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ON THE SCATTERING OF THERMAL NEUTRONS BY BOUND PROTONS

BY

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INTRODUCTION

After FERMI's discovery[1] of the possibility of producing slow neutrons by surrounding a source of fast neutrons by hydrogeneous substances such as paraffin wax, the problem of the mechanism of the collision between neutrons and protons has become important for the study of the properties of slow neutrons. The problem has already been treated by FERMI himself[2], who describes the slowing-down process in the following way. Neglecting first the fact that the protons in the paraffin are bound chemically, the fast neutrons which come from the source will make elastic collisions with the protons giving up on the average half of their kinetic energy at every collision. In this way they will soon reach thermal energies, where they will remain for a relatively long time, because now the chance that a neutron will get by a collision with a proton some of the thermal energy of the latter is about the same as that it will lose energy by the collision. The neutron will therefore diffuse round in the paraffin until it is finally captured by a proton. So long as the neutron energy is large compared with the oscillation energy of the proton it is legitimate to consider the latter as free. As the highest oscillation frequency of the proton in paraffin is of the order 3000 cm^{-1}

1) E. FERMI, and coll., Proc. Roy. Soc. **149**, 522 (1935).

2) E. FERMI, Ric. scient. VII. II. 13 (1936). See also H. A. BETHE, Rev. of Mod. Phys., **9**, No. 2 1937.

corresponding to an energy of 0.37 volt[1] it will be correct to treat the protons as free for neutron energies down to about one volt.

Classically the total cross-section for the scattering should be the same above and below one volt, as the cross-section is classically always the geometrical area of the proton. In a quantum treatment, however, the binding of the proton has a large influence, as first pointed out by FERMI[2], who showed that one may use the BORN approximation in calculating cross-sections for the slow neutrons. In this approximation the cross-section is proportional to the square of the reduced mass[3], and as this is equal to the neutron mass when the proton is bound strongly compared with the neutron energy but equal to half the neutron mass when the proton is free, it is seen that the cross-section in the first extreme case will be four times as large as in the second extreme case. For intermediate cases this chemical factor, as it is called, will lie between one and four. FERMI found by his model for the binding the value 3.3 in the case of the *C*-neutrons.

Because of this quantum effect we have therefore different stages in the slowing-down process. In the first stage, fast neutrons with energies of the order some million volts, the cross-section is experimentally found to be of the order $1-2\times 10^{-24}\,\mathrm{cm}^2$ [4] corresponding to a mean free path in paraffin of about 5 cm. Owing to the collisions the energy will soon decrease and the cross-section will therefore in-

[1] $(\hbar\omega)_{\mathrm{volt}} = \frac{hc}{1.59\cdot 10^{-12}}\,(\tilde{\nu})_{\mathrm{cm}^{-1}} = 1.233\cdot 10^{-4}\,(\tilde{\nu})_{\mathrm{cm}^{-1}}$.

[2] Loc. cit.

[3] cf. eq. (1) p. 12.

[4] J. CHADWICK, Proc. Roy. Soc. **142**, 1 (1933) and J. R. DUNNING and coll., Phys. Rev. **48**, 265 (1935).

crease[1] until the energy is small compared with the energy of the excited state of the deuteron. In this second stage the cross-section will be independent of the energy and it is found[2] to be about 13×10^{-24} cm^2 corresponding to a mean free path of 1 cm for neutron energies from about 10 000 volts down to resonance energies of the order of some volts. In the third stage when the energy gets below one volt the chemical binding becomes noticeable and the cross-section increases to about 48×10^{-24} cm^2 for thermal energies[2], so that the mean free path decreases to about 0.3 cm.

For the two first stages FERMI has obtained the energy distribution of the neutrons[3] which in the second stage, where the mean free path is a constant, turns out to be proportional to $\frac{dE}{E}$. In the third stage, neutron energies below one volt, the problem of the energy distribution has neither as yet been solved theoretically, nor is it known accurately from experiments.[4]

For this last problem and for further problems connected with the slowing-down process, such as temperature effects, it is of interest to determine theoretically the effect of the chemical binding on the scattering cross-sections. Recently attempts have been made to connect such calculations with a still more extended range of problems: it has been proposed[5] to adopt for the cross-section of free protons — which is of considerable importance for the determination of the

[1] Cf. e. g. H. A. BETHE and R. F. BACHER, Rev. of Mod. Phys., **8**, No. 2 (1936) eq. (62).

[2] M. GOLDHABER and G. H. BRIGGS, Proc. Roy. Soc. **162**, 127 (1937) and O. R. FRISCH, H. v. HALBAN jun. and J. KOCH, Kgl. Danske Vidensk. Selsk. Skr. Mat.-fys. Med. XV, No. 10 (1938).

[3] Loc. cit.

[4] cf. later p. 9.

[5] BETHE, loc. cit.

neutron- and radiation width of excited nuclear levels[1] as well as for the theory of the deuteron and the discussion of the relation between proton-proton and proton-neutron forces[2] — instead of the direct experimental value which is not very accurate, the quotient of the thermal cross-section and a calculated chemical factor. It would, however, be much preferable for the above purposes to have a more exact experimental determination of the free proton cross-section as it is only possible to base such calculations on very rough models for the binding of the protons in paraffin and similar hydrogeneous substances. In spite of this fact it is, as we have seen, of interest to get some rough ideas about the influence of the binding, and we shall in this paper treat the problem by help of a model for the binding which we shall discuss in § 1.

§ 1. Discussion of a simplified model for the binding of the protons.

The scattering cross-section and the energy loss can be calculated exactly if the proper function for the nuclear motion in the molecules concerned is known. Theoretically it is possible from an analysis of the molecular spectra to obtain the frequencies of the vibrations and the normal coordinates which determine the form of the different normal vibrations. For the more complicated molecules, however, such as paraffin which is mostly used for the purpose of slowing down the neutrons, the resulting expressions would indeed be very complicated and unmanageable, quite apart from the fact that for these complicated molecules not all

[1] H. A. Bethe and G. Placzek, Phys. Rev. **51**, 450 (1937).
[2] G. Breit and J. R. Stehn, Phys. Rev. **52**, 396 (1937).

the data needed are accurately known. Simpler molecules, like water for instance, have on the other hand so far only been used in the liquid state, and in this the interaction between the molecules which is of considerable importance for our problem cannot easily be treated quantitatively. We shall therefore in the present paper only discuss a very schematic model for the binding.

I. Instead of the normal vibrations we assume each proton to oscillate independently in a harmonic potential, which we shall assume to be anisotropic, since it can be deduced from molecular spectra that the protons oscillate with larger frequencies in the direction of the valency-bond than in the perpendicular directions. For the frequencies we shall take $\nu_z = 3000\ \text{cm}^{-1} = 0.37$ volts, $\nu_x = \nu_y = \gamma \nu_z$ with $\gamma = 0.4$ so that $\nu_x = \nu_y = 1200\ \text{cm}^{-1} = 0.148$ volts.

II. As we have already mentioned the binding has no influence classically on the scattering. This is also true if we do not consider the motion as a whole but only the separate degrees of freedom. Now we know that the nuclear motions in the molecules have also in addition to the larger frequencies which we have accounted for by the assumption I, a spectrum extending to quite small frequencies. These small frequencies we will take into consideration by assuming that the protons and their potentials can move freely like gas molecules with a MAXWELL velocity distribution, so that we substitute for the energy exchange between the neutrons and the small frequencies the exchange of kinetic energy between the neutrons and these "molecules". So long as the neutron energy can be considered large compared with the energies corresponding to these frequencies we can namely, as we have just seen, consider these separate degrees of freedom as unbound, only the fact that

they are connected with the other degrees of freedom with the large frequencies must be accounted for. This we do by ascribing an effective mass to the "molecules" consisting of proton and potential, and for this effective mass we choose the value 14 times the neutron mass, which is the mass of a CH_2 group. This figure is rather arbitrary and corresponds to the conception that the energy taken up in the neutron collision by a proton is transferred to a single carbon atom in the hydrocarbon chain rather than to several of them.[1)]

Our two assumptions are of course very arbitrary and certainly not fulfilled in nature. No account is taken of interference effects, and apart from this it is known, for instance, that the frequency of the *C-C* vibrations in Æthan (C_2H_6) and other heavy carbon molecules is of the order of 1000 cm^{-1}, which is about five times the energy of thermal neutrons at room temperature[2)], so that these vibrations cannot at all be considered small. The model described is on the other hand the next simplest after that chosen by FERMI[3)], the isotropic oscillator with infinite mass, and it is certainly a better approximation than his[4)]. Taking now our model for granted, we shall first see which con-

1) It must be emphasized that this model is in no way identical with a gas of CH_2 groups. Firstly, in a CH_2 group the positions of the hydrogen atoms depend on each other; this gives rise to important interference effects which we do not consider in our model; secondly, the slowing-down process by free CH_2 groups would — apart from the slowing-down by elastic collisions — take place by energy transfer to the three proper vibrations of the group and the three rotations of the group as a whole, while in our case we have two times three vibrations and no rotation.

2) For $T = 290°$ abs we have $kT = 0.025$ volts $= 203\ cm^{-1}$.

3) Loc. cit.

4) After the conclusion of our calculations a discussion of the effect of the anharmonic binding on somewhat similar lines has been published by BETHE, loc. cit., where, however, the influence of the thermal motions are not considered (cf. the §§ 4—6 of the present paper).

clusions regarding the influence of the binding we can draw from the model, and next we shall use the results to estimate the effect of temperature variation on the mean free path.

In order to obtain definite results regarding the last problem it is necessary to know the energy ranges of the neutrons with which we are dealing. We shall assume these to be the so-called C-neutrons, that is the neutrons which are strongly absorbed in cadmium. The range of strong absorption in *Cd* extends from 0 to about 0.3 volts.[1] Further we must know the energy distribution of the *C*-neutrons. This is not exactly known; its theoretical determination is just one of the aims of the theoretical study of the slowing-down process with which we are dealing in the present paper. Two methods of investigation have been used to determine the energy distribution of the *C*-neutrons experimentally. First the method of the mechanical velocity selector[2]. By this method it is found that at room temperature the energy distribution has a maximum for an energy of the order of kT. Second the method of absorption in Boron[3]. As the capture cross-section in Boron is assumed to follow the $\frac{1}{v}$ law[4] it is possible by absorption experiments in this element to compare the mean value of $\frac{1}{v}$ for different kinds of neutrons. If for instance the *C*-neutrons were in thermal equilibrium with the slowing-down medium this mean value and hence the Boron absorption should vary with the absolute temperature of the medium as $T^{-\frac{1}{2}}$. While between

1) Cf. e. g. J. G. HOFFMAN and H. A. BETHE, Phys. Rev. **51**, 1021, (1937).

2) J. R. DUNNING and coll. Phys. Rev. **48**, 704 (1935). Cf. also BETHE, loc. cit.

3) For a survey of the literature cf. FRISCH, HALBAN and KOCH loc. cit.

4) R. FRISCH and G. PLACZEK, Nature **137**, 357 (1936). D. F. WEEKES, M. S. LIVINGSTON and H. A. BETHE, Phys. Rev. **49**, 471 (1936).

400° and room temperature no deviation from this $T^{-\frac{1}{2}}$ law has been found the increase of the Boron absorption between room- and liquid air temperature, and still more between liquid air and liquid hydrogen temperature, is much less than would follow from a $T^{-\frac{1}{2}}$ law. This proves that at least for temperatures of liquid air and downwards the energy distribution of the C-neutrons cannot be represented by a MAXWELL distribution with the temperature of the slowing-down medium. The question how far their energy distribution can be represented by a MAXWELL distribution corresponding to a higher temperature or by a mixture between a maxwellian and a non-maxwellian part shall not be discussed here. In view of these possibilities, however, it remains interesting to investigate the energy dependence of the scattering cross-section for a MAXWELL beam of neutrons. We shall therefore for the purpose of the following calculations assume the C-neutrons to obey the MAXWELL law throughout. A consequence of this assumption together with the assumptions made about the binding mechanism is, however, that we cannot expect a direct comparison of the results of our calculations with experiment to give a quantitative agreement.

§ 2. General theoretical remarks.

As first proved by FERMI[1] it is possible to find a "rectangular hole" potential V' with radius $\varrho' << \lambda$ and depth D', which substituted for the neutron-proton potential will give correct cross-sections in the BORN approximation so long as the following conditions are satisfied:

[1] Loc. cit. Cf. also BETHE, loc. cit. Part B p. 123.

I. The DE BROGLIE wave-length, λ,[1] for the neutron relative to the proton must be large compared with the range of the neutron-proton force, ϱ:

$$\lambda >> \varrho .$$

II. The total cross-section, Q, must be small compared with the square of the wave-length:

$$Q << \lambda^2 .$$

III. For I to be satisfied one can deduce[2] that the dimension of the proton wave function, a, must be large compared with the range of the neutron-proton force:

$$a >> \varrho .$$

For slow neutrons and protons bound in paraffin all these conditions are certainly satisfied, as for such neutrons λ is of the order of 10^{-9} cm or more, Q is of the order of 48×10^{-24} cm^2 and we further know that ϱ and a are respectively of the order of 10^{-13} cm and at least 10^{-9} cm.

For the differential cross-sections per unit solid angle $d\omega$, $I_{mn}(\theta, \varphi)$, where $I_{mn}(\theta, \varphi)\, d\omega$ is defined as the number of neutrons which are scattered, after having excited the proton from its m'th into its n'th state, into the solid angle $d\omega$ in the direction θ, φ per unit time and per scatterer, if there in the incident beam is one neutron crossing unit area per unit time at the place of the scatterer, we have now in the BORN approximation the well known expressions[3]

[1] This is for non-relativistic energies given by $(\lambda)_{\mathrm{cm}} = \dfrac{h}{(2 m_N E_N)^{1/2}} = 2.85 \times 10^{-9}\, E_N^{-1/2}$ when $E_N = \dfrac{1}{2} m_N v_{rel}^2$ is measured in volts, v_{rel} being the velocity of the neutron relative to the proton.

[2] For instance by FOURIER analyzing the wave function of the proton in respect to velocity.

[3] Cf. e. g. MOTT and MASSEY, "Theory of Atomic Collisions", p. 100, eq. (21). (The equation is erroneous, the factor $\dfrac{k_{mn}}{k_0}$ missing). It will be seen that in this approximation I depends on θ only, not on φ.

$$I_{mn}(\theta) = \left. \begin{aligned} &= \frac{k_{mn}}{k_0} \left| \frac{2M_N}{4\pi\hbar^2} \int d\tau_N \int d\tau_P \exp(i\boldsymbol{k}''_{mn}\boldsymbol{r})\, V'(|\boldsymbol{r}_N - \boldsymbol{r}_P|)\, \psi_n^*(\boldsymbol{r}_P)\, \psi_m(\boldsymbol{r}_P) \right|^2 \\ &\boldsymbol{k}''_{mn} = \boldsymbol{k}_0 - \boldsymbol{k}_{mn},\ k_0^2 = \left(\frac{2\pi}{\lambda_0}\right)^2 = \frac{2M_N E_0}{\hbar^2}, \\ &k_{mn}^2 = \frac{2M_N}{\hbar^2}(E_0 - (E_n - E_m)), \quad \theta = \sphericalangle(\boldsymbol{k}_0, \boldsymbol{k}_{mn}) \end{aligned} \right\} \quad (1)$$

where ψ_m and ψ_n are the wave functions of the proton before and after the collision, $\boldsymbol{k}_o$ and $\boldsymbol{k}_{mn}$ the initial and final wave vectors [1] of the neutron, and M_N, E_o the *reduced* mass and energy of the neutron.

In this expression V' only depends on the distance between the neutron and the proton, so taking $\boldsymbol{r}_N - \boldsymbol{r}_P$ as a new variable in the $d\tau_N$ integration we can at once perform this and using that the exponential is equal to unity by this integration due to $\lambda >> \varrho'$ we get

$$I_{mn}(\theta) = q \cdot \frac{k_{mn}}{k_0} \left| \int \psi_n^*(\boldsymbol{r}_P) \exp(i\boldsymbol{k}''_{mn}\boldsymbol{r}_P)\, \psi_m(\boldsymbol{r}_P)\, d\tau_P \right|^2 \quad (2)$$

$$q = \frac{M_N^2}{4\pi^2\hbar^4} \left| \int V'\, d\tau \right|^2 = \frac{4}{9} \frac{M_N^2}{\hbar^4} (D'\varrho'^3)^2 \quad (3)$$

Equation (3) we can write in the following way using the expression for the total cross-section for scattering between a neutron and a free proton [2] which we shall denote by Q_{free}

$$\pi q = \left(\frac{M_N}{m_N}\right)^2 \cdot Q_{\text{free}}. \quad (4)$$

We emphasize here that the expression (1) or (2) is calculated in coordinates relative to the center of gravity of the system in which the proton is bound and as this fact sometimes gives

[1] The wave vector is just the momentum vector divided by $\hbar$.
[2] Cf. Note 1, eq. (N 5).

rise to a little confusion we shall briefly give the definitions here, the transformation formulae being derived in Note 2. In the theory for two-body collisions *three* different coordinate systems are used.[1] First the system where the one particle is at rest before the collision, which we shall call the *rest* system and denote by R.[2] (All variables denoted by capital letters). Next the system where the center of gravity of the two particles is at rest both before and after the collision, which we shall call the *center of gravity* system and denote by C. (All variables denoted by small letters with an asterisk). Finally the system which has its origin in the center of gravity of the one particle both before and after the collision, which we shall call the *relative* system and denote by r. (All variables denoted by small letters). Let the two particles have masses m_1, m_2 and coordinate vectors $\boldsymbol{R}_1$, $\boldsymbol{R}_2$, then the center of gravity, $\boldsymbol{R}_c$, is defined by

$$m_1\boldsymbol{R}_1 + m_2\boldsymbol{R}_2 = (m_1 + m_2)\boldsymbol{R}_c \tag{5}$$

The coordinates referred to the center of gravity are next defined by

$$\boldsymbol{r}_1^* = \boldsymbol{R}_1 - \boldsymbol{R}_c, \qquad \boldsymbol{r}_2^* = \boldsymbol{R}_2 - \boldsymbol{R}_c \tag{6}$$

Putting (6) into (5) we get

$$\boldsymbol{r}_1^* = -\frac{m_2}{m_1}\boldsymbol{r}_2^* \quad \text{or} \quad r_1^* = \frac{m_2}{m_1}r_2^*, \quad \theta_1^* = \pi - \theta_2^*, \quad \varphi_1^* = \varphi_2^* + \pi \tag{7}$$

if we introduce polar coordinates. Finally the relative coordinates are defined by

$$\boldsymbol{r}_2 = \boldsymbol{R}_2 - \boldsymbol{R}_1 = \boldsymbol{r}_2^* - \boldsymbol{r}_1^*, \qquad \boldsymbol{r}_1 = 0 \tag{8}$$

the particle with index one being taken as the particle initially resting in the R system. Using (5) we then have, introducing the reduced mass

$$\left.\begin{aligned} M &= \frac{m_1 \cdot m_2}{m_1 + m_2} \\ \boldsymbol{r}_2^* &= \frac{M}{m_2}\boldsymbol{r}_2 \quad \text{or} \quad r_2^* = \frac{M}{m_2}r_2, \quad \theta_2^* = \theta_2, \quad \varphi_2^* = \varphi_2 \\ \boldsymbol{r}_1^* &= -\frac{M}{m_1}\boldsymbol{r}_2 \quad \text{or} \quad r_1^* = \frac{M}{m_1}r_2, \quad \theta_1^* = \pi - \theta_2, \quad \varphi_1^* = \varphi_2 + \pi. \end{aligned}\right\} \tag{9}$$

[1] The following also applies to the case where one or both of the two particles are complex, consisting of more parts. In this case the mass is the total mass and the coordinatevector is the one of the center of gravity.

[2] It should be noted that this system it not always identical with the coordinatesystem in which we make the observations, cf § 4.

We see from (9) the important fact that the angle of the colliding particle is the same in the C system and in the r system, due to

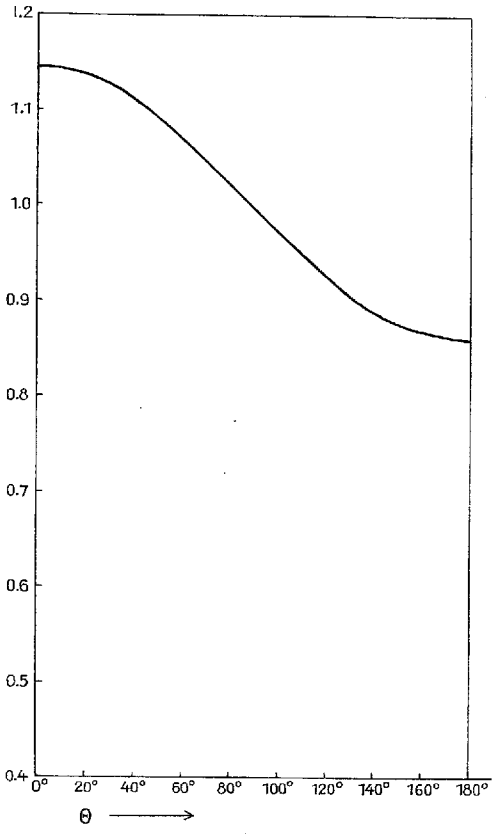

FIG. 1. *Angular distribution of scattered neutrons in the rest system corresponding to isotropic distribution in the center of gravity system, for* $m_S = 14\, m_N$.

which circumstance the formula (1) is often said to be derived in the C system in spite of the fact that it is really derived in the r system.

From the formulae (2) and (4) we can at once deduce that, as was already mentioned in the introduction, the total cross-section will be nearly four times as large as the

one for a free proton when the proton is strongly bound, which means that the space in which the proper function of the proton is different from zero is very small compared with the wave-length of the neutron. We can then put the exponential equal to one, so that we get quite independent of the form of the proper function of the proton

$$I_{mn}(\theta) = q \cdot \frac{k_{mn}}{k_0} \delta_{mn} = q \cdot \delta_{mn} \tag{10}$$

which means that only elastic scattering can occur and that this is spherical symmetric in the *relative* system just as is the case for scattering by a free proton.[1] In the *rest* system, however, we will no longer get the cos Θ law[2] due to the mass of the scatterer being now larger than the neutron mass. In FIG. 1 we have plotted in units of q the curve for (10) transformed to the *rest* system[3] for the mass of the scatterer, m_S, equal to 14 m_N.

For the total elastic cross-sections we get from (10)

$$Q_{nn} = 4\pi q = 4\left(\frac{M_N}{m_N}\right)^2 Q_{\text{free}} \tag{11}$$

using (4). For the case $M_N = m_N$ i. e. $m_S = \infty$ the factor of Q_{free} in (11) reduces to the factor 4 first obtained by FERMI.[4] We have in this work taken $m_S = 14\, m_N$ throughout so that

$$4\left(\frac{M_N}{m_N}\right)^2 = 4\left(\frac{14}{15}\right)^2 = 4 \cdot 0.871 = 3.48 \tag{12}$$

which makes a considerable difference.

1) Cf. Note 1.
2) Cf. Note 2 eq. (N 19).
3) Cf. Note 2 eq. (N 18).
4) loc. cit.

§ 3. The anisotropic oscillator.

We now in (2) put the wave functions for our anisotropic oscillator and as these are products of three wave functions for a one-dimensional harmonic oscillator, the matrix element will be a product of the matrix elements of the type given by eq. (N 24) in Note 3. Using the formulae (N 32) and (N 35) in Note 3 we have at once for the $0 \to 0$ transition, which is the only one we shall treat here

$$\left.\begin{aligned} I_{00} &= q \cdot \exp\left(-\frac{1}{2} a_z^2 \left[\frac{1}{\gamma}\left(k''^2_{00_x} + k''^2_{00_y}\right) + k''^2_{00_z}\right]\right) \\ a_z &= \left(\frac{\hbar}{M_P \omega_z}\right)^{1/2}, \quad \frac{1}{2} k''^2_{00} a_z^2 = \varepsilon \frac{E_0}{\hbar \omega_z} 4 \sin^2 \frac{\theta}{2}, \\ \varepsilon &= \frac{M_N}{M_P}^{1)}, \quad E_0 = \frac{1}{2} M_N v_{rel}^2 \end{aligned}\right\} \quad (13)$$

M_P being the reduced mass of the *proton*, $\omega_z = 2\pi\nu_z$, ν_z the frequency of the oscillation in the direction of the z-axis and θ the angle between $\mathbf{k}_0$ and $\mathbf{k}_{00}$ i. e. the scattering angle of the neutron.

Further we must take the mean value of (13) over all directions of the oscillator. This we do by taking the axes of the oscillator as coordinate system and averaging over all directions of $\mathbf{k}''_{00}$ in respect to this system, the length of $\mathbf{k}''_{00}$ being kept constant. In this way we get, denoting the mean value by $\overline{I_{00}}|_{\text{osc}}$ [2)]

$$\overline{I_{00}}|_{\text{osc}} = q \cdot \exp\left(-\frac{k''^2_{00} a_z^2}{2\gamma}\right) \int_0^1 \exp\left(\frac{1}{2} k''^2_{00} a_z^2 \left(\frac{1}{\gamma} - 1\right) t^2\right) dt \quad (14)$$

We introduce as new variable the dimensionless quantity

1) Cf. eq. (N 36).

2) A mean value we shall in this paper always denote by this symbol.

$$W = \varepsilon \frac{E_0}{\hbar\,\omega_z} = \frac{E_N}{\hbar\,\omega_z} \frac{\varepsilon}{1+\frac{m_N}{m_S}}, \quad E_N = \frac{1}{2} m_N V_N^2 \qquad (15)$$

where V_N is the velocity of the neutrons in the *rest* system, and then we can write (14) in the following form, due to

$$\frac{1}{2} k_{00}''^2 a_z^2 = 4\,W \sin^2 \frac{\theta}{2} \quad \text{(by (13))}$$

$$\overline{I_{00}}\Big|_{\text{osc}} = q \cdot \frac{\exp\left(-\frac{1}{\gamma} 4\,W \sin^2 \frac{\theta}{2}\right)}{\left(\frac{1}{\gamma}-1\right)^{1/2} 2\,W^{1/2} \sin \frac{\theta}{2}} \int_0^{\left(\frac{1}{\gamma}-1\right)^{1/2} 2\,W^{1/2} \sin \frac{\theta}{2}} \exp(t^2)\,dt \qquad (\gamma \leqq 1). \quad (16)$$

For $\gamma = 1$ we get the cross-section for the isotropic oscillator[1]

$$I_{00}^{is} = q \cdot \exp\left(-4W \sin^2 \frac{\theta}{2}\right). \qquad (17)$$

In FIG. 2 we have plotted in units of q the curve (16)[2] transformed to the *rest* system[3] for two different values of W, $W = 0.0697$ (full line) and $W = 0.0156$ (dotted line) which correspond to $\hbar w_z = 0.37$ volts, $\gamma = 0.4$, $m_S = 14\,m_N$ and E_N equal to the effective energy of neutrons at room respectively at liquid air temperature, i. e. 90° abs.[4] It is seen that even at liquid air temperature there is still a considerable deviation from the spherical symmetry which is always assumed in calculations about the diffusion of thermal neutrons.[5]

1) Cf. FERMI, loc. cit., and Note 3 eq. (N 34).

2) The function $\int_0^x \exp(t^2)\,dt$ is tabulated in JAHNKE-EMDE "Tables of Functions", p. 106.

3) Cf. Note 2 eq. (N 18)

4) Cf. § 6 p. 38.

5) Cf. FERMI and BETHE, loc. cit.

The curves in Fig. 2 can also be represented by the function

$$I_{00}^{is''} = \frac{1}{3} I_{00}^{is} + \frac{2}{3} I_{00}^{is'}, \qquad I_{00}^{is'} = I_{00}^{is}\left(\frac{1}{\gamma} W\right), \tag{18}$$

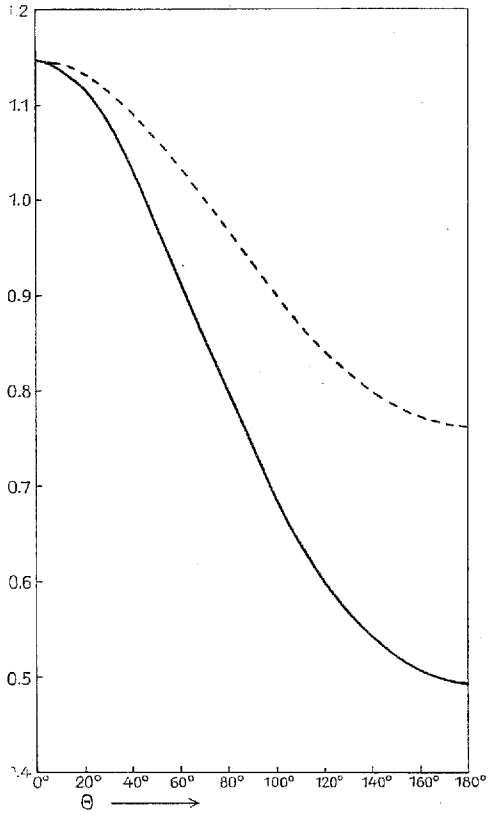

Fig. 2. *Angular distribution in the rest system of neutrons scattered by anisotropic oscillator. Full line corresponds to* $W = 0.0697$, *dotted line to* $W = 0.0156$, W *given by (15).*

where I_{00}^{is} is given by (17) and $I_{00}^{is'}$ stands for the same function with $\gamma\omega$ substituted for ω, which we can write as in (18) with W given by (15). The reason why the curves (16) and (18) are so like is easily seen analytically by expanding in powers of W. We then get

$$\left.\begin{aligned}\overline{I_{00}}_{\text{osc}} &= q\cdot\exp\left(-\frac{1}{\gamma}4\,W\sin^2\frac{\theta}{2}\right)\left[1+\frac{1}{3}[\,]+\frac{1}{10}[\,]^2+\frac{1}{42}[\,]^3+\cdots\right]\\ I_{00}^{is''} &= q\cdot\exp\left(-\frac{1}{\gamma}4\,W\sin^2\frac{\theta}{2}\right)\left[1+\frac{1}{3}[\,]+\frac{1}{6}[\,]^2+\frac{1}{18}[\,]^3+\cdots\right]\\ [\,] &= \left(\frac{1}{\gamma}-1\right)4\,W\sin^2\frac{\theta}{2}\end{aligned}\right\}\quad(19)$$

so that the two curves have the same starting point and starting tangent and the difference comes first in the second power of W.

From (14) we can now by integrating over θ and φ get the mean value of the total cross-section. The result is

$$\overline{Q_{00}}_{\text{osc}} = \int_0^{2\pi}d\varphi\int_0^{\pi}\overline{I_{00}}_{\text{osc}}\sin\theta\,d\theta = \pi q\cdot\int_0^1\frac{1-\exp\left(-4\,W\frac{1}{\gamma}\left[1-(1-\gamma)t^2\right]\right)}{W\frac{1}{\gamma}\left[1-(1-\gamma)t^2\right]}\,dt. \quad(20)$$

For $\gamma = 1$ we get the well-known formula for the isotropic oscillator[1]

$$Q_{00}^{is} = \pi q\cdot\frac{1-\exp(-4\,W)}{W}. \quad(21)$$

In FIG. 3 we have in the full curve plotted (20) in units of Q_{free}[2] for $\gamma = 0.4$ and $m_S = 14\,m_N$. Also we have in the same figure in the dotted curve plotted the curve analogous to (18),

$$Q_{00}^{is''} = \frac{1}{3}Q_{00}^{is}+\frac{2}{3}Q_{00}^{is'}, \qquad Q_{00}^{is'} = Q_{00}^{is}\left(\frac{1}{\gamma}W\right), \quad(22)$$

As we know from (19) the two curves have the same starting point and starting tangent. This can also be seen by direct expanding in powers of W

1) Cf. FERMI, loc. cit. Cf. also Note 4.
2) Cf. eqs. (4) and (12).

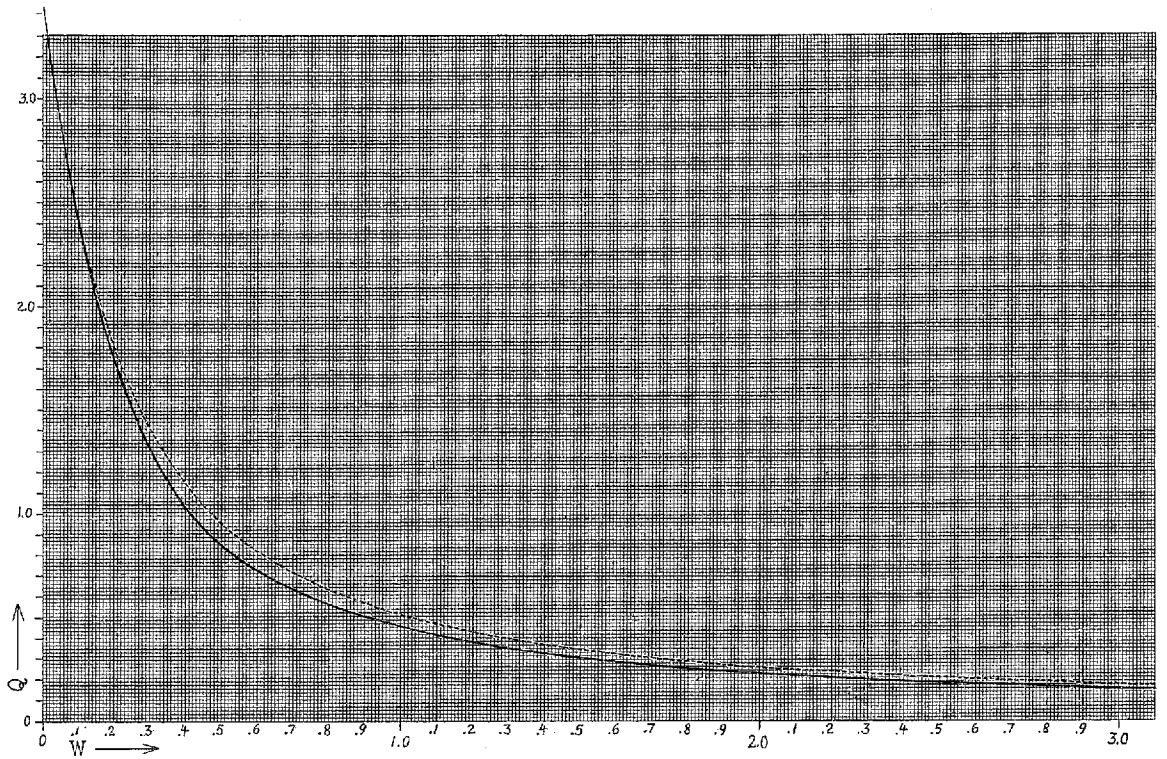

Fig. 3. *Energy dependence of total elastic scattering cross-section for anisotropic oscillator with* $\gamma = 0.4$ *(full line). Dotted line corresponding mean value for isotropic oscillators (see text). (The unit on the ordinate axis is* Q_{free} *whereas the curves are plotted in units (=* 50 mm*) of* $\pi q = 0.871 \cdot Q_{free}$ *(cf. eq. (11)) so one ordinate unit equals* $50/0.871 = 57.4$ mm.*)*

$$\left.\begin{aligned}
\overline{Q_{00}}|_{\mathrm{osc}} &= \pi q \cdot 4 \sum_{n=0}^{\infty} \frac{\left(4\,W\frac{1}{\gamma}\right)^n}{(n+1)!}(-1)^n \sum_{s=0}^{n}\binom{n}{s}(-1)^s \frac{(1-\gamma)^s}{2s+1} = \\
&= \pi q \cdot 4\left[1-2\,W\frac{1}{\gamma}\left(\frac{1}{3}\gamma+\frac{2}{3}\right)+\frac{8}{3}\,W^2\frac{1}{\gamma^2}\left(\frac{1}{5}\gamma^2+\frac{4}{15}\gamma+\frac{8}{15}\right)+\cdots\right]^{1)} \\
Q_{00}^{is''} &= \pi q \cdot 4 \sum_{n=0}^{\infty} \frac{\left(4\,W\frac{1}{\gamma}\right)^n}{(n+1)!}(-1)^n\left(\frac{1}{3}\gamma^n+\frac{2}{3}\right) = \\
&= \pi q \cdot 4\left[1-2\,W\frac{1}{\gamma}\left(\frac{1}{3}\gamma+\frac{2}{3}\right)+\frac{8}{3}\,W^2\frac{1}{\gamma^2}\left(\frac{1}{3}\gamma^2+\frac{2}{3}\right)+\cdots\right].
\end{aligned}\right\} \quad (23)$$

Since $\gamma \leqq 1-(1-\gamma)\,t^2 \leqq 1$ (due to $\gamma = 0.4 < 1$) in the integration range of t in (20) we can for large W neglect the exponential and we find then after elementary integration

$$\left.\begin{aligned}
\overline{Q_{00}}|_{\mathrm{osc}} &= \pi q \cdot \frac{\gamma}{2(1-\gamma)^{1/2}} ln\left(\frac{1+(1-\gamma)^{1/2}}{1-(1-\gamma)^{1/2}}\right)\cdot W^{-1} \\
Q_{00}^{is''} &= \pi q \cdot \left(\frac{1}{3}+\frac{2}{3}\gamma\right)\cdot W^{-1}
\end{aligned}\right\} \quad (W >> 1) \quad (24)$$

For $\gamma = 0.4$ the two coefficients are respectively 0.531 and 0.6. That $I_{00}^{is''}$ and $Q_{00}^{is''}$ are very nearly equal to $\overline{I_{00}}|_{\mathrm{osc}}$ and $\overline{Q_{00}}|_{\mathrm{osc}}$ is also physically plausible. $I_{00}^{is''}$ and $Q_{00}^{is''}$ we can namely interprete as the average cross-sections for scattering in a substance consisting to one third of oscillators with energy $\hbar\omega_z$ and to two thirds of oscillators with energy $\gamma\hbar\omega_z$, while we by $\overline{I_{00}}|_{\mathrm{osc}}$ and $\overline{Q_{00}}|_{\mathrm{osc}}$ are averaging over all directions of one oscillator with one degree of freedom oscillating with an energy $\hbar\omega_z$ and two degrees of freedom oscillating with an energy $\gamma\hbar\omega_z$, so that one would think that the two kinds of averaging would give nearly the same result, which is

1) This series is, as is easily verified, identical with BETHE loc. cit. Part B eq (463), if we put $m_S = \infty$, as then our $\pi q \to \sigma_0$, $W \to \varepsilon_1$, $W\frac{1}{\gamma} \to \varepsilon_2$ by BETHE.

in fact found to be the case as we have just seen. Due to the expression (20) being far more complicated than the expression (22), we shall in the following use $Q_{00}^{is''}$ instead of $\overline{Q_{00}}_{\text{osc}}$, the error being negligible especially as we shall only be interested in that part of (20) which belongs to small values of W.

§ 4. Influence of the temperature motion of the scattering centers.

We must now take the second feature of our binding model into consideration. At the same time we shall define a new scattering cross-section which can be directly measured. The cross-section is as a rule determined experimentally by measuring the absorption in varying thicknesses of paraffin.[1] If now the scatterer does not rest but moves with a velocity $\boldsymbol{v}_S$ relative to the coordinate system in which we are measuring, it is clear that another number per unit time of neutrons will be turned out of the beam and so we shall find another absorption coefficient. This number of neutrons expelled from the beam we can easily get by using the fact that the total cross-section is the same in all GALILEI systems[2] and so the total number scattered per unit time and per scatterer or the probability for a scattering process is just

$$P = \varrho v_{\text{rel}} Q$$

where ϱ is the density of the neutron beam, i. e. number per unit volume, $\boldsymbol{v}_{\text{rel}}$ the velocity of the neutrons relative to the scatterer and Q the total cross-section calculated in the *relative* system. In an experiment, however, we can

[1] Cf. e. g. E. AMALDI and E. FERMI, Phys. Rev. **50**, 899 (1936).
[2] Cf. Note 2.

only measure the velocity of the neutrons relative to our *observing* system, $\boldsymbol{v}_N$, and not the one relative to the scatterer, $\boldsymbol{v}_{\text{rel}}$, and so we must define an experimental cross-section Q_{exp} by the equation

$$P = \varrho\, v_N\, Q_{\text{exp}} \tag{25}$$

so that the experimental cross-section is given in terms of the usual one by

$$Q_{\text{exp}} = \frac{v_{\text{rel}}}{v_N}\, Q. \tag{26}$$

Now we can take our second assumption about the binding model into consideration, the velocity $\boldsymbol{v}_S$ of the scatterer not being constant, but distributed according to some probability law, $F(\boldsymbol{v}_S)$, the probability for finding the scatterer with a velocity between $\boldsymbol{v}_S$ and $\boldsymbol{v}_S + d\boldsymbol{v}_S$ being just equal to $F(\boldsymbol{v}_S)\, d\boldsymbol{v}_S$. So on the average we shall find the scattering probability, which we shall denote by $\overline{P}|_S$, equal to

$$\overline{P}|_S = \frac{\int P F(\boldsymbol{v}_S)\, d\boldsymbol{v}_S}{\int F(\boldsymbol{v}_S)\, d\boldsymbol{v}_S}$$

and so the average experimental cross-section, $\overline{Q_{\text{exp}}}|_S$, will be given by

$$\overline{Q_{\text{exp}}}|_S = \frac{\int \frac{v_{\text{rel}}}{v_N}\, Q\, F(\boldsymbol{v}_S)\, d\boldsymbol{v}_S}{\int F(\boldsymbol{v}_S)\, d\boldsymbol{v}_S} \tag{27}$$

For $F(\boldsymbol{v}_S)\, d\boldsymbol{v}_S$ we have assumed the MAXWELL distribution

$$F(\boldsymbol{v}_S)\, d\boldsymbol{v}_S = \left(\frac{\mu}{\pi}\right)^{3/2} \exp\left(-\mu\, v_S^2\right) d\boldsymbol{v}_S, \qquad \mu = \frac{m_S}{2\,k\,T_S} \tag{28}$$

where m_S is the mass of the scatterer (which we in this paper have chosen to be equal to 14 times the neutron mass), k is the BOLTZMANN constant[1], T_S is the absolute temperature, and the constant is chosen so that $\int F(\boldsymbol{v}_S)\, d\boldsymbol{v}_S = 1$.

For Q we ought to take the expression (20), but as we are not interested in temperatures much higher than room temperature, the main part of the integral in (27) will come from that part of Q which belongs to values of the energy not much higher than $\frac{3}{2}kT$ which means that our variable W [2] will be of the order 0.1 due to the value of $\hbar\omega_z$ having been chosen equal to 0.37 volts. For small values of W, however, we have seen that (20) can be approximated by $Q_{00}^{is''}$ defined in (22), so that we can safely put $Q_{00}^{is''}$ instead of the Q from (20) into (27). We have therefore first to put (21) into (27) and we get then using (15) and (28)

$$\left.\begin{aligned}\overline{Q_{\exp}^{is}}\Big|_S = \pi q \cdot \frac{\left(\frac{\mu}{\pi}\right)^{3/2}}{\mu' v_N}\int |\boldsymbol{v}_S - \boldsymbol{v}_N| \left[\frac{1-\exp\left(-4\mu' |\boldsymbol{v}_S-\boldsymbol{v}_N|^2\right)}{|\boldsymbol{v}_S-\boldsymbol{v}_N|^2}\right] \times \\ \times \exp\left(-\mu v_S^2\right) d\boldsymbol{v}_S, \qquad \mu' = \frac{1}{2}\frac{M_N \varepsilon}{\hbar\omega_z}\end{aligned}\right\} \quad (29)$$

Taking $\boldsymbol{v}_S - \boldsymbol{v}_N = \boldsymbol{v}$ as new variable and choosing a polar coordinate system with $\boldsymbol{v}_N$ as polar axis the integration can be worked out and we get[3]

$$\overline{Q_{\exp}^{is}}\Big|_S = \pi q \cdot \mu'^{-1} v_N^{-2} \times$$
$$\times \left[\Phi(\mu^{1/2} v_N) - \exp\left(-\frac{4\mu'\mu}{4\mu'+\mu} v_N^2\right)\left(\frac{\mu}{4\mu'+\mu}\right)^{1/2} \Phi\left(\frac{\mu v_N}{(4\mu'+\mu)^{1/2}}\right)\right] \quad (30)$$

where $\Phi(x)$ is the GAUSS error function defined in eq. (N 49).

[1] $k = 1.371 \times 10^{-16}$ erg grad^{-1} $= 8.623 \times 10^{-5}$ volts grad^{-1}. For room temperature, $T = 290°$ abs, we have $kT = 0.0250$ volts.

[2] Cf. eq. (15).

[3] Cf. Note 5.

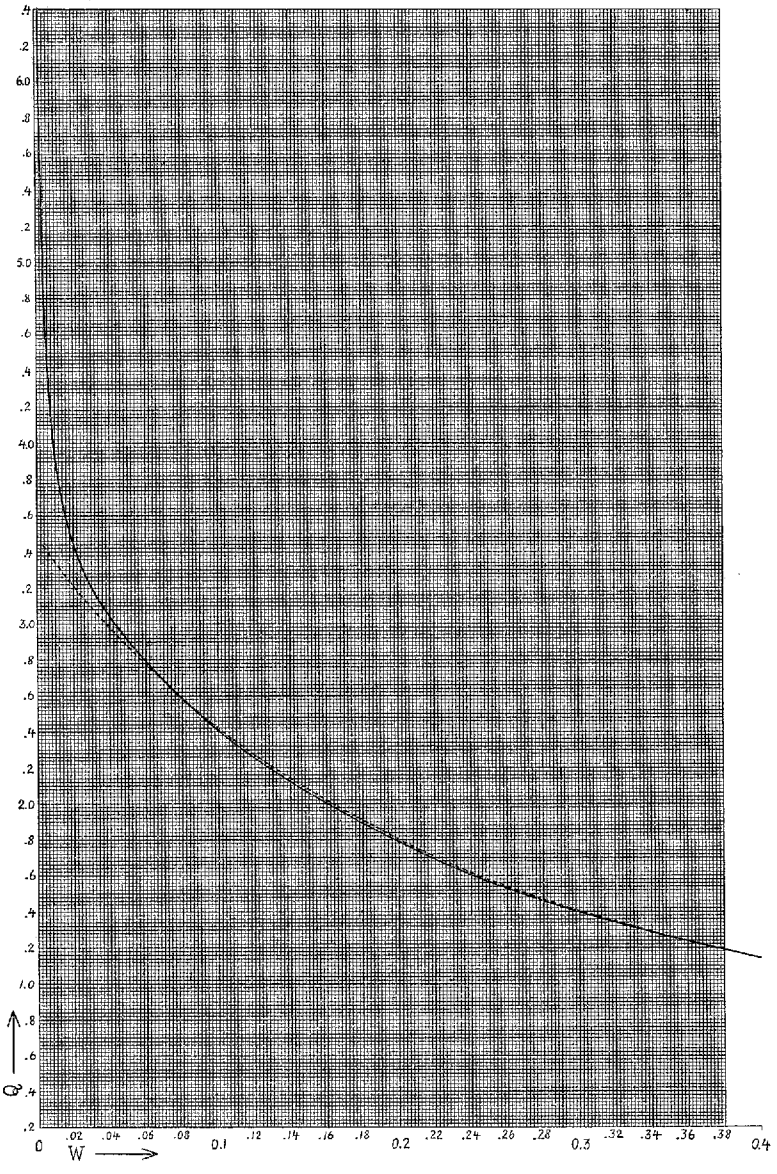

FIG. 4. *Energy dependence of scattering cross-section (in units of* Q_{free}*) for scattering centers at room temperature, i. e. 290° abs (full line). Dotted line corresponds to resting scattering centers.*

We introduce now the new dimensionless variables W [1] and s defined by

$$\left.\begin{aligned} W &= \mu' v_N^2 = \frac{\frac{1}{2} m_N v_N^2}{\hbar \omega_z} \frac{\varepsilon}{1+\frac{m_N}{m_S}}, \\ s &= \frac{\mu'}{\mu} = \frac{k T_S}{\hbar \omega_z} \frac{\varepsilon}{1+\frac{m_S}{m_N}}, \quad \varepsilon = \frac{M_N}{M_P} \text{ (cf. eq. (N 36))} \end{aligned}\right\} \quad (31)$$

Putting (31) into (30) we finally find, due to $\mu v_N^2 = \frac{1}{s} W$

$$\overline{Q^{is}_{\text{exp}}}\Big|_S = \pi q \cdot W^{-1} \times$$

$$\times \left[\Phi\left(\left(\frac{1}{s} W\right)^{1/2}\right) - (1+4s)^{-1/2} \exp\left(-\frac{4W}{1+4s}\right) \Phi\left(\left(\frac{\frac{1}{s} W}{1+4s}\right)^{1/2}\right)\right]. \quad (32)$$

For $\overline{Q^{is'}_{\text{exp}}}\Big|_S$ we find the same formula only with $\frac{1}{\gamma} W$ and $\frac{1}{\gamma} s$ substituted for W and s. In Fig. 4 we have in the full curve plotted

$$\overline{Q^{is''}_{\text{exp}}}\Big|_S = \frac{1}{3} \overline{Q^{is}_{\text{exp}}}\Big|_S + \frac{2}{3} \overline{Q^{is'}_{\text{exp}}}\Big|_S \quad (33)$$

in units of Q_{free} [2] as a function of W for $m_S = 14\, m_N$, $\hbar \omega_z = 0.37$ volts, $\gamma = 0.4$ and $T_S = 290°$ abs which makes $s = \frac{1}{222}$. Also we have plotted the curve for $Q^{is''}$ [3] and it is seen that for $W > 0.1$ the two curves are identical. The reason for this can easily be seen analytically from eq.

[1] The W here is formally equal to W in (15) only the E_N there is now the kinetic energy in the *observation* system and not in the *rest* system as in (15). Only for $T_S = 0°$, i. e. resting scattering centers, these two systems are identical.

[2] Cf. eqs. (4) and (12).

[3] Cf. eq. (22).

(32). For x greater than 2.5 $\Phi(x) = 1$ and so we get, due to $s << 1$

$$\overline{Q^{is}_{\text{exp}}}\Big|_S = \pi q \cdot W^{-1}(1 - \exp(-4W)) = Q^{is}, \quad \left(\frac{1}{s}W\right)^{1/2} > 2.5$$

and so under the same condition

$$\overline{Q^{is''}_{\text{exp}}}\Big|_S = Q^{is''}.$$

This result we also get if we take the temperature of the scatterer $T_S = 0°$ which means that the scatterer is resting, and we should therefore as cross-section find just $Q^{is''}$ which is in fact the case.

For neutrons of room temperature $E_{\text{kin}} = kT = 0.025$ volts we have $W = 0.063$ and we see from the curve that the corresponding $\overline{Q^{is''}_{\text{exp}}}\Big|_S$ is equal to $2.76 \cdot Q_{\text{free}}$. If we take $E_{\text{kin}} = \frac{3}{2}kT$ we get $W = 0.095$ and $\overline{Q^{is''}_{\text{exp}}}\Big|_S = 2.46 \cdot Q_{\text{free}}$.[1)]

§ 5. Maxwell distribution of the incident neutrons.

From the formulae (32) and (33) we can already draw conclusions about the temperature effects. In order to be able, however, to compare the results with experiments, we must take into consideration that the beams of thermal neutrons which can be produced in praxis, e. g. by slowing down fast neutrons in paraffin, are never homogeneous but have some energy distribution. As discussed in § 1 this is not known quite exactly, but we shall here approximate it by the Maxwell distribution. If $e(E)$ is the Maxwell distribution for the *current*, that means that the

1) As will be seen later, the effective energy is 1.103 kT at room temperature, cf. § 6 especially p. 38. The correct value is therefore $2.69 \cdot Q_{free}$ (cf. also Table 1, $T_S = T_N = 290°$).

probability for the neutron which hits the scatterer having an energy between E and $E+dE$ is $e(E)\,dE$, then the cross-section which would be measured should just be the average value of $\overline{Q_{\text{exp}}|_S}$

$$\frac{\int \overline{Q_{\text{exp}}|_S}\, e(E)\, dE}{\int e(E)\, dE}.$$

In praxis, however, this is not the value measured due to the fact that the Boron detector which is mostly used to measure the intensity of the neutron beam is not equally sensible for all neutron energies, but absorbs according to the $\frac{1}{v}$ law. If we then by $I(E)$ denote the sensibility of the detector, that means the fraction of the neutrons hitting the detector which it records, then what is really measured is obviously the following average value of the cross-section $\overline{Q_{\text{exp}}|_S}$ which we shall denote by $\overline{\overline{Q_{\text{exp}}|_S}|_N}$

$$\overline{\overline{Q_{\text{exp}}|_S}|_N} = \frac{\int \overline{Q_{\text{exp}}|_S}\, e(E)\, I(E)\, dE}{\int e(E)\, I(E)\, dE} \tag{34}$$

and this we shall now calculate.

The Maxwell distribution for the *current*, $e(E)$, is proportional to $v_N F(\boldsymbol{v}_N)\, d\boldsymbol{v}_N$, $F(\boldsymbol{v}_N)\, d\boldsymbol{v}_N$ being given in (28) if we substitute N for S, or transformed from velocity to energy, proportional to

$$E_N^{1/2} \cdot 2\pi^{-1/2} (kT_N)^{-3/2} E_N^{1/2} \exp\left(-\frac{E_N}{kT_N}\right) dE_N.$$

Due to the $\frac{1}{v}$ law we have further that

$$I(E) = \alpha E^{-1/2}$$

where α is some constant characteristic for the detector used. So we get that

$$e(E)\,I(E)\,dE = \alpha\,F(E)\,dE, \qquad \int F(E)\,dE = 1$$

and as the factor α drops out in (34), what we have to calculate is in fact only the mean value in regard to the MAXWELL distribution for the *density*

$$\overline{\overline{Q_{\exp}|_S}|_N} = \int \overline{Q_{\exp}}|_S\, F(E)\,dE. \tag{35}$$

For $\overline{Q_{\exp}}|_S$ we have now to put $\overline{Q^{is''}_{\exp}}|_S$ given by (32) and (33), and we must therefore first calculate $\overline{\overline{Q^{is}_{\exp}}|_S}|_N$. If we define a new dimensionless quantity, n, by

$$n = \frac{kT_N}{\hbar\,\omega_z}\,\frac{\varepsilon}{1+\dfrac{m_N}{m_S}}, \qquad \varepsilon = \frac{M_N}{M_P} \text{ (cf. eq. (N 36))} \tag{36}$$

we can write

$$F(E)\,dE = G(W)\,dW = 2\pi^{-1/2} n^{-3/2} W^{1/2} \exp\left(-\frac{1}{n} W\right) dW$$

and putting this and (32) into (35), we get

$$\overline{\overline{Q^{is}_{\exp}}|_S}|_N = \pi q \cdot 2\pi^{-1/2} n^{-3/2} \times$$

$$\times \int_0^\infty W^{-1}\left[\Phi\left(\left(\frac{1}{s} W\right)^{1/2}\right) - (1+4s)^{-1/2} \exp\left(-\frac{4W}{1+4s}\right) \Phi\left(\left(\frac{\dfrac{1}{s} W}{1+4s}\right)^{1/2}\right)\right] \times \tag{37}$$

$$\times\, W^{1/2} \exp\left(-\frac{1}{n} W\right) dW.$$

Both integrals are here of the same type

$$\int_0^\infty W^{-1/2} \exp(-\alpha^2 W)\, \Phi(\beta\, W^{1/2})\, dW = 2\pi^{-1/2} \alpha^{-1} \operatorname{Arctg} \frac{\beta}{\alpha}$$

which formula is proved in Note 6.

Putting in the correct values for α and β we get after an elementary calculation

$$\left.\begin{aligned}\overline{\overline{Q^{is}_{\exp}|_S}}|_N &= \pi q \cdot 4\pi^{-1} n^{-1} \times \\ \times \left[\operatorname{Arctg}\left(\frac{n}{s}\right)^{1/2} - (1+4(n+s))^{-1/2} \operatorname{Arctg}\left(\frac{1}{1+4(n+s)} \cdot \frac{n}{s}\right)^{1/2}\right] & \\ s = \frac{kT_S}{\hbar\omega_z} \frac{\varepsilon}{1+\frac{m_S}{m_N}}, \quad n = \frac{kT_N}{\hbar\omega_z} \frac{\varepsilon}{1+\frac{m_N}{m_S}}, & \\ \frac{n}{s} = \frac{T_N}{T_S}\frac{m_S}{m_N}, \quad \varepsilon = \frac{M_N}{M_P} = \frac{m_S^2}{m_S^2 - m_N^2} \text{ (cf. eq. (N 36))} &\end{aligned}\right\} \quad (38)$$

For $T_S = 0°$ we have found[1] that $\overline{Q^{is}_{\exp}|_S} = Q^{is}$ so that we can obtain $\overline{Q^{is}}|_N$ by putting $s = 0$ in (38)

$$\overline{Q^{is}}|_N = \lim_{s \to 0} \overline{\overline{Q^{is}_{\exp}|_S}}|_N = \pi q \cdot 2 n^{-1} (1 - (1+4n)^{-1/2})$$

due to $\operatorname{Arctg} \infty = \frac{\pi}{2}$.

For $\overline{\overline{Q^{is'}_{\exp}|_S}}|_N$ we get the same formula with $\frac{1}{\gamma}s$ and $\frac{1}{\gamma}n$ substituted for s and n respectively, and so finally

$$\overline{\overline{Q^{is''}_{\exp}|_S}}|_N = \frac{1}{3}\overline{\overline{Q^{is}_{\exp}|_S}}|_N + \frac{2}{3}\overline{\overline{Q^{is'}_{\exp}|_S}}|_N \cong \overline{\overline{Q_{\exp}|_S}}|_N. \quad (39)$$

In FIG. 5 we have plotted the curve (39) in units of Q_{free}[2] as a function of T_N for various values of T_S with $m_S = 14\, m_N$, $\hbar\omega_z = 0.37$ volts, and $\gamma = 0.4$. The values are also given in TABLE 1. We see that for $T_S = T_N = 290°$ the cross-section is 2.7 times larger than the free cross-section. AMALDI and FERMI[3] find experimentally for the ratio of the two cross-sections the value 3.7. The experi-

1) Cf. p. 27.
2) Cf. eqs. (4) and (12).
3) loc. cit.

mental value for the free proton cross-section measured with resonance neutrons is, however, very inaccurate, as already mentioned in § 1.

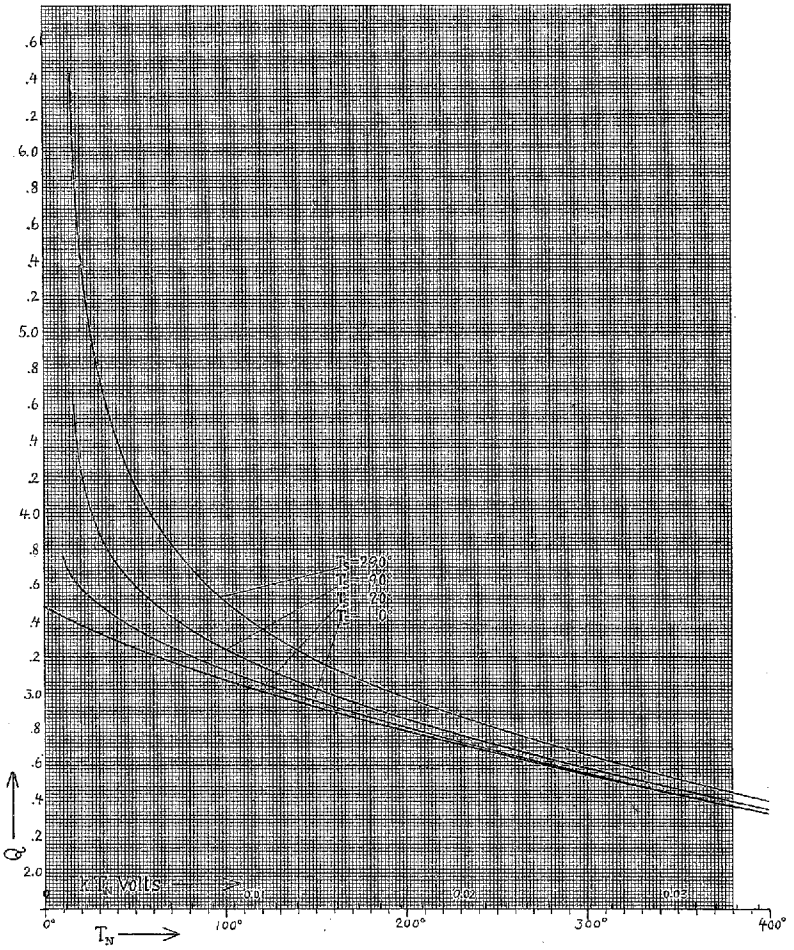

FIG. 5. *Scattering cross-sections (in units of Q_{free}) for neutron beams with* MAXWELL *distribution as function of neutron temperature for different values of the temperature of the scattering centers.*

We see further that for liquid air temperature the cross-section is 34 % higher than for room temperature, the scat-

Table 1. *The total elastic scattering cross-section in units of Q_{free} as a function of T_S and T_N given by eq. (39) with $m_S = 14\, m_N$ i. e. $\pi q = 0.871 \cdot Q_{free}$ (cf. p. 15).*

T_S \ T_N	20°	90°	290°	kT volts	
0°	3.39	3.11	2.58	0	
20°	3.59	3.17	2.58	0.00172	liquid hydrogen temperature
90°	4.26	3.30	2.61	0.00776	liquid air —
290°	5.49	3.61	2.69	0.0250	room —

terer being kept at room temperature. The agreement with the experimental value of 26 % found by Fink[1] is even better than can be expected in view of the rough assumptions of our model[2]. The values for liquid hydrogen temperature (20° abs) are only given for the sake of illustration, as for temperatures as low as these our model loses every justification. In this case, infinite effective mass would be the more appropriate approximation.

In order to see how much of the variation in our curves comes from the special form of the cross-section of the anisotropic oscillator and how much from the motion of the scattering centers $\left(\text{the factor } \frac{v_{\text{rel}}}{v_N} \text{ in (26)}\right)$ we have to compare the curves with the curve for $T_S = 0$, as the latter contains only the first influence. We see that the difference is negligible for room temperature but gets important for liquid air temperature. Another way of studying the influence of the motion of the scattering centers consists in calculating $\overline{Q_{\text{exp}}}\big|_{S}\big|_{N}$ for Q equal to a constant. Putting this into (27) we find, proceeding exactly as in the

[1] G. A. Fink, Phys. Rev. **50**, 738 (1936). A similar value was found by Frisch, Halban and Koch, loc. cit.

[2] Cf. § 1.

calculation of $\overline{Q_{\exp}^{is}}\Big|_S$, the only new formula needed being given in Note 7,

$$\overline{Q_{\exp}^{\text{const}}}\Big|_S = Q\cdot\frac{s}{2W}\left[\Phi\left(\left(\frac{W}{s}\right)^{1/2}\right)\left(1+\frac{2W}{s}\right)+\frac{2}{\pi^{1/2}}\left(\frac{W}{s}\right)^{1/2}\exp\left(-\frac{W}{s}\right)\right] \quad (40)$$

where W and s are given by eqs. (31). It is seen that (40) is only a function of $\frac{W}{s}$ as it must be, since $\hbar\omega_z$ does not enter into the problem considered here. Also it is seen that, due to $\Phi(\infty)=1$, we get for $s=0$ or for $\left(\frac{W}{s}\right)^{1/2} >> 1$ that, as is physically obvious,

$$\overline{Q_{\exp}^{\text{const}}}\Big|_S = Q.$$

Putting further (40) into (35) we get, after elementary calculations, the only new integral needed being given in Note 8,

$$\overline{\overline{Q_{\exp}^{\text{const}}}\Big|_S}\Big|_N = Q\cdot 2\pi^{-1}n^{-1}\left[(ns)^{1/2}+(n+s)\operatorname{Arctg}\left(\frac{n}{s}\right)^{1/2}\right] \quad (41)$$

n being given by (36). Also here $\hbar\omega_z$ drops out, as it must, since (41) only depends on $\frac{n}{s}$. Further we get for $T_S = 0°$

$$\lim_{s\to 0}\ \overline{\overline{Q_{\exp}^{\text{const}}}\Big|_S}\Big|_N = Q$$

as we must get, since $\lim_{s\to 0}\ \overline{Q_{\exp}^{\text{const}}}\Big|_S = \text{const}$, and $\overline{\text{const}}\Big| = \text{const}$. In FIG. 6 we have plotted the curves (41) as function of the two temperatures for $m_S = 14\,m_N$ and Q being taken equal to $\overline{\overline{Q_{\exp}^{is''}}\Big|_S}\Big|_N$ for $T_S = 0°$ and $T_N = 290°$, that is $Q = 2.58\cdot Q_{\text{free}}$[1], so that we can directly compare these

[1] Cf. TABLE 1.

curves with the curves in Fig. 5. It is seen that the general character of the curves is the same, coming from the common integrations, but that the curves in Fig. 5 have another

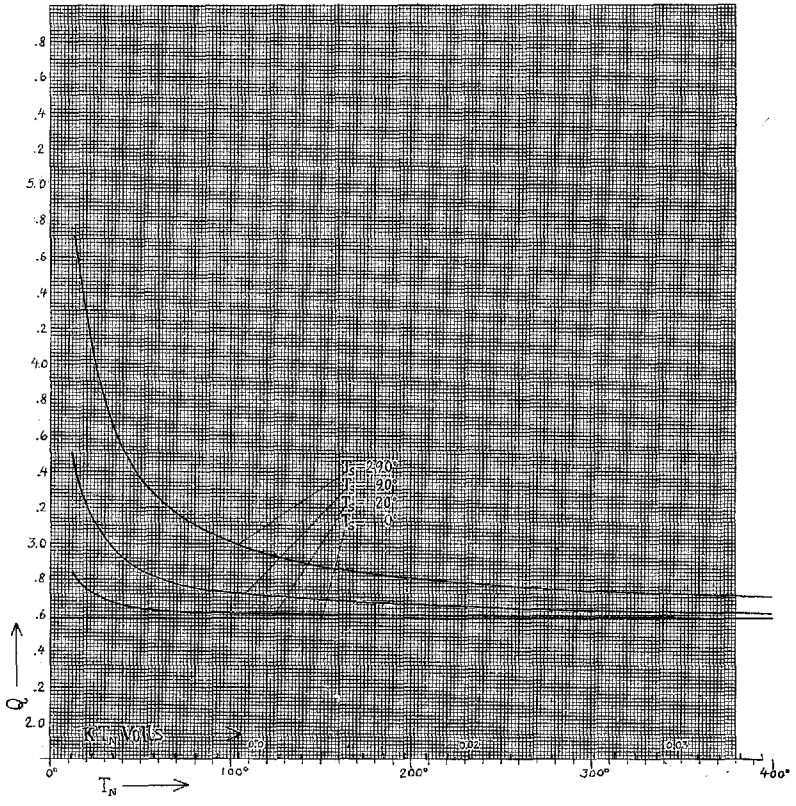

Fig. 6. *Same as Fig. 5 for a scattering cross-section which does not depend on neutron energy in the center of gravity system.*

asymptote coming from the special function chosen for Q in (27).

§ 6. Effective neutron energy.

With the help of the curves in Fig. 4 and Fig. 5 we can now treat the problem of the effective neutron energy. By

this we mean that energy, $\bar{E}$, which a homogeneous beam of neutrons must have in order that the *scattering* cross-section shall be equal to that of a MAXWELL beam of temperature T_N. For $\bar{E}$ we therefore have the equation

$$\overline{Q_{\exp}|_S}\,(\bar{E}) = \overline{\overline{Q_{\exp}|_S}|_N}\,(kT_N). \tag{42}$$

Now our expressions for the Q's are not given directly as functions of the energies but of the variables W, n and s. We have, however,[1]

$$\left.\begin{gathered} W = \alpha E, \qquad n = \alpha E_N, \qquad s = \beta E_S \\ \alpha = \frac{1}{\hbar\omega_z}\frac{\varepsilon}{1+\frac{m_N}{m_S}} = 2.52 \text{ volts}^{-1}, \quad \beta = \frac{1}{\hbar\omega_z}\frac{\varepsilon}{1+\frac{m_S}{m_N}} = 0.180 \text{ volts}^{-1} \end{gathered}\right\} \tag{43}$$

for

$$\hbar\omega_z = 0.37 \text{ volts} \quad \text{and} \quad m_S = 14\, m_N$$

where E is the energy of the homogeneous neutron beam in the coordinate system of the *observer* and E_N, E_S are equal to kT_N and kT_S respectively. So we can solve the equation (42) in terms of W, n and s. This can, however, only be done analytically in a few special cases.

I. case: $\left(\frac{n}{s}\right)^{1/2} << 1.$

As the highest temperature we are interested in is less than say $1000° = 0.0862$ volts, we see that both n and s are small compared to unity and so we can in this case expand everything in (38) with the result that

[1] Cf. eqs. (31) and (36).

$$\overline{Q^{is''}_{\exp}}\Big|_{sN} = \pi q \cdot \frac{16}{\pi}\left(\frac{s}{n}\right)^{1/2}. \tag{44}$$

From this formula we see that Q is large for $\left(\frac{n}{s}\right)^{1/2} << 1$ and from the curve for $\overline{Q^{is''}_{\exp}}\Big|_{s}$ we can conclude that the corresponding value of $\left(\frac{W}{s}\right)^{1/2}$ is also small. By expanding in eq. (32) we then find

$$\overline{Q^{is''}_{\exp}}\Big|_{s} = \pi q \cdot \frac{8}{\pi^{1/2}}\left(\frac{s}{W}\right)^{1/2}. \tag{45}$$

Putting (44) and (45) into (42) we readily find that the effective energy is given by

$$\bar{E} = \frac{\pi}{4} kT_N \qquad \left(\text{for } \left(\frac{n}{s}\right)^{1/2} << 1,\ n << 1,\ s << 1\right) \tag{46}$$

independent of the temperature of the scattering centers. This value is also the effective energy of a MAXWELL beam in regard to *absorption* in Boron[1] (while we define the effective energy in regard to *scattering*) because the cross-sections in both cases vary as $\frac{1}{v}$ (cf. eq. (45)).

If we take also higher powers in the expansions into consideration, we are able to get information about the starting tangent of the curve $\frac{\bar{E}}{kT_N} = f(kT_N)$. The calculations are however lengthy and we shall therefore only give the result found, namely that for small values of E

$$\frac{\bar{E}}{kT_N} = \frac{\pi}{4}\ (1 - \varkappa^2 kT_N) \tag{47}$$

[1] Cf. e. g. H. H. GOLDSMITH and F. RASETI, Phys. Rev. **50**, 328 (1936). Cf. also BETHE, loc. cit. Part B p. 136.

$\varkappa$ being a numerical constant depending on T_S, so that the curve is seen to decrease at the beginning when kT_N is increased.

II. case: $\left(\frac{n}{s}\right)^{1/2} >> 1$.

This we cannot fulfil for all values of s if we still want both s and n to be small compared to unity. If, however, all three conditions are fulfilled, we can put Φ and Arctg equal to 1 and $\frac{\pi}{2}$ respectively and we get then from (32) and (38)

$$\overline{Q^{is}_{\text{exp}}}\Big|_S = \pi q \cdot 4(1-2W), \quad \overline{\overline{Q^{is}_{\text{exp}}}\Big|_S}\Big|_N = \pi q \cdot 4(1-3n). \quad (48)$$

For $Q^{is'}_{\text{exp}}$ we get the same expressions with $\frac{1}{\gamma} W$ and $\frac{1}{\gamma} n$ substituted for W and n respectively. From (42) we then easily obtain

$$\bar{E} = \frac{3}{2} kT_N \quad \left(\text{for } \left(\frac{n}{s}\right)^{1/2} >> 1,\ n << 1,\ s << 1\right). \quad (49)$$

This is the classical relation that the mean energy of a MAXWELL beam is equal to $\frac{3}{2} kT$, which result we would also expect to turn out under the conditions stated above.

III. case: $\left(\frac{n}{s}\right)^{1/2} >> 1,\ n >> 1$.

In this extreme case we would find independently of s

$$\overline{Q^{is}_{\text{exp}}}\Big|_S = \pi q \cdot W^{-1}, \quad \overline{\overline{Q^{is}_{\text{exp}}}\Big|_S}\Big|_N = \pi q \cdot 2n^{-1} \quad (50)$$

and so from (42)

$$\bar{E} = \frac{1}{2} kT_N \quad \left(\text{for } \left(\frac{n}{s}\right)^{1/2} >> 1,\ n >> 1\right). \quad (51)$$

This case is, however, not of much physical importance, as we cannot neglect the inelastic scattering for energies which make $n >> 1$.

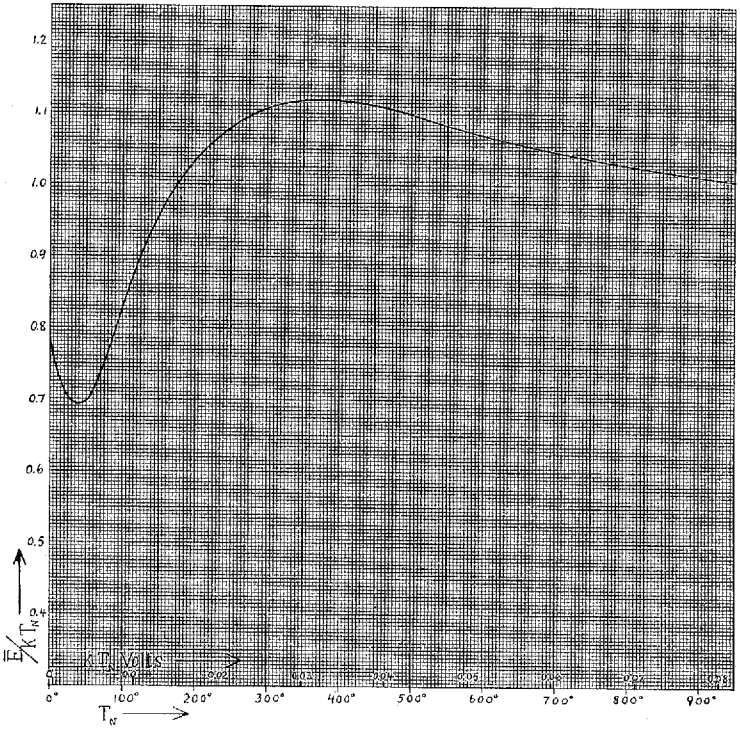

Fig. 7. *Effective neutron energy as function of neutron temperature.*

In Fig. 7 we give the curve for $\frac{\bar{E}}{kT_N}$ as a function of kT_N for $T_S = 290°$ found numerically from the curves in Figs. 4 and 5. This "pipe" like curve we have already used in § 2[1] to obtain that the effective neutron energy at room and at liquid air temperature is equal to $1.103\, kT_N$ and $0.795\, kT_N$ respectively.

[1] Cf. p. 17.

Note 1.

For our potential V' we have assumed $\lambda >> \varrho'$ [1], so we know that all the phases will be negligible except the first one, this being given by [2]

$$\eta_0 = \operatorname{arctg}\left(\frac{k_0}{k'} \operatorname{tg} k'\varrho'\right) - k_0\varrho'. \qquad \text{(N 1)}$$

Further [3]

$$I(\theta,\varphi) = \left|\frac{1}{2ik_0}\left(\exp(2i\eta_0) - 1\right)\right|^2 = \frac{\eta_0^2}{k_0^2} \qquad \text{(N 2)}$$

since $\eta_0 << 1$. As in our case $k'\varrho'$ and $k_0\varrho'$ are both small, we can expand the tg and arctg in (N 1) with the result

$$\eta_0 = \frac{1}{3}k_0\varrho'^3(k'^2 - k_0^2) = \frac{1}{3}k_0\varrho'^3 \cdot \frac{m_N D'}{\hbar^2}. \qquad \text{(N 3)}$$

Putting (N 3) into (N 2) we find

$$I = \frac{1}{9}\frac{m_N^2}{\hbar^4}(D'\varrho'^3)^2 \qquad \text{(N 4)}$$

which shows that I is independent of both the angle and the velocity of the neutrons so that the scattering is spherically symmetric. [4]

Since $Q = \int I\,d\omega$ we finally get for the total cross-section for scattering of slow neutrons by free protons, that this is a constant given by

$$Q = \frac{4\pi}{9}\frac{m_N^2}{\hbar^4}(D'\varrho'^3)^2. \qquad \text{(N 5)}$$

Note 2.

As the transformation formulae between different coordinate systems are often used but seldom given in full, we shall here compile them for reference. Firstly let us consider two coordinate systems K and K^* so that K^* has axes parallel to the axes of K and further moves along the positive x-axis of K with constant velo-

[1] Cf. p. 10.

[2] Cf. e. g. MOTT and MASSEY, "Theory of atomic collisions" eq. (30), p. 29. (The mass there is equal to the reduced mass, $\frac{m_N}{2}$).

[3] MOTT and MASSEY, loc. cit. eq. (17), p. 24.

[4] It should be remarked that this means that in the *rest* system the differential cross-section is proportional to $\cos\Theta$ cf. Note 2, eq. (N 19).

city u. A particle is moving with velocity v^* in the system K^* forming an angle θ^* with the x^*-axis. In the non-relativistic case which we are here considering, $u << c$, we have then that in the K system the particle moves with velocity $v = u + v^*$, the angle θ between v and the x-axis being determined by

$$\left.\begin{aligned} \operatorname{tg}\theta = \frac{b}{a} &= \frac{\sin\theta^*}{\cos\theta^* + \frac{u}{v^*}}, \quad \cos\theta = \frac{a}{v} = \frac{\cos\theta^* + \frac{u}{v^*}}{\left(1 + \frac{u^2}{v^{*2}} + 2\frac{u}{v^*}\cos\theta^*\right)^{1/2}}, \\ \sin\theta &= \frac{b}{v} = \frac{\sin\theta^*}{\left(1 + \frac{u^2}{v^{*2}} + 2\frac{u}{v^*}\cos\theta^*\right)^{1/2}} \end{aligned}\right\} \quad \text{(N 6)}$$

which formulae are at once deduced from FIG. 8.

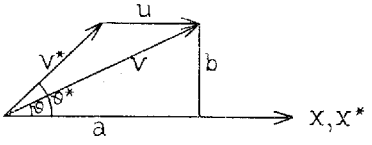

FIG. 8.

For the K- and K^*-system we now take the R and C systems[1] and shall obtain the transformation formulae (N 6) for this case, when the particle with mass m_2 moves with constant velocity V_2 along the x-axis before the collision. From (5) we find, due to $V_1 = 0$

$$u = V_2 \cdot \frac{m_2}{m_1 + m_2} \quad \text{(N 7)}$$

and as we assume that no outer forces are acting, this velocity is the same before and after the collision. To obtain $v_1'^*$ and $v_2'^*$ (the dashes referring to the state after the collision) and so the transformation formulae for the two scattering angles, θ_1^* and θ_2^*, we only need to use the conservation law for the energy

$$\frac{1}{2} m_1 v_1^{*2} + \frac{1}{2} m_2 v_2^{*2} - (E_n - E_m) = \frac{1}{2} m_1 v_1'^{*2} + \frac{1}{2} m_2 v_2'^{*2} \quad \text{(N 8)}$$

where $E_n - E_m$ is the excitation energy given up by the particle 2 in order to excite the particle 1 from its m'th to its n'th state

[1] Cf. p. 13.

during the collision. This energy can be positive or negative; for $m = 0$ it is positive for all n. Using (7) and $v_1^* = -u$, $v_2^* = V_2 - u$, we get from (N 8) due to (N 7)

$$v'^{*2}_2 = V_2^2 \left(\frac{m_1}{m_1 + m_2}\right)^2 - 2\frac{m_1}{m_2}\frac{(E_n - E_m)}{(m_1 + m_2)} \qquad \text{(N 9)}$$

(N 6) is now fully determined by (N 7) and (N 9), but only in the case of elastic scattering, $E_n - E_m = 0$, we get simple analytic expressions. In this case we get, using (7)

$$v_2'^* = \frac{m_1}{m_1 + m_2} \cdot V_2, \quad v_1'^* = \frac{m_2}{m_1} v_2'^* = u \qquad \text{(N 10)}$$

so that (N 6) becomes, independently of V_2, using (9)

$$\left.\begin{aligned} \operatorname{tg} \Theta_2 &= \frac{\sin \theta_2}{\cos \theta_2 + \frac{m_2}{m_1}}, \quad \cos \Theta_2 = \frac{\cos \theta_2 + \frac{m_2}{m_1}}{\left(1 + \frac{m_2^2}{m_1^2} + 2\frac{m_2}{m_1}\cos \theta_2\right)^{1/2}}, \\ \sin \Theta_2 &= \frac{\sin \theta_2}{\left(1 + \frac{m_2^2}{m_1^2} + 2\frac{m_2}{m_1}\cos \theta_2\right)^{1/2}} \\ \Theta_1 &= \frac{1}{2}(\pi - \theta_2), \quad \Phi_1 = \varphi_2 + \pi, \quad \Phi_2 = \varphi_2 \end{aligned}\right\} \qquad \text{(N 11)}$$

so $0 \leqq \Theta_2 \leqq \pi$ when $m_1 \neq m_2$, but $0 \leqq \Theta_2 \leqq \frac{\pi}{2}$ when $m_1 = m_2$ because we then simply get

$$\Theta_2 = \frac{1}{2}\theta_2$$

which, combined with (N 11), gives the well-known relation

$$\Theta_1 + \Theta_2 = \frac{\pi}{2}.$$

Solving (N 11) for θ_2 we find

$$\sin(\theta_2 - \Theta_2) = \frac{m_2}{m_1} \sin \Theta_2 \qquad \text{(N 12)}$$

which for $m_1 >> m_2$ reduces to

$$\theta_2 = \Theta_2 + \frac{m_2}{m_1} \sin \Theta_2.$$

Further we can, using the conservation laws, deduce the formulae for the energies before and after the collision in the *rest* system:

$$\left.\begin{aligned} E_2 &= \frac{1}{2} m_2 V_2^2, \quad E_1 = 0 \\ E_2' &= E_2 \left(1 - \frac{4 m_1 m_2}{(m_1 + m_2)^2} \cos^2 \Theta_1\right), \\ E_1' &= E_2 \frac{4 m_1 m_2}{(m_1 + m_2)^2} \cos^2 \Theta_1 \end{aligned}\right\} \quad \text{(N 13)}$$

By definition we have for the cross-sections that

$$I_{mn}(\Theta, \Phi)\, d\Omega = I_{mn}(\theta^*, \varphi^*)\, d\omega^* = I_{mn}(\theta, \varphi)\, d\omega \qquad \text{(N 14)}$$

(dropping the index 2) which we can write, due to $\Phi = \varphi^* = \varphi$

$$I_{mn}(\Theta, \Phi) = I_{mn}(\theta, \varphi) \frac{\sin \theta}{\sin \Theta} \frac{d\theta}{d\Theta}. \qquad \text{(N 15)}$$

For the special case of elastic scattering, $m = n$, we get from (N 11)

$$\frac{\sin \theta}{\sin \Theta} = \left(1 + \frac{m_2^2}{m_1^2} + 2 \frac{m_2}{m_1} \cos \theta\right)^{1/2} \qquad \text{(N 16)}$$

$$\frac{d\Theta}{\cos^2 \Theta} = \frac{1 + \frac{m_2}{m_1} \cos \theta}{\left(\cos \theta + \frac{m_2}{m_1}\right)^2} d\theta. \qquad \text{(N 17)}$$

Putting (N 16) and (N 17) into (N 15) we finally get using (N 11)

$$\left.\begin{aligned} I_{nn}(\Theta, \Phi) = I_{nn}(\theta, \varphi) \frac{\left(1 + \frac{m_2^2}{m_1^2} + 2 \frac{m_2}{m_1} \cos \theta\right)^{3/2}}{\left(1 + \frac{m_2}{m_1} \cos \theta\right)} &= I_{nn}(\theta, \varphi) \cdot g(\theta) \\ \left(1 - \frac{m_2}{m_1}\right)^2 \leq g(\theta) &\leq \left(1 + \frac{m_2}{m_1}\right)^2. \end{aligned}\right\} \quad \text{(N 18)}$$

For $m_1 >> m_2$ we get

$$g(\theta) = 1 + 2 \frac{m_2}{m_1} \cos \theta = 1 + 2 \frac{m_2}{m_1} \cos \Theta.$$

In the special case $m_1 = m_2$ we find the well-known formula

$$g(\theta) = 4 \cos \frac{\theta}{2} = 4 \cos \Theta. \qquad \text{(N 19)}$$

The function $g(\theta)$ we can transform to Θ by (N 11) and we find then

$$g(\theta) = \frac{\left(\frac{m_2}{m_1}\cos\Theta + \left(1 - \frac{m_2^2}{m_1^2}\sin^2\Theta\right)^{1/2}\right)^2}{\left(1 - \frac{m_2^2}{m_1^2}\sin^2\Theta\right)^{1/2}} \qquad \text{(N 20)}$$

which for $m_1 >> m_2$ just reduces to $1 + 2\frac{m_2}{m_1}\cos\Theta$.

From (N 14) we can at once deduce that the *total* cross-sections in the two coordinate systems are identical. This is, however, only a special case of the more general theorem that the *total* cross-section is the same in all GALILEI systems.[1] This is readily seen from the definition. Calling the probability for the scattering process P, that is the number per unit time and per scatterer scattered out of the beam, the cross-section Q is defined as the ratio between P and the number in the incident beam crossing unit area per unit time *at the place of the scatterer*, so that we have

$$Q = \frac{P}{\varrho\, v_{\text{rel}}} \qquad \text{(N 21)}$$

where ϱ is the density of the beam, that is the number of particles per unit volume, and v_{rel} is the velocity of the beam relative to the scatterer. Since P, ϱ and v_{rel} are the same in all GALILEI systems, Q is at once seen to be invariant.[2]

From (N 14) we can also deduce the transformation formulae for the differential cross-sections from one GALILEI system K to another K^*. The formulae for the angles are, when u is the velocity of the system K^*, measured in K

$$\left.\begin{aligned}
\cos\Theta &= \frac{\cos\Theta^* + \frac{u}{v_2^*}\cos\theta_1^* + \frac{u}{v_1^*}\cos\theta_2^* + \frac{u^2}{v_1^*\, v_2^*}}{\left(1 + \frac{u^2}{v_1^{*2}} + 2\frac{u}{v_1^*}\cos\theta_1^*\right)^{1/2}\left(1 + \frac{u^2}{v_2^{*2}} + 2\frac{u}{v_2^*}\cos\theta_2^*\right)^{1/2}} \\
\sin\Phi &= \frac{\sin(\varphi_1^* - \varphi_2^*)\sin\theta_2^*}{(1-\cos^2\Theta)^{1/2}\left(1 + \frac{u^2}{v_2^{*2}} + 2\frac{u}{v_2^*}\cos\theta_2^*\right)^{1/2}}
\end{aligned}\right\} \qquad \text{(N 22)}$$

[1] By a GALILEI system is understood a coordinate system which moves with constant velocity along a straight line.

[2] We have only considered the non-relativistic case. In the relativistic case Q is also invariant so long as the coordinate systems move in the direction of the current.

(with the expression for cos Θ inserted from the above formula)

$$\theta_{1,2}^* = \sphericalangle(\boldsymbol{v}_{1,2}^*, \boldsymbol{u}), \quad \varphi_1^* - \varphi_2^* = \sphericalangle([\boldsymbol{v}_1^* \times \boldsymbol{u}], [\boldsymbol{v}_2^* \times \boldsymbol{u}])$$
$$\Theta^* = \sphericalangle(\boldsymbol{v}_1^*, \boldsymbol{v}_2^*), \qquad \Phi^* = \sphericalangle([\boldsymbol{v}_1^* \times \boldsymbol{v}_2^*], [\boldsymbol{v}_1^* \times \boldsymbol{u}])$$

(and analogous formulae for the quantities without stars) which formulae are obtained from the general formulae of spherical trigonometry, using (N 6), the direction $\boldsymbol{u}$ being taken as the polar axis in a polar coordinate system. Theoretically we can from (N 14) and (N 22) obtain the transformation formulae for the differential cross-sections, but in praxis the resulting expressions are so complicated that they are quite unmanageable, with the exception of the special case where $\boldsymbol{u}$ has the same direction as one of the $\boldsymbol{v}^*$'s, in which case (N 22) reduces to (N 6). Further, if $\boldsymbol{u}$ lies in the plane of $\boldsymbol{v}_1^*$ and $\boldsymbol{v}_2^*$ we get $\Phi = \Phi^* = 0$, the formula for Θ being the same.

Note 3.[1]

For the eigenfunction of the one-dimensional oscillator we have[2]

$$\left.\begin{aligned} \psi_n(x) &= a^{-1/2}\pi^{-1/4}2^{-\frac{n}{2}}(n!)^{-1/2}\exp\left(-\frac{1}{2}\frac{x^2}{a^2}\right)H_n\left(\frac{x}{a}\right) \\ a &= \left(\frac{\hbar}{M_P\,\omega}\right)^{1/2}, \quad E_n = \left(n+\frac{1}{2}\right)\hbar\,\omega \end{aligned}\right\} \text{(N 23)}$$

where M_P is the reduced mass of the proton, ω the cyclical frequency of the proton and H_n the n'th HERMITE polynomial. With these wave functions we shall now calculate the matrix element

$$\begin{aligned} &(n \mid \exp(ik''_{mn_x}x) \mid m) = \\ &\int \psi_n^* \cdot \exp \cdot \psi_m\,dx = \pi^{-1/2}2^{-\frac{n+m}{2}}(n!\,m!)^{-1/2}e_{nm} \end{aligned} \qquad \text{(N 24)}$$

$$\left.\begin{aligned} e_{nm} = \bar{e}_{mn} &= \int_{-\infty}^{+\infty} H_n(y)\exp(ib\,y)\,H_m(y)\exp(-y^2)\,dy \\ b &= a\,k''_{mn_x}, \quad y = \frac{x}{a}. \end{aligned}\right\} \text{(N 25)}$$

[1] The matrix elements given in this note have been given previously by the author, cf. Nordiska Naturforskarmötet i Helsingfors 1936. The reports, p. 248.

[2] Cf. e. g. RUARK and UREY: "Atoms, Molecules and Quanta", p. 533.

By definition we have[1]

$$H_m(y) = (-1)^m \exp(y^2) \left(\frac{d}{dy}\right)^m \exp(-y^2). \qquad \text{(N 26)}$$

If we put (N 26) into (N 25) and integrate by parts, we get due to the fact that $\exp(-y^2)$ and all its derivatives vanish in $\pm\infty$ faster than every power of y

$$e_{nm} = 2n e_{n-1,\,m-1} + ib e_{n,\,m-1} \qquad \text{(N 27)}$$

using $H_n' = 2n H_{n-1}$, $n \geqq 1$.[2] Using this recurrence formula l times on itself, we can show by induction that the result is

$$e_{nm} = \sum_{s=0}^{l} 2^{l-s} \binom{l}{s} \frac{n!}{(n-l+s)!} (ib)^s e_{n-l+s,\,m-l} \qquad \text{(N 28)}$$

Here l is restricted by the condition that $n-l$ and $m-l$ must both be positive or zero, that is $l \leqq \min(m, n)$. Assuming $m \leqq n$ we can therefore put $l = m$. By reducing $e_{n-m+s,0}$ in the same way, we get

$$e_{n-m+s,0} = (ib)^{n-m+s} \cdot e_{00} = (ib)^{n-m+s} \pi^{1/2} \exp\left(-\frac{b^2}{4}\right) \qquad \text{(N 29)}$$

due to

$$e_{00} = \int_{-\infty}^{+\infty} \exp(iby) \exp(-y^2)\, dy =$$

$$= \exp\left(-\frac{b^2}{4}\right) \int_{-\infty}^{+\infty} \exp\left(-\left(y - \frac{ib}{2}\right)^2\right) dy = \exp\left(-\frac{b^2}{4}\right) \pi^{1/2}.$$

Putting now (N 29) into (N 28) with $l = m$, we get

$$e_{nm} = \pi^{1/2} 2^m (ib)^{n-m}\, n!\, m! \exp\left(-\frac{b^2}{4}\right) \sum_{s=0}^{m} \frac{(ib)^{2s}}{2^s\, s!\, (m-s)!\, (n-m+s)!} \qquad \text{(N 30)}$$

$$(m \leqq n).$$

For $n < m$ we get the same formula with n and m interchanged. (N 30) into (N 24) finally gives us

[1] Cf. COURANT-HILBERT: "Methoden der mathematischen Physik", p. 78.

[2] Cf. COURANT-HILBERT, loc. cit. p. 78.

$$\left.\begin{aligned}\left(n\left|\exp\left(ik''_{mn_x}x\right)\right|m\right) = \\ = 2^{-\frac{n+m}{2}}(n!\,m!)^{1/2}(ib)^{|n-m|}\exp\left(-\frac{b^2}{4}\right)\sum_{s=0}^{l}{}'\frac{(-1)^s b^{2s}}{2^s s!\,(l-s)!\,(|n-m|+s)!} \\ l = \min(n,m),\quad b = ak''_{mn_x}.\end{aligned}\right\} \text{(N 31)}$$

For the one state being just the groundstate we get

$$\left(n\left|\exp\left(ik''_{on_x}x\right)\right|0\right) = (n!)^{-1/2}2^{-\frac{n}{2}}(ib)^n\exp\left(-\frac{b^2}{4}\right). \quad \text{(N 32)}$$

For the 3-dimensional isotropic oscillator we can at once get the cross-sections from (N 32) since the eigenfunctions are only the products of three of the type (N 23). Due to the states being degenerate with the multiplicity $g_n = \frac{(n+1)(n+2)}{2}$ we must form

$$I_{mn} = q\cdot\frac{1}{g_n}\frac{k_{mn}}{k_0}\sum_{m_x+m_y+m_z=m}{}'\quad\sum_{n_x+n_y+n_z=n}{}'\left|\left(n_x n_y n_z\left|\exp\left(ik''_{mn}r\right)\right|m_x m_y m_z\right)\right|^2 \text{(N 33)}$$

This is very complicated unless $m = 0$ in which case we can at once perform the summations if we only choose the (arbitrary) coordinate system so that $\boldsymbol{k}''_{mn}$ is along one of the axes

$$\begin{aligned}I_{on} &= q\cdot\frac{k_{on}}{k_0}\sum_{n_x+n_y+n_z=n}{}'\left|\left(n_x\left|\exp\left(ik''_{on}x\right)\right|0\right)\left(n_y|1|0\right)\left(n_z|1|0\right)\right|^2 = \\ &= q\cdot\frac{k_{on}}{k_0}\frac{1}{n!}\left(\frac{b^2}{2}\right)^n\exp\left(-\frac{b^2}{2}\right)\end{aligned} \quad \text{(N 34)}$$

$$\frac{b^2}{2} = \frac{1}{2}k''^2_{on}a^2 = \frac{M_N}{M_P}\frac{E_0}{\hbar\omega}\left[2-\frac{E_n-E_0}{E_0}-2\left(1-\frac{E_n-E_0}{E_0}\right)^{1/2}\cos\theta\right] \text{(N 35)}$$

by (1). This is the formula found by FERMI[1] apart from the factor $\left(\frac{M_N}{m_N}\right)^2$ in q (cf. eq. (4)) and from the factor $\frac{M_N}{M_P}$ in eq. (N 35) which by FERMI (and by BETHE) are both put equal to unity:

[1] loc. cit. Cf. also BETHE, loc. cit. Part B eq. (455). It should be noted that by the authors quoted m_S it put equal to infinity throughout.

$$\varepsilon = \frac{M_N}{M_P} = \left(\frac{m_N m_S}{m_N + m_S}\right)\left(\frac{m_P + m_B}{m_P m_B}\right) = \frac{{m_S}^2}{{m_S}^2 - {m_N}^2} \geqq 1 \quad \text{(N 36)}$$

where m_B is the mass of the binding center $m_B = m_S - m_P$, $m_P \cong m_N$. For our value of $m_S = 14\, m_N$, ε has the value 1.0051, which can be safely replaced by unity, so that we apart from the important factor in q are left with FERMI's formula.

It might be of interest to note that the formula (N 34) can also be obtained by writing the wave functions in polar coordinates. We only give the formulae for reference[1]

$$\left.\begin{aligned} \psi_{nlm}(r,\theta,\varphi) &= \left[\frac{(2l+1)}{4\pi}\frac{(l-|m|)!}{(l+|m|)!}\right]^{1/2} \exp(im\varphi)\, P_l^{|m|}(\cos\theta) \times \\ \times a^{-3/2} &\left[\frac{2\left(\frac{n+l+1}{2}\right)!}{\left(\frac{n-l}{2}\right)!}\right]^{1/2} \frac{\exp\left(-\frac{r^2}{2a^2}\right)}{\left(l+\frac{1}{2}\right)!} \cdot {}_1F_1\left(-\frac{n-l}{2};\ l+\frac{1}{2}+1;\ \frac{r^2}{a^2}\right) \\ -l \leq m \leq +l, &\quad l = n,\ n-2,\ n-4, \cdots \geqq 0, \quad E_n = \left(n+\frac{3}{2}\right)\hbar\omega \end{aligned}\right\} \text{(N 37)}$$

P_l^m is the ordinary associated LEGENDRE polynomial[2] and ${}_1F_1$ the confluent hypergeometric function.[3] We find

$$\left.\begin{aligned} &(n\,l\,m \mid \exp(i\boldsymbol{k}''_{on}\boldsymbol{r}) \mid 000) = \\ &= \delta_{0m}\,\pi^{-1/4}\, i^l\, 2^{\frac{l+1}{2}} \left[\frac{(2l+1)\left(\frac{n+l+1}{2}\right)!}{\left(\frac{n-l}{2}\right)!}\right]^{1/2} \frac{\left(\frac{n+l}{2}\right)!}{(n+l+1)!}\, b^n \exp\left(-\frac{b^2}{4}\right) \\ &b = a k''_{on}. \end{aligned}\right\} \text{(N 38)}$$

Putting (N 38) into the formula analogous to (N 33) we just find the expression in (N 34) for I_{on} due to the formula proved by the author[4]

$$\sum_{l=n,\ n-2 \cdots \geqq 0}^{\infty} \pi^{-1/2}\, 2^{2l+1} \left[\frac{(2l+1)\left(\frac{n+l+1}{2}\right)!}{\left(\frac{n-l}{2}\right)!}\right]\left[\frac{\left(\frac{n+l}{2}\right)!}{(n+l+1)!}\right]^2 = \frac{1}{2^n\, n!} \quad \text{(N 39)}$$

[1] The author, loc. cit.

[2] COURANT-HILBERT, loc. cit., p. 282.

[3] Cf. e. g. MOTT and MASSEY, loc. cit., p. 36.

[4] See Matematisk Tidsskrift, Copenhagen 1937. "To Prof. H. BOHR on his 50th birthday", p. 42.

In the calculation leading to formula (N 38) we also get the matrix elements for the fixed rotator with two degrees of freedom whose eigenfunctions are just the first part of (N 37) multiplied by $\delta(r - r_0)$, r_0 being the dimension of the rotator. We find

$$(jm|\exp(i\boldsymbol{k}''_{oj}\boldsymbol{r})|00) = \delta_{om}\left(\frac{\pi}{2}\right)^{1/2}(2j+1)^{1/2}\, i^j (k''_{oj}r_o)^{-1/2} J_{j+1/2}(k''_{oj}r_o) \quad \text{(N 40)}$$

$J_{j+1/2}$ being the Bessel function of order $j+\frac{1}{2}$.

Note 4.

It may be interesting to note that the formula for the total elastic cross-section for the isotropic oscillator can also be deduced by direct calculation of the BORN phases and their summation which is indeed very seldom possible. We have, since the phases are all small,[1]

$$Q = 4\pi k_0^{-2} \sum_{n=0}^{\infty}{}' (2n+1)\zeta_n^2 \quad \text{(N 41)}$$

$$\zeta_n = -\frac{2M_N k_0}{\hbar^2}\int_0^\infty V'_{00}\left[\left(\frac{\pi}{2k_0 r}\right)^{1/2} J_{n+1/2}(k_0 r)\right]^2 r^2\, dr \quad \text{(N 42)}$$

$$V'_{00} = \int |\psi_0|^2\, V'\, d\tau = |\psi_0(\boldsymbol{r}_N)|^2 (-D')\,\frac{4\pi}{3}\varrho'^3 \quad \text{(N 43)}$$

due to $\varrho' << a$. If we put (N 43) into (N 42) using (N 37), we get

$$\zeta_n = \frac{M_N D'}{\hbar^2}\frac{4}{3}\pi^{1/2}\frac{\varrho^3}{a^3}\int_0^\infty \exp\left(-\frac{r^2}{a^2}\right) J^2_{n+1/2}(k_0 r)\, r\, dr. \quad \text{(N 44)}$$

Now

$$\int_0^\infty \exp(-p^2 t^2) J_\nu(at) J_\nu(bt)\, t\, dt = \frac{1}{2p^2}\exp\left(-\frac{a^2+b^2}{4p^2}\right) I_\nu\left(\frac{ab}{2p^2}\right)^{2)} \quad \text{(N 45)}$$

$$I_\nu(x) = \exp\left(-\frac{\pi}{2}i\nu\right) J_\nu(ix)^{3)} \quad \text{(N 46)}$$

[1] Cf. MOTT and MASSEY, loc. cit. eq. (5), p. 138 and eq. (12), p. 90.
[2] WATSON: "Bessel functions", eq. (1). p. 395.
[3] WATSON, loc. cit., eq. (2). p. 77.

$$\sum_{n=0}^{\infty}{}' \left(n+\frac{1}{2}\right) I^2_{n+1/2}(x) = -i \sum_{n=0}^{\infty}{}' \left(n+\frac{1}{2}\right)(-1)^n J^2_{n+1/2}(ix) = \tag{N 47}$$

$$= -i\frac{\sin 2ix}{2\pi} = \frac{e^{2x}-e^{-2x}}{4\pi}. \quad ^{1)}$$

Putting (N 42) — (N 47) into (N 41) we get, using $\frac{1}{2}k_0^2 a^2 = W$ [2]

$$Q = \left(\frac{M_N}{m_N}\right)^2 \frac{4\pi}{9}\frac{m_N^2}{\hbar^4}(D'\varrho'^3)\frac{1-\exp(-4W)}{W}$$

which is just eq. (21) remembering eq. (4).

Note 5.

We first prove the formula

$$\int_0^{\infty}\left[\exp\left(-\alpha^2(x-\beta)^2\right)-\exp\left(-\alpha^2(x+\beta)^2\right)\right]dx = \pi^{1/2}\alpha^{-1}\Phi(\alpha\beta) \tag{N 48}$$

where $\Phi(x)$ is the GAUSS error function given by

$$\Phi(x) = 2\pi^{-1/2}\int_0^x \exp(-t^2)\,dt. \tag{N 49}$$

Taking in the first part $y = x-\beta$, in the second $y = x+\beta$ as new variable we get

$$\int_0^{\infty} = \int_{-\beta}^{\infty} - \int_{+\beta}^{\infty} = 2\int_0^{\beta}\exp(-\alpha^2 y^2)\,dy$$

so (N 48) follows at once. We can now work out the integral in (29). The two angle integrations being performed, we are left with

$$\frac{\pi q}{\mu' v_N^2}\left(\frac{\mu}{\pi}\right)^{1/2}\exp(-\mu v_N^2)\int_0^{\infty}\left[\exp(-\mu v^2)-\exp\left(-(\mu+4\mu')v^2\right)\right]\times$$

$$\times\left[\exp(2\mu v_N v)-\exp(-2\mu v_N v)\right]dv.$$

The two integrals here are just of the type (N 48). In the first we have $\alpha^2 = \mu$, $\beta = v_N$, in the second $\alpha^2 = (\mu+4\mu')$, $\beta = \frac{\mu v_N}{\mu+4\mu'}$. Putting in these values in (N 48) we easily find (30).

1) WATSON, loc. cit. eq. (3). p. 152.

2) Cf. eqs. (1) and (15).

Note 6.

We prove here the formula

$$\int_0^\infty W^{-1/2} \exp(-\alpha^2 W)\, \Phi(\beta W^{-1/2})\, dW = 2\pi^{-1/2}\alpha^{-1} \operatorname{Arctg} \frac{\beta}{\alpha} \qquad \text{(N 50)}$$

where $\Phi(x)$ is defined in eq. (N 49).

We put first $t = \beta W^{1/2}$ as new variable and get

$$\int_0^\infty = 2\beta^{-1} \int_0^\infty \exp\left(-\frac{t^2}{x^2}\right) \Phi(t)\, dt = 2\beta^{-1} f(x), \quad x = \frac{\beta}{\alpha}. \qquad \text{(N 51)}$$

If we now differentiate the function $f(x)$ we get

$$f'(x) = x^{-1} \int_0^\infty \left[\exp\left(-\frac{t^2}{x^2}\right) \frac{2t}{x^2}\right] [t\Phi(t)]\, dt.$$

Integrating by parts we can get the inhomogeneous differential equation for $f(x)$

$$f'(x) = x^{-1} f(x) + \pi^{-1/2} \frac{x}{1+x^2}$$

which by the ordinary methods can be solved to

$$f(x) = \pi^{-1/2} \cdot x \cdot \operatorname{Arctg} x + (\text{constant} \cdot x).$$

The constant can be determined to be equal to zero by expanding $\Phi(t)$ and integrating term by term. For $x^2 < 1$ the resultant series is convergent to just $\pi^{-1/2} \cdot x \cdot \operatorname{Arctg} x$. This in (N 51) then proves (N 50).

Note 7.

We prove the formula

$$\begin{aligned} &\int_0^\infty x^2 \left[\exp(-\alpha^2 (x-\beta)^2) - \exp(-\alpha^2 (x+\beta)^2)\right] dx = \\ &\quad = \Phi(\alpha\beta) \left[\frac{\pi^{1/2}\beta^2}{\alpha} + \frac{\pi^{1/2}}{2\alpha^3}\right] + \frac{\beta}{\alpha^2} \exp(-\alpha^2\beta^2) \end{aligned} \qquad \text{(N 52)}$$

where $\Phi(x)$ is defined in eq. (N 49).

Putting $y = x \pm \beta$ we get

$$\int_0^\infty = \int_{-\beta}^{+\beta} y^2 \exp(-\alpha^2 y^2)\, dy + 2\beta \int_{-\beta}^{+\beta} y \exp(-\alpha^2 y^2)\, dy +$$

$$+ 4\beta \int_\beta^\infty y \exp(-\alpha^2 y^2)\, dy + \beta^2 \int_{-\beta}^{+\beta} \exp(-\alpha^2 y^2)\, dy.$$

The second integral is zero, the two last ones can be performed at once and the first one by integration by parts. The result is

$$\int_0^\infty = \left[-\frac{\beta}{\alpha^2} \exp(-\alpha^2\beta^2) + \frac{\pi^{1/2}}{2\alpha^3} \Phi(\alpha\beta)\right] + \frac{2\beta}{\alpha^2} \exp(-\alpha^2\beta^2) + \frac{\pi^{1/2}\beta^2}{\alpha} \Phi(\alpha\beta)$$

which immediately proves (N 52).

Note 8.

We prove the formula

$$\int_0^\infty W^{1/2} \exp(-\alpha^2 W)\, \Phi(\beta W^{1/2})\, dW = \pi^{-1/2} \alpha^{-3} \left[\frac{\alpha\beta}{\alpha^2+\beta^2} + \operatorname{Arctg}\frac{\beta}{\alpha}\right] \quad \text{(N 53)}$$

where $\Phi(x)$ is defined in eq. (N 49).

We take as new variable $t = \beta W^{1/2}$ and get using (N 49).

$$\int_0^\infty = 2\beta^{-3} 2\pi^{-1/2} \int_0^\infty dt \int_0^t du\, t^2 \exp\left(-\frac{\alpha^2 t^2}{\beta^2}\right) \exp(-u^2).$$

Now

$$\int_0^\infty dt \int_0^t du = \int_0^\infty du \int_u^\infty dt$$

and so we get, performing the $\int_u^\infty dt$ by integrating by parts

$$\int_0^\infty = 2\beta^{-3} 2\pi^{-1/2} \int_0^\infty du \exp(-u^2) \times$$

$$\times \left\{\frac{\beta^3}{\alpha^3} \frac{1}{2} \left[\frac{\alpha u}{\beta} \exp\left(-\frac{\alpha^2 u^2}{\beta^2}\right) + \frac{\pi^{1/2}}{2}\left(1 - \Phi\left(\frac{\alpha u}{\beta}\right)\right)\right]\right\}.$$

Here all integrations can be performed, using eq. (N 50). The result is

$$\int_0^\infty = \pi^{-1/2}\alpha^{-3}\left[\frac{\alpha\beta}{\alpha^2+\beta^2}+\frac{\pi}{2}-\operatorname{Arctg}\frac{\alpha}{\beta}\right]$$

which proves (N 53) because we have the elementary identity

$$\frac{\pi}{2}-\operatorname{Arctg}\frac{1}{x}=\operatorname{Arctg} x.$$

Summary.

In the present paper we discuss the scattering of thermal neutrons in hydrogeneous substances. In § 1 we discuss the binding model for the protons. We assume the protons to be bound independently in an anisotropic oscillator taking the largest oscillation energy equal to 0.37 volts, and the others equal to 0.4 times that. Further we take the lower frequencies into consideration by ascribing an effective mass, which we have chosen equal to fourteen times the neutron mass, to the system consisting of proton plus potential and assuming these "molecules" to move freely like gas molecules with a MAXWELL distribution. In §§ 2 and 3 the cross-sections are calculated. In §§ 4 and 5 we discuss the temperature effects. Firstly it is found that when both the neutrons and the scattering substance have room temperatures, the cross-section is 2.7 times larger than the free cross-section. Secondly it is found that the cross-section for neutrons at liquid air temperature i. e. 90° abs is 34 % higher than at room temperature. These figures are compared with the experiments. Finally we in § 6 discuss which effective energy must be attributed to a beam of MAXWELL neutrons in regard to the *scattering* cross-section. It is found that for our model this effective energy lies between 0.7 kT and 1.1 kT depending on the tempera-

ture. In the mathematical notes we have further compiled various formulae for transformation of coordinate systems, matrix elements and integrals used in the text.

In conclusion I wish to thank Prof. NIELS BOHR for his kind interest in this work and to express my appreciation to Prof. G. PLACZEK for suggesting the problem to me and for many valuable and helpful discussions in the course of the calculations. Further I wish to thank Dr. F. KALCKAR, Dr. C. MØLLER and Dr. V. WEISSKOPF for many stimulating discussions.

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