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Hg-DYNAMICS I

THEORY OF THE LAMINAR FLOW OF AN ELECTRICALLY CONDUCTIVE LIQUID IN A HOMOGENEOUS MAGNETIC FIELD

BY

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Introduction.

Hg-Dynamics. In 1918 the author designed a new pump, the electromagnetic pump, for use in connection with the so-called jet-wave rectifier. The pump is shown in fig. 1. Between the pole pieces *NS* of a strong electromagnet a gap is formed. The surfaces of the pole pieces, which form the walls of the gap, are covered with an insulating coating, and the gap is closed above and below by two electrodes $E_1 E_2$. We will assume that the channel thus formed is inserted at such a point in a hydrodynamic circuit with mercury, that it is always kept filled with this liquid. If, then, an electric current is passed through the gap from one electrode to the other the interaction between this current and the field will produce a "hydromotive force" in the direction of the axis of the channel, i. e. the apparatus will act as a pump. It is, as a matter of fact, at the same time an electromotor and a pump, the armature of the motor being the liquid to be transported by the pump.

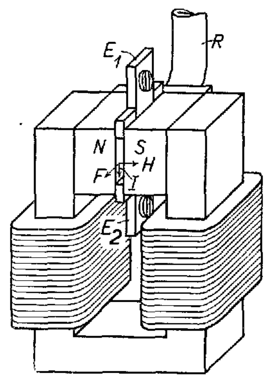

Fig. 1. Electromagnetic Pump.

The invention is, as will be seen, no very ingenious one, the principle utilized being borrowed directly from a well known apparatus for measuring strong magnetic fields. Neither does the device represent a particularly effective pump, the efficiency being extremely low due mainly to the large resistivity of mercury and still more to the contact resistance between the electrodes and the mercury. In spite hereof considerable interest was in the course of time bestowed on the apparatus, firstly because of a good many practical applications in cases where the efficiency is of small moment and then, during later years, owing to its inspiring nature. As a matter of fact the study of the pump revealed to the author what he considered a new field of

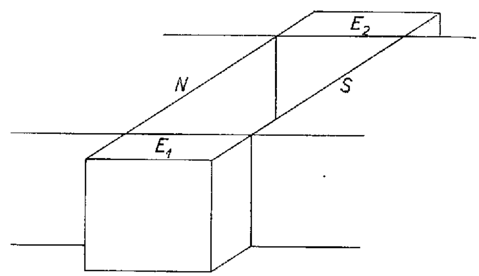

Fig. 2. Perpendicular Combination of homogeneous hydrodynamic, electric and magnetic Fields of Flow.

investigation, that of the flow of a conductive liquid in a magnetic field, a field for which the name Hg-dynamics was suggested[1].

In the electromagnetic pump of the simple form described above we have to do with the system indicated in fig. 2. In the space confined between the pole pieces NS and the electrodes $E_1 E_2$ two homogeneous fields perpendicular to each other may be set up: a magnetic field and an electric current field. If the space forms part of a hydrodynamic circuit with mercury a third homogeneous — or nearly homogeneous — field perpendicular to the two others may be created, viz. a liquid current field. The theory of the electromagnetic pump is the theory of the

[1] Fourth International Congress for Applied Mechanics. Cambridge 1934. Abstracts of Papers p. 40.

interaction of these three fields and Hg-Dynamics in its general form is the theory of the same three fields in every possible combination, each of the fields being free to vary as well in space as in time.

It will at once be gathered that the complete analytical solution of the problems of Hg-dynamics is as a rule precluded. The distribution over the liquid medium of velocity, pressure, electromotive force, electric current, magnetic field intensity etc. is given by the combination of the ordinary hydrodynamical equations: — the equations of momentum and the equation of continuity — with the general equations of electrodynamics: — Faraday's Law of induction, Biot and Savart's Law, Ohm's Law, the Induction Law for moving conductors — the distribution being furthermore governed by the various mechanical, electric and magnetic boundary conditions. In spite of the tremendous intricacy of the analysis certain problems can be solved — exactly or approximately — because of their comparatively great simplicity. Other problems, while not susceptible of any theoretical analysis, nevertheless demand solution — owing to the interest attaching to them — and consequently attain solution through experimental analysis.

One of the problems which is capable of a fairly complete theoretical analysis under certain simple conditions is the laminar flow of a conductive liquid in a homogeneous magnetic field. An example of the other class of problems not accessible to such analysis is the turbulent flow of a conductive liquid in a homogeneous magnetic field[1]. Both problems have a bearing on the electromagne-

[1] It may here be stated that it was precisely the search for a means of influencing the turbulence of a liquid flow that led to the realisation of the existence of the new field of research, the field of Hg-Dynamics.

tic pump. The former is dealt with in the present theoretical paper. The experimental verification of the theory will be given in a following publication.

The author is indebted to several of his collaborators for suggestions in connection with the discussion of the problem dealt with in the present paper and other related problems. He particularly desires to thank Mr. Lögstrup Jensen for valuable assistance. He acknowledges with gratitude the receipt of financial aid granted by the Trustees of the Carlsberg Foundation in connection with the completion of the present work.

1. Differential Equation for the Velocity Distribution across a rectangular Channel with conductive Top and Bottom.

A channel with a rectangular cross-section, fig. 3, is considered. The width $2a$ of the section is assumed small compared to the height h. In accordance herewith the

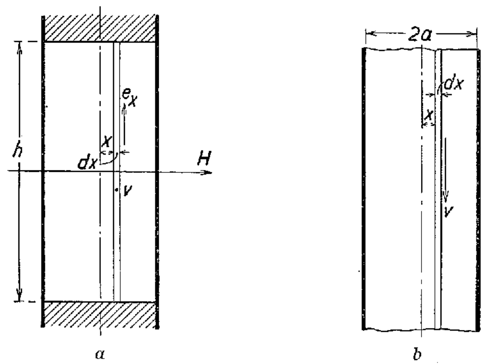

Fig. 3 *a*—*b*. Channel with rectangular Cross-Section.

effect of the upper and lower walls on the flow is neglected. These walls are formed of a highly conductive material, while the liquid of the flow is mercury. The magnetic field H is supposed to be perpendicular to the side walls of the channel.

The theory of the laminar flow with no field put on is well known. One cm.^{2} of a layer of thickness dx will be acted on by a resultant frictional force $\eta \frac{d^2 v}{dx^2} \cdot dx$ in the direction of the flow, v being the velocity of the liquid and η the viscosity of the same. This force is compensated by a pressure drop $\frac{p}{L} dx$, L being the length of the channel and p the total pressure required to drive the liquid through the channel. So

(1) $$\eta \frac{d^2 v}{dx^2} = -\frac{p}{L}$$

from which

(2) $v = \frac{1}{2} \frac{p}{\eta L} (a^2 - x^2)$, (2a) $V = \int_{-a}^{+a} v h \, dx = \frac{2}{3} \frac{p}{\eta L} h a^3$,

V being the volume flow of the fluid. Hence there would be a parabolic velocity distribution across the channel.

Now, with the field on, electromotive forces are induced in the various layers, the force in an arbitrary layer being

(3) $$e_x = 10^{-8} v_x h H \text{ Volt.}$$

The electromotive forces will give rise to a current distribution determined as follows. Let the voltage difference between the bottom and the top of the channel be E, then for an arbitrary layer the current $I_x dx$, I_x being the current density, is determined by

(4) $$E + e_x = I_x dx \cdot L \cdot \varkappa \cdot \frac{h}{L dx} = I_x \varkappa h,$$

$\varkappa$ being the specific resistance of the liquid.

Again the total electric current flow from the bottom to the top is zero, i. e.

(5) $$\int_{-a}^{+a} I_x dx = 0.$$

Introducing the expression for I_x taken from (4) into (5) we get

(6) $$\int_{-a}^{+a} (E + e_x) \, dx = 0$$

or

$$E = -\frac{1}{2a}\int_{-a}^{+a} e_x\,dx = -e_m, \tag{7}$$

e_m being the mean value of the electromotive force. So

$$I_x = \frac{1}{\varkappa h}(e_x - e_m) = 10^{-8}\,\frac{H}{\varkappa}(v_x - v_m)\ \text{Amp./cm}^2 \tag{8}$$

where

$$v_m = \frac{1}{2a}\int_{-a}^{+a} v_x\,dx. \tag{9}$$

Now the pressure gradient to which the interaction of the current and the field gives rise is

$$\dot{p}_e = \frac{1}{10}\,I_x H = 10^{-9}\,\frac{H^2}{\varkappa}(v_x - v_m) \tag{10}$$

and so the total pressure drop in the layer is $\dot{p}_e L$ i. e.

$$p_e = 10^{-9}\,\frac{LH^2}{\varkappa}(v_x - v_m). \tag{11}$$

It is tending to hamper the motion of the flow while the frictional force tends (formally) to speed up the motion. So it will be gathered that the pressure required to maintain the flow is now determined by

$$p = 10^{-9}\,\frac{LH^2}{\varkappa}(v_x - v_m) - \eta L\,\frac{d^2 v_x}{dx^2} \tag{12}$$

or by

$$p = 10^{-9}\,\frac{LH^2}{\varkappa}\left(v - \frac{1}{2a}\int_{-a}^{a} v\,dx\right) - \eta L\,\frac{d^2 v}{dx^2}, \tag{13}$$

v being written instead of v_x. This equation determines the steady velocity distribution after the magnetic field is put on.

We will assume that with no field put on there is a steady flow characterised by

$$(14)\qquad v = \frac{1}{2}\,\frac{p}{\eta L}\,(a^2 - x^2).$$

Then, suddenly, the magnetic field is set on. We ask for the current distribution at the first moment when the velocity distribution is still unaltered. The current distribution is determined by

$$(15)\qquad I = 10^{-8}\,\frac{H}{\varkappa}\,(v - v_m)$$

where

$$(16)\qquad v_m = \frac{1}{2a}\int_{-a}^{+a} v\,dx = p\cdot\frac{a^2}{3\eta L}.$$

Introducing v and v_m taken from the first and the third formula into (15), we find

$$(17)\qquad I = 10^{-8}\,\frac{H}{2\varkappa\eta}\cdot\frac{p}{L}\left(\frac{a^2}{3} - x^2\right).$$

Finally the electromagnetic pressure gradient is determined by

$$(18)\qquad \dot{p}_e = 10^{-9}\,\frac{H^2}{2\varkappa\eta}\cdot\frac{p}{L}\left(\frac{a^2}{3} - x^2\right).$$

We may consider an arrangement for which $a =$ 0.125 cm., $\varkappa = 10^{-4}$ Ohm·cm., $\eta = 0.0159$ (c. g. s.), $H = 10^4$ Gauss (Liquid: Mercury). With these values the formulae (14), (15) and (18) become

$$v = 31.4\,\frac{p}{L}\cdot(1.56\cdot 10^{-2} - x^2)\ \text{cm./sec.}$$

$$I = 31.4\,\frac{p}{L}\cdot(0.520\cdot 10^{-2} - x^2)\ \text{Amp./cm.}^2$$

$$\dot{p}_e = 31.4\cdot 10^3\,\frac{p}{L}\cdot(0.520\cdot 10^{-2} - x^2)\ \text{dyne/cm.}^2.$$

In the middle plane $x = 0$ all the three quantities are maximum. Their values are

$$v_{\max} = 0.490 \frac{p}{L} \text{ cm./sec.},$$

$$I_{\max} = 0.163 \frac{p}{L} \text{ Amp./cm.}^2,$$

$$\dot{p}_{e,\max} = 163 \frac{p}{L} \text{ dyne/cm}^2.$$

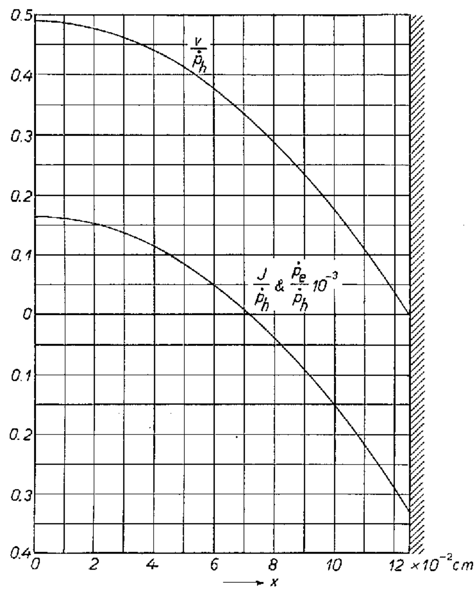

Fig. 4. Distribution of Velocity, Current and electromagnetic Pressure Gradient at the Moment the Field is set on.

Thus, it is seen, that at the first moment the electromagnetic pressure gradient is 163 times greater than the hydrodynamic pressure gradient required to maintain the viscous flow. Fig. 4 shows graphically the distribution of the quantities $v/\dot{p}_h$, $I/\dot{p}_h$ and $\dot{p}_e/\dot{p}_h$, where $\dot{p}_h = p/L$.

2. Solution of the Differential Equation for the Velocity Distribution.

The differential equation determining the velocity distribution across the channel was

$$(1) \qquad p = 10^{-9}\,\frac{L H^2}{\varkappa}\left(v - \frac{1}{2a}\int_{-a}^{+a} v\,dx\right) - \eta L\,\frac{d^2 v}{dx^2},$$

or

$$(2) \qquad \frac{d^2 v}{dx^2} - 10^{-9}\,\frac{H^2}{\eta\varkappa}\cdot v = -\frac{p}{\eta L} - 10^{-9}\,\frac{H^2}{\eta\varkappa}\cdot\frac{1}{2a}\int_{-a}^{+a} v\,dx.$$

Substituting

$$(3) \qquad 10^{-9}\,\frac{H^2}{\eta\varkappa} = A^2,$$

$$(4) \qquad \frac{p}{\eta L} = B,$$

$$(5) \qquad 10^{-9}\,\frac{H^2}{\eta\varkappa}\cdot\frac{1}{2a}\int_{-a}^{+a} v\,dx = \frac{A^2}{2a}\int_{-a}^{+a} v\,dx = C,$$

where C is an unknown constant, (2) may be written

$$(6) \qquad \frac{d^2 v}{dx^2} - A^2 v = -B - C,$$

the solution of which is

$$(7) \qquad v = \frac{B+C}{A^2} + C_1 e^{Ax} + C_2 e^{-Ax}.$$

From the two conditions

$$(8) \qquad v = 0, \qquad \text{when } x = \pm a$$

it is found that

$$(9) \qquad C_1 = C_2 = -\frac{B+C}{A^2}\cdot\frac{1}{e^{Aa} + e^{-Aa}},$$

hence

$$(10) \qquad v = \frac{B+C}{A^2}\left(1 - \frac{e^{Ax} + e^{-Ax}}{e^{Aa} + e^{-Aa}}\right).$$

Inserting this expression into (5), the following equation for the determination of C is derived

$$(11)\qquad \frac{A^2}{2a}\int_{-a}^{+a}\frac{B+C}{A^2}\left(1-\frac{e^{Ax}+e^{-Ax}}{e^{Aa}+e^{-Aa}}\right)dx = C,$$

from which

$$(12)\qquad C = B\left(Aa\cdot\frac{e^{Aa}+e^{-Aa}}{e^{Aa}-e^{-Aa}}-1\right).$$

Introducing into (10) we obtain the final formula for v:

$$(13)\qquad v = \frac{Ba}{A}\cdot\frac{e^{Aa}+e^{-Aa}}{e^{Aa}-e^{-Aa}}\left(1-\frac{e^{Ax}+e^{-Ax}}{e^{Aa}+e^{-Aa}}\right),$$

or in hyperbolic functions

$$(14)\qquad v = \frac{Ba}{A}\cdot\frac{\cosh Aa-\cosh Ax}{\sinh Aa}.$$

3. Influence of the magnetic Field on the Velocity Distribution.

We may now, first of all, discuss the influence of the magnetic field on the velocity distribution across the rectangular duct. Assuming in the first instance that $Aa = z_0$ is markedly smaller than unity we may in the formula (13) of the preceding paragraph develop the exponential terms in series. Neglecting terms of a higher power than the fourth we find

$$(1)\qquad v = \frac{Ba^2}{z_0}\cdot\frac{z_0^2-z^2+\frac{1}{12}(z_0^4-z^4)}{2z_0+\frac{z_0^3}{3}}$$

where z stands for Ax. This formula may be written

$$(2)\qquad v = \frac{p'}{2\eta L}\cdot(a^2-x^2)\left(1-\frac{A^2}{12}(a^2-x^2)\right)$$

from which the average velocity over the cross-section is found to be

(3) $$v_m = \frac{p}{3\eta L} \cdot a^2 \left[1 - \frac{1}{15}(Aa)^2\right].$$

On comparing formula (2) to formula (2) of paragraph 1 it is seen that the effect of the magnetic field, when $Aa << 1$, is to reduce the velocity of each layer of the flow slightly (and so also the average velocity). Let $Aa = 0.5$, then the reduction of the central velocity ($x = 0$) is, as will be seen, about 2 p. c. and that of the average velocity about 1.7 p. c. Now $A^2 a^2 = 10^{-9} \frac{H^2}{\eta \varkappa} \cdot a^2$. So, with $Aa = 0.5$, $H^2 a^2 = \frac{\eta \varkappa}{4} \cdot 10^9$. With mercury $\varkappa = 10^{-4}$ Ohm · cm. approximately and $\eta = 0.0160$ c. g. s. In case of $a = 0.125$ cm. we find that H must be equal to 160 Gauss to make $Aa = 0.5$. With a 10 p. c. solution of NaCl $\eta = 0.01$ c. g. s. and $\varkappa = 8.25$ Ohm · cm. approximately from which $Ha = 4.55 \cdot 10^3$ with $Aa = 0.5$. So, if H is taken to be 1000 Gauss a must be chosen equal to 0.445 cm. in order to make $Aa = 0.5$.

Now, if on the other hand Aa is very large so that e^{-Aa} is quite negligible compared to e^{Aa}, a great change in the average velocity of the flow and at the same time in the velocity distribution may ensue. The average velocity $v_{m \cdot 2}$ is equal to $\frac{C}{A^2}$ (paragraph 2, equation (5)). So from the formula (12) of paragraph 2 it follows that, under the conditions referred to, we have

(4) $$v_{m \cdot 2} = \frac{B}{A^2}(Aa - 1)$$

or approximately

(5) $$v_{m \cdot 2} = \frac{Ba}{A} = \frac{p}{\eta L} \cdot \frac{\sqrt{\varkappa \eta}}{H \sqrt{10^{-9}}} \cdot a.$$

The average velocity with no field on is

$$v_{m\cdot 1} = \frac{p}{3\eta L} \cdot a^2. \tag{6}$$

Hence

$$\frac{v_{m\cdot 1}}{v_{m\cdot 2}} = \frac{a}{3\sqrt{\eta\varkappa}} \cdot \frac{H}{10^4\sqrt{10}}. \tag{7}$$

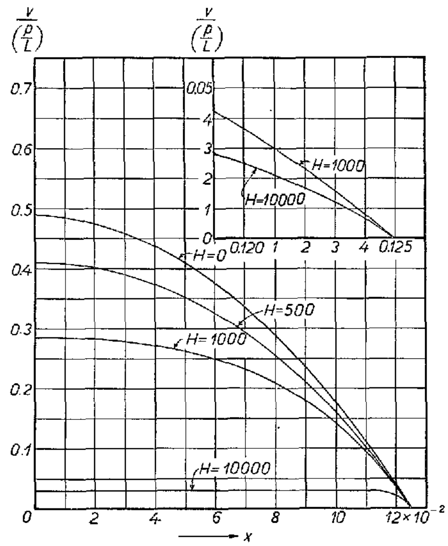

Fig. 5. Velocity Distributions when the Flow has become steady.

With mercury, with $a = 0.125$ cm. and $H = 10000$ Gauss $Aa = 31.3$ and $e^{Aa} = 38.0 \cdot 10^{12}$ while e^{-Aa} is extremely small. So the condition for the approximation is amply fulfilled and we find $\frac{v_{m\cdot 1}}{v_{m\cdot 2}} = 10.45$. Thus in the case considered the average velocity is reduced to $\frac{1}{10}$ by the field, it being assumed that the pressure gradient is kept unchanged. At the same flow of liquid $\frac{p_1/L_1}{p_2/L_2} = \frac{v_{m\cdot 2}}{v_{m\cdot 1}}$. Thus in our example the pressure gradient must with the field on be 10 times what it is with no field on in order to produce the same flow.

Finally fig. 5 represents the velocity distribution over one half of a 0.250 cm. wide channel corresponding to four values of the magnetic field. It is assumed that the liquid is mercury and that the flow is laminar. The velocities correspond to unit pressure gradient. It is seen that the effect of the field is to reduce the average velocity and to flatten out the velocity distribution curve. At a field $H =$ 10000 Gauss the velocity has become constant over most of the section.

4. The electromagnetic Viscosity.

Returning to the general expression for v, equation (13) paragraph 2, we may now derive the corresponding formula for the average velocity over the cross-section. It is found to be

$$v_m = \frac{1}{2a}\int_{-a}^{+a} v\,dx = \frac{Ba}{A}\cdot\frac{e^{Aa}+e^{-Aa}}{e^{Aa}-e^{-Aa}}\left[1-\frac{1}{Aa}\cdot\frac{e^{Aa}-e^{-Aa}}{e^{Aa}+e^{-Aa}}\right]$$

(1)

$$= \frac{p}{\eta L}a^2\cdot\frac{1-\dfrac{tanh\,z_0}{z_0}}{\dfrac{tanh\,z_0}{z_0}}\cdot\frac{1}{z_0^2}.$$

The volume flow through the duct is

$$V = v_m\cdot 2ah = \frac{2p}{\eta}\cdot\frac{a^3h}{L}\cdot\frac{1-\dfrac{tanh\,z_0}{z_0}}{\dfrac{tanh\,z_0}{z_0}}\cdot\frac{1}{z_0^2}. \tag{2}$$

Comparing (2) with formula (2a) paragraph 1 it is seen that the volume flow with the magnetic field on is the same that would ensue without the field, but with the same pressure drop, if the viscosity were

(3) $$\eta_e' = \eta \cdot \frac{z_0^2}{3} \cdot \frac{\dfrac{\tanh z_0}{z_0}}{1 - \dfrac{\tanh z_0}{z_0}}.$$

For η_e' the term apparent or virtual viscosity may be used. Assuming $z_0 = Aa$ sufficiently small compared to 1 we may use the approximation

$$\frac{\tanh z_0}{z_0} = 1 - \frac{z_0^2}{3} + \frac{2}{15} z_0^4, \quad (z_0 << 1)$$

giving

(4) $$\eta_e' = \eta + \frac{1}{15} 10^{-9} \frac{H^2 a^2}{\varkappa} = \eta + \eta_e$$

where

(5) $$\eta_e = \frac{1}{15} 10^{-9} \frac{H^2 a^2}{\varkappa}$$

may be termed the electromagnetic viscosity. Formula (4) obviously represents the variation of the apparent viscosity with the field intensity H, when H is small (or the channel very narrow).

If, on the other hand, z_0 is large compared to 1 (strong fields), then we may write

(6) $$\eta_e' = \eta \cdot \frac{z_0^2}{3} \cdot \frac{1}{z_0 - 1} = \eta \frac{z_0}{3} \left(1 + \frac{1}{z_0}\right)$$

or

(6′) $$\eta_e' = \frac{1}{3} \cdot 10^{-5} \sqrt{10} \cdot \frac{Ha}{\sqrt{\varkappa}} \cdot \sqrt{\eta} + \frac{\eta}{3} = \frac{\sqrt{15}}{3} \sqrt{\eta \eta_e} + \frac{\eta}{3}.$$

So while with feeble fields the apparent viscosity — or rather the electromagnetic viscosity — varies in a parabolic manner with the field intensity H, the apparent viscosity varies linearly with H when H is large. These features of the electromagnetic viscosity are clearly elucidated by a graph of the function

$$f(z_0) = z_0^2 \cdot \frac{\dfrac{tanh\, z_0}{z_0}}{1 - \dfrac{tanh\, z_0}{z_0}}.$$

In fig. 6 such a graph is shown.

Finally we may draw attention to a peculiar interpretation which may be given of the electromagnetic viscosity

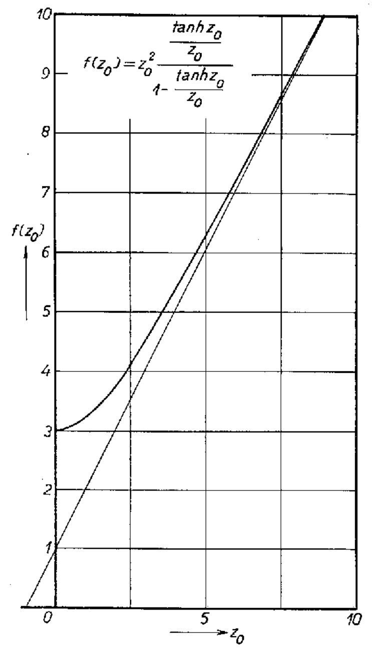

Fig. 6. Graph of the Function $f(z_0) = z_0^2 \dfrac{\dfrac{tanh\, z_0}{z_0}}{1 - \dfrac{tanh\, z_0}{z_0}}$.

η_e (equation (5)). This quantity is $\dfrac{H^2 a^2}{\varkappa}$ multiplied by a dimensionless factor. Let us assume a beam cut out of the liquid across the channel in the way illustrated in fig. 7. Let the width and the height of the beam both be 1 cm. and let these two dimensions be in the direction of the axis of the channel and perpendicular to the same respectively. The length of the beam is $2a$. Now the volume of the beam is obviously $2a$ so $2H^2a$ is proportional to the magnetic energy stored up in the space occupied by the beam. The resistance met by the induced current in the beam is $\frac{\varkappa}{2a}$. So the virtual viscosity is simply proportional to the ratio of the magnetic energy in the beam considered and the electric resistance of the fluid in the

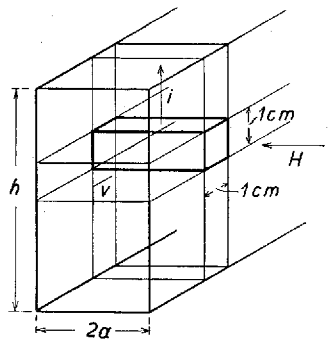

Fig. 7. Diagram illustrating the Nature of the electromagnetic Viscosity.

same beam perpendicularly to the magnetic field and to the flow. Replacing $\frac{H^2a^2}{\varkappa}$ by $\frac{H^2a^2h}{\varkappa h}$ it is seen that the virtual viscosity is also proportional to the ratio of the magnetic energy in a slice of 1 cm. thickness cut out of the flow and the electric resistance of the fluid in this slice taken perpendicularly to the motion and to the field.

5. The Channel short-circuited through a Conductor containing Resistance and electromotive Force.

We shall now imagine the top and the bottom of the channel in fig. 3 to be connected through a conductor containing a resistance R and an electromotive force E. Then the current density in an arbitrary layer of the flow is determined by

$$e + E = I \varkappa h + RL \int_{-a}^{+a} I \, dx, \tag{1}$$

where

$$e = 10^{-8} h H v. \tag{2}$$

Again it is assumed that a pressure gradient $\dot{p}$ is maintained in the flow. Hence we have

$$\dot{p} = \frac{1}{10} IH - \eta \frac{d^2 v}{dx^2}. \tag{3}$$

Introducing e and I taken from (2) and (3) respectively into equation (1) this equation assumes the shape

$$\begin{aligned} 10^{-8} h H v + E &= \frac{10}{H} \dot{p} \varkappa h + \frac{10}{H} \eta \varkappa h \frac{d^2 v}{dx^2} \\ &+ \frac{10}{H} \dot{p} RL \cdot 2a + \frac{10}{H} \eta RL \left[\left(\frac{dv}{dx}\right)_a - \left(\frac{dv}{dx}\right)_{-a} \right]. \end{aligned} \tag{4}$$

Noting that $\left(\frac{dv}{dx}\right)_{-a} = -\left(\frac{dv}{dx}\right)_a$ we may write (4) in the form

$$\frac{d^2 v}{dx^2} - \frac{10^{-9} H^2}{\varkappa \eta} v + \frac{\dot{p}}{\eta} + \frac{\dot{p}}{\eta} \cdot \frac{R}{R_c} - \frac{EH}{10 \eta \varkappa h} + 2 \frac{RL}{\varkappa h} \cdot \left(\frac{dv}{dx}\right)_a = 0 \tag{5}$$

where $R_c = \frac{\varkappa h}{2aL}$ is the electric resistance in the channel between top and bottom. Substituting

$$10^{-9} \frac{H^2}{\eta \varkappa} = A^2, \tag{6}$$

$$\frac{\dot{p}}{\eta} = B, \tag{7}$$

$$\frac{\dot{p}}{\eta} \cdot \frac{R}{R_c} - \frac{EH}{10 \eta \varkappa h} = D, \tag{8}$$

$$\text{(9)} \qquad \frac{2RL}{\varkappa h} \cdot \left(\frac{dv}{dx}\right)_a = C$$

where A and B thus stand for the same qualities as in paragraph 2, equation (5) may be written

$$\text{(10)} \qquad \frac{d^2 v}{dx^2} - A^2 v + B + D + C = 0.$$

Noting that $v = 0$ for $x = a$ and $x = -a$ the solution is found to be

$$\text{(11)} \qquad v = \frac{B+D+C}{A^2}\left(1 - \frac{e^{Ax} + e^{-Ax}}{e^{Aa} + e^{-Aa}}\right).$$

C is now found by differentiating equation (11):

$$\text{(12)} \qquad \frac{dv}{dx} = -\frac{B+D+C}{A} \cdot \frac{e^{Ax} - e^{-Ax}}{e^{Aa} + e^{-Aa}}.$$

Introducing $x = a$ we find

$$\text{(13)} \qquad \left(\frac{dv}{dx}\right)_a = -\frac{B+D+C}{A} \cdot \frac{e^{Aa} - e^{-Aa}}{e^{Aa} + e^{-Aa}} = \frac{C}{2RL}\varkappa h$$

from which

$$\text{(14)} \qquad C = -\frac{\dfrac{B+D}{A} \cdot \dfrac{e^{Aa} - e^{-Aa}}{e^{Aa} + e^{-Aa}}}{A^2\left(\dfrac{\varkappa h}{2RL} + \dfrac{1}{A} \cdot \dfrac{e^{Aa} - e^{-Aa}}{e^{Aa} + e^{-Aa}}\right)}.$$

Introducing finally into (11) we arrive at the formula:

$$\text{(15)} \qquad v = \frac{B+D}{A^2} \, \frac{\dfrac{\varkappa h}{2RL} A}{\dfrac{\varkappa h}{2RL} \cdot A + \dfrac{e^{Aa} - e^{-Aa}}{e^{Aa} + e^{-Aa}}} \left(1 - \frac{e^{Ax} + e^{-Ax}}{e^{Aa} + e^{-Aa}}\right).$$

Comparing with formula (13) in paragraph 2 it is readily seen that the shape of the velocity distribution curve is not altered by connecting the top and the bottom by an external conductor. The shape is determined solely by the value of Aa. The external current branch and what is in it influence the intensity of the flow, i. e. the average value of v, only.

6. Discussion of the Theory for the electrically short-circuited Channel.

For a fuller discussion of formula (15) in the preceding paragraph we may introduce the values of B and D. Further it should be noted that $\frac{\varkappa h}{2RL} \cdot A = \frac{R_c}{R} Aa$. Finally, in expression (8) paragraph 5 for D, the quality $\frac{E}{\varkappa h}$ is obviously the current density I_0 which the electromotive force E would create in the channel if the latter were filled with liquid at rest and if the external conductor had no resistance. So the last term in (8) is simply $\frac{\dot{p}_e}{\eta}$ where $\dot{p}_e$ is the pressure gradient to which I_0 would give rise. Taking these facts into account it is readily seen that expression (15) of paragraph 5 can be written:

$$(1) \quad v = \frac{\frac{\dot{p}}{\eta}\left(1+\frac{R}{R_c}\right)-\frac{\dot{p}_e}{\eta}}{A^2} \cdot \frac{\frac{R_c}{R} Aa}{\frac{R_c}{R} Aa + \frac{e^{Aa}-e^{-Aa}}{e^{Aa}+e^{-Aa}}} \cdot \left(1-\frac{e^{Ax}+e^{-Ax}}{e^{Aa}+e^{-Aa}}\right).$$

If $R = \infty$ and $E = 0$ we have returned to the case considered in paragraphs (1)—(4) and it is readily seen that (1) becomes identical with formula (13) of paragraph (2). Again if $E = 0$, $R = 0$ we simply have the channel short-circuited by an external conductor without resistance. Then (1) assumes the shape

(2) $$v = \frac{B}{A^2}\left(1 - \frac{e^{Ax} + e^{-Ax}}{e^{Aa} + e^{-Aa}}\right).$$

The effect of the short-circuiting is a reduction of the velocity. Comparing with formula (13) of paragraph 2 it is seen that the ratio of the velocities with and without short-circuiting is

(3) $$\frac{v_s}{v_0} = \frac{1}{Aa} \cdot \frac{e^{Aa} - e^{-Aa}}{e^{Aa} + e^{-Aa}}.$$

In the case of a channel of width $2a = 0.250$ cm. and with a flow of mercury in a field of 10000 Gauss $Aa = 31.3$. As with this value the last factor in (3) is 1, $\frac{v_s}{v_0} = \frac{1}{Aa} = \frac{1}{31.3}$. So with short-circuiting the velocity is reduced to about 3 p. c. of what it would be without short-circuiting. From this it will be gathered that the short-circuited channel may represent an extremely effective "brake" to the motion of the fluid.

The "braking" effect may be controlled by a resistance in the external conductor. The effect of the resistance is clearly seen from equation (1). Assuming $p_e = 0$, i. e. $E = 0$, we find for the ratio of the velocities with and without resistance in the external conductor:

(4) $$\frac{v_R}{v_S} = \left(1 + \frac{R}{R_c}\right) \frac{\frac{R_c}{R} Aa}{\frac{R_c}{R} Aa + \frac{e^{Aa} - e^{-Aa}}{e^{Aa} + e^{-Aa}}}.$$

If e^{Aa} is large compared to e^{-Aa} (4) may be written

(5) $$\frac{v_R}{v_S} = \frac{1 + \frac{R}{R_c}}{1 + \frac{1}{Aa} \cdot \frac{R}{R_c}}.$$

In the example above $Aa = 31.1$. In this case, if $R = R_c$, $\frac{v_R}{v_S} = 2$ approximately and if $R = nR_c$, $\frac{v_R}{v_S}$ increases as n, approximately, as long as n is a smaller number than 1, 2 or 3. With increasing n $\frac{v_R}{v_S}$ approaches the limit Aa.

Finally we may consider the general case in which there is both resistance and electromotive force in the external branch of the electric circuit. Then it is seen from (1) that the flow may be stopped altogether by applying such an electromotive force (in the direction of that induced by the flow) that

$$p_e = p\left(1 + \frac{R}{R_e}\right) \tag{6}$$

or equal to the impressed pressure gradient proper if R is small compared to R_c. If, now, p_e is increased beyond the value given by (6) then the flow will change its direction and the channel with its circuit and field will act as an electromagnetic pump. So the expression (1) forms part of the theory of this apparatus and shows that the velocity distribution in the channel is quite independent of the velocity of the flow and only dependent on the quantity

$$Aa = \frac{H}{10^4 \sqrt{10\varkappa\eta}} \cdot a. \tag{7}$$

With large values of Aa the velocity is as seen from fig. 5 almost constant over the whole width of the channel. This information was derived on the supposition of a laminar flow. The very effective smoothing out caused by a strong magnetic field suggests that the results of our discussion may also prove applicable to a flow which without the magnetic field would be turbulent, while with the field it is made laminar.

7. Characteristics of the electromagnetic Pump.

The volume flow V through an electromagnetic pump may be derived from the expression (1) in paragraph 6. Seeing that the average value of the last factor of this expression is determined by

$$(1)\qquad \frac{1}{2a}\int_{-a}^{+a}\left(1-\frac{e^{Aa}+e^{-Aa}}{e^{Aa}+e^{-Aa}}\right)dx = 1-\frac{1}{Aa}\,\frac{e^{Aa}-e^{-Aa}}{e^{Aa}+e^{-Aa}}$$

we find for the volume flow

$$(2)\quad V = 2ha\cdot\frac{\frac{\dot p}{\eta}\left(1+\frac{R}{R_c}\right)-\frac{\dot p_e}{\eta}}{A^2}\cdot\frac{\frac{R_c}{R}\cdot Aa}{\frac{R_c}{R}\cdot Aa+\frac{e^{Aa}-e^{-Aa}}{e^{Aa}+e^{-Aa}}}\left(1-\frac{1}{Aa}\,\frac{e^{Aa}-e^{-Aa}}{e^{Aa}+e^{-Aa}}\right)\ \text{cm.}^3/\text{sec.}$$

If here R is small compared to R_c as it ought to be and if further Aa is large, we may replace (2) by

$$(3)\qquad V = 2ha\cdot\frac{\frac{\dot p}{\eta}-\frac{\dot p_e}{\eta}}{A^2}\cdot\left(1-\frac{1}{Aa}\right)\ \text{cm.}^3/\text{sec.}$$

or if Aa is large compared to 1

$$(4)\qquad V = \frac{2ha}{\eta A^2}\,(\dot p-\dot p_e)\ \text{cm.}^3/\text{sec.}$$

from which

$$(5)\qquad \dot p = \dot p_e + V\cdot\frac{\eta A^2}{2ha}.$$

The total pressure difference p between the inlet and outlet of the pump is $\dot p L$ (apart from the losses at the inlet and outlet). Hence

$$p = \dot{p}_e L + V \cdot \frac{\eta A^2}{2ha} \cdot L$$
(6)
$$= \dot{p}_e L + 10^{-9} \frac{H^2}{2ha\varkappa} \cdot LV.$$

Seeing now that with the pump V is to be taken as negative and that $\dot{p}_e L$ is "the manometric pressure" p_0 of the pump we have

(7) $$p = p_0 - 10^{-9} \frac{H^2}{2ha\varkappa} \cdot LV$$

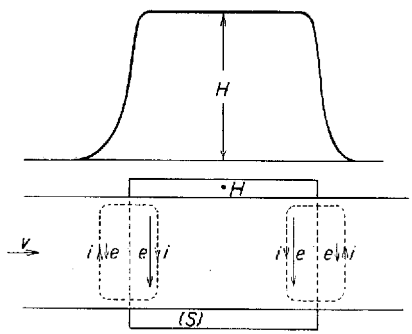

Fig. 8. Origin of the Pressure Loss at the Boundaries of the magnetic Field.

V being the numerical value of the flow. So the p-V characteristic should be a straight line or the pressure drop $p_{h+e} = p_0 - p$ should be so:

(8) $$p_{h+e} = p_0 - p = 10^{-9} \frac{H^2}{2ha\varkappa} \cdot LV.$$

With the actual pump there is an additional pressure loss at the boundaries of the magnetic field. This will readily be understood from fig. 8. Here the channel is seen located within and outside a magnetic field produced between two pole pieces *HS*, one (*S*) in front of and the other behind the channel. The field distribution curve too is shown, it extends somewhat beyond the pole-pieces. The flow gives rise to electromotive forces which are smaller outside the pole-pieces than within. So circulating currents i must arise at the boundaries of the magnetic field having the directions indicated in the figure. Obviously the currents will cause hydromotive forces to be set up which at both boundaries will have a resultant opposing the flow of the liquid.

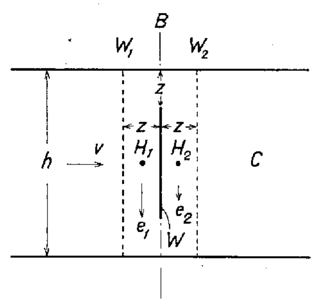

Fig. 9. To the Theory of the Pressure Loss at the Boundaries of the magnetic Field.

Now in order to develop an expression for the pressure loss at the two boundaries we may consider a greatly simplified model, fig. 9.

Here C is the channel, B is one of the boundary lines of the pole-pieces. To the left and right of B are two zones of width z in the direction of the flow. Within the left-hand zone the magnetic field has a constant value H_1, within the right-hand zone the field is H_2. We shall imagine three walls W_1, W and W_2 to be arranged. Of these W_1 and W_2 cover the full area of cross-section of the flow while W leaves two openings above and below, the width of these openings being z. The three walls possess the rather peculiar quality of permitting the hydraulic flow to pass without obstruction, while forming an absolute hindrance to the electric currents. To the left of W_1 the field is constant, equal to H_1, to the right of W_2 it is zero.

With this imaginary system a circulating current is set up the density I of which is obviously determined by

$$(9) \qquad I = \frac{e_1 - e_2}{2hz} = \frac{10^{-8}v}{2z} \cdot (H_1 - H_2) \text{ Amp./cm.}^2.$$

The corresponding pressure gradients within the left and right hand zone are respectively $-\frac{1}{10} H_1 I$ and $\frac{1}{10} H_2 I$. So the pressure loss is

$$\Delta p = \frac{1}{10} z I (H_1 - H_2) = 10^{-9} \frac{z}{2\varkappa} \cdot (H_1 - H_2)^2 v \frac{\text{dyne}}{\text{cm.}^2}. \tag{10}$$

Assuming now that $H_2 = 0$ and introducing the volume flow determined by $V = 2ah \cdot v$, (10) may be written

$$\Delta p = 10^{-9} \frac{z H^2}{2 a h \varkappa} \cdot V. \tag{11}$$

Twice this quantity — corresponding to the two boundaries — should now be added to the pressure drop occurring inside the field and given by formula (8). So the total electromagnetic-hydraulic pressure drop in the pump is represented by

$$p_0 - p + \Delta p = 10^{-9} \frac{H^2}{2 a h \varkappa} (L + z) V. \tag{12}$$

Thus it will be seen that the boundary effect may probably be taken into account by simply adding a certain length z to the actual length L of the channel or by replacing L by an effective length L_e somewhat greater than L. The correction z is likely to be proportional to the height h of the channel. Otherwise z also depends on the shape of the magnetic field curve and on the design of the outlet and inlet end of the pump when these ends are within the domain of the field or close to its boundaries.

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