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FOURTH ORDER VACUUM POLARIZATION

BY

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CONTENTS

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The real and imaginary parts of the kernel for the fourth order vacuum polarization are calculated for all values of the four-dimensional energy momentum vector. If an expansion in powers of the square of this quantity is used, the first coefficient agrees with a result previously obtained by BARANGER *et. al.*

I. Introduction.

In a previous paper, one of us[1] has developed a formulation of renormalized quantum electrodynamics that is slightly different from the standard techniques used by most authors. This modification was introduced because of its convenience in discussions of general principles. It has been applied, for example, to a discussion, avoiding perturbation theory, of the magnitude of the renormalization constants.[2] In the present paper, we wish to show that the new method can also be used with advantage in practical calculations in which perturbation theory is applied, and, as an illustration, the fourth order vacuum polarization has been chosen. BARANGER, DYSON and SALPETER[3] have computed those terms in this effect which are important in the Lamb shift. They present, however, only the result and very few intermediary steps of the calculation. On the other hand, we attempt to give a fairly detailed account of our calculations, and compute not only the terms of immediate experimental interest, but also the complete vacuum polarization kernel as a function of the four-dimensional momentum. As will be seen later, our calculation is simplified to a certain extent by the fact that we can use the result of an earlier calculation of the lowest-order radiative corrections to the current operator[4] and thereby avoid some

[1] G. KÄLLÉN, Helv. Phys. Acta **25**, 417 (1952), in the following quoted as I.
[2] G. KÄLLÉN, Dan. Mat. Fys. Medd. **27**, no. 12 (1953).
[3] M. BARANGER, F. J. DYSON, and E. E. SALPETER, Phys. Rev. **88**, 680 (1952).
[4] J. SCHWINGER, Phys. Rev. **76**, 790 (1949).

integrations. Since the main work involved in the calculation of a high-order effect is connected with the integrations over the so-called "Feynman auxiliary variables", a simplification at this point is not without interest. A further advantage of our method is that the questions of regularization[1] and of the so-called "overlapping divergences"[2] are completely avoided. Finally, due to the application of the known expression for the current operator, we need not carry out any explicit mass renormalization in our calculations.

II. General Outline of the Method.

We start from the following formulae given in I:

$$\langle o|\overset{\text{ind}}{j_\mu}(x)|o\rangle = \frac{1}{(2\pi)^4}\int dp\, e^{ipx}\left(-\overline{\Pi}(p^2)+\overline{\Pi}(0)-i\pi\,\varepsilon(p)\,\Pi(p^2)\right)\overset{\text{ext}}{j_\mu}(p), \tag{1}$$

$$\overline{\Pi}(p^2)-\overline{\Pi}(0) = -p^2\int_0^\infty \frac{da\,\Pi(-a)}{a(a+p^2)}, \tag{2}$$

$$\Pi(p^2) = \frac{V}{-3p^2}\sum_{p^{(z)}=p}\langle 0|j_\mu|z\rangle\langle z|j_\mu|0\rangle. \tag{3}$$

The notation is the same as in I and will be used here without further explanation. If the matrix elements of the current operator are expanded in powers of e,

$$j_\mu = ej_\mu^{(0)} + e^2 j_\mu^{(1)} + e^3 j_\mu^{(2)} + \cdots, \tag{4}$$

the first non-vanishing contribution to the function $\Pi(p^2)$ will be

$$\Pi^{(0)}(p^2) = \frac{Ve^2}{-3p^2}\sum_{p^{(z)}=p}\langle 0|j_\mu^{(0)}|z\rangle\langle z|j_\mu^{(0)}|0\rangle. \tag{5}$$

[1] W. Pauli and F. Villars, Rev. Mod. Phys. **21**, 434 (1949). The regularization of the fourth order vacuum polarization has been discussed by E. Karlson, Arkiv f. Fysik **7**, 221 (1954).

[2] A. Salam, Phys. Rev. **82**, 217 (1951). For the special problem of fourth order vacuum polarization, the overlapping divergences have been discussed by R. Jost and J. M. Luttinger, Helv. Phys. Acta **23**, 201 (1949).

This expression can be computed easily and gives

$$\Pi^{(0)}(p^2) = \frac{e^2}{12\pi^2}\left(1-\frac{2m^2}{p^2}\right)\sqrt{1+\frac{4m^2}{p^2}}\,\theta(-p^2-4m^2), \tag{6}$$

$$\theta(x) = \frac{1}{2}\left[1+\frac{x}{|x|}\right], \tag{6a}$$

$$\left.\begin{aligned}&\overline{\Pi}^{(0)}(p^2)-\overline{\Pi}^{(0)}(0) = \\ &-\frac{e^2}{12\pi^2}\left[\frac{4m^2}{p^2}-\frac{5}{3}+\left(1-\frac{2m^2}{p^2}\right)\sqrt{1+\frac{4m^2}{p^2}}\log\frac{1+\sqrt{1+\frac{4m^2}{p^2}}}{\left|1-\sqrt{1+\frac{4m^2}{p^2}}\right|}\right].\end{aligned}\right\} \tag{7}$$

The subsequent term in the expansion of the function $\Pi(p^2)$ is of order e^4, and contains the following terms:

$$\Pi^{(1)}(p^2) = \Pi_a^{(1)}(p^2)+\Pi_b^{(1)}(p^2), \tag{8}$$

$$\Pi_a^{(1)}(p^2) = \frac{Ve^4}{-3p^2}\sum_{p^{(z)}=p}\langle 0|j_\mu^{(1)}|z\rangle\langle z|j_\mu^{(1)}|0\rangle, \tag{9}$$

$$\Pi_b^{(1)}(p^2) = \frac{Ve^4}{-3p^2}\sum_{p^{(z)}=p}\langle 0|j_\mu^{(2)}|z\rangle\langle z|j_\mu^{(0)}|0\rangle + \text{complex conjugate}. \tag{10}$$

The expansion of the current operator has been computed earlier.[1] From these results it can be seen that the term (9) gets contributions from states with one in-coming pair and one in-coming photon.[2] These matrix elements are

$$\left.\begin{aligned}\langle 0|j_\mu^{(1)}|q,q',k\rangle &= \frac{e^2}{V^{3/2}}\frac{1}{\sqrt{2\omega}}\bar{u}^{(-)}(-q')\left[\gamma_\mu\frac{i\gamma(q+k)-m}{2qk-\mu^2}\gamma e\right.\\ &\left.-\gamma e\frac{i\gamma(q'+k)+m}{2q'k-\mu^2}\gamma_\mu\right]u^{(+)}(q).\end{aligned}\right\} \tag{11}$$

The notation in the last expression is self-explanatory, except possibly for the quantities $u^{(\pm)}(q)$. These are the normalized

[1] Cf., e. g., G. Källén, Arkiv f. Fysik **2**, 371 (1950).

[2] For the definition of particle numbers for these physical states, cf., *e. g.*, G. Källén, Physica **19**, 850 (1953).

plane-wave solutions of the free-particle Dirac equation. The index (+) refers to solutions with positive energy and the index (—) to solutions with negative energy. The vector e is the polarization vector of the photon; V is the volume of periodicity and μ a small photon mass introduced to handle infrared divergences. In the computation of the function $\Pi_a^{(1)}(p^2)$, we must "square" the expression (11) and sum over all states where $k+q+q'=p$. Using well-known properties of the functions u, and taking the limit $V\to\infty$, we can write this sum as an integral

$$\left.\begin{aligned} & e^4 V \sum_{q+q'+k=p} \langle 0 | j_\mu^{(1)} | q, q', k\rangle \langle k, q', q | j_\mu^{(1)} | 0\rangle = \\ & -\frac{e^4}{(2\pi)^6}\int dk\, dq\, dq'\, \delta(p-q-q'-k)\,\delta(q^2+m^2)\,\theta(q)\,\delta(q'^2+m^2)\,\theta(q') \\ & \times\delta(k^2+\mu^2)\,\theta(k)\, Sp\left[(i\gamma q'+m)\left(\gamma_\mu\frac{i\gamma(q+k)-m}{2\,qk-\mu^2}\gamma_\lambda\right.\right. \\ & \left.-\gamma_\lambda\frac{i\gamma(q'+k)+m}{2q'k-\mu^2}\gamma_\mu\right)(i\gamma q-m)\left(\gamma_\mu\frac{i\gamma(q'+k)+m}{2\,q'k-\mu^2}\gamma_\lambda\right. \\ & \left.\left.-\gamma_\lambda\frac{i\gamma(q+k)-m}{2\,qk-\mu^2}\gamma_\mu\right)\right]. \end{aligned}\right\} \quad (12)$$

The evaluation of this integral, which is the main task in our computation, is given in a later paragraph.

The first approximation to the current, $j_\mu^{(0)}$, has matrix elements which connect the vacuum only to states with one in-coming pair. Hence, the expression (10) will reduce to a sum over states with one in-coming pair

$$\Pi_b^{(1)}(p^2) = \frac{Ve^4}{-3\,p^2}\sum_{q+q'=p}\langle 0|j_\mu^{(2)}|q,q'\rangle\langle q',q|j_\mu^{(0)}|0\rangle + \text{complex conjugate.} \quad (13)$$

As has been mentioned in the introduction, the matrix elements $\langle 0|j_\mu^{(2)}|q,q'\rangle$ have been computed by SCHWINGER.[1] We write his result as

$$\left.\begin{aligned} & e^2\langle 0|j_\mu^{(2)}|q,q'\rangle = [-\overline{\Pi}^{(0)}(p^2)+\overline{\Pi}^{(0)}(0)+\overline{R}(p^2)-\overline{R}(0)+\overline{S}(0)-i\pi(\Pi^{(0)}(p^2) \\ & -R(p^2))]\langle 0|j_\mu^{(0)}|q,q'\rangle-\frac{1}{2\,m}(q_\mu-q'_\mu)[\overline{S}(p^2)+i\pi S(p^2)]\langle 0|\overline{\psi}^{(0)}\psi^{(0)}|q,q'\rangle, \end{aligned}\right\} \quad (14)$$

[1] Footnote 4, p. 3.

$$R(p^2) = \frac{e^2}{8\pi^2}\left\{\frac{1+\frac{2m^2}{p^2}}{1+\frac{4m^2}{p^2}}\log\left(-\frac{p^2+4m^2}{\mu^2}\right)-\frac{3}{2}\right\}\sqrt{1+\frac{4m^2}{p^2}}\;\theta(-p^2-4m^2), \quad (15)$$

$$\begin{aligned} R(p^2)-\overline{R}(0) = \frac{e^2}{4\pi^2}\Bigg\{&\frac{1+\frac{2m^2}{p^2}}{\sqrt{1+\frac{4m^2}{p^2}}}\left[-\Phi\left(-\frac{1-\sqrt{1+\frac{4m^2}{p^2}}}{1+\sqrt{1+\frac{4m^2}{p^2}}}\right)+\frac{\pi^2}{4}\theta(-p^2)\right.\\ &\left.-\frac{1}{4}\log^2\left|\frac{1+\sqrt{1+\frac{4m^2}{p^2}}}{1-\sqrt{1+\frac{4m^2}{p^2}}}\right|+\log\left|\frac{1+\sqrt{1+\frac{4m^2}{p^2}}}{1-\sqrt{1+\frac{4m^2}{p^2}}}\right|\cdot\log\frac{2\sqrt{1+\frac{4m^2}{p^2}}}{1+\sqrt{1+\frac{4m^2}{p^2}}}\right]\\ &+\frac{3}{4}\sqrt{1+\frac{4m^2}{p^2}}\log\left|\frac{1+\sqrt{1+\frac{4m^2}{p^2}}}{1-\sqrt{1+\frac{4m^2}{p^2}}}\right|-2\\ &+\log\frac{m}{\mu}\left[1-\frac{1+\frac{2m^2}{p^2}}{\sqrt{1+\frac{4m^2}{p^2}}}\log\left|\frac{1+\sqrt{1+\frac{4m^2}{p^2}}}{1-\sqrt{1+\frac{4m^2}{p^2}}}\right|\right]\Bigg\}, \end{aligned} \quad (16)$$

$$S(p^2) = -\frac{e^2}{4\pi^2}\frac{m^2}{p^2}\frac{\theta(-p^2-4m^2)}{\sqrt{1+\frac{4m^2}{p^2}}}, \quad (17)$$

$$\overline{S}(p^2) = \frac{e^2}{8\pi^2}\frac{2\frac{m^2}{p^2}}{\sqrt{1+\frac{4m^2}{p^2}}}\log\left|\frac{1+\sqrt{1+\frac{4m^2}{p^2}}}{1-\sqrt{1+\frac{4m^2}{p^2}}}\right|. \quad (18)$$

The connection between the functions $R(p^2)$ and $\overline{R}(p^2)$, and between $S(p^2)$ and $\overline{S}(p^2)$, is the same as the connection between $\Pi(p^2)$ and $\overline{\Pi}(p^2)$, which is given in Eq. (2). This is a consequence of the "causal" structure of the theory which says that the value of the current in one point x can depend only on the previous history of the system inside the retarded light-cone belonging to x. If Eq. (14) is written in x-space, we get a relation of the form

$$\left.\begin{aligned} \langle 0 | j_\mu^{(2)}(x) | q, q' \rangle = \int dx' F(x-x') \langle 0 | j_\mu^{(0)}(x') | q, q' \rangle \\ + \int G(x-x') \langle 0 | \frac{\partial \overline{\psi}^{(0)}(x')}{\partial x'_\mu} \psi^{(0)}(x') - \overline{\psi}^{(0)}(x') \frac{\partial \psi^{(0)}(x')}{\partial x'_\mu} | q, q' \rangle . \end{aligned}\right\} \quad (14a)$$

Causality requires $F(x)$ and $G(x)$ to vanish if $x_0 < 0$ and this gives, in a well-known way, the relations involving the Hilbert transformations. This offers a new possibility of computing the matrix element under discussion by first computing the "imaginary parts" $R(p^2)$ and $S(p^2)$, which can be obtained by integrating over finite domains in momentum space and, subsequently, computing the "real parts" with the aid of Hilbert transformations. Actually, a calculation of this kind has been performed. However, it has not been found to be much simpler than the standard methods for this problem. On the other hand, arranging the computation in this way is certainly not a more complicated procedure. We will not insist on this point here, but accept the results (14)—(18) as they stand. Consequently, the computation of the function $\Pi_b^{(1)}(p^2)$ will be reduced to simple algebraic manipulations of these expressions. The function $\Phi(x)$ in (16) is defined by the integral

$$\Phi(x) = \int_1^x \frac{dt}{t} \log|1+t|. \quad (19)$$

Hereby it is supposed that the argument x is real, *i. e.* that $1 + \frac{4m^2}{p^2} > 0$. This will be sufficient at this stage. The integral $\Phi(x)$ has many interesting properties which will be of some use in our calculation and that are discussed in the Appendix.

We now write the function $\Pi_b^{(1)}(p^2)$ as

$$\left.\begin{aligned} \Pi_b^{(1)}(p^2) = 2\,\Pi^{(0)}(p^2)\,[-\overline{\Pi}^{(0)}(p^2) + \overline{\Pi}^{(0)}(0) + \overline{R}(p^2) \\ - \overline{R}(0) + \overline{S}(0)] + 2\,\overline{S}(p^2)\,X(p^2), \end{aligned}\right\} \quad (20)$$

where

$$X(p^2) = \frac{4e^2}{3\,p^2}\frac{1}{(2\,\pi)^3}\int dq\, dq'\, \delta(p-q-q') \times \delta(q^2+m^2)\,\theta(q)\,\delta(q'^2+m^2)\,\theta(q')\,(qq'+m^2). \tag{21}$$

The last expression is easily computed

$$X(p^2) = \frac{e^2}{24\,\pi^2}\left(1+\frac{4\,m^2}{p^2}\right)^{3/2}\theta(-p^2-4\,m^2). \tag{22}$$

Collecting all these results, we have

$$\begin{aligned}\Pi_b^{(1)}(p^2) = \frac{e^4}{48\,\pi^4}\Bigg[&\frac{\delta}{3}(3-\delta^2)\left(-\frac{17}{3}+\delta^2\right)+\frac{\delta^2}{2}\left(7-3\,\delta^2+\frac{1}{3}\delta^4\right)\log\frac{1+\delta}{1-\delta}\\ &+\log\frac{m}{\mu}\,\delta(3-\delta^2)\left(1-\frac{1+\delta^2}{2\,\delta}\log\frac{1+\delta}{1-\delta}\right)-\frac{1}{2}(3-\delta^2)(1+\delta^2)\Bigg(\Phi\left(-\frac{1-\delta}{1+\delta}\right)\\ &-\frac{\pi^2}{4}+\frac{1}{4}\log^2\frac{1+\delta}{1-\delta}-\log\frac{1+\delta}{1-\delta}\cdot\log\frac{1+\delta}{2\,\delta}\Bigg)\Bigg]\theta(1-\delta),\end{aligned} \tag{23}$$

where

$$\delta = \sqrt{1+\frac{4\,m^2}{p^2}} > 0. \tag{24}$$

III. Discussion of the Part $\Pi_a^{(1)}(p^2)$.

The remaining part of the function $\Pi(p^2)$, the integral (12), can be treated in the following way. We first compute the trace of the γ-matrices. This is a straightforward calculation and the necessary work can be considerably reduced by performing first the summations over the indices μ and λ. This can be done with the aid of the well-known formulae

$$\gamma_\lambda\gamma_{\nu_1}\gamma_{\nu_2}\cdot\cdot\cdot\cdot\,\gamma_{\nu_{2n+1}}\gamma_\lambda = -2\,\gamma_{\nu_{2n+1}}\cdot\cdot\cdot\cdot\,\gamma_{\nu_2}\gamma_{\nu_1} \tag{25}$$

$$\gamma_\lambda\gamma_{\nu_1}\gamma_{\nu_2}\gamma_\lambda = 4\,\delta_{\nu_1\nu_2}. \tag{26}$$

The complete trace can then be written

$$Sp\,[\cdots] = \frac{S^{(1)}(q, q')}{(2\,qk-\mu^2)^2} + \frac{S^{(1)}(q', q)}{(2\,q'k-\mu^2)^2} + \frac{S^{(2)}(q, q') + S^{(2)}(q', q)}{(2\,qk-\mu^2)(2q'k-\mu^2)}, \quad (27$$

$$S^{(1)}(q, q') = -32\,[kq\cdot kq' + 2\,m^2\,qk + m^2\,q'k + m^2\,(qq' - 2\,m^2)], \quad (28$$

$$S^{(2)}(q, q') = -16\,[2\,(qq')^2 - 4\,m^2\cdot qq' + 2\,(kq + kq')\,qq' - m^2\,(kq + kq')]. \quad (29$$

Terms containing μ^2 have been dropped in (28) and (29), as they will obviously vanish in the limit $\mu \to 0$.

Our next task is to compute an integral of the form

$$\left.\begin{array}{l} J = \int dk\,dq\,dq'\,\delta(p-k-q-q')\,\delta(q^2+m^2)\,\delta(q'^2+m^2)\,\delta(k^2+\mu^2) \\ \qquad \times\,\theta(q)\,\theta(q')\,\theta(k)\,F(qk, q'k, qq'). \end{array}\right\} \quad (30$$

This can conveniently be done in two steps. We first consider

$$\left.\begin{array}{l} I(p'^2, kp') = \int dq\,\delta(q^2+m^2)\,\theta(q)\,\delta((p'-q)^2+m^2)\,\theta(p'-q) \\ \qquad \times\,F(qk, p'k-qk, p'q-q^2). \end{array}\right\} \quad (31$$

This is a *finite* integral and we compute it in the special coordinate system where the space-like components of the vector p' vanish. We then obtain

$$I(-p_0'^2, -k_0p_0') = \pi \int_{x_1}^{x_2} \frac{dx}{2\,p_0'\,|\bar{k}|}\,F\left(x, -k_0p_0' - x, m^2 - \frac{1}{2}p_0'^2\right)\theta(p_0'^2 - 4\,m^2), \quad (32$$

$$x_{1,\,2} = -\frac{1}{2}\,k_0\,p_0' \mp |\bar{k}|\sqrt{\frac{1}{4}p_0'^2 - m^2}. \quad (32\,\mathrm{a})$$

We now write this result in an invariant way, as

$$I(p'^2, kp') = \frac{\pi}{2\sqrt{(kp')^2 + \mu^2 p'^2}}\int_{x_1}^{x_2} dx\,F\left(x, kp' - x, m^2 + \frac{1}{2}p'^2\right)\theta(-p'^2 - 4\,m^2), \quad (33$$

$$x_{1,2} = \frac{1}{2} kp' \mp \frac{1}{2} \sqrt{(kp')^2 + \mu^2 p'^2} \cdot \sqrt{1 + \frac{4m^2}{p'^2}}, \tag{33a}$$

and treat the next integration similarly. The result is

$$\left.\begin{aligned} J &= \int dk\, \delta(k^2+\mu^2)\, \theta(k)\, I((p-k)^2, kp+\mu^2) = \\ &-\frac{\pi^2}{2p^2} \int_{\mu\sqrt{-p^2}}^{-\frac{1}{2}(p^2+4m^2)} dy \int_{-\xi}^{+\xi} dz\, F\left(\frac{z-y+\mu^2}{2}, \frac{-z-y+\mu^2}{2}, y+m^2+\frac{p^2-\mu^2}{2}\right), \end{aligned}\right\} \tag{34}$$

$$\xi = \sqrt{1 + \frac{4m^2}{p^2-\mu^2+2y}} \cdot \sqrt{y^2+\mu^2 p^2}. \tag{34a}$$

Applying this technique to the integral (12), we get

$$\Pi_a^{(1)}(p^2) = \frac{e^2}{12\pi^4} \frac{1}{p^4} (A+B)\, \theta(-p^2-4m^2), \tag{35}$$

$$A = \int_{\mu\sqrt{-p^2}}^{-\frac{1}{2}(p^2+4m^2)} dy \int_{-\xi}^{+\xi} dz \left[-\frac{1}{2} + \frac{m^2-y}{z-y} + \frac{y}{y^2-z^2}(p^2-m^2)\right], \tag{36}$$

$$B = \int_{\mu\sqrt{-p^2}}^{-\frac{1}{2}(p^2+4m^2)} dy \int_{-\xi}^{+\xi} dz\, (p^2-2m^2) \left[\frac{m^2}{(z-y)^2} + \frac{1}{2}\frac{p^2+2m^2}{y^2-z^2}\right]. \tag{37}$$

To obtain these expressions, we have introduced the quantities y and z into (27), which becomes

$$\left.\begin{aligned} Sp[\cdots] = -32 \Big\{ &-\frac{1}{2} + \frac{m^2-y}{z-y} + \frac{m^2(p^2-2m^2)}{(z-y)^2} \\ &+ \frac{1}{y^2-z^2}\left(y(p^2-m^2) + \frac{1}{2}(p^4-4m^4)\right)\Big\}. \end{aligned}\right\} \tag{38}$$

The quantity A will stay finite in the limit $\mu \to 0$ and can be expressed in elementary functions. After some straightforward calculations we get the result

$$A = \frac{3}{64} p^4 \left[2\delta\,(5 - 3\,\delta^2) - (5 + 6\,\delta^2 - 3\,\delta^4)\log\frac{1+\delta}{1-\delta} \right]. \tag{39}$$

The quantity B is a little more tricky to handle and the limit $\mu \to 0$ cannot be performed in all terms. We write (37) as

$$B = \frac{1}{8} p^4 (3 - \delta^2)\left[-(1 - \delta^2)\, B^{(1)} + (1 + \delta^2)\, B^{(2)} \right], \tag{40}$$

where

$$B^{(1)} = 2\, I\,(1), \tag{41}$$

$$B^{(2)} = \int_{-1}^{+1} I\,(z)\, dz, \tag{42}$$

$$I\,(z) = \int_{\mu\sqrt{-p^2}}^{-\frac{p^2}{2}\delta^2} \frac{\xi\, dy}{y^2 - \xi^2 z^2} = \int_{\varepsilon}^{\delta^2} dy \frac{\sqrt{y^2 - \varepsilon^2}\cdot\sqrt{1 - \frac{1-\delta^2}{1-y}}}{y^2 - z^2 (y^2 - \varepsilon^2)\left[1 - \frac{1-\delta^2}{1-y}\right]}, \tag{43}$$

$$\varepsilon = \frac{2\,\mu}{\sqrt{-p^2}}. \tag{43a}$$

The term containing the logarithmic dependence on μ in $I(z)$ can be split off in the following way:

$$I\,(z) = \int_{\varepsilon}^{\delta^2} \frac{\sqrt{y^2 - \varepsilon^2}\cdot\delta\, dy}{y^2 - \delta^2 z^2 (y^2 - \varepsilon^2)} + \int_0^{\delta^2} \frac{dy}{y}\left[\frac{\sqrt{1 - \frac{1-\delta^2}{1-y}}}{1 - z^2\left(1 - \frac{1-\delta^2}{1-y}\right)} - \frac{\delta}{1 - \delta^2 z^2} \right]. \tag{44}$$

In the first integral, we make the transformation $1 - \frac{\varepsilon^2}{y^2} = t^2$ and rewrite it as

$$\left.\begin{aligned} \int_{\varepsilon^2}^{\delta^2} \frac{dy}{y}\, \frac{\delta\cdot\sqrt{1 - \varepsilon^2/y^2}}{1 - z^2\delta^2 (1 - \varepsilon^2/y^2)} &= \frac{\delta}{1 - z^2\delta^2} \int_0^{\sqrt{1-\varepsilon^2/\delta^2}} \frac{dt}{1-t} \\ &+ \delta \int_0^1 \frac{dt}{1-t}\left[\frac{2\,t^2}{(1+t)(1 - z^2\delta^2 t^2)} - \frac{1}{1 - z^2\delta^2} \right]. \end{aligned}\right\} \tag{45}$$

The remaining integrations can now be performed without difficulty, and $I(z)$ is found to be

$$(z) = \frac{1}{1-z^2}\left(z \log\frac{1+\delta z}{1-\delta z} - \log\frac{1+\delta}{1-\delta}\right) - \left(\frac{1}{2z} + \frac{3\delta^2 z}{2(1-\delta^2 z^2)}\right)\log\frac{1+\delta z}{1-\delta z} + \frac{\delta}{1-\delta^2 z^2}\log\frac{8\delta^2}{\varepsilon(1-\delta^2)}. \tag{46}$$

The last integration in (42) introduces again the function $\Phi(x)$. With the aid of the formulae given in the Appendix we can write the result as

$$^{(2)} = -3\,\Phi\left(-\frac{1-\delta}{1+\delta}\right) - 2\,\Phi\left(\frac{1-\delta}{1+\delta}\right) - \frac{3}{4}\pi^2 + \log\frac{1+\delta}{1-\delta}\left[\log\frac{m}{\mu} + \frac{1}{4}\log\frac{1+\delta}{1-\delta} + \log\frac{(1+\delta)^2}{4\delta}\right]. \tag{47}$$

The remaining part of the calculation is purely algebraic in nature. Collecting previous results, we get

$$\begin{aligned} T_a^{(1)}(p^2) = \frac{e^4}{48\pi^4}\Bigg[&\frac{\delta}{8}(39-17\delta^2) + \frac{1}{16}(33-10\delta^2+\delta^4)\log\frac{1+\delta}{1-\delta} \\ &-\frac{\delta}{2}(3-\delta^2)\log\frac{64\delta^4}{(1-\delta^2)^3} - \delta(3-\delta^2)\left[1-\frac{1+\delta^2}{2\delta}\log\frac{1+\delta}{1-\delta}\right]\cdot\log\frac{m}{\mu} \\ &-\frac{(3-\delta^2)(1+\delta^2)}{2}\left(3\,\Phi\left(-\frac{1-\delta}{1+\delta}\right) + 2\,\Phi\left(\frac{1-\delta}{1+\delta}\right) + \frac{3}{4}\pi^2 - \frac{1}{4}\log^2\frac{1+\delta}{1-\delta} \right. \\ &\left. -\log\frac{1+\delta}{1-\delta}\log\frac{(1+\delta)^2}{4\delta}\right)\Bigg]\,\theta(1-\delta). \end{aligned} \tag{48}$$

Adding (48) and (23), we find that the terms depending on μ cancel. In this way we obtain

$$\begin{aligned} T^{(1)}(p^2) = \frac{\alpha^2}{3\pi^2}\Bigg\{&\delta\left[-\frac{19}{24} + \frac{55}{72}\delta^2 - \frac{\delta^4}{3} - \frac{3-\delta^2}{2}\log\frac{64\delta^4}{(1-\delta^2)^3}\right] \\ &+\log\frac{1+\delta}{1-\delta}\left[\frac{33}{16} + \frac{23}{8}\delta^2 - \frac{23}{16}\delta^4 + \frac{1}{6}\delta^6 + \left(\frac{3}{2} + \delta^2 - \frac{\delta^4}{2}\right)\log\frac{(1+\delta)^3}{8\delta^2}\right] \\ &-\left(\frac{3}{2} + \delta^2 - \frac{1}{2}\delta^4\right)\left[4\,\Phi\left(-\frac{1-\delta}{1+\delta}\right) + 2\,\Phi\left(\frac{1-\delta}{1+\delta}\right) + \frac{\pi^2}{2}\right]\Bigg\}\,\theta(1-\delta), \end{aligned} \tag{49}$$

$$\alpha = \frac{e^2}{4\pi}. \tag{49a}$$

This is our expression for the imaginary part of the kernel (1). According to (2), the real part is obtained after a Hilbert transformation of this expression. This will be discussed in the next paragraph.

IV. The Real Part of the Vacuum Polarization Kernel.

So far, all our results are given as functions of the quantity δ defined in (24). It is therefore convenient to introduce a new variable of integration instead of a in (2). If we put

$$1 - \frac{4m^2}{a} = z^2, \tag{50}$$

we get

$$\overline{\Pi}^{(1)}(p^2) - \overline{\Pi}^{(1)}(0) = 2\int_0^1 \frac{z\,dz}{z^2 - \delta^2}\,\Pi^{(1)}(\delta = z). \tag{51}$$

Not all the integrations in (51) can be carried out explicitly, with the result expressed by elementary functions or by the function $\Phi(x)$. The new integrals which appear can be written in the standard form

$$F(x, y) = \int_0^1 \frac{dt}{t} \log|1 + xt| \cdot \log|1 + yt|. \tag{52}$$

All the necessary integrals over the function $\Phi(x)$ can be expressed in terms of this $F(x, y)$

$$P\int_0^a \frac{dz}{z + b}\,\Phi(z) = \Phi(a)\log\left|1 + \frac{a}{b}\right| - F\left(a, \frac{a}{b}\right). \tag{53}$$

In our final result, one of the variables in $F(x, y)$ has only a very small number (three) of different values. We therefore introduce the following three integrals, each of which depends on only one variable:

$$F(x) = \int_{-1}^{+1} \frac{dt}{t} \log(1+t) \log\left|1 - \frac{t^2}{x}\right|, \tag{54}$$

$$G(x) = \int_{-1}^{+1} \frac{dt}{1+t} \log\frac{1-t}{2} \cdot \log\left|1 - \frac{t^2}{x}\right|, \tag{55}$$

$$H(x) = \int_{-1}^{+1} \frac{dt}{1+t} \log|t| \log\left|1 - \frac{t^2}{x}\right|. \tag{56}$$

The remaining integrations are then straightforward, and not too time consuming. The result can be written as

$$\begin{aligned} \overline{\Pi}^{(1)}(p^2) - \overline{\Pi}^{(1)}(0) = \frac{\alpha^2}{3\pi^2}\Bigg\{ & -\frac{13}{108} + \frac{11}{72}\delta^2 - \frac{1}{3}\delta^4 + \delta\left(\frac{19}{24} - \frac{55}{72}\delta^2 \right. \\ & \left. - \frac{1}{3}\delta^4\right)\log\frac{1+\delta}{|1-\delta|} - \left(\frac{33}{32} + \frac{23}{16}\delta^2 - \frac{23}{32}\delta^4 + \frac{\delta^6}{12}\right)\left(\log^2\frac{1+\delta}{|1-\delta|} - \pi^2\theta(1-\delta)\right) \\ & + \delta(3-\delta^2)\Bigg[\Phi\left(\frac{1-\delta}{1+\delta}\right) + 2\Phi\left(-\frac{1-\delta}{1+\delta}\right) + \frac{\pi^2}{4} - \frac{3}{4}\pi^2\theta(1-\delta) \\ & - \frac{3}{4}\log^2\frac{1+\delta}{|1-\delta|} + \frac{1}{2}\log\frac{1+\delta}{|1-\delta|}\log\frac{64\delta^4}{|1-\delta^2|^3}\Bigg] \\ & + (3 + 2\delta^2 - \delta^4)\left[F(\delta^2) + \frac{3}{2}G(\delta^2) - H(\delta^2)\right]\Bigg\}. \end{aligned} \tag{57}$$

If this expression is expanded in powers of δ^{-1}, the first non-vanishing term will be of order δ^{-2}. The same conclusion can also be obtained from a study of Eq. (51). If this expression is expanded in powers of δ^{-1}, we get immediately

$$\overline{\Pi}^{(1)}(p^2) - \overline{\Pi}^{(1)}(0) = -\frac{2}{\delta^2}\int_{-1}^{+1} z\,dz\,\Pi^{(1)}(\delta = z) + \cdots. \tag{58}$$

The numerical coefficient of the first power of δ^{-2} has been computed from Eq. (57) and with the aid of the integration indicated in Eq. (58). The agreement of the results serves as a check on the calculations. In either way we obtain

$$\overline{\Pi}^{(1)}(p^2)-\overline{\Pi}^{(1)}(0) = -\frac{1}{\delta^2}\cdot\frac{\alpha^2}{\pi^2}\cdot\frac{82}{81}+\cdots = -\frac{p^2}{m^2}\frac{\alpha^2}{\pi^2}\frac{41}{162}+\cdots. \tag{59}$$

This also agrees with the result obtained by BARANGER, DYSON, and SALPETER.[1]

In Eq. (57) it is supposed that δ is real, that is, p^2 is either positive or less than $-4\,m^2$. For $0 < -p^2 < 4\,m^2$, δ will be purely imaginary. In this case we have to substitute arctangent functions for logarithms, according to the following rules:

$$\delta \log\frac{1+\delta}{|1-\delta|} \rightarrow 2\eta \arctang\frac{1}{\eta}, \quad (\eta = i\,\delta > 0), \tag{60 a}$$

$$\log^2\frac{1+\delta}{|1-\delta|} - \pi^2\,\theta\,(1-\delta) \rightarrow -4\arctang^2\frac{1}{\eta}, \tag{60 b}$$

$$\begin{aligned}\delta\Bigg[\Phi\left(\frac{1-\delta}{1+\delta}\right) + 2\,\Phi\left(-\frac{1-\delta}{1+\delta}\right) + \frac{\pi^2}{4} - \frac{3}{4}\pi^2\,\theta\,(1-\delta) - \frac{3}{4}\log^2\frac{1+\delta}{|1-\delta|}\\ + \frac{1}{2}\log\frac{1+\delta}{|1-\delta|}\log\frac{64\,\delta^4}{|1-\delta^2|^3}\Bigg] \rightarrow \eta\Bigg[\psi\left(2\arctang\frac{1}{\eta}\right) - 2\,\psi\,(2\arctang\eta)\\ + \arctg\frac{1}{\eta}\log\frac{64\,\eta^4}{(1+\eta^2)^3}\Bigg],\end{aligned} \tag{60 c}$$

$$\psi\,(x) = \sum_1^\infty \frac{\sin\,(nx)}{n^2}. \tag{60 d}$$

At the point $p^2 = -4\,m^2$, or $\delta = 0$, the expression (57) has a logarithmic singularity. If, during the calculation, the photon mass μ had been kept different from zero in *all* places, our result would have shown a finite peak at this point. For practical applications, the weak logarithmic infinity will not be very harmful, as one is in general interested in convolution integrals involving the function $\overline{\Pi}(p^2)-\overline{\Pi}(0)$. In such expressions, the result (57) will be sufficient. For large values of $|p^2/m^2|$, our function behaves as $\log^2|p^2/m^2|$. Fig. 1 gives a qualitative idea of the behaviour of the fourth approximation of the vacuum polarization kernel as a function of $-p^2/m^2$. A figure of the corresponding behaviour of the functions $\Pi^{(0)}(p^2)$ and $\overline{\Pi}^{(0)}(p^2)-\overline{\Pi}^{(0)}(0)$ would be rather

[1] Footnote 3, page 3.

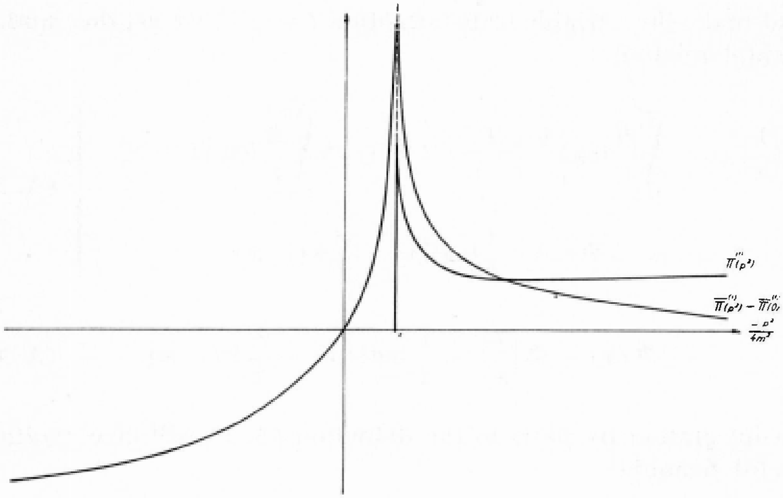

Fig. 1. Qualitative behaviour of the e^4 approximation of the real and of the imaginary parts of the vacuum polarization kernel.

similar to Fig. 1. The only qualitative difference would be that the function $\Pi^{(0)}(p^2)$ vanishes at the point $-p^2 = 4\,m^2$ and that the function $\overline{\Pi}^{(0)}(p^2) - \overline{\Pi}^{(0)}(0)$ has a finite peak at this point.

Appendix.

In the following are given some formulae involving the function $\Phi(x)$, defined in Eq. (19). Although practically all these expressions can be found in the literature,[1] we add this summary for the reader's convenience.

If x is real, our function is defined by

$$\Phi(x) = \int_1^x \frac{dz}{z} \log|1+z|. \qquad \text{(A. 1)}$$

If we consider

$$\Phi\left(\frac{1}{x}\right) = \int_1^{1/x} \frac{dz}{z} \log|1+z|, \qquad \text{(A. 2)}$$

[1] Cf., *e.g.*, K. Mitchell, Phil. Mag. **40**, 351 (1949) and W. Gröbner, N. Hofreiter, Integraltafeln, Wien and Innsbruck, 1950.

and make the variable transformation $t = z^{-1}$, we get the fundamental relation

$$\left.\begin{aligned}\Phi\left(\frac{1}{x}\right) &= -\int_1^x \frac{dt}{t}\log\left|\frac{1+t}{t}\right| - 2\,\theta(-x)\int_{-1}^{+1}\frac{dz}{z}\log|1+z| \\ &= -\Phi(x) + \frac{1}{2}\log^2|x| - \frac{\pi^2}{2}\theta(-x)\end{aligned}\right\} \quad (A.\ 3)$$

or

$$\Phi(x) + \Phi\left(\frac{1}{x}\right) = \frac{1}{2}\log^2|x| - \frac{\pi^2}{2}\theta(-x). \quad (A.\ 4)$$

An integration by parts in the definition (A. 1) will give another useful formula

$$\left.\begin{aligned}\Phi(x) &= \Big[\log|z|\log|1+z|\Big]_1^x - \int_1^x \frac{dz}{1+z}\log|z| \\ &= \log|x|\cdot\log|1+x| - \int_2^{1+x}\frac{dz}{z}\log|1-z|,\end{aligned}\right\} \quad (A.\ 5)$$

or

$$\Phi(x) + \Phi(-1-x) = -\frac{\pi^2}{3} + \log|x|\cdot\log|1+x|. \quad (A.\ 6)$$

Besides (A. 4) and (A. 6), we also mention the formula

$$\left.\begin{aligned}\Phi(x) + \Phi(-x) &= \int_1^x \frac{dz}{z}\log|1-z^2| + \int_{-1}^{+1}\frac{dz}{z}\log|1-z| \\ &= \frac{1}{2}\Phi(-x^2) - \frac{\pi^2}{8}.\end{aligned}\right\} \quad (A.\ 7)$$

Another relation which has been of some use in the calculations can be obtained in the following way:

$$\Phi(x) - \Phi(-x) = \int_1^x \frac{dz}{z}\log\left|\frac{1+z}{1-z}\right| + \frac{\pi^2}{4}. \quad (A.\ 8)$$

The transformation $1 + t = \frac{1 - z}{1 + z}$ transfers this integral to

$$\left.\begin{aligned} \Phi(x) - \Phi(-x) &= \frac{\pi^2}{4} - \int_{-1}^{-\frac{2x}{1+x}} dt \left[\frac{1}{t} - \frac{1}{2+t}\right] \log|1+t| \\ -\theta(-1-x)\int_{-\infty}^{+\infty} \frac{2\,dt}{t(1+t)} \log|1+t| &= \frac{\pi^2}{4} - \pi^2\,\theta(-1-x) \\ &\quad - \Phi\left(\frac{-2x}{1+x}\right) + \Phi\left(\frac{-2}{1+x}\right). \end{aligned}\right\} \quad (A.\ 9)$$

Using (A. 6), we can write (A. 9) as

$$\left.\begin{aligned} \Phi(x) - \Phi(-x) &= \frac{\pi^2}{4} - \pi^2\,\theta(-1-x) + \Phi\left(-\frac{1-x}{1+x}\right) \\ &\quad - \Phi\left(\frac{1-x}{1+x}\right) + \log|x| \cdot \log\left|\frac{1+x}{1-x}\right|. \end{aligned}\right\} \quad (A.\ 10)$$

For complex values of x we can still define the function $\Phi(x)$ as the integral (A. 1), making this definition unique with the aid of a cut along the real axis below the point -1. This function fulfils an equation similar to (A. 4),

$$\Phi(x) + \Phi\left(\frac{1}{x}\right) = \frac{1}{2}\log^2 x, \quad (A.\ 11)$$

where the definition of the logarithm is made unique by the prescription just mentioned. From (A. 11), we conclude that

$$Re\,\Phi(e^{i\vartheta}) = -\frac{1}{4}\vartheta^2. \quad (A.\ 12)$$

For $|x| \leq 1$, we have the power series expansion

$$\Phi(x) = -\frac{\pi^2}{12} + \sum_{1}^{\infty} \frac{(-1)^{n+1} \cdot x^n}{n^2}. \quad (A.\ 13)$$

From (A. 13), it follows that

$$Im\,\Phi(-e^{i\vartheta}) = -\sum_{1}^{\infty}\frac{\sin(n\vartheta)}{n^2} = -\psi(\vartheta). \qquad \text{(A. 14)}$$

Numerical values of $\Phi(x)$ for real x can be obtained from the paper by MITCHELL. The function $\psi(\vartheta)$ in (A. 14) has been tabulated by CLAUSEN[1].

[1] T. CLAUSEN, Jour. f. Math. (Crelle) 8, 298 (1832).

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