. We quote it after Be-

SICOVITCH and BOHR [I] or BESICOVITCH [II] where proofs of the main theorems in the S^p-a. p. and W^p-a. p. cases are also to be found.

A set of real numbers is called satisfactorily uniform if there exists a positive number L such that the ratio of the maximum number of numbers from the set included in an interval of length L to the minimum number is less than 2. A B^p-a. p. function can then be characterized as a p-integrable function for which to every $\varepsilon > 0$ can be found a satisfactorily uniform set of B^p-translation numbers $\{\tau_n\}$, $n = 0, \pm 1, \pm 2, \cdots$, i. e. numbers with

$$\| f(x + \tau_n) - f(x) \|_{B^p} \leqq \varepsilon,$$

and such that further for every $c > 0$

$$\overline{\lim_{T \to \infty}} \frac{1}{2T} \int_{-T}^{T} \left(\overline{\lim_{n \to \infty}} \frac{1}{2n+1} \sum_{\nu = -n}^{n} \frac{1}{c} \int_{x}^{x+c} | f(\xi + \tau_\nu) - f(\xi) |^p d\xi \right) dx \leqq \varepsilon^p.$$

§ 2. Summary of the Paper.

Our counter example from § 1 shows that by a structural characterization of the B^p-a. p. functions a stronger uniformity property must be imposed upon the translation numbers belonging to a given ε than that of being merely B^p-translation numbers belonging to the same ε. The structural characterization of Besicovitch and Bohr shows one way of choosing this uniformity property. By the W^p-a. p. functions the situation was a similar one. In my efforts to find a simpler structural characterization than that of Besicovitch-Bohr I was therefore led to consider a special class of functions which will be called K-a. p. functions.

A function $f(x)$ is called K-a. p. if it is finite outside a K-zero set Z, i. e. a set[1] with

$$\overline{m}_B Z = \overline{\lim_{T \to \infty}} \frac{1}{2T} m(Z; -T, T) = 0,^{2)}$$

1) We operate only with measurable sets.

2) By $(Z; a, b)$ we denote the part of the set Z which is included in the interval $[a, b]$.

and to every $\varepsilon > 0$ there exists a relatively dense set of numbers $\tau = \tau(\varepsilon)$ and a set $E = E(\varepsilon) \supseteqq Z$ with

$$\overline{m}_B E = \overline{\lim_{T \to \infty}} \frac{1}{2T} m(E; -T, T) \leqq \varepsilon$$

such that

$$|f(x+\tau) - f(x)| \leqq \varepsilon$$

when both $x \varepsilon CE$ and $x + \tau \varepsilon CE$ (where CE denotes the complement of E).

My hope was that this class of functions should be identical with the class of functions $f(x)$ which can be K-approximated by trigonometric polynomials, i. e. for which to every $\varepsilon > 0$ there exists a trigonometric polynomial $s(x)$ and a set E with $\overline{m}_B E \leqq \varepsilon$ such that

$$|f(x) - s(x)| \leqq \varepsilon$$

when $x \varepsilon CE$, and as we shall see in § 3 this is really the case. The latter class of functions has first been considered by A. S. Kovanko who gave a structural characterization for the functions of the class analogous to that of Besicovitch-Bohr, upon which his proof was also based (Kovanko [I]).

Starting from the approximation properties of the K-a. p. functions it is easy to show that a B^p-a. p. function means the same thing as a K-a. p. function for which $\|f(x) - (f(x))_N\|_{B^p} \to 0$ for $N \to \infty$. Here $(f(x))_N$ denotes the function $f(x)$ "cut off" at the number N, i. e. the function

$$(f(x))_N = \begin{cases} f(x) \text{ for } |f(x)| \leqq N \\ N \operatorname{sign} f(x) \text{ for } |f(x)| \geqq N. \end{cases}$$

Instead of the additional condition mentioned one may also—as Kovanko has done it—use the additional condition $\|f_E(x)\|_{B^p} \to 0$ for $\overline{m}_B E \to 0$ where $f_E(x)$ denotes the function which is equal to $f(x)$ for x lying in (the arbitrary set) E and equal to 0 elsewhere.

In § 4 instead of the additional condition $\|f(x) - (f(x))_N\|_{B^p} \to 0$ for $N \to \infty$ we consider the simpler condition $\|f(x)\|_{B^p} < \infty$, i. e. we investigate the B^p-bounded K-a. p. functions. It is shown that such a function means the same thing as a function which can be written in the form

$$f(x) = g(x) + j(x)$$

where $g(x)$ is B^p-a. p. and $j(x)$ is a B^p-bounded K-zero function, i. e. a function with $\|j(x)\|_{B^p} < \infty$ and $\overline{m}_B [|j(x)| > \varepsilon] = 0$ for every $\varepsilon > 0$. Denoting a set of functions which differ from one and the same B^p-bounded K-a. p. function by B^p-bounded K-zero functions as a B^p-bounded K-a. p. point, we can therefore say that in every B^p-bounded K-a. p. point there is lying a B^p-a. p. point, and it is easy further to show that there can only lie one B^p-a. p. point.

If $p > 1$ a B^p-bounded K-a. p. function is $B^{p'}$-a. p. for $1 \leqq p' < p$.

A B^p-bounded K-a. p. function need not have a mean value in the Besicovitch sense if $p = 1$, but the mean value notion can be generalized as to comprehend these functions by putting

$$M^* \{f(x)\} = \lim_{N \to \infty} M \{(f(x))_N\}.$$

After that one can in the usual way associate to every B^p-bounded K-a. p. function a Fourier series which will be the same for all functions in the surrounding B^p-bounded K-a. p. point and therefore identical with the Fourier series of the B^p-a. p. point in that point. Hence the Bochner-Fejér polynomials will B^p-converge to a function which only differs from $f(x)$ by a B^p-bounded K-zero function.

We can say that we almost have a structural characterization of a B^p-a. p. function in that of being a B^p-bounded K-a. p. function, viz. in the sense that *the set of B^p-bounded K-a. p. functions gives rise to just the same Fourier series as the set of B^p-a. p. functions.*

§ 3. The Main Theorem for the K-a. p. Functions. The Additional Condition $\|f(x) - (f(x))_N\|_{B^p} \to 0$.

We repeat the definition of a K-a. p. function given in § 2.

Definition. *A (measurable) function $f(x)$ is called K-a. p. if it is finite outside a K-zero set Z, i. e. a set with*

$$\overline{m}_B Z = \overline{\lim_{T \to \infty}} \frac{1}{2T} m(Z; -T, T) = 0, ^{1)}$$

1) By $(Z; a, b)$ we denote the part of the set Z which is included in the interval $[a, b]$.

and to every $\varepsilon > 0$ *there exists a relatively dense set of numbers* $\tau = \tau(\varepsilon)$ *and a set* $E = E(\varepsilon) \supseteqq Z$ *with*

$$\overline{m}_B E = \overline{\lim_{T\to\infty}} \frac{1}{2T} m(E; -T, T) \leqq \varepsilon$$

such that

$$|f(x+\tau) - f(x)| \leqq \varepsilon$$

when both $x \varepsilon CE$ *and* $x + \tau \varepsilon CE$. The numbers τ and the set E are called (joined) translation numbers and exception set of the function $f(x)$ belonging to ε.

We are to show the following

Theorem 1. Main Theorem. *A K-a.p. function may be characterized as a (measurable) function which can be K-approximated by trigonometric polynomials, i.e. for which to every* $\varepsilon > 0$ *there exists a trigonometric polynomial* $s(x)$ *and a set* E *with* $\overline{m}_B E \leqq \varepsilon$ *such that*

$$|f(x) - s(x)| \leqq \varepsilon$$

for $x \varepsilon CE$. The trigonometric polynomial $s(x)$ and the set E are called (joined) approximating polynomial and exception set of the function $f(x)$ belonging to ε.[1)]

We first show the simple part of the theorem, viz. that a function $f(x)$ which can be K-approximated by trigonometric polynomials is K-a.p.. It is obvious that $Z = [|f(x)| = \infty]$ is a K-zero set, for a trigonometric polynomial being finite this set must be contained in the exception set $E(\varepsilon)$ belonging to an arbitrarily small $\varepsilon > 0$, and $E(\varepsilon)$ has $\overline{m}_B E(\varepsilon) \leqq \varepsilon$. Next we prove the translation properties. Let $\varepsilon > 0$ be arbitrarily given. Belonging to ε we choose a trigonometric polynomial $s(x)$ and an exception set E $(\supseteqq Z)$. Then

1) Incidentally the K-approximation can be defined by help of the K_N-norm

$$\| f(x) \|_{K_N} = \| (f(x))_N \|_{B^p}$$

where $(f(x))_N$ denotes the function $f(x)$ cut off at the number N (see p. 7) and $N > 0$ is an arbitrarily chosen fixed number. As easily seen the K_N-norm fulfills the triangel inequality

$$\| f(x) + g(x) \|_{K_N} \leqq \| f(x) \|_{K_N} + \| g(x) \|_{K_N}.$$

$$|f(x) - s(x)| \leq \varepsilon$$

for $x \varepsilon CE$, and $\overline{m}_B E \leq \varepsilon$. The trigonometric polynomial $s(x)$ being an ordinary almost periodic function it has a relatively dense set of O-translation numbers τ belonging to ε, so that

$$|s(x+\tau) - s(x)| \leq \varepsilon$$

for all x. Hence

$$|f(x+\tau) - f(x)| \leq |f(x+\tau) - s(x+\tau)| + |s(x+\tau) - s(x)| + |s(x) - f(x)| \leq 3\varepsilon$$

when both $x \varepsilon CE$ and $x + \tau \varepsilon CE$. This proves the first part of the main theorem.

In order to prove the second part of the main theorem we set up four lemmas.

Lemma 1. *Let $f(x)$ be a K-a.p. function. To every $\varepsilon > 0$ it is then possible to choose a number $\Gamma = \Gamma(\varepsilon)$ and a set $E^* = E^*(\varepsilon)$ with $\overline{m}_B E^* \leq \varepsilon$ such that*

$$|f(x)| \leq \Gamma$$

for $x \varepsilon CE^*$.

Proof. Let Z denote the set $[|f(x)| = \infty]$. Then $\overline{m}_B Z = 0$. Belonging to $\frac{1}{12}\varepsilon$ we choose a relatively dense set $\{\tau\}$ of translation numbers and an exception set E of $f(x)$. The set $\{\tau\}$ being relatively dense we can choose a length L so large that every interval of this length contains a translation number τ. Next we choose a length $L_0 \left(\geq \frac{1}{2} L\right)$ so large that $\frac{L}{2L_0} \leq \frac{1}{3}\varepsilon$, $\frac{1}{2L_0} m(Z; -L_0, L_0) \leq \frac{1}{12}\varepsilon$, and $\frac{1}{2L_0} m(E; -L_0, L_0) \leq \frac{1}{6}\varepsilon$. As is well-known in every bounded set a finite function is bounded outside a set of arbitrarily small measure. In the interval $\left[-L_0 + \frac{1}{2}L, L_0 - \frac{1}{2}L\right]$ the function $f(x)$ is finite outside the set $E_1 = \left(Z; -L_0 + \frac{1}{2}L, L_0 - \frac{1}{2}L\right)$ with $\frac{1}{2L_0} m E_1 \leq \frac{1}{12}\varepsilon$, and in the interval mentioned $f(x)$ is therefore bounded, $|f(x)| \leq K$, outside a set E_2 with $\frac{1}{2L_0} mE_2 \leq \frac{1}{6}\varepsilon$. In the interval $2nL_0 - \frac{1}{2}L < x < 2nL_0 + \frac{1}{2}L$ $(n = \pm 1, \pm 2, \cdots)$ we choose a translation number τ_n from

$\{\tau\}$. We form the set E_3 obtained by performing the translations $\tau_0 = 0, \tau_1, \tau_{-1}, \tau_2, \tau_{-2}, \cdots$ on the set $E_2 \dot{+} \left(E; -L_0 + \frac{1}{2}L, L_0 - \frac{1}{2}L\right)$ and next forming the sum of all the translated sets. The set obtained by the translation τ_n lying entirely in the interval $(2n-1)L_0 < x < (2n+1)L_0$ we have

$$\overline{m}_B E_3 = m_B E_3 \leqq \frac{1}{2L_0} mE_2 + \frac{1}{2L_0} m\left(E; -L_0 + \frac{1}{2}L, L_0 - \frac{1}{2}L\right) \leqq$$
$$\frac{1}{2L_0} mE_2 + \frac{1}{2L_0} m(E; -L_0, L_0) \leqq \frac{1}{6}\varepsilon + \frac{1}{6}\varepsilon = \frac{1}{3}\varepsilon.$$

Let E_4 denote the complement of the set obtained from performing the translations $\tau_0 = 0, \tau_1, \tau_{-1}, \tau_2, \tau_{-2}, \cdots$ on the interval $\left[-L_0 + \frac{1}{2}L, L_0 - \frac{1}{2}L\right]$ and next forming the sum of all the translated intervals. The interval obtained by the translation τ_n lying entirely in the interval $(2n-1)L_0 < x < (2n+1)L_0$, we have

$$\overline{m}_B E_4 = m_B E_4 = \frac{L}{2L_0} \leqq \frac{1}{3}\varepsilon.$$

Putting

$$E^* = E \dot{+} E_3 \dot{+} E_4$$

obviously

$$\overline{m}_B E_3 \leqq \overline{m}_B E + \overline{m}_B E_3 + \overline{m}_B E_4 \leqq \frac{1}{12}\varepsilon + \frac{1}{3}\varepsilon + \frac{1}{3}\varepsilon < \varepsilon.$$

Further for every $x \varepsilon CE^*$ we have

$$|f(x)| \leqq K + \frac{1}{12}\varepsilon,$$

for as $x \varepsilon CE_4$ for some n the argument x will lie in the interval obtained from the interval $\left[-L_0 + \frac{1}{2}L, L_0 - \frac{1}{2}L\right]$ by means of the translation τ_n, and as $x \varepsilon CE_3$ the number $x - \tau_n$ will lie in CE_2 and in CE, so that

$$|f(x)| \leqq |f(x-\tau_n)| + |f(x) - f(x-\tau_n)| \leqq K + \frac{1}{12}\varepsilon.$$

This accomplishes the proof of lemma 1.

The lemma 1 may also be expressed by saying that for every K-a.p. function $f(x)$

$$\overline{m}_B\,[\,|f(x)-(f(x))_N|>0\,]\to 0 \quad \text{when} \quad N\to\infty,$$

where $(f(x))_N$ as usual denotes the cut-off function (see p. 7). Hence the cut-off function $(f(x))_N$ will K-converge towards $f(x)$ for $N\to\infty$. A simple geometric consideration shows that for every τ we have

$$|(f(x+\tau))_N-(f(x))_N|\leqq|f(x+\tau)-f(x)|,$$

and from this it is seen that *if $f(x)$ is K-a.p. so is $(f(x))_N$*, viz. with the same translation numbers and the same exception sets. These two properties of $(f(x))_N$ show that it is sufficient to prove the second part of our main theorem in the case of bounded K-a.p. functions.

Lemma 2. *Let $f(x)$ be a bounded K-a.p. function. Then the "smoothed" function*

$$\varphi_\delta(x)=\frac{1}{\delta}\int_x^{x+\delta}f(\xi)\,d\xi \qquad \delta>0$$

will K-converge towards $f(x)$ for $\delta\to 0$.

Proof. Belonging to ε^2 we choose a relatively dense set $\{\tau\}$ of translation numbers and an exception set E of $f(x)$. The set $\{\tau\}$ being relatively dense we can choose a length L so large that every interval of this length contains a translation number τ. Next we choose an L_0 $\left(\geqq\frac{1}{2}(L+2)\right)$ so large that $\frac{L+2}{2\,L_0}\leqq\varepsilon$ and $\frac{1}{2\,L_0}\,m\,(E;-L_0,L_0)\leqq 2\,\varepsilon^2$. In the interval $[-L_0,L_0]$ the smoothed function $\varphi_\delta(x)$ will converge towards $f(x)$ almost everywhere, and we can therefore determine a δ_0, $0<\delta_0<1$, such that for every δ, $0<\delta<\delta_0$, there exists a set $E_1=E_1(\delta)$ inside $[-L_0,L_0]$ with $\frac{1}{2\,L_0}\,mE_1\leqq\varepsilon$ such that

$$|f(x)-\varphi_\delta(x)|\leqq\varepsilon$$

when $x\,\varepsilon\,(CE_1;-L_0,L_0)$. For an arbitrarily fixed δ, $0<\delta<\delta_0$ (<1), the interval $\left[-L_0+\frac{1}{2}L, L_0-\frac{1}{2}L\right]$ is divided beginning at the left in as many subintervals of the length δ as possible. Let these subintervals fill out the interval $\left[-L_0+\frac{1}{2}L, L_1\right]$. The number δ being <1 the interval $\left[-L_0+\frac{1}{2}L, L_1\right]$ has a length which is $\geqq 2L_0-L-1$. In the interval $2nL_0-\frac{1}{2}L<x<2nL_0+\frac{1}{2}L$ $(n=\pm 1, \pm 2, \cdots)$ we choose a translation number τ_n from $\{\tau\}$. We then consider the set E_2 obtained by performing the translations $\tau_0=0, \tau_1, \tau_{-1}\tau_2, \tau_{-2}, \cdots$ on the set $\left(E; -L_0+\frac{1}{2}L, L_1\right)$ and next forming the sum of all the translated sets. The set obtained by the translation τ_n lying entirely in the interval $(2n-1)L_0 <x<(2n+1)L_0$ we have

$$\overline{m}_B E_2 = m_B E_2 = \frac{1}{2L_0} m \left[E; -L_0+\frac{1}{2}L, L_1\right] \leqq \frac{1}{2L_0} m\,(E; -L_0, L_0) \leqq 2\varepsilon^2.$$

Putting

$$E_3 = E \dotplus E_2$$

we obviously have

$$\overline{m}_B E_3 \leqq \overline{m}_B E + \overline{m}_B E_2 \leqq \varepsilon^2 + 2\varepsilon^2 = 3\varepsilon^2.$$

Corresponding to the division of $\left[-L_0+\frac{1}{2}L, L_1\right]$ in subintervals of length δ we consider the division in subintervals of length δ of all the intervals translated by $\tau_0=0, \tau_1, \tau_{-1}, \tau_2, \tau_{-2}, \cdots$. Let E_4 denote the set consisting of all the subintervals whose intersection with E_3 have relative measures $>\varepsilon$. Then $\overline{m}_B E_4 \leqq 3\varepsilon$. Let further E_5 denote the complement of the set obtained by performing the translations $\tau_0=0, \tau_1, \tau_{-1}, \tau_2, \tau_{-2}, \cdots$ on the interval $\left[-L_0+\frac{1}{2}L, L_1\right]$ and next forming the sum of all the translated intervals. The interval obtained by the translation τ_n lying entirely in the interval $(2n-1)L_0<x<(2n+1)L_0$ we have

$$\overline{m}_B E_5 = m_B E_5 \leqq \frac{L+1}{2L_0}.$$

Let E_4' denote the set E_4 after being expanded δ to the left, i.e. the set of points whose distances in right-hand direction to

E_4 are $\leqq \delta$. Analogously let E_5' denote the set E_5 after being expanded δ to the left. Then

$$\overline{m}_B E_4' \leqq 6\varepsilon \quad \text{and} \quad \overline{m}_B E_5' \leqq \frac{L+2}{2L_0} \leqq \varepsilon.$$

For $x \varepsilon C(E_4' \dotplus E_5')$ the interval $[x, x+\delta]$ is lying entirely in $C(E_4 \dotplus E_5)$ and hence entirely in the sum of two successive subintervals of the length δ whose intersection with E_3 have a relative measure $\leqq \varepsilon$. The intersection $P(x)$ of $[x, x+\delta]$ with E_3 therefore has a relative measure $\leqq 2\varepsilon$. For a suitable value of n the interval $[x, x+\delta]$ by the translation $-\tau_n$ will be brought over in an interval $[x-\tau_n, x-\tau_n+\delta]$ included in the interval $\left[-L_0+\frac{1}{2}L, L_1\right]$, and if ξ is lying in $[x, x+\delta]$ outside $P(x)$ the number $\xi-\tau_n$ in $[x-\tau_n, x-\tau_n+\delta]$ will lie outside E. Hence

$$|\varphi_\delta(x)-\varphi_\delta(x-\tau_n)| = \left|\frac{1}{\delta}\int_x^{x+\delta} f(\xi)\,d\xi - \frac{1}{\delta}\int_{x-\tau_n}^{x-\tau_n+\delta} f(\xi)\,d\xi\right| \leqq \frac{1}{\delta}\int_x^{x+\delta} |f(\xi)-f(\xi-\tau_n)|\,d\xi$$

$$\leqq \frac{1}{\delta}\int_{P(x)} |f(\xi)-f(\xi-\tau_n)|\,d\xi + \frac{1}{\delta}\int_{(CP(x);\,x,\,x+\delta)} |f(\xi)-f(\xi-\tau_n)|\,d\xi \leqq 2\Gamma\cdot 2\varepsilon+\varepsilon^2 = 4\Gamma\varepsilon+\varepsilon^2$$

where Γ denotes the upper bound of $|f(x)|$. Let E_6 denote the set obtained by performing the translations $\tau_0 = 0$, τ_1, τ_{-1}, τ_2, τ_{-2}, $\cdots$ on the set $\left(E_1; -L_0+\frac{1}{2}L, L_1\right)$ and next forming the sum of all the translated sets. Then

$$\overline{m}_B E_6 = m_B E_6 = \frac{1}{2L_0} m\left(E_1; -L_0+\frac{1}{2}L, L_1\right) \leqq \frac{1}{2L_0} m E_1 \leqq \varepsilon.$$

Finally let

$$E_7 = E_7(\delta) = E_3 \dotplus E_4' \dotplus E_5' \dotplus E_6.$$

Then in the first place

$$\overline{m}_B E_7 \leqq \overline{m}_B E_3 + \overline{m}_B E_4' + \overline{m}_B E_5' + \overline{m}_B E_6 \leqq 3\varepsilon^2 + 6\varepsilon + \varepsilon + \varepsilon = 8\varepsilon + 3\varepsilon^2.$$

And secondly for $x \varepsilon CE_7(\delta)$ where $0 < \delta < \delta_0$—determining τ_n as above—we have

$$|\varphi_\delta(x) - f(x)| \leq |\varphi_\delta(x) - \varphi_\delta(x-\tau_n)| + |\varphi_\delta(x-\tau_n) - f(x-\tau_n)| +$$
$$+ |f(x-\tau_n) - f(x)| \leq (4\Gamma\varepsilon + \varepsilon^2) + \varepsilon + \varepsilon^2 = 4\Gamma\varepsilon + \varepsilon + 2\varepsilon^2.$$

Letting $\varepsilon \to 0$ both $8\varepsilon + 3\varepsilon^2$ and $4\Gamma\varepsilon + \varepsilon + 2\varepsilon^2$ will converge towards 0. This accomplishes the proof of lemma 2.

Lemma 3. *Let $f(x)$ be a bounded K-a.p. function and let us consider the smoothed function*

$$\varphi_\delta(x) = \frac{1}{\delta}\int_x^{x+\delta} f(\xi)\,d\xi$$

for a fixed value af $\delta > 0$. Then to every $\varepsilon > 0$ there exists a relatively dense set $I = I(\varepsilon)$ of intervals with the same length $2\zeta_0 = 2\zeta_0(\varepsilon)$ the one of which has its central point in 0 as well as a set $E^* = E^*(\varepsilon)$ with $\overline{m}_B E^* \leq \varepsilon$ such that for every τ from the intervals mentioned

$$|\varphi_\delta(x+\tau) - \varphi_\delta(x)| \leq \varepsilon$$

when both $x \varepsilon CE^*$ and $x + \tau \varepsilon CE^*$.

Proof. The function $f(x)$ being bounded, $|f(x)| \leq \Gamma$, the smoothed function $\varphi_\delta(x)$ is obviously a uniformly continuous function, for

$$|\varphi_\delta(x+\zeta) - \varphi_\delta(x)| = \left| \frac{1}{\delta}\int_{x+\zeta}^{x+\zeta+\delta} f(\xi)\,d\xi - \frac{1}{\delta}\int_x^{x+\delta} f(\xi)\,d\xi \right| =$$
$$\left| \frac{1}{\delta}\int_{x+\delta}^{x+\zeta+\delta} f(\xi)\,d\xi - \frac{1}{\delta}\int_x^{x+\zeta} f(\xi)\,d\xi \right| \leq \frac{2|\zeta|\Gamma}{\delta}$$

which $\to 0$ for $\zeta \to 0$. Belonging to ε^2 we choose a relatively dense set $\{\tau\}$ of translation numbers and an exception set E of $f(x)$. We divide the axis in subintervals of the length δ. Let E_1 denote the set consisting of all the subintervals whose intersections with E have relative measures $> \varepsilon$. Then $\overline{m}_B E_1 \leq \varepsilon$. Let E_2 denote the set E_1 after being expanded 2δ to the left and δ to the right (see p. 13). Then $\overline{m}_B E_2 \leq 4\varepsilon$. The function $\varphi_\delta(x)$ being uniformly continuous we can choose an ζ_0, $0 < \zeta_0 < \delta$, such that

$$|\varphi_\delta(x+\zeta)-\varphi_\delta(x)|\leqq\varepsilon$$

for $|\zeta|\leqq\zeta_0$. Now let τ be a number from $\{\tau\}$, let $|\zeta|\leqq\zeta_0$, and let both x and $x+\tau+\zeta$ lie in CE_2. Then the intervals $[x, x+\delta]$ and $[x+\tau,\ x+\tau+\delta]$ will lie entirely in CE_1 and their intersections $P(x)$ and $P(x+\tau)$ with E will therefore have relative measures $\leqq 2\varepsilon$. Hence, denoting by $P(x+\tau)_{-\tau}$ the set $P(x+\tau)$ translated $-\tau$, we have

$$|\varphi_\delta(x+\tau+\zeta)-\varphi_\delta(x)|\leqq|\varphi_\delta(x+\tau+\zeta)-\varphi_\delta(x+\tau)|+|\varphi_\delta(x+\tau)-\varphi_\delta(x)|\leqq$$

$$\varepsilon+\left|\frac{1}{\delta}\int_{x+\tau}^{x+\tau+\delta}f(\xi)\,d\xi-\frac{1}{\delta}\int_{x}^{x+\delta}f(\xi)\,d\xi\right|\leqq\varepsilon+\frac{1}{\delta}\int_{x}^{x+\delta}|f(\xi+\tau)-f(\xi)|\,d\xi\leqq$$

$$\varepsilon+\frac{1}{\delta}\int_{P(x)\dotplus P(x+\tau)_{-\tau}}|f(\xi+\tau)-f(\xi)|\,d\xi+\frac{1}{\delta}\int_{(C(P(x)\dotplus P(x+\tau)_{-\tau});\,x,\,x+\delta)}|f(\xi+\tau)-f(\xi)|\,d\xi\leqq\varepsilon+8\Gamma\varepsilon+\varepsilon^2.$$

All numbers in the ζ_0-neighbourhoods of our translation numbers τ are therefore "fine" translation numbers of $\varphi_\delta(x)$ with a "small" exception set E_2. Further all numbers from the ζ_0-neighbourhood of 0 are fine translation numbers (even O-translation numbers) of $\varphi_\delta(x)$. This accomplishes the proof of lemma 3.

Lemma 4. *Let $f(x)$ be a K-a.p. function. Then to every $\varepsilon>0$ there exists a relatively dense set $I=I(\varepsilon)$ of intervals with the same length $2\zeta_0=2\zeta_0(\varepsilon)>0$ the one of which has its central point in 0 as well as a set $E^*=E^*(\varepsilon)\subseteqq[|f(x)|=\infty]$ with $\overline{m}_B E^*\leqq\varepsilon$ such that for every τ from the intervals mentioned*

$$|f(x+\tau)-f(x)|\leqq\varepsilon$$

when both $x\varepsilon CE^*$ and $x+\tau\,\varepsilon\, CE^*$.

Proof. Corresponding to $\frac{1}{3}\varepsilon$ we may on account of lemma 1 choose a number $N>0$ and a set $E_1\supseteqq[|f(x)|=\infty]$ with $\overline{m}_B E_1\leqq\frac{1}{3}\varepsilon$ such that

$$f(x)=(f(x))_N$$

for $x\varepsilon CE_1$. For $(f(x))_N$ we form the smoothed function

$$\varphi_{\delta}(x) = \frac{1}{\delta}\int_{x}^{x+\delta}(f(\xi))_N \, d\xi$$

and may on account of lemma 2 choose $\delta > 0$ and a set E_2 with $\overline{m}_B E_2 \leqq \frac{1}{3}\varepsilon$ such that

$$|(f(x))_N - \varphi_{\delta}(x)| \leqq \frac{1}{3}\varepsilon$$

for $x \varepsilon CE_2$. Finally on account of lemma 3 there exists a relatively dense set I of intervals with the same length $2\zeta_0 > 0$ the one of which has its central point in 0 as well as a set E_3 with $\overline{m}_B E_3 \leqq \frac{1}{3}\varepsilon$ such that for every τ from the intervals mentioned

$$|\varphi_{\delta}(x+\tau) - \varphi_{\delta}(x)| \leqq \frac{1}{3}\varepsilon$$

when both $x \varepsilon CE_3$ and $x + \tau \varepsilon CE_3$. Putting

$$E^* = E_1 \dotplus E_2 \dotplus E_3$$

we have

$$\overline{m}_B E^* \leqq \overline{m}_B E_1 + \overline{m}_B E_2 + \overline{m}_B E_3 \leqq \frac{1}{3}\varepsilon + \frac{1}{3}\varepsilon + \frac{1}{3}\varepsilon = \varepsilon.$$

Further for every τ from I we have

$$|f(x+\tau) - f(x)| = |(f(x+\tau))_N - (f(x))_N| \leqq$$

$$|f(x+\tau)_N - \varphi_{\delta}(x+\tau)| + |\varphi_{\delta}(x+\tau) - \varphi_{\delta}(x)| + |\varphi_{\delta}(x) - (f(x))_N| \leqq$$

$$\frac{1}{3}\varepsilon + \frac{1}{3}\varepsilon + \frac{1}{3}\varepsilon = \varepsilon$$

when both $x \varepsilon CE^*$ and $x + \tau \varepsilon CE^*$. This accomplishes the proof of lemma 4.

We now pass to the proper proof of the second part of the main theorem. As mentioned before (p. 12) we need only consider bounded K-a. p. functions. Let then $f(x)$ be a bounded K-a. p. function, $|f(x)| \leqq \Gamma$, for all x. In the following we may assume $\Gamma \leqq 1$, otherwise we only consider the function $\frac{1}{\Gamma}f(x)$. We are to show that $f(x)$ can be K-approximated by trigonometric polynomials or—what on account of the main theorem in the theory of the O-a. p. functions amounts to the same thing—by ordinary almost periodic functions. Let $\varepsilon > 0$ be arbitrarily given. We are to indicate an ordinary almost periodic function $\varphi(x)$ and a set E^* with $\overline{m}_B E^* \leqq \eta_1(\varepsilon)$ such that

$$|f(x) - s(x)| \leqq \eta_2(\varepsilon)$$

for $x \varepsilon CE^*$ where $\eta_1(\varepsilon) \to 0$ and $\eta_2(\varepsilon) \to 0$ for $\varepsilon \to 0$. For the function $f(x)$ and belonging to ε^2 we may on account of lemma 4 choose a relative dense set I' of intervals consisting of translation numbers and with the same length δ as well as an exception set E. The length L is chosen so large that every interval of this length contains an interval of length δ from I'. Next L_0 $(\geqq L + \delta)$ is chosen so large that $\frac{L+\delta}{L_0} \leqq \varepsilon$. In every one of the intervals $nL_0 < x < nL_0 + L$ $(n = 0, \pm 1, \pm 2, \cdots)$ we choose an interval of translation numbers from I' with the length δ. We consider the total set I of these translation numbers. Let $m_B I = \eta$. Next we define a function $K(t)$ by

$$K(t) = \begin{cases} \frac{1}{\eta} \text{ for } t \varepsilon I \\ 0 \text{ for } t \varepsilon CI. \end{cases}$$

Then

$$M\{K(t)\} = 1.$$

We consider the expression

$$\frac{1}{2T}\int_{-T}^{T} f(x+t) K(t)\, dt.$$

The modulus of this expression is $\leqq \frac{1}{\eta}$. Denoting by $x_1, x_2, \cdots$ an enumerable everywhere dense set on the whole axis and by $0 < T_1 < T_2 < \cdots \to \infty$ an arbitrary sequence, by the diagonal procedure we can extract a subsequence—again denoted T_n—such that for the new sequence T_n

$$\frac{1}{2T_n}\int_{-T_n}^{T_n} f(x+t) K(t)\, dt$$

is convergent for x equal to $x_1, x_2, \cdots$. We first show that the expression then converges for all x. Let $\varepsilon_0 > 0$ be arbitrarily given. We choose a set E^* with $\overline{m}_B E^* \leqq \varepsilon_0 \eta$ and a positive ζ_0 such that

$$|f(x+\zeta) - f(x)| \leqq \varepsilon_0 \eta$$

for $|\zeta| \leqq \zeta_0$ when both $x \varepsilon CE^*$ and $x + \zeta \varepsilon CE^*$ (lemma 4). Next we choose an x_m from our enumerable set above such that $|x - x_m| \leqq \zeta_0$. Letting E^*_{-x} denote the set E^* after being translated $-x$ we have

$$\frac{1}{2T_n}\int_{-T_n}^{T_n} |f(x+t) - f(x_m+t)|\, dt = \frac{1}{2T_n}\int_{(E^*_{-x} \dot{+} E^*_{-x_m};\, -T_n,\, T_n)} |f(x+t) - f(x_m+t)|\, dt +$$

$$\frac{1}{2T_n}\int_{(C(E^*_{-x} \dot{+} E^*_{-x_m});\, -T_n,\, T_n)} |f(x+t) - f(x_m+t)|\, dt \leqq \frac{1}{2T_n}\, 2m\left(E^*_{-x} \dot{+} E^*_{-x_m}; -T_n, T_n\right) + \varepsilon_0\eta \leqq$$

$$2\,\frac{1}{2T_n}\, m\left(E^*_{-x}; -T_n, T_n\right) + 2\,\frac{1}{2T_n}\, m\left(E^*_{-x_m}; -T_n, T_n\right) + \varepsilon_0\eta$$

which for n sufficiently large, $n \geqq n_0$, is

$$\leqq 4\varepsilon_0\eta + 4\varepsilon_0\eta + \varepsilon_0\eta = 9\varepsilon_0\eta.$$

Hence for $n \geqq n_0$

$$\left|\frac{1}{2T_n}\int_{-T_n}^{T_n} f(x+t)\, K(t)\, dt - \frac{1}{2T_n}\int_{-T_n}^{T_n} f(x_m+t)\, K(t)\, dt\right| \leqq$$

$$\frac{1}{2T_n}\int_{-T_n}^{T_n} |f(x+t) - f(x_m+t)|\, K(t)\, dt \leqq \frac{1}{\eta}\,\frac{1}{2T_n}\int_{-T_n}^{T_n} |f(x+t) - f(x_m+t)|\, dt \leqq 9\varepsilon_0.$$

This together with the fact that the expression

$$\frac{1}{2T_n}\int_{-T_n}^{T_n} f(x_m+t)\, K(t)\, dt$$

has a limit for $n \to \infty$ shows that for n_1 and n_2 sufficiently large

$$\left|\frac{1}{2T_{n_1}}\int_{-T_{n_1}}^{T_{n_1}} f(x+t)\, K(t)\, dt - \frac{1}{2T_{n_2}}\int_{-T_{n_2}}^{T_{n_2}} f(x+t)\, K(t)\, dt\right| \leqq 19\varepsilon_0,$$

and hence the limit

$$\lim_{n\to\infty} \frac{1}{2\,T_n}\int_{-T_n}^{T_n} f(x+t)\,K(t)\,dt$$

exists for our arbitrarily chosen x. We then consider the function

$$\varphi(x) = \lim_{n\to\infty} \frac{1}{2\,T_n}\int_{-T_n}^{T_n} f(x+t)\,K(t)\,dt$$

which—as we shall see—will have the properties desired.

We first show that $\varphi(x)$ is an ordinary almost periodic function. Let $\varepsilon_0 > 0$ be arbitrarily given. Belonging to $\varepsilon_0\eta$ we choose a relatively dense set $\{\tau\}$ of translation numbers which contains an interval with central point 0, as well as an exception set E^* of $f(x)$ (lemma 4). For each such τ

$$|\varphi(x+\tau)-\varphi(x)| \leq \overline{\lim_{n\to\infty}} \frac{1}{2\,T_n}\int_{-T_n}^{T_n} |f(x+t+\tau)-f(x+t)|\,K(t)\,dt \leq$$

$$\overline{\lim_{n\to\infty}} \frac{1}{2\,T_n}\int_{(E^*_{-x}\dot{+}E^*_{-x-\tau};\,-T_n,\,T_n)} |f(x+t+\tau)-f(x+t)|\,K(t)\,dt +$$

$$\overline{\lim_{n\to\infty}} \frac{1}{2\,T_n}\int_{(C(E^*_{-x}\dot{+}E^*_{-x-\tau});\,-T_n,\,T_n)} |f(x+t+\tau)-f(x+t)|\,K(t)\,dt \leq$$

$$\frac{2}{\eta}\,\overline{m}_B\left(E^*_{-x}\dot{+}E^*_{-x-\tau}\right)+\frac{1}{\eta}\,\varepsilon_0\eta \leq \frac{2}{\eta}\,\overline{m}_B E^*_{-x}+\frac{2}{\eta}\,\overline{m}_B E^*_{-x-\tau}+\varepsilon_0 \leq$$

$$2\,\varepsilon_0+2\,\varepsilon_0+\varepsilon_0 = 5\,\varepsilon_0 .$$

All our numbers τ thus being O-translation numbers of $\varphi(x)$ belonging to $5\,\varepsilon_0$ we conclude that $\varphi(x)$ is an ordinary almost periodic function.

We now come to the salient point of the proof, viz. the demonstration that $\varphi(x)$ only differs "a little" from $f(x)$ outside an exception set with "small" $\overline{m}_B$. As $M\{K(t)\} = 1$ we have

$$f(x) = \lim_{n\to\infty} \frac{1}{2\,T_n} \int_{-T_n}^{T_n} f(x)\, K(t)\, dt,$$

and hence

$$\varphi(x) - f(x) = \lim_{n\to\infty} \frac{1}{2\,T_n} \int_{-T_n}^{T_n} (f(x+t) - f(x))\, K(t)\, dt.$$

Letting T_n for abbreviation also denote an arbitrary subsequence of the above sequence T_n we get the estimation

$$|\varphi(x) - f(x)| \leq \overline{\lim_{n\to\infty}} \frac{1}{2\,T_n} \int_{-T_n}^{T_n} |f(x+t) - f(x)|\, K(t)\, dt =$$

$$\overline{\lim_{n\to\infty}} \frac{1}{2\,T_n} \int_{(I;\,-T_n,\,T_n)} |f(x+t) - f(x)|\, K(t)\, dt,$$

and for $x \varepsilon CE$ this is

$$\leq \overline{\lim_{n\to\infty}} \frac{1}{2\,T_n} \int_{(IE_{-x};\,-T_n,\,T_n)} |f(x+t) - f(x)|\, K(t)\, dt + \overline{\lim_{n\to\infty}} \frac{1}{2\,T_n} \int_{(ICE_{-x};\,-T_n,\,T_n)} |f(x+t) - f(x)|\, K(t)\, dt \leq$$

$$\frac{2}{\eta}\, \overline{m}_B(IE_{-x}) + \varepsilon^2$$

where $\overline{m}_B$ is formed through the subsequence T_n. Now $m_B\, I = \eta$ and the problem thus is only to prove that the intersection IE_{-x} of I and E_{-x} for a suitable subsequence T_n will have an $\overline{m}_B$ (formed through the subsequence T_n) which is essentially smaller than the m_B of I for all x outside a set with small $\overline{m}_B$. Instead of IE_{-x} we may as well consider the set $I_x E$ obtained by the translation x, because $\overline{m}_B(IE_{-x}) = \overline{m}_B(I_x E)$. We investigate $I_x E$ when x runs through one of the intervals $[nL_0, (n+1)\, L_0]$ and we are to show that for a suitable subsequence T_n (the same for all intervals $[nL_0, (n+1)\, L_0]$) the measure $\overline{m}_B(I_x E)$ (formed through this subsequence T_n) will be essentially smaller than η for x in the interval mentioned outside a set with small relative measure. All intervals being treated analogously we may content ourselves with considering detailed only the interval $[0, L_0]$. When x runs through the interval $[0, L_0 - L]$ every one of the small intervals in I_x will decribe an interval lying entirely in one of the intervals $[mL_0, (m+1) L_0]$. The interval $[0, L_0 - L]$ is divided beginning at the left in as many subintervals of length δ as possible. Let $N\delta$ denote the last point of division. Then $L_0 - L - N\delta < \delta$. Corresponding to the points of division $0, \delta, 2\delta, \cdots,$ $N\delta$ we consider the sets $I, I_\delta, I_{2\delta}, \cdots, I_{N\delta}$. Any two of these sets are non-intersecting (with exception of endpoints of intervals) and all sets together fill out the set being decribed of I_x when x runs through the interval $[0, N\delta]$. We now choose our subsequence T_n such that

$$\lim_{n \to \infty} \frac{1}{2\, T_n}\, m\, (I_{q\delta} E; - T_n, T_n)$$

exists for $q = 1, 2, \cdots, N$ and such that the analogous limits exist for all the remaining intervals $[nL_0, (n+1) L_0]$. For this subsequence T_n we will use the above estimation of $|\, \varphi(x) - f(x)\,|$. We first ask: How many of the sets $I_{q\delta}$ have an intersection with E with relative measure (formed through the subsequence T_n) which is $> \varepsilon\eta$. As the sets are non-intersecting and $\overline{m}_B E \leqq \varepsilon^2$ the number A in question must satisfy the relation

$$A\, \varepsilon\eta \leqq \varepsilon^2,$$

i.e. $A \leqq \frac{\varepsilon}{\eta}$. For x lying in $[0, N\delta]$ the set I_x has only points in common with $I_{q\delta}$ for $|\, x - q\,\delta\,| \leqq \delta$. Letting x in $[0, N\delta]$ avoid the intervals $|\, x - q\,\delta\,| \leqq \delta$ corresponding to the A sets $I_{q\delta}$ above with $m_B(I_{q\delta} E) > \varepsilon\eta$ (m_B formed through the subsequence T_n) the sets I_x must be contained in the sum of two neighbour-$I_{q\delta}$'s with $m_B\,(I_{q\delta} E) \leqq \varepsilon\eta$. Together the intervals which x must avoid have a relative measure $\leqq A\, \frac{2\,\delta}{L_0} \leqq \frac{\varepsilon}{\eta}\, \frac{2\,\delta}{L_0} = \frac{\varepsilon}{\eta}\, 2\,\eta = 2\,\varepsilon$. Then $\overline{m}_B\,(I_x E) \leqq 2\,\varepsilon\eta$ ($\overline{m}_B$ formed through T_n) and this quantity is essentially smaller than η. Considering the whole interval $[0, L_0]$ the argument x must further avoid the interval $[N\delta, L_0]$ of a length $\leqq L + \delta$. Hence the relative measure of the intervals which x must avoid in $[0, L_0]$ is $\leqq \frac{L+\delta}{L_0} + 2\,\varepsilon \leqq \varepsilon + 2\,\varepsilon = 3\,\varepsilon$. Analogous results are obtained for the remaining intervals $[nL_0, (n+1)L_0]$. Letting E_1 denote the sum of exception sets from all the intervals $[nL_0, (n+1)L_0]$ we get

$$\overline{m}_B(I_x E) \leqq 2\varepsilon\eta \qquad (\overline{m}_B \text{ formed through } T_n)$$

for $x \varepsilon CE_1$, where $\overline{m}_B E_1 \leqq 3\varepsilon$. Inserting in the estimation on page 21 we get for $x \varepsilon C(E \dot{+} E_1)$

$$|\varphi(x) - f(x)| \leqq \frac{2}{\eta} 2\varepsilon\eta + \varepsilon^2 = 4\varepsilon + \varepsilon^2$$

and further

$$\overline{m}_B(E \dot{+} E_1) \leqq \overline{m}_B E + \overline{m}_B E_1 \leqq \varepsilon^2 + 3\varepsilon.$$

Letting $\varepsilon \to 0$ both $4\varepsilon + \varepsilon^2$ and $3\varepsilon + \varepsilon^2$ will converge towards 0. This accomplishes the proof of the main theorem.

Remark. The main theorem is still valid if instead of the "B-measure" $\overline{m}_B$ we use an S-measure or a W-measure. This is easily seen from the fact that the cut-off function still in the corresponding sense will converge to the function considered, in connection with the fact that the K-a. p. properties of the cut-off function (with S or W instead of B) immediately involve that it possesses the S^p-a. p. respectively W^p-a. p. properties mentioned in the introduction. For the cut-off function can then be S-, respectively W-approximated by trigonometric polynomials and this involves that the cut-off function and with it the original function itself can be K-approximated by trigonometric polynomials. In the S-case we obviously need not claim the exception set to be the same for all τ belonging to the same ε, cp. W. Stepanoff [I], especially definition I and lemma II.

The connection between the K-a. p. and the B^p-a. p. functions appears from the following simple

Theorem 2. *A B^p-a. p. function means the same as a K-a. p. function $f(x)$ for which* $\| f(x) - (f(x))_N \|_{B^p} \to 0$ for $N \to \infty$.

This theorem has the corollary

Theorem 3. *A bounded K-a.p. function is B^p-a.p. for all p.*

If $f(x)$ is K-a.p. so is $(f(x))_N$ (see p. 12). The corollary thus gives:

Theorem 4. *If $f(x)$ is K-a.p. the cut-off function $(f(x))_N$ is B^p-a.p. for all p.*

We now turn to the proof of theorem 2.

1°. If $f(x)$ is B^p-a.p. then as is well-known we shall have $\|f(x)-(f(x))_N\|_{B^p} \to 0$ for $N \to \infty$. Further a sequence $s_n(x)$ of trigonometric polynomials can be found such that $\|f(x)-s_n(x)\|_{B^p} \to 0$, and from this follows that

$$\overline{m}_B[|f(x)-s_n(x)|>\varepsilon] \to 0$$

for every fixed $\varepsilon > 0$; hence $s_n(x)$ K-converges towards $f(x)$ which is therefore K-a.p.

2°. Let $f(x)$ be K-a.p. and let $\|f(x)-(f(x))_N\|_{B^p} \to 0$ for $N \to \infty$. From the first assumption follows that we can find a sequence of ordinary almost periodic functions $\varphi_n(x)$ which K-converges towards $f(x)$. As

$$|(f(x))_N-(\varphi_n(x))_N| \leqq |f(x)-\varphi_n(x)|$$

the sequence $(\varphi_n(x))_N$ (N fixed) of almost periodic functions K-converges towards $(f(x))_N$. The functions $(f(x))_N$ and $(\varphi_n(x))_N$ being uniformly bounded $(\varphi_n(x))_N$ (N fixed) will also B^p-converge towards $(f(x))_N$; for to every $\varepsilon > 0$ we can choose a number n_0 such that

$$|(f(x))_N-(\varphi_n(x))_N| \leqq \varepsilon \quad \text{for} \quad x \,\varepsilon\, CE_n \quad \text{and} \quad n \geqq n_0$$

where $\overline{m}_B E_n \leqq \varepsilon$ and from this follows that

$$\|(f(x))_N-(\varphi_n(x))_N\|_{B^p}^p \leqq \varepsilon^p + (2N)^p \varepsilon \quad \text{for} \quad n \geqq n_0.$$

Hence $(f(x))_N$ is B^p-a.p. The second assumption $\|f(x)-(f(x))_N\|_{B^p} \to 0$ then involves that $f(x)$ is B^p-a.p., too.

Instead of the additional condition $\|f(x)-(f(x))_N\|_{B^p} \to 0$ for $N \to \infty$ one may also—as Kovanko has done it—use the additional condition $\|f_E(x)\|_{B^p} \to 0$ for $\overline{m}_B E \to 0$ where $f_E(x)$ denotes the function which is equal to $f(x)$ in (the arbitrary set)

E and equal to 0 outside E. This is easily seen by means of lemma 1.

Remark. If we substitute for $\overline{m}_B$ either $\overline{m}_S$ or $\overline{m}_W$ and at the same time for the B^p-norm respectively the S^p_L-norm or the W^p-norm the theorems 2—4 are obviously still valid.

§ 4. The Connection between the B^p-bounded K-a. p. Functions and the B^p-a. p. Functions.

Instead of the additional condition $\| f(x) - (f(x))_N \|_{B^p} \to 0$ for $N \to \infty$ we shall in this section consider the simpler additional condition $\| f(x) \|_{B^p} < \infty$, i. e. we shall investigate the B^p-bounded K-a. p. functions. In the proofs given or referred to in this section one always uses the characterization by approximation of the K-a. p. functions. The connection between B^p-bounded K-a. p. functions and the B^p-a. p. functions is expressed in the following

Theorem 5. *A B^p-bounded K-a. p. function $f(x)$ can be written in the form*

$$f(x) = g(x) + j(x)$$

where $g(x)$ is B^p-a. p. and $j(x)$ is a B^p-bounded K-zero function, i. e. a function for which $\| j(x) \|_{B^p} < \infty$ and $\overline{m}_B[|j(x)| > \varepsilon] = 0$ for every $\varepsilon > 0$. Conversely every function of this form is a B^p-bounded K-a. p. function.

Proof. 1°. The last part of the theorem is obvious.

2°. For the proof of the first part of the theorem we refer to the proof of an analogous theorem in Bohr and Følner [I], p. 99, viz. the theorem that a B^1-a. p. point which contains a B^p-bounded function also contains a B^p-a. p. function. This theorem relies on a theorem of Jessen [I] stating that every real B^1-a. p. function has an asymptotic distribution function. Further Jessen and Wintner [I] have shown that every real K-a. p. function has an asymptotic distribution function. By means of this theorem it is possible—just as in Bohr and Følner [I]—to show that for a B^p-bounded K-a. p. function $f(x)$ the cut-off functions $(f(x))_N$ $(N = 1, 2, \cdots)$ will form a B^p-fundamental sequence which on account of theorem 4 and the completeness of the B^p-a. p. spaces will B^p-converge towards a B^p-a. p.

function $g(x)$. The sequence $(f(x))_N$ K-converging both towards $f(x)$ and $g(x)$ the function $j(x) = f(x) - g(x)$ is a K-zero function and $f(x)$ and $g(x)$ being both B^p-bounded so is $j(x)$.

Remark. This theorem is not valid if $\overline{m}_B$ is replaced by $\overline{m}_S$ and the B^p-norm by the S^p-norm (in which case functions of the type $j(x)$ is 0 almost everywhere). For $p > 1$ we have a counter example in Bohr and Følner [I], main example 2, p. 70, and for $p = 1$ an analogous example may easily be constructed. The theorem is valid no more if $\overline{m}_B$ is replaced by $\overline{m}_W$ and the B^p-norm by the W^p-norm. For $p > 1$ we have a counter example in Bohr and Følner [I], main example 3, p. 83 and for $p = 1$ an analogous example may easily be constructed.

By a B^p-bounded K-point we shall understand the set of functions which only differ from one and the same B^p-bounded function by B^p-bounded K-zero functions. If function one in the points is a (B^p-bounded) K-a. p. function so are all functions in the point and the point is called a B^p-bounded K-a. p. point. The non obvious part of our theorem may then be expressed: In every B^p-bounded K-a. p. point there is lying a B^p-a. p. point. On the other hand only one B^p-a. p. point can be lying in a B^p-bounded K-a. p. point, for if $j(x)$ is a K-zero function and B^p-a. p. then $(j(x))_N$ is a $B^{p'}$-zero function for all p', specially a B^p-zero function, and $\|j(x) - (j(x))_N\|_{B^p} \to 0$ for $N \to \infty$, so that $j(x)$ is also a B^p-zero function.

Theorem 6. *For $p > 1$ a B^p-bounded K-zero function $j(x)$ is a $B^{p'}$-zero function for $1 \leq p' < p$.*

Proof. We have

$$\|j(x)\|_{B^{p'}} = \overline{\lim_{T\to\infty}} \sqrt[p']{\frac{1}{2T}\int_{-T}^{T} |j(x)|^{p'} dx} \leq \overline{\lim_{T\to\infty}} \sqrt[p']{\frac{1}{2T}\int_{([|j(x)| \leq \varepsilon]; -T, T)} |j(x)|^{p'} dx} + \overline{\lim_{T\to\infty}} \sqrt[p']{\frac{1}{2T}\int_{([|j(x)| > \varepsilon]; -T, T)} |j(x)|^{p'} dx} = I_1 + I_2.$$

Here $I_1 \leq \varepsilon$. For I_2 we have

$$I_2 = \| j(x)\, e(x) \|_{B^{p'}}$$

where $e(x) = 1$ in $[|j(x)| > \varepsilon]$ and $e(x) = 0$ outside $[|j(x)| > \varepsilon]$ By means of Hölder's inequality we get

$$I_2^{p'} = \overline{\lim_{T\to\infty}} \frac{1}{2T} \int_{-T}^{T} |j(x)|^{p'} e(x)^{p'} dx <$$

$$\left(\overline{\lim_{T\to\infty}} \frac{1}{2T} \int_{-T}^{T} |j(x)|^{p} dx\right)^{\frac{p'}{p}} \left(\overline{\lim_{T\to\infty}} \frac{1}{2T} \int_{-T}^{T} e(x)^{\frac{1}{\frac{1}{p'}-\frac{1}{p}}} dx\right)^{1-\frac{p'}{p}} =$$

$$\| j(x) \|_{B^p}^{p'} \left(\overline{m}_B [|j(x)| > \varepsilon]\right)^{1-\frac{p'}{p}} = 0$$

so that $I_2 = 0$. Hence $\| j(x) \|_{B^{p'}} \leqq \varepsilon$ for every $\varepsilon > 0$, i. e. $\| j(x) \|_{B^{p'}} = 0$.

Moreover the following theorem is valid:

Theorem 7. *For $p > 1$ a B^p-bounded K-a. p. function $f(x)$ is a $B^{p'}$-a. p. function for $1 \leqq p' < p$.*

Proof. This is a consequence of theorem 5 and theorem 6. The theorem, however, may also be proved directly by means of Hölder's inequality which shows that $\| f(x) - (f(x))_N \|_{B^{p'}} \to 0$ for $N \to \infty$.

As is easily seen a B^1-bounded K-zero function (which a fortiori is a K-a. p. function) need not possess a mean value in the Besicovitch sense. We are to prove, however, that the mean value notion can be generalized such that every B^1-bounded K-a. p. function gets a mean value and a B^1-bounded K-zero function especially the mean value 0. For a B^1-bounded K-a. p. function $f(x)$ we simply define the generalized mean value $M^*\{f(x)\}$ by

$$M^*\{f(x)\} = \lim_{N\to\infty} M\{(f(x))_N\}.$$

The mean value $M\{(f(x))_N\}$ exists since $(f(x))_N$ — on account of theorem 4 — is a B^1-a. p. function. And as mentioned in the proof of theorem 5 the sequence $(f(x))_N$ $(N = 1, 2, \cdots)$ is a B^1-fundamental sequence from which follows that $M\{(f(x))_N\}$ is a fundamental sequence since

$$|M\{(f(x))_{N_1}\} - M\{(f(x))_{N_2}\}| \leqq M\{|(f(x))_{N_1} - (f(x))_{N_2}|\} = \\ \|(f(x))_{N_1} - (f(x))_{N_2}\|_{B^1}$$

and hence $\lim_{N\to\infty} M\{(f(x))_N\}$ exists. If $f(x)$ is a B^1-a. p. function then $\|f(x) - (f(x))_N\|_{B^1} \to 0$ and an estimation analogous to the above one gives

$$M^*\{f(x)\} = M\{f(x)\}.$$

Let $f(x)$ be a B^p-bounded (and a fortiori B^1-bounded) K-a. p. function. From theorem 5 follows that

$$f(x) = g(x) + j(x)$$

where $g(x)$ is B^p-a. p. and $j(x)$ is a B^p-bounded K-zero function. We will show that

$$M^*\{f(x)\} = M\{g(x)\}.$$

Proof. For $x \varepsilon [|j(x)| \leqq \varepsilon]$ we have

$$|(g(x) + j(x))_N - (g(x))_N| \leqq |j(x)| \leqq \varepsilon$$

and as $\overline{m}_B[|j(x)| > \varepsilon] = 0$ we get

$$|M\{(g(x) + j(x))_N\} - M\{(g(x))_N\}| \leq \varepsilon$$

so that

$$M\{(g(x) + j(x))_N\} = M\{(g(x))_N\}.$$

Letting $N \to \infty$ we get

$$M^*\{f(x)\} = M^*\{g(x) + j(x)\} = M\{g(x)\}, \text{ q. e. d.}$$

Now let

$$g(x) \sim \sum A_n e^{i\Lambda_n x}.$$

Together with $f(x)$ the function $f(x)e^{-i\lambda x}$ is also a B^p-bounded K-a. p. function, and together with $j(x)$ the function $j(x)e^{-i\lambda x}$ is also a B^p-bounded K-zero function. We then get

$$M^*\{f(x)e^{-i\lambda x}\} = M^*\{g(x)e^{-i\lambda x} + j(x)e^{-i\lambda x}\} = M\{g(x)e^{-i\lambda x}\}.$$

By means of the new mean value notion we can in the usual manner ascribe to $f(x)$ a Fourier series which will be identical with the Fourier series of $g(x)$:

$$f(x) \sim \sum A_n e^{i \Lambda_n x}.$$

The Bochner-Fejér polynomials of the series will B^p-converge towards a B^p-a. p. function (viz. $g(x)$) which only differs from $f(x)$ by a B^p-bounded K-zero function.

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