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ON THE LOGARITHMIC DERIVATIVES OF THE GAMMA FUNCTION

BY

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1. Introduction. The difference calculus has led to the introduction into analysis of new classes of functions defined as solutions of equations of the type

$$\underset{\omega}{\Delta} F(z) = \varphi(z)$$

or of difference equations of higher order. Among the simplest and most important of the functions defined in this manner is $\psi(z)$, the logarithmic derivative of the gamma function.

The central role played by $\psi(z)$ in the difference calculus, as well as its importance for analysis in general, would seem to justify a detailed study of the properties of this function. Most of these have been known a long time, but there are still some problems outstanding. In the present paper we undertake an investigation of the distribution of the values taken on by $\psi(z)$ and of the corresponding conformal mapping. This problem requires a detailed study of the properties of $\psi'(z)$ and in particular of the zeros of this function. In Part I of the paper we are chiefly concerned with a determination of regions in the plane where the real part of $\psi(z)$ is positive. The study of $\psi'(z)$ follows in Part II; the main problem is attacked in Part III.[1]

[1] The present investigation was undertaken at the suggestion of Professor N. E. Nörlund. I should like to use this opportunity to express my gratitude to Professor Nörlund and to all the Copenhagen mathematicians for their friendly interest and for the cordial reception which they have given me.

Part I.

A Preliminary Study of $\psi(z)$.

2. Formal properties of $\psi(z)$. The function $\psi(z)$ is defined as that principal solution of the equation

$$\Delta F(z) = \frac{1}{z}$$

which assumes the value $-C$ for $z = +1$, where $C = 0.5772156649\ldots$ is Euler's constant. We have

$$\psi(z) = -C + \sum_{n=0}^{\infty} \left[\frac{1}{n+1} - \frac{1}{n+z}\right]. \tag{1}$$

Of the many relations satisfied by $\psi(z)$ we notice the following

$$\psi(z+1) = \psi(z) + \frac{1}{z}, \tag{2}$$

$$\psi(1-z) = \psi(z) + \pi \cot \pi z, \tag{3}$$

$$m\,\psi(mz) = \sum_{n=0}^{m-1} \psi\left(z + \frac{n}{m}\right) + m \log m, \tag{4}$$

$$\lim_{\varrho \to \infty} \left[\psi(z) - \log z\right] = 0. \tag{5}$$

Here m is a positive integer and $\log z$ denotes the principal determination of the logarithm; ϱ is the least distance of z from the negative real axis. Let us write

$$\psi(x+iy) = R(x, y) + i\,I(x, y), \tag{6}$$

where

$$R(x, y) = -C + \sum_{n=0}^{\infty} \left[\frac{1}{n+1} - \frac{n+x}{(n+x)^2 + y^2}\right], \tag{7}$$

$$I(x, y) = y \sum_{n=0}^{\infty}{}' \frac{1}{(n+x)^2+y^2}. \tag{8}$$

In view of formulas (2)—(4) these functions satisfy the following relations:

$$R(x+1, y) = R(x, y) + \frac{x}{x^2+y^2}, \tag{9}$$

$$I(x+1, y) = I(x, y) - \frac{y}{x^2+y^2}, \tag{10}$$

$$\begin{cases} R(1-x, -y) = R(1-x, y) = \\ = R(x, y) + \pi \cot \pi x \dfrac{\coth^2 \pi y - 1}{\cot^2 \pi x + \coth^2 \pi y}, \end{cases} \tag{11}$$

$$\begin{cases} I(1-x, -y) = -I(1-x, y) = \\ = I(x, y) - \pi \coth \pi y \dfrac{\cot^2 \pi x + 1}{\cot^2 \pi x + \coth^2 \pi y}, \end{cases} \tag{12}$$

$$m R(mx, my) = \sum_{n=0}^{m-1}{}' R\left(x + \frac{n}{m}, y\right) + m \log m, \tag{13}$$

$$m I(mx, my) = \sum_{n=0}^{m-1}{}' I\left(x + \frac{n}{m}, y\right), \tag{14}$$

$$\lim_{\varrho \to \infty} \left[R(x, y) - \log |z|\right] = 0, \tag{15}$$

$$\lim_{\varrho \to \infty} \left[I(x, y) - \arg z\right] = 0. \tag{16}$$

For particular values of x we can express $I(x, y)$ in terms of elementary functions. Thus

$$I(0, y) = \frac{\pi}{2} \coth \pi y + \frac{1}{2y}, \tag{17}$$

$$I\left(\frac{1}{2}, y\right) = \frac{\pi}{2} \operatorname{th} \pi y. \tag{18}$$

The former relation is obtainable from (10) and (12) by letting $x \to 0$, the latter from (12) by putting $x = \frac{1}{2}$.[1]

For purposes of numerical calculation we shall use the following relation[2]

$$(19) \quad \psi(z) = \log z - \frac{1}{2z} - \sum_{\nu=1}^{m} \frac{B_{2\nu}}{2\nu z^{2\nu}} + \int_0^\infty \frac{\bar{B}_{2m}(t)\, dt}{(t+z)^{2m+1}}.$$

Here $B_2, B_4, \ldots$ are the Bernoullian numbers; $\bar{B}_{2m}(t)$ is that periodic function of period unity which on the interval (0, 1) coincides with $B_{2m}(t)$, the Bernoullian polynomial of order $2m$. We shall use this formula for purely imaginary values of z. Setting $z = iy$ we get

$$(20) \quad R(0, y) = \log|y| + \sum_{\nu=1}^{m} \frac{|B_{2\nu}|}{2\nu y^{2\nu}} + \Re \int_0^\infty \frac{\bar{B}_{2m}(t)\, dt}{(t+iy)^{2m+1}},$$

where the absolute value of the remainder is less than

$$(21) \quad \operatorname{Max} |B_{2m}(t)| \int_0^\infty \frac{dt}{|t+iy|^{2m+1}} = \frac{2 \cdot 4 \cdot 6 \ldots (2m-2)}{1 \cdot 3 \cdot 5 \ldots (2m-1)} \cdot \frac{|B_{2m}|}{y^{2m}}.$$

Finally we shall have some use for the following factorial series

$$(22) \quad \psi(z+h) - \psi(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} \frac{h(h-1) \ldots (h-n)}{z(z+1) \ldots (z+n)},$$

which converges when $\Re(z) > 0$ and $\Re(z+h) > 0$.[3]

[1] I am indebted to Professor N. E. Nörlund for formula (18) which will be found useful below.

[2] See N. E. Nörlund: Vorlesungen über Differenzenrechnung, Berlin, J. Springer, 1924, p. 106. All the fundamental formulas for $\psi(z)$ which we use in the present paper are to be found in this book, chiefly in Chapter Five.

[3] See Nörlund, l. c. p. 251.

3. Properties of $R(x, y)$ and $I(x, y)$. It follows from (7) that

$$I(x, y) \gtreqless 0 \text{ according as } y \gtreqless 0. \tag{23}$$

Hence all the zeros of $\psi(z)$ are real. As

$$\psi'(x) = \sum_{n=0}^{\infty} \frac{1}{(n+x)^2} > 0$$

for all real values of x, we conclude that $\psi(z)$ vanishes once and only once on each of the intervals $(-n-1, -n)$, $n = 0, 1, 2, \ldots$ and in addition once on the positive real axis. The positive zero x_0 lies between 1 and 2; it was computed by Gauss and Legendre who found $x_0 = 1.46163\ldots$

Substituting $z = -n-\frac{1}{2}$ in (3) we find that

$$\psi\left(-n-\frac{1}{2}\right) = \psi\left(n+\frac{3}{2}\right) > 0 \quad \text{for } n = 0, 1, 2, \ldots \tag{24}$$

It follows that the zero x_n of $\psi(z)$ on the interval $(-n, -n+1)$ lies on the left half of this interval. With the aid of (3) in conjunction with (5) we conclude that

$$x_n \sim -n + \frac{1}{\log n}. \tag{25}$$

All these facts are of course well known. We shall now take up a detailed discussion of $R(x, y)$. It follows from (7) that

$$R(x, y_1) > R(x, y_2) \text{ when } x \geqq 0 \text{ and } |y_1| > |y_2|. \tag{26}$$

Hence in particular

$$R(x, y) > 0 \quad \text{when} \quad x \geqq x_0, \ z \neq x_0. \tag{27}$$

Using formula (9) we conclude that

$$(28) \qquad R(x, y) \lesseqgtr R(x+1, y) \text{ according as } x \gtreqless 0.$$

Further, formula (11) implies that

$$(29) \quad R(x, y) \begin{cases} > R(1-x, y) & \text{when} \quad n+\frac{1}{2} < x < n+1, \\ = R(1-x, y) & \text{when} \quad x = n+\frac{1}{2} \text{ or } n+1, \\ < R(1-x, y) & \text{when} \quad n < x < n+\frac{1}{2}. \end{cases}$$

Here n is an arbitrary integer including zero. Suppose that $n \leqq -2$, then (29) together with (27) implies that

$$(30) \quad R(x, y) > 0 \text{ when } -n-\frac{1}{2} \leqq x \leqq -n, \ n = 1, 2, 3, \ldots$$

If we set $n = -1$ in (29) we merely get that $R(x, y) > 0$ when $-\frac{1}{2} \leqq x \leqq 1-x_0$.

The result stated in formula (30) can be improved upon; in fact we have

$$(31) \quad R(x, y) > 0 \text{ when } x_{n+1} \leqq x \leqq -n, \ n = 1, 2, 3, \ldots,$$

where, as above, x_{n+1} denotes the zero of $\psi(z)$ on the interval $(-n-1, -n)$. It is evidently sufficient to prove that $R(x, y)$ is positive for $x_{n+1} \leqq x \leqq -n-\frac{1}{2}$, as the remainder of the interval is already taken care of. But this follows from formula (11). We have

$$R(x, y) = R(1-x, y) - \pi \cot \pi x \frac{\coth^2 \pi y - 1}{\coth^2 \pi y + \cot^2 \pi x}.$$

Let x be fixed on the interval $\left(-n-1, -n-\frac{1}{2}\right)$. The first term on the right hand side is always positive and increases with $|y|$. The second term is also positive, but decreases when $|y|$ increases. Consequently

$$(32)\quad \begin{cases} R(x, y_1) > R(x, y_2) \text{ when } -n-1 \leqq x \leqq -n-\dfrac{1}{2} \\ \qquad\qquad \text{and } |y_1| > |y_2|. \end{cases}$$

If we set $y_2 = 0$ in (32) and assume $x_{n+1} \leqq x \leqq -n-\frac{1}{2}$, then the right hand side is positive; this suffices to prove (31).

Next we proceed to prove that

$$(33)\qquad R(x, y) > 0 \text{ when } x \leqq -\frac{3}{4},\ |y| \geqq \frac{1}{2}.$$

For this purpose we again use formula (11). Let us give y a fixed positive value and vary x, then

$$(34)\quad -\frac{\pi}{sh\, 2\pi y} \leqq \pi \cot \pi x \frac{\coth^2 \pi y - 1}{\coth^2 \pi y + \cot^2 \pi x} \leqq \frac{\pi}{sh\, 2\pi y}.$$

If $|y| \geqq \frac{1}{2}$, (34) implies that $R(x, y)$ differs from $R(1-x, y)$ by at most $\dfrac{\pi}{sh\,\pi} = 0.2704$. But if $x \leqq -\frac{3}{4}$ and $|y| \geqq \frac{1}{2}$, $R(1-x, y) \geqq R\left(\frac{7}{4}, \frac{1}{2}\right)$. In fact, the least value of $R(1-x, y)$ in the region in question must be reached on the boundary. In view of (26) the least value on the vertical boundary is to be found at the lowest point. The horizontal boundary remains. Consider formula (7) with $x \geqq \frac{7}{4}$ and $y = \frac{1}{2}$. All the terms

$$\frac{n+x}{(n+y)^2 + \frac{1}{4}} \qquad (n = 0, 1, 2, \ldots)$$

will then be decreasing functions of x when x increases. Hence the least value of $R\left(1-x, \frac{1}{2}\right)$ for $x \leqq -\frac{3}{4}$ will be reached at $x = -\frac{3}{4}$. It is difficult to estimate the size of $R\left(\frac{7}{4}, \frac{1}{2}\right)$ without computation so we use the computed value 0.3136 (>0.2704), to be found in Table I on p. 53. Hence (33) is true. The same type of argument can be used in order to show that $R(x, y) > 0$ when $x \leqq 0$, $|y| > 1$. There is some doubt whether or not $R(x, y)$ will take on negative values on the line segment from $-\frac{3}{4}+\frac{i}{2}$ to $-\frac{1}{2}+\frac{i}{2}$.

Now let us assume that $x \leqq -n-\frac{1}{2}$ where n is an integer $\geqq 3$, and that $|y| \geqq y_0 > 0$. In view of (11) and (34) we have that

$$\begin{aligned} R(x, y) &\geqq R\left(n+\frac{3}{2},\ y_0\right)-\frac{\pi}{sh\, 2\pi y_0} \\ &> R\left(n+\frac{3}{2},\ 0\right)-\frac{\pi}{sh\, 2\pi y_0} \\ &= -C-2\log 2+2\left[1+\frac{1}{3}+\dots+\frac{1}{2n+1}\right]-\frac{\pi}{sh\, 2\pi y_0} \\ &> -C-\log 2+\log\left(n+\frac{3}{2}\right)-\frac{\pi}{sh\, 2\pi y_0}. \end{aligned}$$

Thus, in order that $R(x, y)$ be positive when $x \leqq -n-\frac{1}{2}$, $|y| \geqq y_0$, it is sufficient that

$$sh\, 2\pi y_0 \geqq \pi\left[\log\left(n+\frac{3}{2}\right)-C-\log 2\right]^{-1}.$$

Hence, a fortiori,

$$(35)\left\{\begin{array}{c} R(x, y) > 0 \text{ when} \\ x \leqq -n-\frac{1}{2} \text{ and } |y| \geqq \frac{1}{2}\left[\log\left(n+\frac{3}{2}\right)-C-\log 2\right]^{-1}. \end{array}\right.$$

Formula (35) gives a better estimate than (34) when $n \geqq 9$. Thus we see that the region in the neighborhood of $z = -n$ where $R(x, y) < 0$, contracts indefinitely when $n \to \infty$. Its maximum diameter is

$$O\left(\frac{1}{\log n}\right).$$

The arcs on which $R = 0$ contract steadily to zero in the following sense. Consider the arc of $R = 0$ on which $-n \leqq x \leqq x_n \, (n \geqq 1)$ which arc we denote by R_n. Let us imagine that R_n be moved parallel to the real axis a distance of one unit to the left. The transferred curve will then completely enclose R_{n+1}, the two curves having only the point $z = -n-1$ in common. This follows from (9). In fact, if $z+1$ is on R_n then $R(x+1, y) = 0$ and

$$R(x, y) = -\frac{x}{x^2+y^2} > 0,$$

i. e. the point z lies outside of R_{n+1} provided $z \neq -n-1$.

Part II.

Investigation of $\psi'(z)$.

4. Formal properties of $\psi'(z)$. In order to continue the discussion profitably we shall need to investigate the derivative of $\psi(z)$ in some detail and especially the location of the points where $\psi'(z) = 0$, i. e. the non-singular points where the mapping ceases to be conformal. We have

$$\psi'(z) = \sum_{n=0}^{\infty} \frac{1}{(z+n)^2}. \tag{36}$$

The most important relations satisfied by $\psi'(z)$ are the following:

$$\psi'(z+1) - \psi'(z) = -\frac{1}{z^2}, \tag{37}$$

$$\psi'(z) + \psi'(1-z) = \frac{\pi^2}{\sin^2 \pi z}, \tag{38}$$

$$\lim_{\varrho \to \infty} \left[\psi'(z) - \frac{1}{z}\right] = 0. \tag{39}$$

$$m^2 \psi'(mz) = \sum_{n=0}^{m-1} \psi'\left(z+\frac{n}{m}\right). \tag{40}$$

We set

$$\psi'(z) = r(x, y) + ij(x, y),$$

with

$$r(x, y) = \sum_{n=0}^{\infty} \frac{(x+n)^2 - y^2}{\left[(x+n)^2+y^2\right]^2}, \tag{41}$$

$$j(x, y) = -2y \sum_{n=0}^{\infty} \frac{x+n}{\left[(x+n)^2+y^2\right]^2}. \tag{42}$$

Of the relations satisfied by $r(x, y)$ and $j(x, y)$ which are a consequence of (37)—(40) we notice the following:

$$r(x+1, y) = r(x, y) - \frac{x^2-y^2}{(x^2+y^2)^2}, \tag{43}$$

$$j(x+1, y) = j(x, y) + \frac{2xy}{(x^2+y^2)^2}, \tag{44}$$

$$r(x, y) = -r(1-x, y) + \pi^2 \frac{\sin^2 \pi x \, ch^2 \pi y - \cos^2 \pi x \, sh^2 \pi y}{[\sin^2 \pi x + sh^2 \pi y]^2}, \tag{45}$$

$$j(x, y) = -j(1-x, y) - \frac{\pi^2}{2} \frac{\sin 2\pi x \, sh\, 2\pi y}{[\sin^2 \pi x + sh^2 \pi y]^2}, \tag{46}$$

$$m^2 r(mx, my) = \sum_{n=0}^{m-1} r\left(x+\frac{n}{m}, y\right), \tag{47}$$

$$m^2 j(mx, my) = \sum_{n=0}^{m-1} j\left(x+\frac{n}{m}, y\right). \tag{48}$$

For certain special purposes we shall need the factorial series

$$\psi'(z) = \sum_{n=0}^{\infty} \frac{n!}{(n+1)\, z(z+1) \ldots (z+n)}, \tag{49}$$

which converges when $\Re(z) > 0$. This series is easily obtainable from the corresponding series for $\psi(z+h) - \psi(z)$ in formula (22) by dividing by h and then letting h tend to zero.[1]

It is trivial to notice but useful to remember that

$$r(x, y) = \frac{\partial}{\partial x} R(x, y) = \frac{\partial}{\partial y} I(x, y) \tag{50}$$

$$j(x, y) = -\frac{\partial}{\partial y} R(x, y) = \frac{\partial}{\partial x} I(x, y). \tag{51}$$

5. $\psi'(z)$ **in the right half-plane.** It is obvious that

$$\operatorname{sgn} j(x, y) = -\operatorname{sgn} y \quad \text{when} \quad x \geqq 0. \tag{52}$$

It is further clear that $\psi'(x)$ is real positive when x is real. From these two observations we conclude that $\psi'(z) \neq 0$ when $\Re(z) \geqq 0$. In the expression

$$\frac{\pi^2}{\sin^2 \pi z} = \sum_{n=-\infty}^{+\infty} \frac{1}{(z+n)^2}$$

we set $z = iy$. The result can be written in the form

$$-\frac{\pi^2}{sh^2 \pi y} = -\frac{1}{y^2} + 2 \sum_{n=0}^{\infty} \frac{n^2 - y^2}{(n^2 + y^2)^2}.$$

Hence we have

$$r(0, y) = -\frac{1}{2y^2} - \frac{\pi^2}{2 sh^2 \pi y} < 0. \tag{53}$$

Thus $w = \psi'(z)$ maps the line $x = 0$ in the z-plane upon a curve J in the w-plane

$$u = r(0, y), \qquad v = j(0, y),$$

[1] See also Nörlund, l. c. p. 243.

which curve lies entirely in the half-plane $u < 0$ except for the point $(0, 0)$, where J is tangent to the v-axis. J does not intersect itself for $r(0, y)$ increases steadily with $|y|$; it consists of two branches symmetric with respect to the negative u-axis, which is the asymptote of both. Let the region outside of J be denoted by $\varDelta$. It will be proved in § 11 that $\psi''(z) \neq 0$ in $\varDelta + J$. Thus $w = \psi'(z)$ maps the half-plane $\Re(z) > 0$ conformally upon $\varDelta$. Thus every value in $\varDelta$ is taken on once and only once by $\psi'(z)$ in the right half-plane. A simple calculation shows that

$$|v| < \frac{\sqrt{3}}{16}\pi^2 = 1.069$$

on J; hence the values not taken on in $\Re(z) > 0$ have negative real part and a numerically small imaginary part.[1]

In the half-plane $\Re(z) \geqq 1$, $r(x, y) > 0$. To see this we notice first that

$$r(1, y) = r(0, y) + \frac{1}{y^2} = \frac{1}{2y^2} - \frac{\pi^2}{2\,sh^2\,\pi y} > 0$$

in view of formulas (43) and (53). Thus the curve $r(x, y) = 0$ does not intersect the line $x = +1$. On the other hand, there are two branches of this curve in the right half-plane which pass through the origin, where they have the slopes $+1$ and -1 respectively, and which admit of the imaginary axis as their asymptote. Hence the branches of $r(x, y) = 0$ which lie in the right half-plane must be enclosed in the strip $0 \leqq x < +1$. It follows from formula (39) that there are no other branches of the curve $r(x, y) = 0$ in the right half-plane. Hence $r(x, y) > 0$ when $x \geqq +1$.

[1] To obtain the estimate given for $|v|$ we replace each term in the series (42) by its maximum value for $x = 0$ and sum these maximum values. The estimate is rather crude; $|v|$ probably does not exceed 0.8.

6. $\psi'(z)$ in the left half-plane. We now turn our attention to the left half-plane. Let k be a positive integer; then

$$(54)\qquad r(-k, y) = \sum_{n=1}^{k}{}' \frac{n^2 - y^2}{(n^2+y^2)^2} - \frac{1}{y^2} + \sum_{n=1}^{\infty}{}' \frac{n^2-y^2}{(n^2+y^2)^2},$$

or

$$(55)\qquad r(-k, y) = \sum_{n=1}^{k}{}' \frac{n^2-y^2}{(n^2+y^2)^2} + r(0, y).$$

In view of (53) we can conclude that $r(-k, y) < 0$ when $|y| \geqq k$. When $|y| < k$ we have

$$\begin{aligned} r(-k, y) &< \sum_{n=1}^{\infty}{}' \frac{n^2-y^2}{(n^2+y^2)^2} + r(0, y) \\ &= \frac{1}{y^2} + 2r(0, y) = -\frac{\pi^2}{sh^2 \pi y} < 0. \end{aligned}$$

Hence

$$(56)\qquad r(-k, y) < 0, \qquad k = 0, 1, 2, \ldots$$

for all values of y. Further

$$(57)\qquad j(-k, y) = -2y \sum_{n=k+1}^{\infty}{}' \frac{n}{(n^2+y^2)^2},$$

$$(58)\qquad j\left(-k-\frac{1}{2}, y\right) = -2y \sum_{n=k+1}^{\infty}{}' \frac{n+\frac{1}{2}}{\left[\left(n+\frac{1}{2}\right)^2 + y^2\right]^2}.$$

Consequently $\psi'(z) \neq 0$ on all the lines $x = -\frac{n}{2}$ $(n = 0, 1, 2, \ldots)$ and

$$\operatorname{sgn} \Re\left[\psi'(-n+iy)\right] = -1,$$

$$(59)\qquad \operatorname{sgn} \Im\left[\psi'\left(-\frac{n}{2}+iy\right)\right] = -\operatorname{sgn} y.$$

7. Introduction of the cells. The lines $x = -\frac{n}{2}(n = 0, 1, 2, \ldots)$ and $y = 0$ divide the left half-plane into an infinite number of cells

$$C_n : -\frac{n}{2} < x < -\frac{n-1}{2}, \quad y > 0,$$

and

$$\overline{C}_n : -\frac{n}{2} < x < -\frac{n-1}{2}, \quad y < 0.$$

Theorem: Each of the cells C_{2k-1} and $\overline{C}_{2k-1}$ contains one and only one complex zero of $\psi'(z)$. The cells C_{2k} and $\overline{C}_{2k}$ do not contain any zeros $(k = 1, 2, 3, \ldots)$.

In order to prove this theorem we trace the image of the boundary of a cell C_n by the transformation $w = \psi'(z)$ avoiding the vertices of the cell at the singular points in the usual manner. For the following discussion consult Fig. 1 which gives a schematic representation of the situation. The line drawn in full corresponds to the case when n is odd and the dotted line to the case when n is even.

Let the image of the line segment $x = -\frac{n}{2}$, $0 \leqq y$ be denoted by J_n. In view of (57) and (58) the curve J_{2k} lies entirely in the third quadrant of the w-plane; it is asymptotic to the negative real axis and tangent to the v-axis at the origin. According to (59) J_{2k-1} lies in the lower half-plane; starting from a point on the positive real axis, it ends in the third quadrant at the origin and tangent to the v-axis. J_{2k-2} and J_{2k-1} intersect at least once in the third quadrant forming a loop together; it is probable that J_{2k-1} and J_{2k} do not intersect each other, also that the curves J_n do not intersect themselves, but this is immaterial for our present purpose.

The lower boundary of the cell is mapped upon a segment of the positive real axis which is in parts covered twice when n is odd. Finally a small circular arc $|z+k+1| = \varrho e^{i\theta}$, $0 \leqq \theta \leqq \frac{\pi}{2}$, is mapped upon a large contour in the

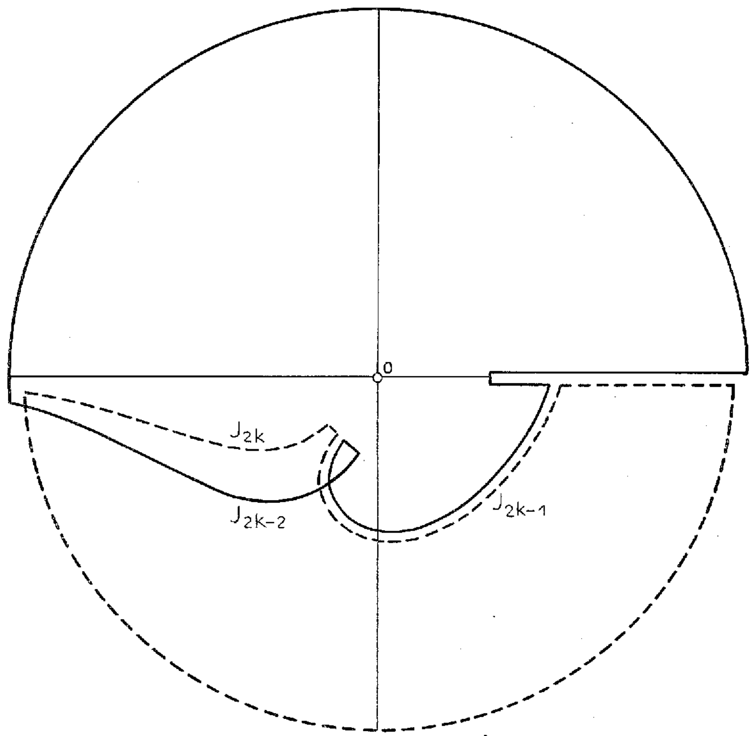

Fig. 1.

lower half-plane, and an arc $|z+k| = \varrho e^{i\theta}$, $\frac{\pi}{2} \leqq \theta \leqq \pi$, is mapped upon a contour in the upper half-plane. Keeping these facts in mind or consulting the figure the reader will see that the argument of $\psi'(z)$ remains unchanged when we trace the boundary of C_{2k} but increases by 2π along the boundary of C_{2k-1}, a result which suffices to prove our theorem.

We shall prove in § 11 that $\psi''(z) \neq 0$ in C_{2k} for all values of k. It follows that $\psi''(z)$ maps the interior of C_{2k} conformally upon a region in the lower half of the w-plane a region which, however, may partly overlap itself. The map of C_{2k-1} is neither conformal in the interior nor on the boundary. Since $\psi''(-k-1+\varepsilon) < 0$, $\psi''\left(-k-\frac{1}{2}\right) < 0$, $\psi''(-k-\varepsilon) > 0$ and $\psi'''(x) > 0$, where k is a positive integer or zero, $\varepsilon > 0$ and x is real, we conclude that $\psi''(z)$ vanishes once and only once in the interval $(-k-1, -k)$ and, in fact, on the right half of this interval. We have also noticed that the curves J_{2k-1} and J_{2k-2} intersect in the third quadrant where they form a loop. This indicates that $\psi''(z)$ vanishes at least once in the interior of C_{2k-1}. Thus we have at least 3 zeros of $\psi''(z)$ in the strip $-k-\frac{1}{2} < x < -k$ for every integral $k \geqq 0$. We shall see later that there are exactly 3 zeros of $\psi''(z)$ in this strip.

8. The curves $r = 0$ and $j = 0$. In order to gain additional information regarding the map corresponding to $w = \psi'(z)$ we consider the curves $r(x, y) = 0$ and $j(x, y) = 0$. The points $z = -n$ $(n \geq 0)$ are double poles of $\psi'(z)$; hence they are double points of the curves $r = 0$ and $j = 0$. The r-curves have the slopes $+1$ and -1 at $z = -n$, the j-curves have the slopes 0 and ∞ at this point.

One of the j-curves through $z = -n$ is the real axis. Let the other j-curve through this point be denoted by j_n. We have already seen that $j(x, y) < 0$ in C_{2n} and > 0 in $\overline{C}_{2n}$ $(n = 1, 2, 3, \ldots)$. This follows also directly from formula (46) which shows that $j(x, y) < 0$ if $-n \leqq x \leqq -n + \frac{1}{2}$, $y > 0$. Consequently j_n lies entirely in $C_{2n-1} + \overline{C}_{2n-1}$. It is a closed curve which intersects the real axis at $z = -n+1$ and at the point where $\psi''(x) = 0$. The curve j_n grows steadily with n in the following sense. Let us imagine that j_n be moved parallel to the real axis a distance of one unit to the left. It will then have a contact with j_{n+1} at $z=-n$; with the exception of this point, the transferred curve lies entirely within j_{n+1}. This follows from formula (44); indeed, if $z+1$ lies on j_n then $j(x+1, y)=0$ and

$$j(x, y) = -\frac{2xy}{(x^2+y^2)^2} > 0, \qquad (y > 0),$$

i. e., the point z lies inside of j_{n+1}.

It is possible to find upper limits for $|y|$ on j_n with the aid of formulas (44), (46) and (48). If y is fixed positive

$$\frac{\pi^2}{2} \frac{\sin 2\pi x \operatorname{sh} 2\pi y}{(\sin^2 \pi x + \operatorname{sh}^2 \pi y)^2} < \pi^2 \frac{\operatorname{ch} \pi y}{\operatorname{sh}^3 \pi y}.$$

The latter expression is less than 0.075 when $y \geqq 1$. On the other hand, we can show by a simple but tedious calculation that $j(3.5, 1) < -0.075$. Further,

$$\frac{\partial j}{\partial x} = 2y \sum_{n=0}^{\infty} \frac{3(x+n)^2 - y^2}{\left[(x+n)^2+y^2\right]^3}.$$

This expression is certainly positive when $0 \leqq y \leqq \sqrt{3}\, x$. Hence $j(x, 1)$ increases with x when $x \geqq \frac{1}{\sqrt{3}}$. Thus $j(x, 1) \leqq j(3.5, 1)$ when $1 \leqq x \leqq 3.5$. We conclude, with the aid of (46), that

$$j(x, 1) < j(3.5, 1) + \pi^2 \frac{\operatorname{ch} \pi}{\operatorname{sh}^3 \pi} < 0$$

when $-2.5 \leqq x \leqq 0$, and we can obviously draw the same conclusion for the larger interval $-3 \leqq x \leqq 0$. Hence, $|y| < 1$ on j_1, j_2 and j_3. We now use (48) with $m=2$, viz.

$$4j(2x, 2y) = j(x, y) + j\left(x+\frac{1}{2}, y\right)$$

and set $-3 \leqq x \leqq 0$ and $y = 1$. It follows that $|y| < 2$ on j_4, j_5 and j_6. Repeating the argument we conclude successively that $|y| < 4$ on $j_7 - j_{12}$, $|y| < 8$ on $j_{13} - j_{24}$, and so on. These limits for $|y|$ on j_n are probably not very good for large values of n; they could be improved upon, but the task is rather laborious.

We now turn our attention to the curves $r = 0$. We have already discussed in § 5 the branches of this curve in the right half-plane. Two arcs of $r = 0$ start at $z = -n$ in the interior of j_n. These arcs cannot remain inside of j_n; if they did so, they would have to intersect on the real axis forming a closed curve which, however, is impossible since $\psi'(z)$ does not have any real zeros. Hence these two arcs have to intersect j_n and obviously at $z = z_n$ and $\bar{z}_n$ where $\psi'(z) = 0$. These two arcs must pass through $z = -n-1$ since there is no other place where they can intersect the line $x = -n-1$, in view of (56), and they cannot wander off to the point at infinity.

We refer the reader to Fig. 2, which gives a schematic representation of the curves $r = 0$ (drawn in full lines) and $j = 0$ (drawn in dotted lines).

It is possible to find limits for the curves $r = 0$ which are somewhat more satisfactory than those found for the j-curves. Let r_n denote that arc of $r = 0$ on which $-n \leqq x \leqq -n+1$ $(n \geqq 1)$ and $y > 0$. We notice first that r_n expands with n just as j_n does. If $z+1$ lies on r_n, then $r(x+1, y) = 0$ and

$$r(x, y) = \frac{x^2 - y^2}{(x^2 + y^2)^2} > 0$$

provided $|y| < -x$. That this proviso is verified follows from formulas (62) below. Hence, we are justified in concluding that the point z lies inside of r_{n+1}, if z does not coincide with one of the poles.

It follows from (41) that $r(x, y) > 0$ if all the following inequalities are simultaneously fulfilled:

$$(60) \qquad (x+n)^2 - y^2 \geqq 0, \quad n = 0, 1, 2, \ldots$$

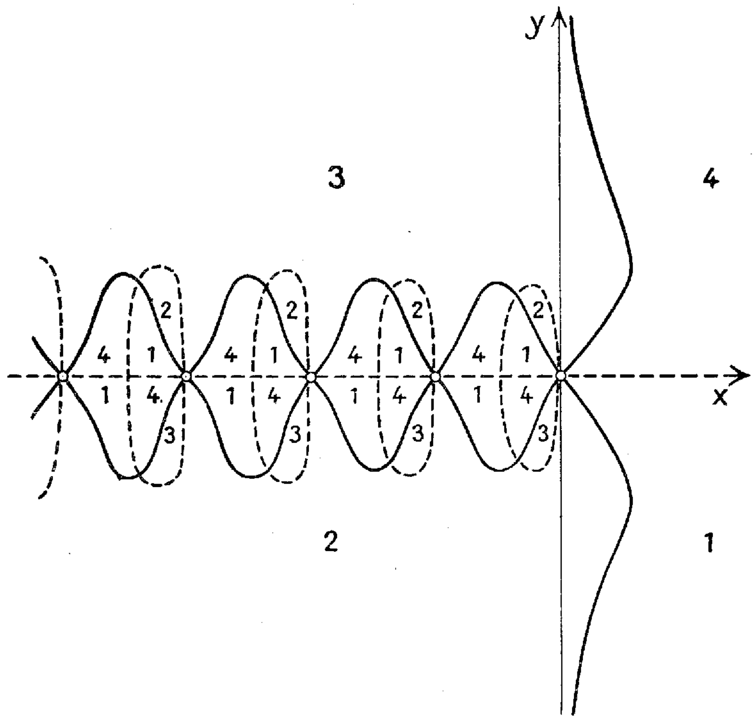

Fig. 2.

These inequalities determine a sector of opening $\frac{\pi}{2}$ in the right half-plane, and, in addition, a set of squares in the left half-plane each square having a line segment $(-n-1, -n)$ as one of its diagonals. Thus r_n lies above the polygonal line joining $z = -n, \; -n + \frac{1}{2} + \frac{i}{2}$ and $-n+1$.

A partial limitation of r_n from above can be found with the aid of (45). If $x \leq 0$, $r(1-x, y) > 0$. Hence,

$r(x, y)$ will be negative when $x \leqq 0$ and $\sin^2 \pi x \operatorname{ch}^2 \pi y - \cos^2 \pi x \operatorname{sh}^2 \pi y \leqq 0$, or

(61) $\quad r(x, y) < 0$ when $x \leqq 0$ and $\tan^2 \pi x \leqq \operatorname{th}^2 \pi y$.

This inequality implies that r_n lies below the corresponding arcs of the curve

$$\tan^2 \pi x = \operatorname{th}^2 \pi y \quad \text{or} \quad y = \frac{1}{2\pi} \log \tan \pi \left(\frac{1}{4} \pm x\right).$$

This curve consists of infinitely many arcs, passing in pairs through the points $z = -n$ where they have slopes equal to ± 1, and having the lines $x = -n \pm \frac{1}{4}$ as asymptotes. This method of course does not give any upper bound for r_n in the interval $-n + \frac{1}{4} \leqq x \leqq -n + \frac{3}{4}$.

In order to fill this gap we use the same method as above for j_n. We have

$$-\frac{\pi^2}{\operatorname{sh}^2 \pi y} \leqq \pi^2 \frac{\sin^2 \pi x \operatorname{ch}^2 \pi y - \cos^2 \pi x \operatorname{sh}^2 \pi y}{[\sin^2 \pi x + \operatorname{sh}^2 \pi y]^2} \leqq \frac{\pi^2}{\operatorname{ch}^2 \pi y},$$

when y is fixed. Let us set $y = +1$ and vary x on the interval $(-k, -1)$ where k is a positive integer which will be chosen below. Then

$$r(x, 1) \leqq -\min r(1 - x, 1) + \frac{\pi^2}{\operatorname{ch}^2 \pi}.$$

We have

$$\frac{\partial r}{\partial x} = 2 \sum_{n=0}^{\infty}{}' (x + n) \frac{3y^2 - (x+n)^2}{[y^2 + (x+n)^2]^3} < 0$$

if $0 \leqq \sqrt{3}\; y \leqq x$. Thus $r(1 - x, 1) \geqq r(1 + k, 1)$ when $-k \leqq x \leqq -1$. Now

$$r(1 + k, 1) = \frac{1}{2} - \frac{\pi^2}{2 \operatorname{sh}^2 \pi} - \frac{3}{25} - \frac{8}{100} - \cdots\cdots - \frac{k^2 - 1}{(k^2 + 1)^2},$$

whence

$$r(x, 1) \leqq \frac{\pi^2}{2\,\mathrm{sh}^2\,\pi} + \frac{\pi^2}{\mathrm{ch}^2\,\pi} + \frac{3}{25} + \frac{8}{100} + \cdots\cdots + \frac{k^2-1}{(k^2+1)^2} - \frac{1}{2}.$$

The expression on the right hand side is negative for $k \leqq 12$. Thus

(62 a) $\qquad r(x, 1) < 0 \quad \text{for} \quad -12 \leqq x \leqq 0.$

Using (47) with $m = 2$ we conclude that

(62 b) $\qquad r(x, 2) < 0 \quad \text{for} \quad -24 \leqq x < -12,$

(62 c) $\qquad r(x, 4) < 0 \quad \text{for} \quad -48 \leqq x < -24$

and so on. These estimates are probably rather crude, but they seem to justify the conclusion that the maximum ordinate on r_n grows considerably slower with n than the maximum ordinate on j_n.

The curves $r = 0$ and $j = 0$ divide the z-plane into an infinity of regions. Four of these are infinite in extent, all the others are finite. All the finite regions and the infinite ones in the right half-plane are mapped conformally and without overlapping upon a complete quadrant of the w-plane by the transformation $w = \psi'(z)$. The numbers plotted in the different regions of the figure indicate which quadrant corresponds to the region in question. The other infinite regions are mapped, the upper one upon the third and the lower one upon the second quadrant, but the map is not conformal and overlaps itself infinitely often since the regions under consideration contain all the complex zeros of $\psi''(z)$.

In order to build up the corresponding Riemann surface we can proceed as in § 17 below. To carry through the discussion properly would, however, require rather elaborate considerations so we restrict ourselves to these indications.

9. Lower limitation of the zeros of $\psi'(z)$. We shall now proceed to a further delimitation of the zeros of $\psi'(z)$. The inequalities obtained for $r(x, y)$ and $j(x, y)$ in the preceding paragraph give upper limits for y_n, the ordinate of the zero z_n of $\psi'(z)$ in the cell C_n. In particular, formulas (62) imply that

$$y_n < \left[\frac{n-1}{12}\right] + 1, \quad n = 1, 2, 3, \ldots \tag{63}$$

where as usual $[u]$ denotes the largest integer less than or equal to u. This estimate is of course rather unsatisfactory for large values of n, but shows nevertheless that y_n grows rather slowly.

A lower limit for y_n can be obtained with the aid of formulas (38) and (49). It follows from (49) that, when $\Re(z) > 1$,

$$|z\,\psi'(z)| \leqq \sum_{n=0}^{\infty} \frac{n!}{(n+1)\,|z+1|\ldots|z+n|} < \sum_{n=0}^{\infty} \frac{1}{(n+1)^2} = \frac{\pi^2}{6},$$

or

$$|(1-z)\,\psi'(1-z)| < \frac{\pi^2}{6} \quad \text{when } \Re(z) < 0. \tag{64}$$

Similarly

$$\left|\psi'(1-z) - \frac{1}{1-z}\right| < \frac{\frac{\pi^2}{3} - 2}{|1-z|\,|2-z|}. \tag{65}$$

In virtue of (38) we have that $\psi'(z) \neq 0$ if

$$\left|\frac{\pi^2}{\sin^2 \pi z}\right| > |\psi'(1-z)|,$$

and, using (64), we see that this is a fortiori the case when

$$\left|\frac{\pi^2}{\sin^2 \pi z}\right| \geq \frac{\pi^2}{6\,|1-z|},$$

or

$$|\sin^2 \pi z| \leqq 6\,|1-z|. \tag{66}$$

Thus the two branches of the curve C

$$\sin^2 \pi x + \mathrm{sh}^2 \pi y = 6\sqrt{(x-1)^2 + y^2} \tag{67}$$

for $x \leqq 0$, together with a connecting segment on the imaginary axis, bound a simply-connected region R such that $\psi'(z) \neq 0$ on $R + C$. A fairly simple reckoning shows that $\dfrac{dy}{dx} < 0$ on the upper half of C, i. e., y decreases when x increases.

We can now obtain a lower limit for y_n as follows. Evidently y_n exceeds the ordinate of the point on C whose abscissa is $-n+1$; this ordinate is determined as the real positive root of the equation

$$\mathrm{sh}\,\pi y = \sqrt{6}\,\sqrt[4]{n^2 + y^2}.$$

This equation implies that

$$\mathrm{sh}\,\pi y > \sqrt{6n},$$

or

$$y > \frac{1}{\pi} \log \left[\sqrt{6n} + \sqrt{6n+1}\right] > \frac{1}{\pi} \log 2\sqrt{6n}.$$

Hence

$$y_n > \frac{1}{\pi} \log 2\sqrt{6n}. \tag{68}$$

In particular, $y_1 > 0.5$. For small values of n, formulas (63) and (68) give comparatively narrow limits for y_n.

10. The asymptotic distribution of the zeros of $\psi'(z)$. We shall now take up the asymptotic distribution of the zeros. We introduce the function

$$\Phi(z) = \frac{\pi^2}{\sin^2 \pi z} - \frac{1}{1-z}, \tag{69}$$

and proceed to prove the following

Theorem: $\Phi(z)$ has exactly one zero in each of the cells C_{2n-1} and $\overline{C}_{2n-1}$ and no zeros in the cells C_{2n} and $\overline{C}_{2n}$. If we denote the zero in C_{2n-1} by ζ_n and set $\zeta_n = \xi_n + i\eta_n$, then $-n+\frac{1}{2} < \zeta_n < -n+\frac{3}{4}$, and

$$(70)\left\{\begin{aligned} \zeta_n = -n+\frac{1}{2}+\frac{\log 2\pi\sqrt{n+\frac{1}{2}}}{2\pi^2\left(n+\frac{1}{2}\right)}+ \\ +\frac{i}{\pi}\left[\log 2\pi\sqrt{n+\frac{1}{2}}-\frac{1}{4\pi^2\left(n+\frac{1}{2}\right)}\right]+O\left(\frac{\log^2 n}{n^2}\right). \end{aligned}\right.$$

Let ζ_n be the center of a circle Γ_n of radius $\frac{3}{n+1}$. Then each circle Γ_n with $n \geqq 11$ contains one and only one zero of $\psi'(z)$.

We postpone the proof of formula (70) until the rest of the theorem has been proved. We readily verify that

$$\pi < \arg \Phi(-n+iy) < \frac{3\pi}{2},$$

$$\Im\, \Phi\left(-n-\frac{1}{2}+iy\right) < 0,$$

when $y > 0$ and $n = 0, 1, 2, \ldots$ Further $\Phi(x) > 0$ for x real and negative. These relations are exactly the same as those satisfied by $\psi'(z)$ on the lines in question; they permit us to repeat the proof given in § 7 with $\psi'(z)$ replaced by $\Phi(z)$; this suffices to prove the statement about the cells.

To verify that $-n+\frac{1}{2} < \xi_n < -n+\frac{3}{4}$ we notice that

$$\Re\left[\sin^2\pi(x+iy)\right] = \frac{1}{2}(1-\cos 2\pi x \operatorname{ch} 2\pi y) \leqq \frac{1}{2}$$

if $\cos 2\pi x \geqq 0$, i. e., if $-n+\frac{3}{4} \leqq x \leqq -n+\frac{5}{4}$. On the other hand,

$$\Re\left[\pi^2(1-z)\right] = \pi^2(1-x) \geqq \pi^2$$

when $x \leqq 0$. Hence ξ_n must be limited in the way just mentioned.

Let $\zeta = \xi + i\eta$ $(\xi < 0,\ \eta > 0)$ be an arbitrary zero of $\Phi(z)$. We shall study $\Phi(\zeta+\omega) = \underset{\omega}{\Delta}\Phi(\zeta)$ when $|\omega| = r$, a fixed number. We have

$$\begin{aligned}\Phi(\zeta+\omega) &= \pi^2 \frac{\sin^2 \pi\zeta - \sin^2 \pi(\zeta+\omega)}{\sin^2 \pi\zeta \sin^2 \pi(\zeta+\omega)} + \frac{\omega}{(1-\zeta)(1-\zeta-\omega)} \\ &= -\frac{\omega}{1-\zeta}\left[\frac{\sin \pi\omega}{\omega}\,\frac{\sin \pi(2\zeta+\omega)}{\sin^2 \pi(\zeta+\omega)} - \frac{1}{1-\zeta-\omega}\right].\end{aligned}$$

We now assume $r \leqq \frac{1}{4}$. Then

$$|\Phi(\zeta+\omega)| \geqq \left|\frac{\omega}{1-\zeta}\right| \left[2\sqrt{2}\,\frac{\operatorname{sh}\pi(2\eta-r)}{\operatorname{ch}^2 \pi(\eta-r)} - \frac{1}{|1-\zeta|-r}\right].$$

The fraction involving the hyperbolic functions increases steadily with η when r is fixed, and decreases when r increases if η is fixed $> \frac{r}{2}$. Thus the fraction will be made as small as possible if we give η its least value and take $r = \frac{1}{4}$.

In order to obtain a suitable lower limit for the bracket we set $\zeta = \zeta_n$ with $n \geqq 11$. This implies $|1-\zeta|-r > 11$, and, since $\eta_n > \frac{1}{\pi}\log 2\pi\sqrt{n}$,

$$\eta > \eta_{11} > \frac{1}{\pi}\log 2\pi\sqrt{11} > 0.96.\text{[1]}$$

[1] In order to obtain this estimate, use the same type of argument as in the proof of formula (68).

With these restrictions upon ζ and ω we find that

$$(71) \qquad |\Phi(\zeta+\omega)| \geqq \frac{K|\omega|}{|1-\zeta|} \text{ where } K > 0.445.$$ [1]

In view of formulas (38) and (65) we have

$$(72)\ \psi'(z) = \Phi(z) + P(z) \text{ where } |P(z)| < \frac{\frac{\pi^2}{3}-2}{|(1-z)(2-z)|},$$

when $\Re(z) < 0$. With each of the points ζ_n, $n \geqq 11$, as center we lay a circle Γ_n of radius r_n. We shall determine r_n in such a manner that

$$(73) \qquad |\Phi(z)| > |P(z)| \quad \text{on} \quad \Gamma_n,$$

and impose in advance the condition $r_n \leqq \frac{1}{4}$. Setting $z = \zeta_n + \omega_n$ $(|\omega_n| = r_n)$ on Γ_n we have from (65) and (71)

$$|\Phi(\zeta_n+\omega_n)| \geqq \frac{K r_n}{|1-\zeta_n|},$$

$$|P(\zeta_n+\omega_n)| < \frac{\frac{\pi^2}{3}-2}{|(1-\zeta_n-\omega_n)(2-\zeta_n-\omega_n)|}.$$

Thus (73) will be fulfilled if

$$r_n \leqq \frac{\frac{\pi^2}{3}-2}{K} \frac{|1-\zeta_n|}{|1-\zeta_n|-r_n} \frac{1}{|2-\zeta_n|-r_n}.$$

We now use the assumptions $n \geqq 11$, $r_n \leqq \frac{1}{4}$ together with the fact that $K > 0.445$. These premises imply that $|1-\zeta_n| > 11.25$, $|2-\zeta_n|-r_n > n+1$, and

[1] By imposing more severe restrictions upon ζ and ω we can get K as near to 2π as we please.

$$\frac{\frac{\pi^2}{3}-2}{K}\,\frac{|1-\zeta_n|}{|1-\zeta_n|-r_n} < 2.97 < 3.$$

Consequently (73) will hold for $n \geqq 11$ when $r_n = \dfrac{3}{n+1}$. But then it follows from the theorem of Rouché that each circle Γ_n contains one and only one zero of $\psi'(z)$.

It remains to prove formula (70). We begin by determining a set of numbers $\zeta_{m,n}$ satisfying the following conditions

$$(74)\quad \begin{cases} \sin \pi \zeta_{m+1,n} = \pi \sqrt{1-\zeta_{m,n}}, \quad \zeta_{1,n} = -n+\dfrac{1}{2}, \\ -n < \Re(\zeta_{m,n}) < -n+1, \quad \Im(\zeta_{m,n}) > 0. \end{cases}$$

Here $m, n = 1, 2, 3, \ldots$ and $\sqrt{1-z}$ means that determination of the square root which equals to $+1$ when $z=0$. We have

$$\zeta_{2,n} = -n+\frac{1}{2}+\frac{i}{\pi}\log\left[\pi\sqrt{n+\frac{1}{2}}+\pi\sqrt{n+\frac{1}{2}-\frac{1}{\pi^2}}\right],$$

$$(75)\quad \begin{cases} \zeta_{3,n} = -n+\dfrac{1}{2}+\dfrac{\log 2\pi\sqrt{n+\frac{1}{2}}}{2\pi^2\left(n+\frac{1}{2}\right)}+ \\ +\dfrac{i}{\pi}\left[\log 2\pi\sqrt{n+\dfrac{1}{2}}-\dfrac{1}{4\pi^2\left(n+\frac{1}{2}\right)}\right]+O\left(\dfrac{\log^2 n}{n^2}\right), \end{cases}$$

. . . .

We easily verify that

$$\sin\frac{\pi}{2}(\zeta_n-\zeta_{m,n}) = -\frac{\pi\,[\zeta_n-\zeta_{m-1,n}]}{2\cos\frac{\pi}{2}(\zeta_n+\zeta_{m,n})\left[\sqrt{1-\zeta_n}+\sqrt{1-\zeta_{m-1,n}}\right]}.$$

For $m=2$ we have

$$|\zeta_n - \zeta_{1,n}| < c_1 \log n, \quad \left|\cos\frac{\pi}{2}(\zeta_n - \zeta_{2,n})\right| > c_2 \sqrt{n},$$

$$\left|\sqrt{1-\zeta_n} + \sqrt{1-\zeta_{1,n}}\right| > c_3 \sqrt{n},$$

where the c's are positive constants independent of n. Hence

$$\left|\sin\frac{\pi}{2}(\zeta_n - \zeta_{2,n})\right| < \frac{c_1}{2\,c_2\,c_3} \cdot \frac{\log n}{n}$$

and

$$|\zeta_n - \zeta_{2,n}| < C_1 \frac{\log n}{n}.$$

Repeating the argument with $m = 3$ we see that

$$|\zeta_n - \zeta_{3,n}| < C_2 \frac{\log n}{n^2}. \tag{76}$$

Combining formulas (75) and (76) we get formula (70).

11. **The zeros of $\psi''(z)$.** In §§ 5 and 7 we made certain statements regarding the zeros of $\psi''(z)$. We shall now prove the following

Theorem: $\psi''(z)$ has exactly three zeros in each of the strips $-n+\frac{1}{2} \leqq x \leqq -n+1$ $(n = 1, 2, 3, \ldots)$ of which one and only one is real. There are no other zeros.

We begin by proving that

$$\operatorname{sgn} \Im\left[\psi''(-n+iy)\right] = -\operatorname{sgn} y \tag{77}$$

for $n = 0, 1, 2, \ldots$ We have from (53) and (55)

$$\Im\left[\psi''(-n+iy)\right] = -\frac{\partial}{\partial y} r(-n, y)$$

$$= 2y \sum_{m=1}^{n} \frac{3m^2 - y^2}{(m^2+y^2)^3} - \frac{1}{y^3} - \pi^3 \frac{\operatorname{ch} \pi y}{\operatorname{sh}^3 \pi y},$$

where the finite sum is to be suppressed when $n = 0$. This expression clearly has opposite sign to that of y when $|y| \geqq n\sqrt{3}$. If $|y| < n\sqrt{3}$ we have

$$\frac{1}{y}\Im\left[\psi''(-n+iy)\right] < 2\sum_{m=1}^{\infty}\frac{3m^2-y^2}{(m^2+y^2)^3} - \frac{1}{y}\left[\frac{1}{y^3}+\pi^3\frac{\operatorname{ch}\pi y}{\operatorname{sh}^3\pi y}\right]$$

$$= -\frac{2\pi^3}{y}\frac{\operatorname{ch}\pi y}{\operatorname{sh}^3\pi y} < 0.$$

This completes the proof.

We shall now prove that the variation of the argument of $\psi''(z)$ is zero when z describes the perimeter of a large square with vertices at the points $n(\pm 1 \pm i)$ avoiding the point $z = -n$ by a small semi-circle to the right of this point. This contour contains n triple poles of $\psi''(z)$, further at least $3n$ zeros, namely, at least three in each of the strips $-m+\frac{1}{2} < x < -m+1$, $m = 1, 2, 3, \ldots, n$, as we have seen in § 7. If we can prove the statement about the variation of the argument then the theorem follows immediately.

In the neighborhood of $z = \infty$ in the sector $|\arg z| \leqq \pi - \varepsilon$ we have

$$\psi''(z) = -\frac{2}{z^2} + O\left(\frac{1}{z^3}\right).$$

Now let us start with z at $+n$ and describe the contour in the positive sense. Then $w = \psi''(z)$ starts with a small negative value, and its argument decreases from π to approximately $-\frac{\pi}{2}$ when z goes from $+n$ to $n(-1+i)$. When z goes from $n(-1+i)$ to $-n+\varepsilon i$, w remains in the lower half-plane in view of (77), and when $z = -n + i\varepsilon$ $|w|$ is large and $\arg w$ is nearly $-\frac{\pi}{2}$ since

$$\psi''(z) = -\frac{2}{(z+n)^3} + \mathfrak{P}(z+n).$$

When z describes the circular arc $|z+n| = \varepsilon$, $0 \leqq \arg(z+n) \leqq \frac{\pi}{2}$, $|w|$ remains large and $\arg w$ increases from $-\frac{\pi}{2}$ to $+\pi$. Consequently $\arg w$ is back to its initial value after we have described the upper half of the contour, and by reasons of symmetry, $\arg w$ will return to the initial value after we have described the lower half of the contour. This completes the proof of the theorem.

Part III.

The conformal correspondence $w = \psi(z)$.

12. The R, I-net. We shall now return to the psi-function itself, and consider the question of how its values are distributed in the plane. We shall attack this problem from two different angles. First, we have obtained in Parts I and II of the present paper a variety of results which permit us to give a rather detailed discussion of the curves

$$\Re\left[\psi(z)\right] = \text{const., and } \Im\left[\psi(z)\right] = \text{const.}$$

We shall give this discussion in §§ 12—15. Secondly, we try to complete the information so obtained by numerical computation of the psi-function for some values of z. Finally, in § 17 we discuss the Riemann surface corresponding to $w = \psi(z)$ in the light of the results obtained in §§ 12—16.

For the whole discussion the reader should consult Fig. 3, which gives a representation of the curves in question. In the upper half-plane the curves

$R(x, y) = c$, with $c = -2, -1.5, -1, \ldots, +1.5$ and $+2$, are traced; in the lower half-plane we have marked the curves

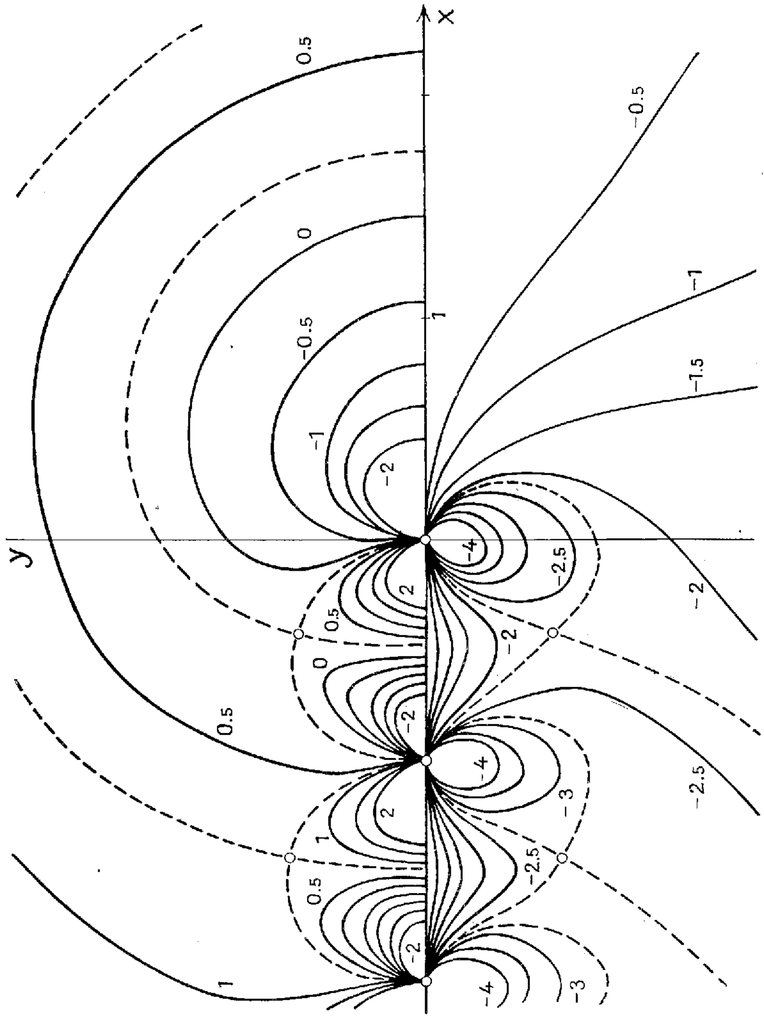

Fig. 3.

$I(x, y) = \gamma$ with $\gamma = -4, -3.5, -3, \ldots, -0.5$ and 0. In addition we have plotted in dotted lines the curves of the two systems which pass through the four zeros of $\psi'(z)$

which are nearest to the origin. The diagram is based upon the results of §§ 12—16, and is believed to give a fairly accurate picture of the situation, but, naturally, it must not be trusted too far.

In the sector $|\arg z| \leqq \pi - \varepsilon$ we have

$$\psi(z) = \log z + O\left(\frac{1}{z}\right).$$

It follows that within this sector and sufficiently far from the origin, the curves $R = c$ correspond to large positive values of c and each curve lies between two circles $|z| = e^c - \delta$ and $|z| = e^c + \delta$. The curves $I = \gamma$ on the other hand, correspond to values of γ between $-\pi + \varepsilon$ and $\pi - \varepsilon$ and are asymptotic to the lines $\arg z = \gamma$.

In the remaining sector we have

$$\psi(z) = \log(1 - z) - \pi \cot \pi z + O\left(\frac{1}{z}\right).$$

Here we have evidently quite a complicated situation; the net corresponding to $\log(1 - z)$ is distorted by the superimposed net due to $-\pi \cot \pi z$.

The points $z = -n$ $(n \geqq 0)$ are simple poles of residue -1 for $\psi(z)$. Let N_ε be a small neighborhood of $z = -n$. Any R-curve in N_ε will pass through $z = -n$ where it will have a vertical tangent. If c is sufficiently large positive (negative) the curve $R = c$ will be closed in N_ε and located to the left (right) of the vertical tangent; further it will be almost circular in shape. Any I-curve in N_ε will pass through $z = -n$ and be tangent to the x-axis. If γ is sufficiently large numerically, the curve $I = \gamma$ will be closed in N_ε and almost circular; it will be above or below the x-axis according as γ is positive or negative. The curves of the two nets have a perfectly definite order in N_ε. Thus, for example, if we describe the upper half of the curve $R = -M$ (M large positive) in N_ε starting from $z = -n + \delta$ (δ real positive) and ending at $z = -n$, then $v = I(x, y)$ will be steadily growing along the curve from the initial value 0 to the final limit $+\infty$, every intermediate value being taken on once and only once. Similarly with the R-curves.

Any curve $R = c$ will consist of an infinity of separate branches, beginning and ending at $z = -n$, one branch for each pole. Any curve $I = \gamma$ will consist of an infinity of branches, which, however, may and as a rule do have end-points in common. Such a branch will join a pole either with itself or with another pole or with the point at infinity.

Through the points z_n where $\psi'(z) = 0$ will pass two and only two branches of each system. If we set

$$\psi(z_n) = w_n = u_n + i v_n, \tag{78}$$

it is two branches of the curve $R = u_n$ and two branches of $I = v_n$ which pass through z_n. These curves are of fundamental importance for the whole discussion and will be considered at length in §§ 14 and 15. No other curve of either system can intersect itself or have a non-singular point in common with any other curve belonging to the same system.

We have discussed the curve $R = 0$ in some detail in § 3. This curve was found to consist of infinitely many separate ovals R_n, one for each pole $z = -n$, $n \geqq 0$, all being outside each other in accordance with the inequalities (30) and (31). Indeed, these inequalities prove the existence in every strip $-n-1 < x < -n$ ($n \geqq 0$) of a substrip where $R(x, y) > 0$. Further, formulas (25) and (35) prove that the oval R_n contracts indefinitely to zero when $n \to +\infty$, and we have shown at the end of § 3 that this contraction process is monotone in a perfectly definite sense.

Let us now turn to a curve $R = c < 0$. This curve clearly consists of separate ovals $R_n(c)$, namely, one and only one oval inside each oval R_n ($n = 0, 1, 2, \ldots$). Thus the ovals $R_n(c)$ are outside of each other when $c \leqq 0$. They will contract indefinitely when $n \to +\infty$ and the process can be shown to be steady or monotone in the sense above mentioned. The same conclusions will hold for sufficiently small positive values of c, but will cease to hold when c is large. Let $R_n(c)$ still denote that branch of $R = c$ which goes through $z = -n$. If c is large we can no longer affirm that the $R_n(c)$ are all outside of each other, but they will have this property for sufficiently large values of n, i. e., if we disregard a finite number of the branches the remaining ones will be outside of each other and of the disregarded branches[1]. Our previous conclusions are valid for the residual infinite set.

13. Differential properties of the net. Now we turn our attention to questions of increase and direction. We have

$$r(x, y) = \frac{\partial}{\partial x} R(x, y) = \frac{\partial}{\partial y} I(x, y),$$

$$j(x, y) = -\frac{\partial}{\partial y} R(x, y) = \frac{\partial}{\partial x} I(x, y).$$

In a region where $r(x, y) > 0$, $R(x, y)$ increases with x and $I(x, y)$ increases with y. In a region where $j(x, y) > 0$,

[1] In order that the $R_n(c)$ be outside of each other for $n \geqq m$ it is necessary and sufficient that $c \leqq u_m$. This follows from the results stated at the end of § 14.

$R(x, y)$ decreases when y increases and $I(x, y)$ increases with y. At a point $z = z_0 = x_0 + iy_0$ where $r(x, y)$ and $j(x, y)$ have the same (opposite) sign, the slope of $R(x, y) = c$ is positive (negative) and the slope of $I(x, y) = \gamma$ is the negative reciprocal of the slope of the R-curve. Let us define as positive direction of the tangent of $R(x, y) = c$ at z_0 that direction in which $I(x, y)$ increases, with a similar definition for the I-curve. This direction is uniquely defined unless z_0 happens to be a zero or a pole of $\psi'(z)$. Let $\varphi_1(z_0)$ be the angle which the positive direction of the tangent of $\Re[\psi(z)] = \Re[\psi(z_0)]$ at $z = z_0$ makes with the positive direction of the real axis, the angle being measured from the axis to the tangent; and let $\varphi_2(z_0)$ be the corresponding angle for the I-curve $\Im[\psi(z)] = \Im[\psi(z_0)]$. Then we have

$$(79) \qquad \varphi_1(z_0) \equiv \frac{\pi}{2} - \arg \psi'(z_0) \qquad (\text{mod. } 2\pi),$$

$$(80) \qquad \varphi_2(z_0) \equiv \quad - \arg \psi'(z_0) \qquad (\text{mod. } 2\pi).$$

Fig. 2 suffices to give us a general notion of the mode of variation of $\arg \psi'(z)$. This figure, it will be remembered, is based upon the discussion of $\psi'(z)$ in §§ 5—8. It will perhaps be useful to collect at this point some of the consequences of this discussion.

It has been noticed that $j(x, y)$ is negative in all the cells C_{2n} and in the first quadrant and positive in the symmetric regions below the x-axis. This implies that $R(x, y)$ grows with $|y|$ in these regions when x is kept fixed. In particular, this will be the case on the boundary of any one of the cells, hence[1]

[1] In order to obtain the lower limits in (81) and (82) use formula (29).

$$R(-n, y) > \psi(n+1), \tag{81}$$

$$R\left(-n-\frac{1}{2}, y\right) \geqq \psi\left(n+\frac{3}{2}\right). \qquad (n = 0, 1, 2, \ldots) \tag{82}$$

In that part of C_{2n} which lies above r_n, we have $\pi < \varphi_1(z_0) < \frac{3\pi}{2}$ and $\frac{\pi}{2} < \varphi_2(z_0) < \pi$; below r_n we have $\frac{\pi}{2} < \varphi_1(z_0) < \pi$ and $0 < \varphi_2(z_0) < \frac{\pi}{2}$.

The R-curves have vertical tangents on j_n, horizontal ones on r_n and $\bar{r}_n$[1]. For the I-curves the situation is of course reversed. Finally, we notice that any vertical line which does not intersect any of the curves j_n, will either intersect an arbitrary curve $R = c$ in two points symmetric to the x-axis or not at all.

14. Qualitative description of the net. We shall now take up the properties of the net in the gross. We aim at a qualitative description of the net which will tell us how the separate branches of the different curves go, what singular points they join, how they separate the plane into regions, and so on. We shall see that the solution of this problem depends essentially upon a special case of the same problem, namely how the critical curves through the zeros of $\psi'(z)$ behave in this or that respect.

We begin by considering the I-curves. Let us inspect the branches of the I-curves which radiate from $z = -n$ $(n \geqq 1)$. One of these curves is the real axis. Now give γ a small positive value. We conclude by reasons of continuity that there is a branch of the curve $I = \gamma$ which joins $z = -n$ with $z = -n+1$ and which lies entirely within a rectangle $-n \leqq x \leqq -n+1$, $0 \leqq y \leqq \delta(\gamma)$ where

[1] We denote the arc of $r = 0$ which lies in the first quadrant by r_0 and let $\bar{r}_n$ mean the curve symmetric to r_n in the lower half-plane $(n = 0, 1, 2, \ldots)$.

$\delta(\gamma) \to 0$ with γ. There is also a branch of the same curve which joins $z = -n$ with $z = -n-1$, but we disregard this arc for the present. Let γ_n be the largest value of γ such that for $\gamma \leqq \gamma_n$ the curve $I = \gamma$ has a branch $I_n(\gamma)$ joining $z = -n$ with $z = -n+1$ without passing through any other singular point. I claim that $I_n(\gamma_n)$ goes through a zero of $\psi'(z)$; to be more specific, I assert that $I_n(\gamma_n)$ goes through $z = z_n$, i. e. $\gamma_n = v_n$. Suppose this were not so and consider that arc of the curve $I = \gamma_n + \delta$ (δ small positive) which starts at $z = -n$ and on which $x+n$ is small positive when y is small positive. Since $\psi'(z) \neq 0$ on $I_n(\gamma_n)$, this arc will be uniformly near to $I_n(\gamma_n)$, i. e., we can find an $\varepsilon = \varepsilon(\delta)$ which tends to zero with δ, such that the distance between the two curves nowhere exceeds ε[1]. But then this branch of $I = \gamma_n + \delta$ must end at $z = -n+1$, which is contrary to the definition of γ_n.

Thus $I_n(\gamma_n)$ goes through a zero of $\psi'(z)$. Suppose that this zero were not z_n. Then $I_n(\gamma_n)$ which joins $z = -n$ with $z = -n+1$, must intersect either the line $x = -n$ or the line $x = -n+1$ in two distinct points with positive ordinates. This, however, is impossible since $I(-m, y)$ is steadily decreasing when y increases, m being a positive integer or 0, in accordance with (56). Hence $I_n(\gamma_n)$ passes

[1] That this is actually the case follows from the following consideration. Leaving out two small arcs at the end-points of $I_n(\gamma_n)$ we can cover the residual arc by a finite number of circles such that: (i) every point on the arc is interior to at least one of the circles, and (ii) the interior of any one of the circles is mapped conformally and without overlapping upon a region in the w-plane by the transformation $w = \psi(z)$. The image of the set of points which belong to at least one of these circles is simply-connected and contains a segment of the line $v = \gamma_n$, hence also a segment of $v = \gamma_n + \delta$ if δ is sufficiently small. This proves the assertion except near the end-points of $I_n(\gamma_n)$. But these do not cause any difficulties since the curves under consideration are tangent to each other at these points. This completes the proof.

through z_n and does not go through any other zero of $\psi'(z)$. Further, $\gamma_n = v_n$. Incidentally we notice that $I_n(v_n)$ does not intersect the lines $x = -n$ or $x = -n+1$ except at the end-points.

Now let $z+1$ trace the arc $I_n(v_n)$ from $-n$ to $-n+1$. Then z traces an arc I_n^* from $-n-1$ to $-n$. On this latter arc $I(x, y) > v_n$ except at the end-points where equality holds, as we see from formula (10). Let D_n^* be the region bounded by I_n^* and the real axis between $-n-1$ and $-n$. The point z_{n+1} may be located (i) within D_n^*, or (ii) on I_n^*, or (iii) outside of D_n^*. Whichever be the actual case, we shall prove that $v_n < v_{n+1}$.

Suppose case (i) be realized, and consider the four arcs of the curve $I = v_{n+1}$ which start at $z = z_{n+1}$. We know that two of these arcs form the branch $I_{n+1}(v_{n+1})$ with end-points at $z = -n-1$ and $-n$. The other two arcs cannot lie completely within D_n^*. If they did, we should have two distinct arcs $I = v_{n+1}$ the ends of which would belong to a small sector $|z+n| < \delta$, $0 < \arg(z+n) < \varepsilon$; this is clearly impossible in view of the order relations between the I-curves in the neighborhood of a pole. Hence these two arcs must intersect I_n^* at a point where $I(x, y) > v_n$, and thus $v_{n+1} > v_n$. In cases (ii) and (iii) we see almost directly that the same conclusion is valid.

Let us now study the I-curves which emanate from $z = 0$ and of which the initial arcs belong to the first quadrant. Such a branch of the curve $I = \gamma$ will be designated by $I_0(\gamma)$. As long as $0 \leqq \gamma \leqq \frac{\pi}{2}$, $I_0(\gamma)$ will remain in the first quadrant and go from $z = 0$ to $z = \infty$, having the line $\arg z = \gamma$ as its asymptote. That $I_0(\gamma)$ cannot intersect the positive imaginary axis follows from formulas (16) and (56), which imply that

(83) $\quad I(-n, y) > \frac{\pi}{2}$ when $y > 0$, $n = 0, 1, 2, \ldots$

When $\gamma > \frac{\pi}{2}$ the arc $I_0(\gamma)$ intersects the imaginary axis and proceeds to the point at infinity as long as $\gamma - \frac{\pi}{2}$ is sufficiently small. There exists a largest value, Γ_0 say, such that $I_0(\gamma)$ ends at infinity for every $\gamma \leqq \Gamma_0$. Just as above we prove that the curve $I_0(\Gamma_0)$ must pass through a zero of $\psi'(z)$, and this zero must be z_1. Suppose contrariwise that it would be z_2 instead. Then $\Gamma_0 = v_2$ and there exists an arc of an I-curve joining $z = 0$ with $z = -1$ on which $I = v_2$. This arc together with the segment of the real axis from 0 to -1 bounds a region D_1 which evidently contains the point z_1 in its interior. Moreover, the four arcs of the curve $I = v_1$ which meet at $z = z_1$ must be enclosed in D_1. But this is impossible since $v_2 > v_1$; indeed, if $\Gamma_0 = v_2 > v_1$ then $I_0(v_1)$ goes from $z = 0$ to $z = \infty$ entirely outside of D_1 in view of the definition of Γ_0. But there are only two arcs of $I = v_1$ which begin or end at $z = 0$ and only two such arcs which begin or end at $z = -1$; if one of the former arcs is outside of D_1, then there is at least one of the four arcs of $I = v_1$ starting at z_1 which does not end in the interior of D_1. We are thus led to a contradiction by assuming that $\Gamma_0 = v_2$; in exactly the same manner we disprove the assumption that $\Gamma_0 = v_n$, $n \neq 1$.

Hence $\Gamma_0 = v_1$. We can now account for all the I-curves which emanate from $z = 0$. As long as $\gamma < v_1$, $I_0(\gamma)$ goes from $z = 0$ to $z = \infty$; when $\gamma > v_1$, $I_0(\gamma)$ is a closed curve beginning and ending at the origin. The two types of curves are separated by arcs of $I = v_1$, namely, $I_0(v_1)$, which goes from $z = 0$ over z_1 to ∞, plus $I_1(v_1)$ which goes from $z = -1$ over z_1 to 0.

We now pass to the second pole at $z = -1$. We designate by $I_1(\gamma)$ that arc of the curve $I = \gamma$ which starts at $z = -1$ and on which $\arg(z+1)$ is small positive when $|z+1|$ is small. As long as $0 \leqq \gamma \leqq v_1$, $I_1(\gamma)$ goes from -1 to 0. Thus if we return for a moment to $z = 0$ we see that the I-curves in a small neighborhood of the origin, $|z| < \delta$, $y \geqq 0$, are either of the type $I_0(\gamma)$, $0 \leqq \gamma$, or of the type $I_1(\gamma)$, $0 \leqq \gamma \leqq v_1$. The former curves begin at $z = 0$, the latter ones end at this point according to our present convention, which is in agreement with our previous way of orienting the curves with the aid of the positive direction of the tangent.

When $0 < \gamma - v_1 < \varepsilon$, $I_1(\gamma)$ joins $z = -1$ with $z = \infty$. There exists a largest value, Γ_1 say, such that $I_1(\gamma)$ has this property for every γ, $v_1 < \gamma \leqq \Gamma_1$. As above we show that $\Gamma_1 = v_2$. Thus the branches $I_1(\gamma)$ join $z = -1$ with $z = \infty$ when $v_1 < \gamma \leqq v_2$, and when $\gamma > v_2$ they are closed curves beginning and ending at $z = -1$. The remaining I-curves which belong to the upper half of an ε-neighborhood of $z = -1$ are curves of the type $I_2(\gamma)$ with $0 \leqq \gamma \leqq v_2$ which start at $z = -2$ and end at $z = -1$. In this manner we can proceed step by step. The situation in the lower half-plane is symmetric to the situation just described.

We notice that

$$\frac{\pi}{2} < \Im\left[\psi(z_n)\right] < \Im\left[\psi(z_{n+1})\right] < \pi. \tag{84}$$

Here the lower limit $\frac{\pi}{2}$ could be raised somewhat; v_1 is certainly greater than 2, — on the other hand

$$(85)\quad v_1 < \operatorname{Max} I\left(-\frac{1}{2}, y\right) = \operatorname{Max}\left[\frac{\pi}{2} \operatorname{th} \pi y + \frac{4y}{1+4y^2}\right] < \frac{\pi}{2} + 1,$$

and in general

$$(86)\quad \begin{cases} v_n < \operatorname{Max} I\left(-n+\frac{1}{2}, y\right) \\ = \operatorname{Max}\left[\frac{\pi}{2} \operatorname{th} \pi y + 4y \sum_{m=1}^{n} \frac{1}{(2m-1)^2+y^2}\right]. \end{cases}$$

These limits are unfortunately not well suited for numerical estimates. That π is the true upper limit in (84) follows from the following consideration. Let ε be arbitrarily small positive and let $\pi - \varepsilon < v < \pi$. There is a unique I-curve which admits of the line $\arg z = v$ as its asymptote, this curve is a branch of $I = v$. We know that any such branch when traced in the negative sense will ultimately lead us to a pole. Suppose that our branch leads to $z = -m$. Then we are dealing with $I_m(v)$ according to the nomenclature adopted above. But if $I_m(\gamma)$ joins $z = -m$ with $z = \infty$ then $v_m < \gamma \leqq v_{m+1}$. Hence the same inequality has to be satisfied by v, i. e. $v_{m+1} > \pi - \varepsilon$.

We now proceed to discuss the fate of the R-curves which emanate from the different poles, and start with $z = 0$. The corresponding arcs of the R-system have been designated by $R_0(c)$ in § 12. As long as $c < -C$, $R_0(c)$ remains in the right half-plane and intersects the positive real axis between 0 and $+1$. There exists a largest value of c, c_1 say, such that all the ovals $R_0(c)$ intersect the positive real axis as long as $c \leqq c_1$. As above, we prove that $R_0(c_1)$ passes through a zero of $\psi'(z)$, and owing to symmetry it will have to pass through two conjugate imaginary zeros. The zero in the upper half-plane must be z_1, i. e. $c_1 = u_1$. Indeed, if $R_0(c_1)$ passed through any other zero but z_1, it would have to intersect the line $x = -\frac{1}{2}$ twice; this is impossible since $R(x, y)$ increases steadily with $|y|$ along this line.

Let the point of intersection of $R_0(u_1)$ with the positive real axis be denoted by P_1. We can find a point $z = p_1$ on the interval $(-1, 0)$ where $\psi(z) = u_1$. Through the latter point passes a branch of $R = u_1$. There is also a branch of the same curve which goes through $z = -1$. These branches must pass through $z = z_1$. In order to see that this is really true, we notice that there are four arcs of the curve $R = u_1$ which meet at $z = z_1$. Two of these have already been accounted for; one joins z_1 with P_1, the other joins z_1 with the origin. Let us follow the remaining two arcs away from $z < z_1$. None of these arcs can intersect the imaginary axis as there is already one arc of the curve $R = u_1$ which does so and $R(0, y)$ increases steadily with $|y|$. Further, none of the arcs in question can wander off to infinity or end at the origin. We are thus sure that one of these arcs will intersect the real axis between $-\frac{1}{2}$ and 0 and the other will intersect the line $x = -\frac{1}{2}$, $y > 0$. As $R(x, y)$ is monotone on both lines there cannot be more than one intersection on each. The arc which intersects the real axis clearly joins $z = z_1$ with $z = p_1$. It follows that

$$u_1 = \psi(p_1) > \psi\left(-\frac{1}{2}\right) = \psi\left(\frac{3}{2}\right) > 0. \tag{87}$$

The arc which intersects $x = -\frac{1}{2}$ remains. This arc will pass through $z = -1$ if we can prove that it cannot intersect the line $x = -1$ at a point of ordinate different from 0. It clearly cannot intersect the real axis between -1 and $-\frac{1}{2}$. In view of (81) it is sufficient for our purpose to prove that

$$u_1 < \psi(2). \tag{88}$$

Let us consider the rectangle whose vertices are $A = 0$, $B = i\eta$, $C = -\frac{1}{2} + i\eta$, $D = -\frac{1}{2}$, where $\eta\,(> 0)$ is to be suitably chosen. It is clear that the curve $R = u_1$ intersects the polygonal line $ABCD$ at least once. Hence $u_1 \leqq \text{Max}\; R(x, y)$ on $ABCD$. Now $R(x, y)$ is monotone increasing on AB and on DC. Hence $u_1 \leqq \text{Max}\; R(x, y)$ on CB. The latter maximum can be estimated with the aid of the methods which we have used in the latter half of § 3. In view of (11) and (34) we have

$$R(x, \eta) \leqq \text{Max}\; R(1-x, \eta) + \frac{\pi}{\operatorname{sh} 2\pi\eta},$$

when z lies on CB. Hence

$$u_1 \leqq R\left(\frac{3}{2}, \eta\right) + \frac{\pi}{\operatorname{sh} 2\pi\eta}, \tag{89}$$

no matter how η be chosen > 0. Now it will be proved in § 15 that it is always possible to find an η such that

$$R\left(n + \frac{1}{2}, \eta\right) + \frac{\pi}{\operatorname{sh} 2\pi\eta} < R(n+1, 0) = \psi(n+1). \tag{90}$$

Hence (88) is actually true[1].

We can now account for all the arcs $R_0(c)$. When $c < u_1$, $R_0(c)$ intersects the positive real axis between 0 and P_1 where $\frac{3}{2} < P_1 < 2$. When $c > u_1$, $R_0(c)$ intersects the negative real axis between p_1 and 0 where $-\frac{1}{2} > p_1$. The two different types of curves are separated and enclosed by lobes of $R_0(u_1)$.

[1] It follows from Table II in § 16 and the corresponding Fig. 4 that u_1 lies between 0.1 and 0.2.

At $z = -1$ we have a similar situation. As long as $c < u_1$, $R_1(c)$ intersects the negative real axis between -1 and p_1 and all these curves are enclosed by a lobe of $R_0(u_1) \equiv R_1(u_1)$. When c is somewhat larger than u_1, $R_1(c)$ intersects the positive real axis beyond P_1. There is a largest value of c, c_2 say, for which this is the case, and we prove in the same manner as above that $c_2 = u_2$. Thus all the curves $R_1(c)$ with $u_1 < c < u_2$ intersect the positive real axis between P_1 and P_2 where $\psi(P_2) = u_2$. When $c > u_2$, $R_1(c)$ intersects the negative real axis to the left of $z = -1$, namely, between p_2 and -1 where $p_2 > -\frac{3}{2}$ and $\psi(p_2) = u_2$.

Finally, in the general case the curves $R_n(c)$ fall into three classes: (i) Curves corresponding to $c < u_n$; these curves intersect the negative real axis to the right of $z = -n$ between $-n$ and p_n, where $-n + \frac{1}{2} < p_n < -n+1$ and where $\psi(p_n) = u_n$. (ii) Curves corresponding to $u_n < c < u_{n+1}$; these curves intersect the positive real axis between P_n and P_{n+1} where $n + \frac{1}{2} < P_n < n+1$ and where $\psi(P_n) = u_n$. (iii) Curves corresponding to $c > u_{n+1}$; these curves intersect the negative real axis to the left of $z = -n$, between p_{n+1} and $-n$. The three different types of curves are separated by lobes of the critical curves $R_n(u_n)$ and $R_n(u_{n+1})$.

15. Inequalities for the critical values. We have thus completed the qualitative description of the R, I-net. It remains to prove formula (90). For this purpose we resort to formula (22), which has not been used in the earlier part of the paper, namely,

$$\psi(z+h)-\psi(z)=\sum_{n=0}^{\infty}{}' \frac{(-1)^n}{n+1}\,\frac{h(h-1)\,\ldots\,(h-n)}{z(z+1)\,\ldots\,(z+n)}.$$

Consequently, if $z=x$ is real and positive

$$\left|\psi(x+h)-\psi(x)-\frac{h}{x}\right|$$

$$\leqq \frac{|h|}{x}\sum_{n=1}^{\infty}{}' \frac{1}{n+1}\,\frac{(|h|+1)\,(|h|+2)\,\ldots\,(|h|+n)}{(x+1)\,(x+2)\,\ldots\,(x+n)}$$

$$\leqq \frac{|h|}{2x}\sum_{n=1}^{\infty}{}' \frac{(|h|+1)\,\ldots\,(|h|+n)}{(x+1)\,\ldots\,(x+n)}=\frac{|h|\,(|h|+1)}{2x\,(x-|h|-1)}$$

provided $x>|h|+1$. Hence

(91) $$\psi(x+h)-\psi(x)=\frac{h}{x}\left[1+\varrho(x,h)\right],$$

where

(92) $$|\varrho(x,h)|<\frac{|h|+1}{2\,(x-|h|-1)}\cdot\text{ when } x>|h|+1.$$

We now choose $x>(|h|+1)\,(2|h|+1)$ and set $h=k+il$. Then

$$|\varrho(x,h)|<\frac{1}{4|h|}$$

and

(93) $R(x+k,l)-R(x,0)=\dfrac{k}{x}+P(x,h)$ where $|P(x,h)|<\dfrac{1}{4x}$.

In (93) we put $x=n+1$, $k=-\dfrac{1}{2}$, $l=\eta$ and obtain

(94) $$R(n+1,0)-R\left(n+\frac{1}{2},\eta\right)>\frac{1}{4(n+1)}$$

provided $n\geqq 2\left(\eta^2+\dfrac{1}{4}\right)+3\sqrt{\eta^2+\dfrac{1}{4}}$, and, a fortiori, if $n\geqq 5\left(\eta^2+\dfrac{1}{4}\right)$ when $\eta\geqq\dfrac{\sqrt{3}}{2}$. It is possible to replace (94) by

$$R(n+1,0) - R\left(n+\frac{1}{2},\eta\right) > \frac{\pi}{\operatorname{sh} 2\pi\eta}$$

as long as

$$\frac{\pi}{\operatorname{sh} 2\pi\eta} < \frac{1}{4(n+1)}.$$

Suppose that we choose n and η subject to the following double inequality

$$5\left(\eta^2+\frac{1}{4}\right) \leqq n < \frac{1}{4\pi} \operatorname{sh} 2\pi\eta - 1, \quad \eta \geqq \frac{\sqrt{3}}{2}; \tag{95}$$

then (94) implies that (90) is fulfilled for such values of n and η. It is now obvious that when we give ourselves an $n \geqq 5$, we can find an η satisfying (95). Thus to every $n \geqq 5$ there exists an η for which (90) holds. We can verify by numerical calculation that (90) holds for $\eta = 1$ when $n = 1, 2, 3$ and 4. We obtain from Table I that

n	1	2	3	4
$R\left(n+\frac{1}{2},1\right)$	0.3480	0.8096	1.1544	1.4386
$\psi(n+1)$	0.4228	0.9228	1.2561	1.5061

Since $\dfrac{\pi}{\operatorname{sh} 2\pi} = 0.0117$ we have verified our statement.

We can obtain an asymptotic expression for $w_n = \psi(z_n)$ for large values of n with the aid of (19) and (70). The result is rather complicated and will not be given here; it permits us to conclude, however, that

$$\psi(P_n) - \psi\left(n+\frac{1}{2}\right) = O\left(\frac{\log^2 n}{n^2}\right);$$

whence

$$P_n - n - \frac{1}{2} = O\left(\frac{\log^2 n}{n}\right), \tag{96}$$

and similarly

$$p_n + n - \frac{1}{2} = O\left(\frac{\log^2 n}{n}\right). \tag{97}$$

The inequalities obtained for the critical values w_n can be summarized as follows:

Theorem: Let z_n and $\bar{z}_n$ $(n = 1, 2, 3, \ldots)$ denote the zeros of $\psi'(z)$ where $-n+\frac{1}{2} < \Re(z_n) < -n+1$ and set $w_n = \psi(z_n) = u_n + i v_n$. Then

$$\psi\left(n+\frac{1}{2}\right) < u_n < \psi(n+1), \tag{98}$$

$$\frac{\pi}{2} < v_n < v_{n+1} < \pi, \tag{99}$$

where $n = 1, 2, 3, \ldots$ Further,

$$u_n = \psi\left(n+\frac{1}{2}\right) + O\left(\frac{\log^2 n}{n}\right), \tag{100}$$

$$v_n = \pi - O\left(\frac{\log n}{n}\right). \tag{101}$$

16. Numerical computation of $\psi(z)$. We can also attack the question of how the values of $\psi(z)$ are distributed with the aid of numerical calculation. Such computations are fairly easy to carry out on the imaginary axis; with the aid of the formulas in § 2 we can afterwards obtain values of the function on other vertical lines.

Formulas (17) and (18), namely,

$$I(0, y) = \frac{\pi}{2} \coth \pi y + \frac{1}{2y}, \quad I\left(\frac{1}{2}, y\right) = \frac{\pi}{2} \operatorname{th} \pi y,$$

enable us to calculate the imaginary part of $\psi(z)$ on the lines $\Re(z) = 0$ and $\frac{1}{2}$. The values on the lines $\Re(z) = n$ and $n+\frac{1}{2}$ are then obtainable with the aid of (10). It does not seem to be possible to get the values of $I(x, y)$ on any other vertical lines with the aid of the formulas in § 2 without the use of (19).

The situation with regard to $R(x, y)$ is rather different. We have

$$R(0, y) = -C + y^2 \sum_{n=1}^{\infty} \frac{1}{n(n^2+y^2)}. \tag{102}$$

This series is not well suited for numerical work, nor does its sum seem to be expressible in terms of elementary functions. To obtain more rapidly convergent series we use transformations of Kummer's type. Writing

$$S_p = \sum_{n=1}^{\infty} \frac{1}{n^p} \qquad (p \geqq 2)$$

we easily see that

$$(103)\left\{\begin{aligned} R(0, y) &= -C + y^2 S_3 - y^4 S_5 + \ldots \\ &+ (-1)^{k-1} y^{2k} S_{2k+1} + (-1)^k y^{2k+2} \sum_{n=1}^{\infty} \frac{1}{n^{2k+1}(n^2+y^2)}. \end{aligned}\right.$$

If $|y| \leqq 1$ and k equals 4 or 5, this expression is quite suited for computations. When $|y| > 1$ we can still apply the same method if we let the transformations apply to the remainder after a suitably chosen term of the original series. For certain values of y the series

$$S_m(y) = \sum_{n=m}^{\infty} \frac{1}{n(n^2-y^2)}$$

has a known value. Thus

$$S_1\left(\frac{1}{2}\right) = 8 \log 2 - 4,\ S_2(1) = \frac{1}{4},\ S_3(2) = \frac{11}{96} \text{ etc.}$$

For such values of y we can obtain a rapidly convergent series in fewer steps, e. g.,

$$R\left(0, \frac{1}{2}\right) = -C + 2\log 2 - 1 - \frac{1}{8} S_5 - \frac{1}{256} S_9 - \frac{1}{2048} \sum_{n=1}^{\infty} \frac{1}{n^9 \left(n^4 - \frac{1}{16}\right)}.$$

The remainder after the first term of the infinite series contributes less than 10^{-7} to the value of $R\left(0, \frac{1}{2}\right)$. The sums S_p which are needed for the computations can be taken from Stieltjes' table in Acta Mathematica, vol. 10.

Formula (103) becomes unmanageable when $|y|$ is larger than about 3. For such values we have to resort to formula (19), which is very convenient for numerical work. Using formulas (20) and (21) with $m = 5$ and substituting the values of the Bernoullian numbers we obtain

$$(104) \quad R(0, y) = \log|y| + \frac{1}{12y^2} + \frac{1}{120y^4} + \frac{1}{252y^6} + \frac{1}{240y^8} + \frac{1}{132y^{10}} + R_{11},$$

$$(104\text{ a}) \qquad |R_{11}| < \frac{64}{2079\, y^{10}}.$$

Using one or the other of these series we have computed the following values

$$R\left(0, \frac{1}{4}\right) = -0.505907, \quad R\left(0, \frac{1}{3}\right) = -0.455210, \quad R\left(0, \frac{1}{2}\right) = -0.328886,$$

$$R\left(0, \frac{2}{3}\right) = -0.186352, \quad R\left(0, \frac{3}{4}\right) = -0.113901, \quad R(0, 1) = 0.094650,$$

$$R\left(0, \frac{3}{2}\right) = 0.444698, \quad R(0, 2) = 0.714592, \quad R\left(0, \frac{9}{4}\right) = 0.827758,$$

$$R(0, 3) = 1.108907, \quad R(0, 4) = 1.391537.$$

The error in these values, barring unfortunate accidents, amounts to less than one unit in the last decimal place.

Knowing $R(0, y)$ and $R(0, 2y)$ we can compute $R\left(\frac{n}{2}, y\right)$ with the aid of (9) and (13). If we know $R(0, 3y)$ in addition, we can obtain $R\left(\frac{n}{3}, y\right)$ with the aid of (9), (11) and (13). Finally, if we know $R(0, y)$, $R(0, 2y)$ and $R(0, 4y)$ we can get $R\left(\frac{n}{4}, y\right)$ with the aid of the same formulas which supply the necessary number of equations.

In the adjoining Table I we have listed the values of $\psi(x+iy)$ for some values of x and y. The sign » in any place of the table indicates that the corresponding value has not been calculated; thus the imaginary part is given for only half of the entries. A last digit set in heavier type indicates that the decimal in question has been raised. The values listed above permit extending the table considerably. In Table II we have listed the real part of $\psi(x+iy)$ at 40 different points in the square $-1 \leq x \leq 0$, $0 \leq y \leq 1$. This table illustrates the run of $R(x, y)$ in the neighborhood of the critical point $z = z_1$. The adjoining Figure 4 is based upon this table; it shows the interpolated curves $R = c$ for $c = -0.5, -0.4, \ldots, 0.5$ and 0.6. In order to avoid crowding the figure we have left out most of the arcs of these curves in the lower half of the diagram. The dots in the figure mark the points where the values of $R(x, y)$ have been calculated. The table and the figure together would seem to suggest that z_1 is near to the point $z = -\frac{1}{3} + \frac{2i}{3}$ and that u_1 is about 0.16.

17. The Riemann surface of $w = \psi(z)$. We can now form a fairly good idea of the structure of the Riemann surface corresponding to $w = \psi(z)$ and its inverse. The singularities of the inverse function $z = \psi^{-1}(w)$ are $w = \infty$, which is a *transcendental critical point*, together with all the points $w = w_n$ and $\overline{w}_n$. The latter points are

Table I.

Values of $\psi(x+iy)$.

$x \backslash y$	0	0.25	0.50	0.75	1
— 2	∞	0.9276 + 4.6859 i	0.9417 + 3.2305 i	0.9644 + 2.7903 i	0.9947 + 2.7767 i
— 1.75	— 2.3028	— 0.4273 + »	0.5709 + »	0.8099 + »	0.8940 + »
— 1.50	0.7032	0.7106 + 1.9720 i	0.7319 + 2.6406 i	0.7667 + 2.7362 i	0.8096 + 2.6025 i
— 1.25	3.7142	1.8340 + »	0.8802 + »	0.7052 + »	0.7181 + »
— 1	∞	0.4353 + 4.6244 i	0.4711 + 3.1128 i	0.5261 + 2.6259 i	0.5947 + 2.5767 i
— 0.75	— 2.8942	0.9873 + »	0.0426 + »	0.3272 + »	0.4633 + »
— 0.50	0.0365	0.0619 + 1.8365 i	0.1319 + 2.4406 i	0.2324 + 2.4695 i	0.3480 + 2.3649 i
— 0.25	2.9142	1.0647 + »	0.1905 + »	0.1169 + »	0.2303 + »
0	∞	— 0.5059 + 4.3891 i	— 0.3289 + 2.7128 i	— 0.1139 + 2.2659 i	0.0947 + 2.0767 i
0.25	— 4.2275	— 2.1873 + »	— 0.8803 + »	— 0.3395 + »	— 0.0167 + »
0.50	— 1.9635	— 1.5381 + 1.0365 i	— 0.8681 + 1.4406 i	— 0.3830 + 1.5424 i	— 0.0520 + 1.5649 i
0.75	— 1.0858	— 0.9353 + »	— 0.6095 + »	— 0.2831 + »	— 0.0050 + »
1	— 0.5772	— 0.5059 + 0.3891 i	— 0.3289 + 0.7128 i	— 0.1139 + 0.9326 i	0.0947 + 1.0767 i
1.25	— 0.2275	— 0.1873 + »	— 0.0803 + »	0.0605 + »	0.2186 + »
1.50	0.0365	0.0619 + 0.2365 i	0.1319 + 0.4406 i	0.2324 + 0.6193 i	0.3480 + 0.7649 i
1.75	0.2475	0.2647 + »	0.3136 + »	0.3836 + »	0.4750 + »
2	0.4228	0.4353 + 0.1538 i	0.4711 + 0.3128 i	0.5261 + 0.4526 i	0.5947 + 0.5767 i

Table II.

Values of $R(x, y)$.

$y \backslash x$	-1	$-\frac{3}{4}$	$-\frac{2}{3}$	$-\frac{1}{2}$	$-\frac{1}{3}$	$-\frac{1}{4}$	0
1	0.5947	0.4633	0.4234	0.3480	0.2720	0.2303	0.0947
$\frac{3}{4}$	0.5261	0.3272	0.2895	0.2324	0.1686	0.1169	− 0.1139
$\frac{2}{3}$	0.5060	»	0.2265	»	0.1578	»	− 0.1864
$\frac{1}{2}$	0.4711	0.0426	0.0311	0.1319	0.2192	0.1905	− 0.3289
$\frac{1}{3}$	0.4448	»	− 0.3727	0.0808	0.5160	»	− 0.4552
$\frac{1}{4}$	0.4353	− 0.9873	»	0.0619	»	1.0647	− 0.5059
0	∞	− 2.8942	− 1.0548	0.0365	1.2590	2.9142	∞

algebraic branch-points in the neighborhood of which two determinations of z are interchanged.

In order to build up the Riemann surface we consider the map of the z-plane corresponding to the transformation $w = \psi(z)$. It is clear that this map will cover itself infinitely often. Thus we have to cut up the z-plane into regions such that each region has a smooth non-overlapping image and then we must piece these different images together. It then becomes a question of how these regions should be chosen. Our previous study of the R, I-net shows that the critical curves through the points z_n and $\bar{z}_n$ give a natural division of the plane into suitable regions. We can choose either the curves $R_n(u_n)$ or the curves $I_n(v_n)$ for this purpose; we select the former curves. We then imagine the plane cut up along those arcs of $R_n(u_n)$ which join z_n and $\bar{z}_n$ with $-n$ and $-n+1$. We do not, however, cut the plane along the remaining arcs of $R_n(u_n)$ which join z_n

with $\bar{z}_n$ over p_n and P_n, respectively. The region outside of all the cuts we denote by D_0. The region inside of the cuts from $z = -n$ over z_n, $-n+1$ and $\bar{z}_n$ back to $-n$ will be denoted by D_n ($n = 1, 2, 3, \ldots$).

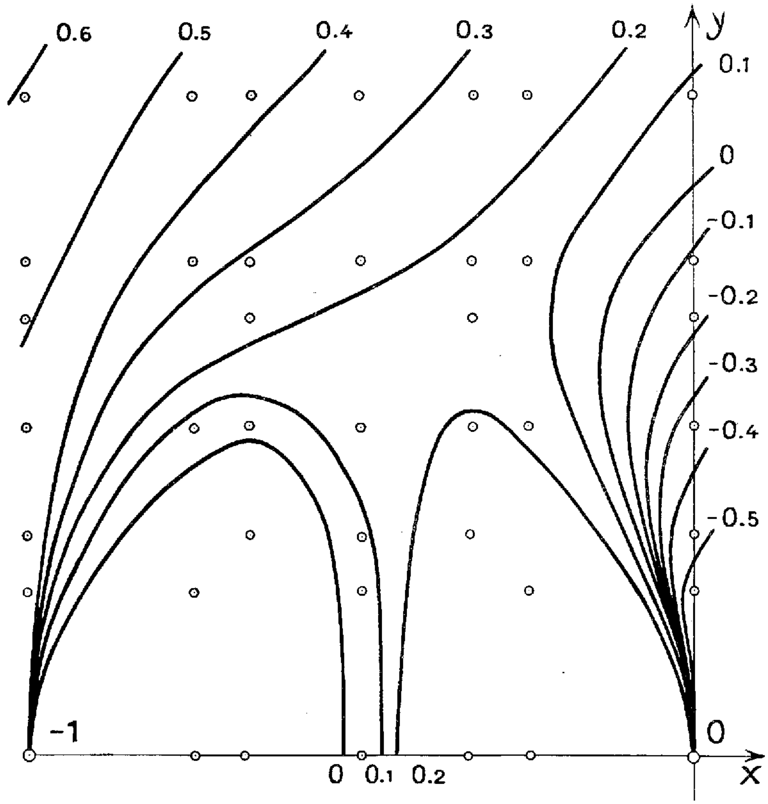

Fig. 4.

We begin by considering D_0. This is a simply-connected region, if we leave out the points $z = -n$ ($n \geqq 1$), in the interior of which $\psi(z)$ is holomorphic and $\psi'(z) \neq 0$. We shall prove that the image of D_0 by the transformation $w = \psi(z)$ is a full plane slit up along the lines $u = u_n$, $v \geqq v_n$ and $u = u_n$, $v \leqq -v_n$ ($n = 1, 2, 3, \ldots$).

In order to see this we shall consider the equation

$$\psi(z) = u + iv.$$

It is not difficult to see that this equation has one and only one solution in the interior of D_0 if $u + iv$ is not on the slits just mentioned, and if $u + iv$ is located on one of the slits there are two solutions on the boundary of D_0. In fact, suppose that $u_m < u < u_{m+1}$.[1] We can then locate $R_m(u)$ in D_0; this curve goes from $z = -m$ back to this point, intersecting the positive real axis between P_m and P_{m+1}. It lies entirely in D_0 and it is the only branch of the curve $R = u$ in D_0, all the other branches are in the excluded regions. If we trace $R_m(u)$ once from $-m$ back to this point going in the positive sense, $I(x, y)$ increases steadily from $-\infty$ to $+\infty$. Thus there is one and only one point on the curve where $I(x, y) = v$ and this point gives the desired solution, which is obviously unique. The case in which $u = u_m$ is easily disposed and will not be considered here. We designate the image of D_0 by Π_0.

In the interior of D_1, $\psi(z)$ takes on every value once and only once with the exception of the values $u = u_1$, $v \geqq v_1$ and $u = u_1$, $v \leqq -v_1$ which are not taken on at all in the interior but twice on the boundary instead. Thus we find that D_1 is mapped upon a full plane slit along the lines $u = u_1$, $v \geqq v_1$ and $u = u_1$, $v \leqq -v_1$. Let this slit plane be denoted by Π_1; Π_0 and Π_1 are evidently connected along the common cuts. In general, the region D_n $(n \geqq 1)$ is mapped upon a full plane Π_n slit along the lines $u = u_n$, $v \geqq v_n$ and $u = u_n$, $v \leqq -v_n$,

[1] We set $u_0 = -\infty$.

and this plane is connected with Π_0 along the common cuts. There is obviously no direct connection between Π_m and Π_n if $mn \neq 0$. The totality of these sheets Π_m constitutes the Riemann surface of $\psi(z)$.

18. Generalizations. In concluding we shall raise the question of the extent to which the results obtained in the present paper may be considered typical for the class of functions defined as principal solutions of equations of the form

$$\underset{\omega}{\Delta} F(z) = \varphi(z). \tag{105}$$

Without pretending to answer this question we shall call attention to a few facts which have an obvious bearing on the situation.

There are many details in the preceeding discussion which are of a highly special nature and which cannot be carried over to a more general case. But the fundamental results of the investigation have been derived either directly from the defining difference equation (2) or from the complementary theorem (3), the multiplication theorem (4) and the asymptotic formula (19). The latter three theorems are all immediate consequences of the difference equation and are not dependent upon the special analytic form of the solution. Now the principal solution of (105) does satisfy a complementary theorem, a multiplication theorem and an asymptotic relation all of a fairly simple nature under very general assumptions on $\varphi(z)$. Further, if $\varphi(z)$ is single-valued the nature and distribution of the singularities of $F(z)$ shows considerable resemblance to the corresponding situation for $\psi(z)$. There is consequently some ground for expecting that also the finer structure of the distribution of the values taken on by the several functions shall show striking resemblances in the special case here treated and the general case mentioned above.

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