Lower Estimates of the Isoperimetric Deficit of Nearly Spherical Domains in $\mathbf{R}^n$ in Terms of Asymmetry

By BENT FUGLEDE

Matematisk-fysiske Meddelelser **44**:1

Det Kongelige Danske Videnskabernes Selskab
The Royal Danish Academy of Sciences and Letters

Commissioner: Munksgaard · Copenhagen 1995

65 pages. DKK 100.-

Abstract

For a convex body K in $\mathbf{R}^n$ with surface area $S(K)$ it is shown that

$$S(K) \geq S(B)(1 + 2n^{-2}c(n)\beta^2 + o(\beta^2)),$$

where B denotes the ball with the same volume as K and centred at the centre of gravity of K (with Lebesgue measure), while β denotes the volume of $K \setminus B$ divided by the volume of K, and the constant $c(n)$ is taken with its biggest possible value. It is shown that $1 < c(n)/(n+1) < 1.4943$ and that

$$c(n) = \min \left\{ \frac{\|\nabla u\|^2 - (n-1)\|u\|^2}{\|u\|_1^2} \;\middle|\; u \in C^1(\Sigma, \mathbf{R}),\ u_0 = u_1 = 0 \right\},$$

where Σ denotes the unit sphere in $\mathbf{R}^n$, ∇ the gradient in the Riemannian sense, $\|\cdot\|$ the L^2-norm and $\|\cdot\|_1$ the L^1-norm on Σ. Finally, u_k denotes (for any L^2-function u on Σ) the projection of u on the eigenspace for (minus) the Laplace-Beltrami operator on Σ corresponding to the kth eigenvalue $\lambda_k = k(k+n-2)$, $k = 0, 1, 2, \ldots$. The following dual characterization of $c(n)$ is obtained:

$$\frac{1}{c(n)} = \max \left\{ \sum_{k=2}^{\infty} \frac{\|f_k\|^2}{\lambda_k - \lambda_1} \;\middle|\; f : \Sigma \to [-1, 1] \text{ measurable} \right\}.$$

It is shown, moreover, that every function f realizing the maximum $1/c(n)$ takes the values ± 1 only, and (at least in dimension $n \leq 4$) that f is even: $f(-\xi) = f(\xi)$. For even $n = 2m$ it is shown that the function $f(\xi) := \mathrm{sgn}(\xi_1^2 + \ldots + \xi_m^2 - 1/2)$ is a stationary solution to the above maximum problem in a natural sense, and it is conjectured that the maximum $1/c(n)$ is attained by this function and essentially by no other. For odd $n = 2m+1$ the constant $1/2$ must be replaced by the solution to a certain transcendental equation involving hypergeometric functions. The stated conjecture is proved valid for $n = 2$, thus recovering a recent result of R. R. Hall, W. K. Hayman, and A. W. Weitsman. The conjecture remains open for $n \geq 3$.

BENT FUGLEDE
Mathematical Institute
University of Copenhagen
Universitetsparken 5
DK-2100 Copenhagen, Denmark

Printed in Denmark by Bianco Lunos Bogtrykkeri A/S
ISSN 0023-3323 ISBN 87-7304-259-5

Contents

1. Introduction

Recently it was shown by Hall, Hayman and Weitsman in [HHW], [HH] that, when f ranges over all measurable functions on $\mathbf{R}$ (mod 2π) taking the values 1 and -1 only, and having the Fourier series $\sum_{k=0}^{\infty}(a_k \cos k\theta + b_k \sin k\theta)$, the quantity

$$\Lambda(f) = \frac{1}{2}\sum_{k=2}^{\infty}\frac{a_k^2 + b_k^2}{k^2 - 1} \tag{1.1}$$

has the biggest possible value

$$\kappa(2) := \max \Lambda(f) = \frac{4}{\pi} - 1, \tag{1.2}$$

attained by the function $\operatorname{sgn}(\cos 2\theta)$ and its translates. From this they derived the following sharp lower bound for the isoperimetric deficit of convex domains K in $\mathbf{R}^2$ (with area $A(K) = A$, perimeter L, and 'asymmetry' α, see (1.4) below):

$$L^2 \geq 4\pi A \left(1 + \frac{\pi}{4-\pi}\alpha^2 + O(\alpha^3)\right) \tag{1.3}$$

as $\alpha \to 0$, the constant $\pi/(4-\pi) = 1/\kappa(2)$ being best possible. They also described a family of convex domains which approach a ball and for which the equality sign holds, [HHW, p. 113].

The *asymmetry* $\alpha = \alpha(K)$ was defined as follows by L. E. Fraenkel (unpublished):

$$\alpha = \alpha(K) := \min_{x \in \mathbf{R}^2} \frac{A(K \setminus B(x, v))}{A(K)} \tag{1.4}$$

as $B(x, v)$ ranges over all discs with the same area as K, i.e., $A(K) = \pi v^2$.

The determination of $\kappa(2)$ in [HH] involved subordination theory from complex analysis. The present paper is an attempt – only partly successful – to obtain similar results in higher dimensions. Our method allows us also to recover (1.2) and (1.3) along with some additional information.

In Section 3 we use the Fraenkel asymmetry α, now in arbitrary dimension n, and also the similar *barycentric* asymmetry β ($\geq \alpha$) defined by fixing the centre x of the ball $B(x, v)$ of equal volume as the barycentre of the domain K (see (2.1) and (2.2) below).

If V denotes the volume and S the surface area of a bounded convex domain K in $\mathbf{R}^n$ we obtain the following slightly weaker n-dimensional analogue of (1.3):

$$\left(\frac{S}{n\omega_n}\right)^n \geq \left(\frac{V}{\omega_n}\right)^{n-1}\left(1 + \frac{2}{n}(n+1)\beta^2 + o(\beta^2)\right), \tag{1.5}$$

where ω_n is the volume of the unit ball in $\mathbf{R}^n$. The function β^2 is sharp in order of magnitude, but the constant $\frac{2}{n}(n+1)$ is no longer best possible (not even in dimension 2). In proving (1.5) we may of course assume that K is normalized so that $V = \omega_n$ and the barycentre b of K is the origin. We then expand the radial function $R = 1 + u$ for K in spherical harmonics, while drawing on results from an earlier paper [F1]. We also obtain more precise information about the remainder term $o(\beta^2)$. An inequality similar to (1.5), but without the remainder term $o(\beta^2)$ and without assuming K to approach a ball, was obtained in [F3], though with a very small constant coefficient (unspecified, but calculable) to β^2.

Writing the biggest possible value of the constant coefficient to β^2 in (1.5) in the form $\frac{2}{n}c(n)$ we thus have $c(n) \geq n+1$. We show that $c(n) > n+1$ and that $c(n)$ is also the biggest possible constant in the Poincaré style quadratic inequality

$$\|\nabla u\|^2 - (n-1)\|u\|^2 \geq c(n)\|u\|_1^2, \tag{1.6}$$

valid for all real-valued C^1-smooth functions u on the unit sphere Σ in $\mathbf{R}^n$ such that

$$\int_\Sigma u\, d\sigma = 0, \quad \int_\Sigma u(\xi)\xi_j\, d\sigma(\xi) = 0 \text{ for } j = 1, \ldots, n. \tag{1.7}$$

Here ∇u denotes the gradient of the function u on Σ in the sense of Riemannian geometry on Σ. Moreover, $d\sigma$ refers to the normalized surface measure on Σ, and $\|\cdot\|$ and $\|\cdot\|_1$ denote the $L^2(\sigma)$-norm, resp. the $L^1(\sigma)$-norm. There exist non-zero functions u satisfying (1.7) such that the equality sign holds in (1.6). One may regard (1.6) as the infinitesimal version of (1.5) corresponding to making the radial function $R = 1 + u$ infinitely close to 1, whereby the side conditions (1.7) express the above normalization $V = \omega_n, b = 0$. The presence of the L^1-norm $\|u\|_1$ in (1.6) (rather than the L^2-norm) makes the precise determination of $c(n)$ difficult.

In Section 4 we consider the following n-dimensional generalization of $\Lambda(f)$ from (1.1):

$$\Lambda(f) := \sum_{k=2}^{\infty} \frac{\|f_k\|^2}{\lambda_k - \lambda_1}, \tag{1.8}$$

where $f = \sum_{k=0}^{\infty} f_k$ is the expansion of a (real-valued) function $f \in L^2(\sigma)$ into spherical harmonics f_k (of degree k), and $\lambda_k = k(k+n-2)$ is the kth eigenvalue of (minus) the Laplace-Beltrami operator Δ on Σ. We give the following dual characterization of $c(n)$:

$$\frac{1}{c(n)} = \kappa(n) := \max\{\Lambda(f) \mid -1 \leq f \leq 1\}. \tag{1.9}$$

It turns out that the maximizing functions f in (1.9) take the values 1 and -1 only (almost everywhere on Σ). This duality result (1.9) is inspired by what is essentially the

2-dimensional case thereof, obtained in [HHW, p. 109–113] where the Fourier expansion of the support function of K was used.

As described in Section 8 the sum $\Lambda(f)$ in (1.8) can be evaluated as an integral as follows:

$$\Lambda(f) = \iint \tilde{G}(\xi\cdot\eta)\, f(\xi) f(\eta)\, d\sigma(\xi)\, d\sigma(\eta),$$

where the kernel $\tilde{G}(t)$, $-1 \le t \le 1$, has been determined explicitly by recursion w.r.t. the dimension n by Berg [Be].

The variational problem of determining the biggest possible constant $c(n)$ in (1.6) under the side conditions (1.7) leads to the following Euler type equation in the distributional sense (after a suitable normalization of u):

$$-\Delta u - (n-1)u = \tilde{f} := \sum_{k=2}^{\infty} f_k, \quad \text{where } f = \operatorname{sgn} u,$$

again under the conditions $u_0 = u_1 = 0$ from (1.7). The presence of $\operatorname{sgn} u$ on the right makes the Euler equation non-linear.

Similarly, let us denote by $\frac{2}{n}c_*(n)$ $(\ge \frac{2}{n}c(n))$ the biggest possible constant coefficient to α^2 in the estimate obtained from (1.5) by replacing β with α. Alternatively, $c_*(n)$ is the biggest possible constant in the inequality obtained from (1.6) by replacing $\|u\|_1$ with the quotient norm $\|\cdot\|_*$ on $L^1(\sigma)/\mathcal{H}_1$, $\mathcal{H}_1$ denoting the space of restrictions to Σ of the linear forms on $\mathbf{R}^n$. (The second side condition in (1.7), amounting to $u_1 = 0$, is unnecessary here.) In analogy with (1.9) we obtain

$$\frac{1}{c_*(n)} = \kappa_*(n) := \max\{\Lambda(f) \mid -1 \le f \le 1,\ f_1 = 0\},$$

and the Euler equation is the same as above, but now with the side conditions $u_0 = f_1 = 0$.

In Theorem 4.4 we show in dimension $n \le 4$ that every maximizing function f for $\kappa(n)$ in (1.9) is *even*: $f(-\xi) = f(\xi)$ (almost everywhere), in particular $f_1 = 0$, and hence

$$\kappa_*(n) = \kappa(n), \quad c_*(n) = c(n) \qquad \text{for } n \le 4.$$

It follows that every minimizing function u for $c(n)$ in (1.6) is likewise even. The proof of these symmetry properties is rather long; it is inspired by a construction due to Hall and Hayman [HH] in the 2-dimensional case. We use spherical harmonics and Legendre polynomials, and spherical potential theory with respect to the operator $\Delta + (n-1)$ on Σ as developed by Berg [Be]. – Although our proof of Theorem 4.4 only seems to work for $n \le 4$, it is conjectured that the result holds in all dimensions.

In Section 5 we treat the case $n = 2$ and prove that $\kappa(2) = \kappa_*(2) = 4/\pi - 1$, by using the corresponding Euler equation and also Theorem 4.4. We further show that the maximum (1.2) remains in force when f is allowed to take arbitrary values in the interval $[-1, 1]$, and moreover that, up to isometries of Σ, this maximum is attained only by the function $\operatorname{sgn}(\cos 2\theta) = \operatorname{sgn}(\xi_1^2 - \frac{1}{2})$.

Section 6 contains an incomplete discussion of the case $n > 2$. (For the complete solution of a related, but more manageable problem, see [F5].) Writing $n = 2m$ for even n and $n = 2m + 1$ for odd n, we consider the function

$$f(n;\, \xi) = \operatorname{sgn}(\xi_1^2 + \ldots + \xi_m^2 - \tau^2) \tag{1.10}$$

of $\xi \in \Sigma$ and show that this function is *stationary* in a certain natural sense for precisely one value of the constant $\tau = \tau(n)$ (> 0), namely $\tau(n) = 1/\sqrt{2}$ for n even, while for n odd $\tau(n)^2$ is the root of a certain transcendental equation involving hypergeometric functions. In terms of the corresponding *stationary value* $\Lambda\big(f(n;\, \cdot)\big)$ we have because $f(n;\, \cdot)$ is an even function on Σ with values in $[-1, 1]$:

$$\Lambda\big(f(n;\, \cdot)\big) \le \kappa_*(n) \le \kappa(n) < \frac{1}{n+1}, \tag{1.11}$$

the last inequality being equivalent to $c(n) > n + 1$, cf. above just before (1.6).

We also consider certain other stationary functions. We conjecture, however, that $f(n;\, \cdot)$ from (1.10) is *maximizing* for $\kappa(n)$, so that the first two inequalities in (1.11) are equalities, but we cannot prove this (except for $n = 2$, cf. above). For even $n = 2m$ we find

$$\Lambda\big(f(2m;\, \cdot)\big) = \frac{1}{2m-1}\left(\frac{\Gamma(\frac{m+1}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{m}{2})}\,\frac{\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})}\,\frac{\Gamma(\frac{m}{2}+\frac{1}{4})}{\Gamma(\frac{m}{2}+\frac{3}{4})} - 1\right) \tag{1.12}$$

(which equals $4/\pi - 1$ for $m = 1$).

For $n = 3$ we have from (1.10) $f(3;\, \xi) = \operatorname{sgn}(\xi_1^2 - \tau^2)$, and we find that

$$\log\frac{1+\tau}{1-\tau} = \frac{2}{1+\tau}, \quad \text{i.e., } \tau \approx 0.5644,$$
$$\Lambda\big(f(3;\, \cdot)\big) = (1-\tau)^2 \approx 0.1898.$$

The conjecture that, with the stated value of τ, the function $\operatorname{sgn}(\xi_1^2 - \tau^2)$ is maximizing for $\kappa(3)$, which then equals $(1-\tau)^2$, has also been proposed in a different form by Richard R. Hall (personal communication).

Stirling's formula applied to (1.12) leads to the following asymptotic formula for the ratio between the lower bound $\Lambda\big(f(n;\, \cdot)\big)$ and the elementary upper bound $1/(n+1)$ in the estimate (1.11) (at least when n is supposed to be *even*):

$$\lim_{n\to\infty} (n+1)\Lambda\big(f(n;\, \cdot)\big) = \frac{1}{\sqrt{\pi}}\frac{\Gamma(1/4)}{\Gamma(3/4)} - 1 \approx 0.6692,$$

the sequence $(n+1)\Lambda(f(n,\cdot))$ being decreasing through even n. In particular, we obtain

$$0.6692 < (n+1)\kappa_*(n) \le (n+1)\kappa(n) < 1 \quad \text{for } n \text{ even.}$$

The same estimates hold for n odd (Theorem 6).

In connection with (1.5) we mention that a different estimate, somewhat similar in spirit, has been obtained by Schneider [Sc] with β replaced by another average measure of non-sphericity of K, defined in terms of the L^2-distance between the support function of K and that of the associated Steiner ball (like in [HHW] for $n = 2$). – For other so-called stability versions of inequalities for convex bodies see [F4] and [GS] (with references) and the survey article [G].

We close this introduction by comparing the results mentioned above with similar results (first in dimension 2) in which the 'average' asymmetries α and β of K are replaced by a stronger 'uniform' measure δ of the deviation of K from circular shape, such as

$$\delta = \frac{r_e - r_i}{v}, \tag{1.13}$$

where r_e denotes the circumradius and r_i the inradius of K, while v as above denotes the radius of a disc with the same area as K. Virtually all work on the present topic has its background in the inequality

$$L^2 \ge 4\pi A\left(1 + \frac{1}{\pi}\delta^2\right) \tag{1.14}$$

obtained by Bonnesen [Bo] for convex domains K in $\mathbf{R}^2$, the coefficient $1/\pi$ to δ^2 being best possible. Actually, Bonnesen's inequality (1.14) holds for arbitrary planar domains K bounded by a simple closed rectifiable curve, [F2]. However, (1.14) does not extend to multiply connected or disconnected domains (not even if we replace δ^2/π by any other positive continuous function of δ approaching 0 as $\delta \to 0$), as one sees by taking for K the difference or the union of the unit disc and a small disc (inside, resp. outside the unit circle).

It is in this connection that the Fraenkel asymmetry α from (1.4) (but not the barycentric asymmetry β) has an advantage over the uniform measure of non-sphericity δ from (1.13). In fact, it was shown in [HHW] that

$$L^2 \ge 4\pi A\left(1 + \tfrac{1}{6}\alpha^2\right)$$

holds for arbitrary planar sets K (of finite area A and finite perimeter L). (The constant $\frac{1}{6}$ is not claimed to be best possible.) It is conjectured that a similar result (with another constant to replace $\frac{1}{6}$) holds in higher dimensions, mutatis mutandis, but this has been proved only in the convex case, see [F3]. On the other hand, for convex domains K in $\mathbf{R}^n$ we also have lower estimates of the isoperimetric deficit (when sufficiently small) in terms of the n-dimensional version of δ from (1.13), the term δ^2/π in (1.14) being then replaced by a constant times $\delta^{\frac{n+1}{2}}$ if $n \ge 4$, and by a constant times $\delta^2/\log(1/\delta)$ if $n = 3$, and these functions of δ are again sharp in order of magnitude, see [F1].

Acknowledgements. I am indebted to my colleague Anton Jensen for help with the numerical calculations leading to the table in Section 6, using the interactive system TERM for numerical computation developed by him. I thank Gerd Grubb for discussions and references concerning regularity of solutions of elliptic equations. The paper has been typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TeX in Times and MathTime fonts, and in this connection I thank Gunnar Forst and Anders Thorup for help in producing the PostScript file.

2. Preliminaries

In Sections 2 and 3 we shall mostly use the same notation as in [F1, §1, p. 622–623]:

K denotes a bounded measurable subset of $\mathbf{R}^n$, $n \geq 2$ (with further properties to be specified later). (The set K was denoted by D in [F1].)

$V = V(K)$ denotes the volume of K (n-dimensional Lebesgue measure).

$S = S(K)$ denotes the surface area of K (i.e., of ∂K), assumed to exist.

$\omega_n = \pi^{n/2}/\Gamma(n/2+1)$ is the volume of the unit ball $\Omega = B(0,1)$ in $\mathbf{R}^n$, hence $n\omega_n$ is the surface area of the unit sphere $\Sigma = \partial\Omega$ in $\mathbf{R}^n$.

$D = D(K)$ denotes the (dimensionless) *isoperimetric deficit* of K. This deficit (denoted by Δ in [F1]) is defined by

$$D = \frac{S}{n\omega_n}\left(\frac{V}{\omega_n}\right)^{-\frac{n-1}{n}} - 1.$$

b denotes the barycentre of K, with jth coordinate $\frac{1}{V}\int_K x_j\,dx$, $j = 1, \dots, n$.

$v = (V/\omega_n)^{1/n}$ is called the *volume radius* of K.

$K_0 = v^{-1}(K - b)$ is called the *normalized* set associated with K.

$d = d(K) = \inf\{t \geq 0 \mid (1-t)_+\Omega \subset K_0 \subset (1+t)\Omega\}$ is the Hausdorff distance between K_0 and Ω. We call d the *spherical deviation* of K (cf. [F1, Definition 2.1]).

Further we consider in Section 3 the *asymmetry* of K in the sense of Fraenkel:

$$\alpha = \alpha(K) = \min_{x\in\mathbf{R}^n} \frac{V(K \setminus B(x,v))}{V(K)} = \min_{x\in\mathbf{R}^n} \frac{V(B(x,v)\setminus K)}{V(K)}, \tag{2.1}$$

and also the following *barycentric asymmetry* of K:

$$\beta = \beta(K) = \frac{V(K\setminus B(b,v))}{V(K)} = \frac{V(B(b,v)\setminus K)}{V(K)}. \tag{2.2}$$

Note that each of the quantities D, d, α, β is the same for K as for the normalized set $K_0 = v^{-1}(K - b)$. Clearly $0 \leq \alpha \leq \beta \leq 1$.

Throughout the paper we denote by σ the normalized surface measure on the unit sphere Σ in $\mathbf{R}^n$. The abbreviation a.e. means: almost everywhere with respect to σ. We consider the usual $L^p(\sigma)$-norms of σ-measurable functions $f : \Sigma \to \mathbf{R}$:

$$\|f\|_p = \left(\int_\Sigma |f(\xi)|^p d\sigma(\xi)\right)^{1/p}, \qquad 1 \leq p < \infty,$$

$$\|f\|_\infty = \min\{t \in \mathbf{R}_+ \mid |f(\xi)| \leq t \ \ \sigma\text{-a.e.}\}.$$

For simplicity we shall mostly write $\|f\|$ in place of $\|f\|_2$.

An important role will be played by the decomposition of $L^2(\sigma)$ into eigenspaces for the Laplace-Beltrami operator Δ on Σ, cf. e.g. [Sp, p. 193 f.] and [M, p. 38]. For any integer $k \geq 0$ we denote by $\mathcal{H}_k$ the vector space of all *spherical harmonics* of order k, i.e., the restrictions to Σ of the harmonic polynomials homogeneous of degree k. These subspaces $\mathcal{H}_k$ of the Hilbert space $L^2(\sigma)$ are mutually orthogonal and span together $L^2(\sigma)$:

$$L^2(\sigma) = \bigoplus_{k=0}^{\infty} \mathcal{H}_k.$$

For any function $f \in L^2(\sigma)$ we denote by f_k the orthogonal projection of f on $\mathcal{H}_k$, and we have the expansions

$$f = \sum_{k=0}^{\infty} f_k, \qquad \|f\|^2 = \sum_{k=0}^{\infty} \|f_k\|^2,$$

the former expansion being convergent in the $L^2(\sigma)$-norm $\|\cdot\|$. Note that

$$f_0 = \int f \, d\sigma,$$

the mean-value of f. In dimension $n = 2$ the above expansion of f is the Fourier expansion because $f_0 = \frac{1}{2}a_0$, $f_k(\cos\theta, \sin\theta) = a_k \cos(k\theta) + b_k \sin(k\theta)$ for $k \geq 1$, in terms of the Fourier coefficients a_k, b_k of f; hence $\|f_k\|^2 = \frac{1}{2}(a_k^2 + b_k^2)$ for $k \geq 1$.

The Laplace-Beltrami operator Δ on Σ (acting in the distribution sense) is a self-adjoint operator on $L^2(\sigma)$ with discrete spectrum, the eigenspaces being $\mathcal{H}_k$ with the corresponding eigenvalues (actually for $-\Delta$)

$$\lambda_k = k(k + n - 2), \qquad k = 0, 1, 2, \ldots,$$

cf. e.g. [M, Lemma 2]. For any function $u = \sum_{k=0}^{\infty} u_k$ in the domain of Δ we thus have

$$-\Delta u = \sum_{k=0}^{\infty} \lambda_k u_k = \sum_{k=1}^{\infty} \lambda_k u_k.$$

For $m = 1$ or 2 we denote by $W^{m,p} = W^{m,p}(\Sigma)$ the Sobolev space of all real distributions u on Σ whose partial derivatives of order m (hence also of orders $\leq m$) in local coordinates on Σ are (locally) in $L^p(\sigma)$. In particular, $W^{1,\infty} = \mathrm{Lip}_1$, the functions on Σ satisfying a Lipschitz condition.

For $u \in W^{1,2}$ we denote by ∇u the gradient of u in the sense of Riemannian geometry on Σ, cf. e.g. [Sp, p. 188], and by $\|\nabla u\|$ the $L^2(\sigma)$-norm of the length $|\nabla u|$ of ∇u.

We denote by dom Δ the domain of Δ as a self-adjoint operator in $L^2(\sigma)$, and similarly for other operators. It is known that dom $\Delta = W^{2,2}$, cf. e.g. [Se, p. 685], or argue as in Remark 4.4 below, using [Hö2, Theorem 17.1.1]. The following lemma is presumably known. (It was used implicitly in [F1, (18), p. 625].)

Lemma 2. *For any $u \in$ dom Δ we have*

$$-\int u\,\Delta u\,d\sigma = \sum_{k=1}^{\infty} \lambda_k \|u_k\|^2 = \|\nabla u\|^2.$$

The latter equation holds more generally for any $u \in W^{1,2}(\Sigma)$.

Proof. The former expression for $-\int u\,\Delta u\,d\sigma$ is obvious since $\lambda_0 = 0$. Because dom $\Delta = W^{2,2} \subset W^{1,2}$ it remains to establish the second equation in the lemma for $u \in W^{1,2}$. The positive self-adjoint operator $-\Delta$ has a positive self-adjoint square root Q, and

$$Qu = \sum_{k=1}^{\infty} \sqrt{\lambda_k}\, u_k, \qquad \|Qu\|^2 = \sum_{k=1}^{\infty} \lambda_k \|u_k\|^2 \tag{2.3}$$

for any $u \in$ dom Q (the domain of Q, characterized by the finiteness of the latter sum in (2.3)). For any $u \in C^2(\Sigma)$ ($\subset$ dom $Q^2 \subset$ dom Q) we have

$$\|Qu\|^2 = \int_\Sigma u\, Q^2 u\, d\sigma = -\int_\Sigma u\,\Delta u\,d\sigma = \|\nabla u\|^2 \tag{2.4}$$

by partial integration. For any $u \in W^{1,2}(\Sigma)$ there exists a sequence of functions $u^{(n)}$ of class $C^2(\Sigma)$ such that

$$\|u^{(n)} - u\| \to 0, \qquad \|\nabla(u^{(n)} - u)\| \to 0.$$

This can be shown by regularization in local coordinates combined with the use of a partition of unity, cf. e.g. [DL, p. 312]. In view of (2.4) the sequence $(Qu^{(n)})$ is Cauchy in $L^2(\sigma)$, and since Q has a closed graph it follows that $u \in$ dom Q and $Qu^{(n)} \to Qu$. From (2.3), (2.4) we therefore conclude that

$$\sum_{k=1}^{\infty} \lambda_k \|u_k\|^2 = \|Qu\|^2 = \lim \|Qu^{(n)}\|^2 = \lim_n \|\nabla u^{(n)}\|^2 = \|\nabla u\|^2. \qquad \square$$

Note that

$$\lambda_1 = n - 1, \quad \lambda_2 - \lambda_1 = n + 1,$$

and if $u_0 = 0$, the expression, important in the sequel,

$$\|\nabla u\|^2 - (n-1)\|u\|^2 = \sum_{k=2}^{\infty} (\lambda_k - \lambda_1)\|u_k\|^2 \tag{2.5}$$

is independent of the linear component $u_1 \in \mathcal{H}_1$. If moreover $u_1 = 0$, the stated expression is $\geq (\lambda_2 - \lambda_1)\|u\|^2 = (n+1)\|u\|^2$, with equality precisely when $u \in \mathcal{H}_2$.

3. The case of strongly starshaped domains

In this section the set K in $\mathbf{R}^n$ is supposed to be *strongly starshaped* with respect to its barycentre b in the sense that the boundary ∂K_0 of the normalized set K_0 can be represented in polar coordinates $R = |x|$, $\xi = x/|x|$, $x \in \mathbf{R}^n \setminus \{0\}$, by

$$R = R(\xi) = 1 + u(\xi), \qquad \xi \in \Sigma,$$

with $R(\cdot)$ of class $\mathrm{Lip}_1 = W^{1,\infty}$, cf. Section 2. Note that $d = \|u\|_\infty$. We may assume that K itself is normalized, i.e., $K = K_0$. As in [F1, p. 623] we then have

$$\begin{aligned} 1 + D = \frac{S}{n\omega_n} &= \int_\Sigma R^{n-2}\sqrt{R^2 + |\nabla R|^2}\, d\sigma \\ &= \int_\Sigma (1+u)^{n-1}\sqrt{1 + (1+u)^{-2}|\nabla u|^2}\, d\sigma, \end{aligned} \tag{3.1}$$

$$\frac{V}{\omega_n} = \int_\Sigma (1+u)^n\, d\sigma \quad \left(= \int_\Sigma 1\, d\sigma = 1\right), \tag{3.2}$$

$$b = \int_\Sigma \big(1 + u(\xi)\big)^{n+1} \xi\, d\sigma(\xi) \quad (= 0). \tag{3.3}$$

Similarly, from (2.2) above,

$$2\beta = \int_\Sigma |(1+u)^n - 1|\, d\sigma. \tag{3.4}$$

In the first approximation, (3.2) and (3.3) imply that $u_0 \approx 0$, $u_1 \approx 0$. More precisely we have, as the spherical deviation $d = \|u\|_\infty$ tends to 0,

$$\|u_0\|_\infty,\ \|u_1\|_\infty = O(1)\|u\|^2 = O(d)\|u\|_1. \tag{3.5}$$

(Here and elsewhere the Landau symbol $O(\cdot)$ is understood to apply uniformly with respect to the strongly starshaped domain K for any prescribed dimension n. In some cases $O(\cdot)$ may take negative values.) From (3.2) we get in fact

$$0 = \int_\Sigma \big((1+u)^n - 1\big)\, d\sigma = nu_0 + O(1)\|u\|^2,$$

whence $\|u_0\|_\infty = O(\|u\|^2)$. Also, $\|u\|^2 \le \|u\|_\infty \|u\|_1 = d\|u\|_1$. From (3.3) and $\int u_1\, d\sigma = 0$ (since $u_1 \in \mathcal{H}_1$) we obtain

$$\begin{aligned}\|u_1\|^2 = \int_\Sigma uu_1\, d\sigma &= \frac{-1}{n+1}\int_\Sigma \big((1+u)^{n+1} - 1 - (n+1)u\big)\, u_1\, d\sigma \\ &= O(1)\|u_1\|_\infty \|u\|^2,\end{aligned}$$

whence $\|u_1\|_\infty = O(\|u\|^2) = O(d)\|u\|_1$ because $\|u_1\|_\infty$ equals a positive constant times $\|u_1\|$.

Definition 3. For any function $u \in L^1(\sigma)$ we write

$$\|u\|_* = \min\big\{\|u - l\|_1 \;\big|\; l \in \mathcal{H}_1\big\},$$

the $L^1(\sigma)$-distance between u and the n-dimensional subspace $\mathcal{H}_1$ of all linear functions (restricted to Σ). Thus $\|u\|_*$ is the quotient norm on $L^1(\sigma)/\mathcal{H}_1$.

Remark 3.1. Clearly $\|u\|_* \le \|u\|_1$. The following estimate in the opposite direction will be used in Remark 3.2 below and in Section 7. Consider any $u \in L^1(\sigma)$ orthogonal to $\mathcal{H}_1 : \int u\xi_j\, d\sigma = 0$ $(j = 1, \dots, n)$, and any minimizing $l \in \mathcal{H}_1$ in the above definition. Then

$$\|l\|_1 \le q\|u\|_* \tag{3.6}$$

with a constant $q = q(n)$ to be determined below. It follows that

$$\|u\|_1 \le \|u - l\|_1 + \|l\|_1 \le (1+q)\|u\|_*. \tag{3.7}$$

In fact,

$$\|l\|_2^2 = \int_\Sigma l(l-u)\, d\sigma \le \|l\|_\infty \|l - u\|_1 = \|l\|_\infty \|u\|_*,$$

and since l is a constant multiple of ξ_1 after a change of coordinates, (3.6) ensues with

$$q = \frac{\|l\|_1 \|l\|_\infty}{\|l\|_2^2} = n\int_\Sigma |\xi_1|\, d\sigma = 1\Big/\int_0^1 (1-t^2)^{\frac{n-1}{2}}\, dt = \frac{2}{\sqrt{\pi}}\,\frac{\Gamma(\frac{n+2}{2})}{\Gamma(\frac{n+1}{2})}.$$

This constant q is best possible in (3.6) as well as in (3.7). We shall not use this fact; it can be shown by taking u (identified with the measure $u\, d\sigma$) weak* close to the measure $\varepsilon_a - \varepsilon_{-a} - 2n\xi_1\, d\sigma$ (orthogonal to $\mathcal{H}_1$), where e.g. ε_a denotes unit mass at $a = (1, 0, \dots, 0)$.

If u is *even*: $u(-\xi) = u(\xi)$ for $\xi \in \Sigma$, then $\|u\|_* = \|u\|_1$. In fact, for any $l \in \mathcal{H}_1$, $2\|u\|_1 \le \|u - l\|_1 + \|u + l\|_1 = 2\|u - l\|_1$ since $l(-\xi) = -l(\xi)$.

Remark 3.2. For any function $u \in L^2(\sigma)$ write

$$\tilde{u} = u - u_0 - u_1 = \sum_{k=2}^{\infty} u_k. \tag{3.8}$$

Returning to strongly starshaped domains K in $\mathbf{R}^n$ we then have (in the notation explained in the beginning of the present section):

$$\|u_0\|_\infty + \|u_1\|_\infty = \|\tilde{u}\|_1 O(d) = \|\tilde{u}\|_* O(d), \tag{3.9}$$

$$\|u\|_1 = \|\tilde{u}\|_1(1 + O(d)), \tag{3.10}$$

$$\|u\|_* = \|\tilde{u}\|_*(1 + O(d)), \tag{3.11}$$

$$\|\nabla u\|^2 - (n-1)\|u\|^2 = \big(\|\nabla \tilde{u}\|^2 - (n-1)\|\tilde{u}\|^2\big)\big(1 + O(d^2)\big). \tag{3.12}$$

From (3.5) we have in fact

$$\|u_0\|_\infty + \|u_1\|_\infty = \|u\|_1 O(d) = (\|\tilde{u}\|_1 + \|u_0\|_\infty + \|u_1\|_\infty) O(d),$$

from which the former equation (3.9) follows, and it implies the latter by application of (3.7) to $\tilde{u}$ (which is indeed orthogonal to $\mathcal{H}_1$). Next, (3.10) and (3.11) follow from (3.9) and the triangle inequality. Finally, (3.12) is obtained by use of Lemma 2:

$$\begin{aligned}\|\nabla u\|^2 - (n-1)\|u\|^2 &= \|\nabla \tilde{u}\|^2 - (n-1)\|\tilde{u}\|^2 - (n-1)\|u_0\|^2 \\ &= \big(\|\nabla \tilde{u}\|^2 - (n-1)\|\tilde{u}\|^2\big)\big(1 + O(d^2)\big),\end{aligned}$$

noting that

$$\|u_0\|^2 = O(d^2)\|\tilde{u}\|_1^2 = O(d^2)\big(\|\nabla \tilde{u}\|^2 - (n-1)\|\tilde{u}\|^2\big)$$

according to (3.9) and the last two lines of Section 2 applied to $\tilde{u}$.

Lemma 3.1. *For strongly starshaped domains K in $\mathbf{R}^n$ we have*

$$\begin{aligned} D &= \tfrac{1}{2}\big(\|\nabla u\|^2 - (n-1)\|u\|^2\big)\big(1 + O\big(d + \|\nabla u\|_\infty^2\big)\big), \\ \beta &= \tfrac{n}{2}\|u\|_1\big(1 + O(d)\big), \\ \alpha &= \tfrac{n}{2}\|u\|_*\big(1 + O(d)\big), \\ |F| &= v\, O(d), \end{aligned}$$

where F denotes the compact set of points x in $\mathbf{R}^n$ realizing the minimum in the definition (2.1) of α, and $|F| := \max_{x \in F} |x|$, while v is the volume radius of K.

In view of Remark 3.2 the stated expressions for D, β, and α remain in force if u is replaced throughout by $\tilde{u} = u - u_0 - u_1$ from (3.8), except in the term $\|\nabla u\|_\infty^2$.

Partial proof. Ad D. The term $-\frac{1}{2}(n-1)\|u\|^2$ arises when the term $\int u\, d\sigma$ is eliminated from $\int(1+u)^{n-1}\, d\sigma$ by use of (3.2), keeping only terms order 2 at most; and the term $\frac{1}{2}\|\nabla u\|^2$ is obvious. See Section 7 for a complete proof.

Ad β. According to (3.4) we have

$$2\beta = \int_\Sigma |(1+u)^n - 1|\, d\sigma = \int_\Sigma \left|\sum_{j=1}^n \binom{n}{j} u^j\right| d\sigma,$$

$$\left|2\beta - n\|u\|_1\right| \leq \int_\Sigma \sum_{j=2}^n \binom{n}{j} d^{j-1}|u|\, d\sigma = O(d)\|u\|_1.$$

Ad α. We may assume that K is normalized. For any $x \in F$ (see the notation at the end of the lemma) the representation of ∂K in polar coordinates centred at x rather than at the barycentre 0 is, in the first approximation, $R = 1 + u(\xi) - l(\xi)$ with $l(\xi) = x \cdot \xi$. This is because $\|x\|$ is small for small d, by the final estimate of the lemma. In view of Definition 3 and the above proof concerning β this explains the main term $\frac{n}{2}\|u\|_*$. See Section 7 for a complete proof.

Ad $|F|$. This estimate is used only in the proof of the above expression for α and will be established in Section 7. □

Remark 3.3. For *convex* domains K the remainder term $O\left(d + \|\nabla u\|_\infty^2\right)$ in the expression for D in Lemma 3.1 can be replaced by $O(d)$ because $\|\nabla u\|_\infty^2 = O(d)$ according to [F1, Lemma 2.2]. Even for non-convex K this replacement can be made in the estimate of D from *above* (i.e., with the equality sign replaced by $\leq$), see the proof in Section 7.

Without discussion of the remainder term, the principal term in the expression for D in Lemma 3.1, expanded in spherical harmonics, was given for $n = 3$ in [PS, p. 33].

Lemma 3.2. *For any C^2-smooth function u on Σ such that $u_0 = u_1 = 0$ there exists a C^2-smooth function $u(t, \xi)$, defined for real t in a neighbourhood of 0 and for $\xi \in \Sigma$, such that $u(0, \xi) = u(\xi)$ and that the set*

$$K(t) := \{r\xi \mid \xi \in \Sigma,\ 0 \leq r \leq 1 + tu(t, \xi)\} \tag{3.13}$$

is convex and normalized (i.e., $K(t)$ has volume ω_n and barycentre 0). For $t \to 0$,

$$\|tu(t, \cdot)\|_\infty,\ \|\nabla(tu(t, \cdot))\|_\infty = O(|t|). \tag{3.14}$$

Proof. For $s = (s_0, s_1, \ldots, s_n) \in \mathbf{R}^{n+1}$ write

$$u_s(\xi) = u(\xi) + s_0 + \sum_{j=1}^{n} s_j \xi_j, \qquad \xi \in \Sigma.$$

Guided by (3.2), (3.3) we consider the following polynomials $f_0, f_1, \ldots, f_n$ in $(s, t) \in \mathbf{R}^{n+2}$, all of which take the value 0 at $(s, t) = (0, 0)$:

$$\begin{aligned} f_0(s,t) &= t^{-1} \int_\Sigma \big((1+tu_s)^n - 1\big)\, d\sigma \quad (\text{for } t \neq 0) \\ &= ns_0 + \sum_{k=2}^{n} \binom{n}{k} t^{k-1} \int_\Sigma (u_s)^k d\sigma, \end{aligned}$$

and for $j = 1, 2, \ldots, n$:

$$\begin{aligned} f_j(s,t) &= t^{-1} \int_\Sigma \big((1+tu_s(\xi))^{n+1} - 1\big)\xi_j d\sigma(\xi) \quad (\text{for } t \neq 0) \\ &= \frac{n+1}{n} s_j + \sum_{k=2}^{n+1} \binom{n+1}{k} t^{k-1} \int_\Sigma (u_s(\xi))^k \xi_j \, d\sigma(\xi), \end{aligned}$$

where we have used that $\int u\, d\sigma = 0$, $\int u\xi_j\, d\sigma = 0$, $\int d\sigma = 1$, $\int \xi_j\, d\sigma(\xi) = 0$, and $\int \xi_i \xi_j\, d\sigma(\xi) = n^{-1}\delta_{ij}$. At $(s, t) = (0, 0)$ we thus have $\partial f_0/\partial s_0 = n$, $\partial f_j/\partial s_j = (n+1)/n$ for $j > 0$, and $\partial f_j/\partial s_k = 0$ for $j \neq k$. By the implicit function theorem the equations $f_j(s, t) = 0$, $j = 0, 1, \ldots, n$, can be solved near the origin in $\mathbf{R}^{n+2}$ in the form

$$s = s(t) = \big(s_0(t), s_1(t), \ldots, s_n(t)\big),$$

where $s(\cdot)$ is analytic in some interval $I = [-\tau, \tau]$, and $s(0) = 0$. Writing

$$u(t, \xi) = u_{s(t)}(\xi) = u(\xi) + s_0(t) + \sum_{j=1}^{n} s_j(t)\xi_j,$$

the function $u(\cdot\,, \cdot)$ is C^2-smooth on $I \times \Sigma$, and $u(0, \xi) = u(\xi)$. The estimates (3.14) are obvious by the compactness of I and Σ. We may therefore take τ small enough so that $1 + t\,u(t, \xi) > 0$ for $(t, \xi) \in I \times \Sigma$. The set $K(t)$ defined in (3.13) is then normalized for each $t \in I$ in view of (3.2), (3.3) because $f_j(s(t), t) = 0$, $j = 0, 1, \ldots, n$. It remains to establish the convexity of $K(t)$ for small $|t|$. It is convenient to extend the function $u(t, \xi)$ to a C^2-smooth function on $I \times \mathbf{R}^n$, likewise denoted by $u(\cdot\,, \cdot)$. Consider any

2-dimensional linear subspace E of $\mathbf{R}^n$, and choose an orthonormal base η, ζ for E. Then $K(t) \cap E$ is given in polar coordinates (r, θ) by $0 \le r \le R(t, \theta)$, where

$$R = R(t, \theta) := 1 + t\, u(t, \eta \cos\theta + \zeta \sin\theta)$$

is of class C^2 on $I \times \mathbf{R}$. Denoting partial differentiation w.r.t. θ by a dash we have

$$R = 1 + O(|t|), \quad R' = O(|t|), \quad R'' = O(|t|)$$

uniformly w.r.t. $\theta \in \mathbf{R}$, $t \in I$, and also w.r.t. E and its orthonormal base η, ζ. This is shown much like (3.14) above by application of the chain rule of differentiation while observing that $\eta \cos\theta + \zeta \sin\theta \in \Sigma$ and that Σ and I are compact. It follows that there exists a number τ_0, $0 < \tau_0 \le \tau$, independent of η, ζ and hence of E, such that

$$R^2 + 2(R')^2 - RR'' = 1 + O(|t|) > 0$$

for every θ provided that $|t| < \tau_0$. In view of the expression for the curvature of a planar curve given in polar coordinates, the above inequality shows that $K(t) \cap E$ has positively curved boundary and hence is convex, provided that $|t| < \tau_0$. Since this holds for any choice of E, $K(t)$ is itself convex when $|t| < \tau_0$. □

Theorem 3. *For strongly starshaped domains K in $\mathbf{R}^n$ we have*

$$\begin{aligned} D &\ge \tfrac{1}{2}(n+1)\|u\|^2\bigl(1 + O\bigl(d + \|\nabla u\|_\infty^2\bigr)\bigr) \\ &\ge \frac{2(n+1)}{n^2}\beta^2\bigl(1 + O\bigl(d + \|\nabla u\|_\infty^2\bigr)\bigr). \end{aligned} \tag{3.15}$$

The constant $\frac{1}{2}(n+1) = \frac{1}{2}(\lambda_2 - \lambda_1)$ in the former inequality is best possible.

The best possible constant $c(n)$, resp. $c_*(n)$, in the ensuing inequality

$$D \ge \frac{2}{n^2}c(n)\beta^2\bigl(1 + O\bigl(d + \|\nabla u\|_\infty^2\bigr)\bigr), \tag{3.16}$$

$$D \ge \frac{2}{n^2}c_*(n)\alpha^2\bigl(1 + O\bigl(d + \|\nabla u\|_\infty^2\bigr)\bigr), \tag{3.17}$$

respectively, for strongly starshaped domains is the same as for convex domains, and is also the best possible constant in the quadratic inequality

$$\|\nabla u\|^2 - (n-1)\|u\|^2 \ge c(n)\|u\|_1^2, \tag{3.18}$$

$$\|\nabla u\|^2 - (n-1)\|u\|^2 \ge c_*(n)\|u\|_*^2, \tag{3.19}$$

respectively, valid for all $u \in W^{1,2}(\Sigma)$ for which $u_0 = u_1 = 0$, i.e.,

$$\int_\Sigma u\, d\sigma = 0, \quad \int_\Sigma u(\xi)\xi_j\, d\sigma(\xi) = 0 \ \text{ for } j = 1, \dots, n.$$

We have

$$c_*(n) \ge c(n) \ge n + 1. \tag{3.20}$$

Proof. As usual we represent the boundary of the normalized domain K_0 in polar coordinates as $x = (1+u(\xi))\xi, \xi \in \Sigma$, whereby $u \in \mathrm{Lip}_1(\Sigma) = W^{1,\infty}(\Sigma)$. Expanding in spherical harmonics we obtain by Lemma 2, taking into account that $\lambda_0 = 0$, $\lambda_1 = n-1$, and $\lambda_k \geq \lambda_2$ for $k \geq 2$:

$$\begin{aligned}\|\nabla u\|^2 - (n-1)\|u\|^2 &= \sum_{k=0}^{\infty}(\lambda_k - \lambda_1)\|u_k\|^2 \\ &\geq \sum_{k=0}^{\infty}(\lambda_2 - \lambda_1)\|u_k\|^2 - \lambda_2\|u_0\|^2 - (\lambda_2 - \lambda_1)\|u_1\|^2 \\ &= (\lambda_2 - \lambda_1)\|u\|^2\big(1 + O(d^2)\big)\end{aligned}$$

in view of (3.5). Since $\lambda_2 - \lambda_1 = n+1$, this leads to the former inequality (3.15) in view of Lemma 3.1. The constant $n+1$ in that inequality is best possible (even for convex K) in view of the final statement in Section 2 together with Lemma 3.2 applied to some non-zero $u \in \mathcal{H}_2$. The second inequality (3.15) follows likewise from Lemma 3.1 since $\|u\| \geq \|u\|_1$. By comparing the ultimate inequality (3.15) with (3.16) we see that $c(n) \geq n+1$, while $c_*(n) \geq c(n)$ follows from $\beta \geq \alpha$, thus establishing (3.20). From the comment after Lemma 3.1 we also see that e.g. (3.18) (applied to $\tilde{u}$) implies (3.16). Invoking also Lemma 3.2, we see that, conversely, (3.16) implies (3.18) in the case where u is C^2-smooth. For general $u \in W^{1,2}$ with $u_0 = u_1 = 0$ we merely approximate u in $W^{1,2}$-norm by C^2-smooth functions v (by regularization). Then $v_0 \to u_0 = 0$ and $v_1 \to u_1 = 0$. It follows that the function $w = v - v_0 - v_1$ is C^2-smooth, $w_0 = w_1 = 0$, and $w \to u$ (in $W^{1,2}$). The validity of (3.18) for u therefore follows from its validity for w. Similarly, (3.17) and (3.19) are equivalent. □

Remark 3.4. The condition $u_1 = 0$ is unnecessary in (3.19) because either member of the inequality remains unchanged if u is replaced by $u - l$ for some $l \in \mathcal{H}_1$. As to the left hand member this is because $\lambda_1 = n-1$, cf. (2.5).

Remark 3.5. For *convex* domains K the remainder term $O\big(d + \|\nabla u\|_\infty^2\big)$ in Theorem 3 can be replaced by $O(d)$ in view of [F1, Lemma 2.2]. For *planar* convex domains $O\big(d + \|\nabla u\|_\infty^2\big)$ may further be replaced by $O(\beta)$ in (3.16) and hence in the ultimate inequality (3.15). In fact, for any convex domain $K \subset \mathbf{R}^2$ such that $D < \frac{1}{2}c(2)\beta^2$ we have from Bonnesen's inequality (see (1.14) in the Introduction): $d = O(\delta) = O(D^{\frac{1}{2}}) = O(\beta)$. (As to the relation $d = O(\delta)$ see [F1, p. 634].) Similarly $O\big(d + \|\nabla u\|_\infty^2\big)$ can be replaced by $O(\alpha)$ in (3.17) in the case of convex domains in $\mathbf{R}^2$; this leads to [HH, Theorem 1], where $c_*(2)$ is found to be $\pi/(4-\pi)$, as will be recovered in Section 5. – For convex domains $K \subset \mathbf{R}^n$ with $n \geq 3$ we may similarly replace $O\big(d + \|\nabla u\|_\infty^2\big)$, e.g. in (3.16), by $O\big(\beta\sqrt{\log(1/\beta)}\big)$ if $n = 3$, and by $O\big(\beta^{\frac{4}{n+1}}\big)$ if $n \geq 4$. (In the above argument replace Bonnesen's inequality by the n-dimensional version of it, obtained in [F1, Theorem 2.3].)

4. The infinitesimal version. Duality

In view of Theorem 3 we are led to investigate the best possible constants $c(n)$, $c_*(n)$ in (3.18), (3.19), respectively; that is (in the notation of Section 2):

$$c(n) = \min\left\{ \frac{\|\nabla u\|^2 - (n-1)\|u\|^2}{\|u\|_1^2} \;\middle|\; u \in W^{1,2}(\Sigma) \setminus \{0\},\; u_0 = u_1 = 0 \right\}, \tag{4.1}$$

$$c_*(n) = \min\left\{ \frac{\|\nabla u\|^2 - (n-1)\|u\|^2}{\|u\|_*^2} \;\middle|\; u \in W^{1,2}(\Sigma) \setminus \{0\},\; u_0 = 0 \right\}, \tag{4.2}$$

where $\|u\|_*$ denotes the quotient norm on $L^1(\sigma)/\mathcal{H}_1$ (Definition 3).

The fact that there are actual minima in (4.1), (4.2) derives from the compactness of the identity map from $W^{1,2}(\Sigma)$ with the Sobolev norm $\|u\|_{1,2} = \sqrt{\|\nabla u\|^2 + \|u\|^2}$ into $L^2(\sigma)$ with the norm $\|u\|$; this is Rellich's theorem [R] (applied in local coordinates on Σ). Also note that, on the relevant subspace (cf. Remark 3.4 in the case of $c_*(n)$)

$$\{u \in W^{1,2}(\Sigma) \mid u_0 = u_1 = 0\},$$

$\|u\|_{1,2}$ and $\left(\|\nabla u\|^2 - (n-1)\|u\|^2\right)^{\frac{1}{2}}$ are equivalent norms because, by Lemma 2,

$$\begin{aligned} \|u\|_{1,2}^2 &= \sum_{k=2}^{\infty} (\lambda_k + 1)\|u_k\|^2 \\ &\le \frac{2n+1}{n+1} \sum_{k=2}^{\infty} (\lambda_k - \lambda_1)\|u_k\|^2 = \frac{2n+1}{n+1}\left(\|\nabla u\|^2 - (n-1)\|u\|^2\right). \end{aligned}$$

Remark 4.1. The minimum in (4.2) remains the same if u is subjected to the further condition $\|u\|_1 = \|u\|_*$. In fact, if $\|u\|_1 > \|u\|_*$ we may replace u by $u + l$ with $l \in \mathcal{H}_1$ so chosen that $\|u + l\|_1 = \|u\|_*$, cf. Definition 3; this substitution leaves u_0, $\|u\|_*$, and $\|\nabla u\|^2 - (n-1)\|u\|^2$ unchanged, cf. Remark 3.4.

Lemma 4.1. *If $u \in L^1(\sigma)$ and $\|u\|_* = \|u\|_1$ then the function f defined by*

$$f(\xi) = \operatorname{sgn} u(\xi) = \begin{cases} 1 & \text{if } u(\xi) > 0 \\ -1 & \text{if } u(\xi) < 0 \end{cases}$$

can be extended to a function $f \in L^\infty(\sigma)$ such that $\|f\|_\infty \le 1$ and $f_1 = 0$ (i.e., $\int fl\, d\sigma = 0$, $l \in \mathcal{H}_1$). In particular, if $u(\xi) \ne 0$ a.e., then $f = \operatorname{sgn} u$ satisfies $f_1 = 0$.

Proof. Write

$$E = \{\xi \in \Sigma \mid u(\xi) = 0\}, \quad E_\varepsilon = \{\xi \in \Sigma \mid |u(\xi)| < \varepsilon\}$$

for $\varepsilon > 0$. We first show, by a variational argument, that

$$\left| \int_{\complement E} (\operatorname{sgn} u) l \, d\sigma \right| \le \int_E |l| \, d\sigma, \qquad l \in \mathcal{H}_1. \tag{4.3}$$

We may assume that $\|l\|_\infty \le 1$ and that $\int_{\complement E} (\operatorname{sgn} u) l \, d\sigma \ge 0$. Then

$$\begin{aligned}
\|u\|_1 = \|u\|_* \le \|u - \varepsilon l\|_1 \\
&= \int_{\complement E_\varepsilon} |u - \varepsilon l| \, d\sigma + \varepsilon \int_E |l| \, d\sigma + \int_{E_\varepsilon \setminus E} |u - \varepsilon l| \, d\sigma \\
&= \int_{\complement E_\varepsilon} (\operatorname{sgn} u)(u - \varepsilon l) \, d\sigma + \varepsilon \int_E |l| \, d\sigma + \int_{E_\varepsilon \setminus E} |u - \varepsilon l| \, d\sigma \\
&= \int_{\complement E} (\operatorname{sgn} u)(u - \varepsilon l) \, d\sigma + \varepsilon \int_E |l| \, d\sigma + O(\varepsilon)\sigma(E_\varepsilon \setminus E) \\
&= \|u\|_1 - \varepsilon \left(\int_{\complement E} (\operatorname{sgn} u) l \, d\sigma - \int_E |l| \, d\sigma \right) + o(\varepsilon),
\end{aligned}$$

which is only possible if (4.3) holds.

If $\sigma(E) = 0$, there is nothing left to be proved, so suppose that $\sigma(E) > 0$. The restriction map $l \mapsto l_{|E}$ of $\mathcal{H}_1$ into $L^1(E, \sigma) = L^1(E)$ is then injective because any $(n-1)$-dimensional subspace of $\mathbf{R}^n$ meets Σ in a null set for σ. We may therefore define a linear form $\varphi : \{l_{|E} \mid l \in \mathcal{H}_1\} \to \mathbf{R}$ by

$$\varphi(l_{|E}) = -\int_{\complement E} (\operatorname{sgn} u) l \, d\sigma, \qquad l \in \mathcal{H}_1.$$

By (4.3), $|\varphi(l_{|E})| \le \int_E |l| \, d\sigma = \|l_{|E}\|_{L^1(E)}$, and so φ extends, by the Hahn-Banach Theorem, to a linear form φ on $L^1(E)$ such that

$$|\varphi(g)| \le \|g\|_{L^1(E)}, \qquad g \in L^1(E).$$

There exists $f \in L^\infty(E)$ with $\varphi(g) = \int_E fg \, d\sigma$ for all $g \in L^1(E)$, and $\|f\|_{L^\infty(E)} = \|\varphi\|_{(L^1(E))^*} \le 1$. In particular,

$$-\int_{\complement E} (\operatorname{sgn} u) l \, d\sigma = \int_E fl \, d\sigma, \qquad l \in \mathcal{H}_1,$$

and so the function f which equals the above f in E, and $\operatorname{sgn} u$ in $\complement E$, satisfies $\|f\|_\infty \le 1$ and $f_1 = 0$. □

The following dual characterization of $c(n)$ and $c_*(n)$ was inspired by [HHW], [HH] (in which $n = 2$).

Theorem 4.1. *We have* $c(n) = 1/\kappa(n)$, $c_*(n) = 1/\kappa_*(n)$, *where*

$$\kappa(n) := \max\left\{\sum_{k=2}^{\infty} \frac{\|f_k\|^2}{\lambda_k - \lambda_1} \;\middle|\; \|f\|_\infty = 1\right\}, \tag{4.4}$$

$$\kappa_*(n) := \max\left\{\sum_{k=2}^{\infty} \frac{\|f_k\|^2}{\lambda_k - \lambda_1} \;\middle|\; \|f\|_\infty = 1,\ f_1 = 0\right\}. \tag{4.5}$$

Proof. First of all, there is an actual maximum in (4.4) and in (4.5). To see this, we define for $f \in L^2(\sigma)$

$$Tf = \sum_{k=2}^{\infty} \frac{f_k}{\lambda_k - \lambda_1} \tag{4.6}$$

(convergent in the $L^2(\sigma)$-norm). Here T is an integral operator with a symmetric kernel $(\xi, \eta) \mapsto \tilde{G}(\xi \cdot \eta)$ determined in Section 8 with reference to [Be]. At the present stage it suffices, however, to note that $Tf \in \operatorname{dom} \Delta$ and that

$$\big(-\Delta - (n-1)\big)Tf = \tilde{f} := \sum_{k=2}^{\infty} f_k = f - f_0 - f_1 \tag{4.7}$$

in the notation employed in Remark 3.2. (This follows from (4.6) because $-\Delta f_k = \lambda_k f_k$ and $\lambda_1 = n - 1$.) Since $\lambda_k - \lambda_1 \to \infty$ as $k \to \infty$, the self-adjoint operator $T : L^2(\sigma) \to L^2(\sigma)$ is compact, and the quadratic form

$$\Lambda(f) := \int_\Sigma (Tf) f \, d\sigma = \sum_{k=2}^{\infty} \frac{\|f_k\|^2}{\lambda_k - \lambda_1} \tag{4.8}$$

is therefore continuous as a function of f in the weak topology on $L^2(\sigma)$, *a fortiori* in the weak* topology on $L^\infty(\sigma)$ viewed as the dual of $L^1(\sigma)$. Because the unit ball in $L^\infty(\sigma)$ is weak* compact, $\Lambda(f)$ has an actual maximum $\kappa(n)$ when considered on this unit ball, and by homogeneity this maximum is attained on the unit sphere $\{f \in L^\infty(\sigma) \mid \|f\|_\infty = 1\}$. Similarly as to $\kappa_*(n)$ because the condition $f_1 = 0$ is equivalent to $\int fl \, d\sigma = 0$ for all $l \in \mathcal{H}_1$, and hence determines a weak* closed subspace of $L^\infty(\sigma)$.

We bring the rest of the proof for the case of $c_*(n)$, $\kappa_*(n)$, the case of $c(n)$, $\kappa(n)$ being similar and slightly easier.

1° $\kappa_*(n)c_*(n) \geq 1$. Consider any non-zero function $u \in W^{1,2}(\Sigma)$ with $u_0 = 0$ such that $\|\nabla u\|^2 - (n-1)\|u\|^2 = c_*(n)\|u\|_*^2$ (briefly: a *minimizing function* for $c_*(n)$, cf. (4.2)). According to Remark 4.1 we may suppose that

$$\|u\|_1 = \|u\|_*.$$

Choose $f \in L^\infty(\sigma)$ as in Lemma 4.1 (i.e., $\|f\|_\infty \le 1$, $f_1 = 0$, and $f(\xi) = \operatorname{sgn} u(\xi)$ for any $\xi \in \Sigma$ with $u(\xi) \neq 0$). Then

$$\|u\|_* = \int f u \, d\sigma = \sum_{k=2}^{\infty} \int \frac{f_k}{\sqrt{\lambda_k - \lambda_1}} u_k \sqrt{\lambda_k - \lambda_1} \, d\sigma, \tag{4.9}$$

$$\begin{aligned} \|u\|_*^2 &\le \sum_{k=2}^{\infty} \frac{\|f_k\|^2}{\lambda_k - \lambda_1} \sum_{k=2}^{\infty} (\lambda_k - \lambda_1)\|u_k\|^2 \\ &\le \kappa_*(n)\big(\|\nabla u\|^2 - (n-1)\|u\|^2\big) \end{aligned} \tag{4.10}$$

by Lemma 2 and the Cauchy-Schwarz inequality applied to the vectors $\sum_2^\infty u_k \sqrt{\lambda_k - \lambda_1}$ and $\sum_2^\infty f_k / \sqrt{\lambda_k - \lambda_1}$ in the Hilbert space $L^2(\sigma) = \bigoplus_{k=0}^\infty \mathcal{H}_k$. It follows from (4.10) that indeed $\kappa_*(n) c_*(n) \ge 1$ because u is minimizing for $c_*(n)$.

2° $\kappa_*(n)c_*(n) \le 1$. Consider any $f \in L^\infty(\sigma)$ with $f_1 = 0$ such that $\Lambda(f) = \kappa_*(n)$ (briefly: a *maximizing function* for $\kappa_*(n)$), and write as in (4.7),

$$\tilde{f} := \sum_{k=2}^{\infty} f_k = f - f_0 - f_1$$

($= f - f_0$ in the present case). Choose $l \in \mathcal{H}_1$ so that $\|Tf + l\|_1 = \|Tf\|_*$ (cf. (4.6) and Definition 3), and write $u = Tf + l$, whereby $\|u\|_1 = \|u\|_*$. Then $u \in \operatorname{dom} \Delta$, and since $\lambda_1 = n - 1$ we obtain by (4.7) and Lemma 2

$$-\Delta u - (n-1)u = \tilde{f}, \tag{4.11}$$

$$\|\nabla u\|^2 - (n-1)\|u\|^2 = \int \tilde{f} u \, d\sigma = \int f u \, d\sigma = \int f(Tf) \, d\sigma = \kappa_*(n)$$

because $u_0 = f_1 = 0$. In view of (4.2) this implies

$$\begin{aligned} \kappa_*(n) = \|\nabla u\|^2 - (n-1)\|u\|^2 &= \int f u \, d\sigma \le \|u\|_1 = \|u\|_* \\ &\le \sqrt{\big(\|\nabla u\|^2 - (n-1)\|u\|^2\big)/c_*(n)} = \sqrt{\kappa_*(n)/c_*(n)}, \end{aligned} \tag{4.12}$$

and consequently $\kappa_*(n) c_*(n) \le 1$. □

Remark 4.2. $\kappa_*(n) \le \kappa(n) < 1/(n+1)$. The former inequality is trivial. In view of Theorem 4.1 above, the latter is a reformulation of (3.20) except for the sharp inequality sign. Here is a direct proof: For any measurable function $f : \Sigma \to [-1, 1]$ (not equal to 0 a.e.) we have from (4.8) because $\lambda_2 - \lambda_1 = n + 1$

$$\Lambda(f) \le \sum_{k=2}^{\infty} \frac{\|f_k\|^2}{\lambda_2 - \lambda_1} \le \frac{1}{n+1}\|f\|^2 \le \frac{1}{n+1}\|f\|_\infty^2.$$

These inequalities cannot all hold with equality, for then f would be of class $\mathcal{H}_2$ and hence $\|f\| < \|f\|_\infty$. An adaptation of this proof leads in principle to a better upper estimate of $\kappa(n)$, e.g. $\kappa(2) < 1/3 - \big((8\sqrt{2}/\pi) - 1\big)/15 \approx 0.3067$, but we shall not go into that because the calculations are complicated for $n > 2$.

Remark 4.3. Every maximizing function f for $\kappa(n)$ or $\kappa_*(n)$ satisfies

$$f(\xi) = \pm 1 \quad \sigma\text{-a.e. on } \Sigma.$$

To establish this, suppose that f is maximizing e.g. for $\kappa_*(n)$, and imagine that the set $F = \{\xi \in \Sigma \mid |f(\xi)| < 1 - \varepsilon\}$ has measure $\sigma(F) > 0$ for some ε, $0 < \varepsilon < 1$. Choose $g \in L^\infty(\sigma)$ with $\|g\|_\infty = 1$ so that $g = 0$ off F, $\int (Tf)g\, d\sigma = 0$ with Tf from (4.6), and finally that $g_0 = 0$, $g_1 = 0$, i.e., $\int gl\, d\sigma = 0$ for all $l \in \mathcal{H}_0 + \mathcal{H}_1$. This is possible since $L^\infty(F, \sigma)$ is infinite dimensional. Then $\|f + \varepsilon g\|_\infty \leq 1$, $(f + \varepsilon g)_1 = 0$, and

$$\int \big(T(f + \varepsilon g)\big)(f + \varepsilon g)\, d\sigma = \int (Tf) f\, d\sigma + \varepsilon^2 \int (Tg) g\, d\sigma > \kappa_*(n),$$

because $\int (Tg) g\, d\sigma = \sum_{k=2}^{\infty} \|g_k\|^2/(\lambda_k - \lambda_1) > 0$. From this contradiction we see that actually $\sigma(F) = 0$ for any choice of ε, and so indeed $f(\xi) = \pm 1$ a.e.

In the next two theorems we establish a bijective correspondence between the set of all (suitably normalized) minimizing functions u for $c(n)$, resp. $c_*(n)$, and the set of all maximizing functions f for $\kappa(n)$, resp. $\kappa_*(n)$. In addition, these theorems contain further properties of the minimizing or maximizing functions in question.

Theorem 4.2. *Any minimizing function u for $c(n)$, resp. $c_*(n)$, is C^1-smooth (after correction on a null set), and*

$$u(\xi) \neq 0 \quad \text{a.e. on } \Sigma. \tag{4.13}$$

Let u be such a minimizing function, normalized so that $c(n)\|u\|_1 = 1$, resp. $c_*(n)\|u\|_* = 1$, and suppose in the case of $c_*(n)$ that $\|u\|_1 = \|u\|_*$. Then $f := \operatorname{sgn} u$ is maximizing for $\kappa(n)$, resp. $\kappa_*(n)$, and

$$u = Tf, \quad \text{resp. } u - u_1 = Tf.$$

Consequently, u is in the domain of Δ and satisfies the Euler type equation

$$-\Delta u - (n-1)u = \tilde{f} := \sum_{k=2}^{\infty} f_k \tag{4.14}$$

($= f - f_0$ in the case of $c_*(n)$).

Proof. Again we bring the proof for the case of $c_*(n)$, $\kappa_*(n)$. Suppose then that u is minimizing for $c_*(n)$ and normalized so that $\|u\|_1 = \|u\|_* = \kappa_*(n)$ (viz. $c_*(n)\|u\|_* = 1$), cf. Remark 4.1. Consider any f as in Lemma 4.1. Then (4.9), (4.10) hold, the latter with equality throughout because $\kappa_*(n) = 1/c_*(n)$. In view of (4.9) there is hence a constant $\gamma > 0$ such that

$$u_k\sqrt{\lambda_k - \lambda_1} = \gamma f_k \big/ \sqrt{\lambda_k - \lambda_1}, \quad k \geq 2.$$

Since $u_0 = 0$ it follows by Lemma 2 (with Tf from (4.6)) that

$$u - u_1 = \sum_{k=2}^{\infty} u_k = \gamma \sum_{k=2}^{\infty} f_k/(\lambda_k - \lambda_1) = \gamma Tf, \tag{4.15}$$

$$c_*(n)\|u\|_*^2 = \|\nabla u\|^2 - (n-1)\|u\|^2 = \sum_{k=2}^{\infty}(\lambda_k - \lambda_1)\|u_k\|^2$$

$$= \gamma \sum_{k=2}^{\infty} \int f_k u_k \, d\sigma = \gamma \int f u \, d\sigma = \gamma \|u\|_1 = \gamma \|u\|_*$$

(in the third last equation we used that $u_0 = f_1 = 0$). Consequently, $\gamma = c_*(n)\|u\|_* = 1$, and

$$\sum_{k=2}^{\infty} \frac{\|f_k\|^2}{\lambda_k - \lambda_1} = \sum_{k=2}^{\infty} \int f_k u_k \, d\sigma = \|u\|_* = \kappa_*(n).$$

Thus f is maximizing for $\kappa_*(n)$. In view of (4.15), u is in the domain of Δ, and the Euler equation (4.11) holds with $\tilde{f} = f - f_0$ by (4.7). Since $f \in L^\infty(\sigma)$ this implies by Remark 4.4 below that u is C^1-smooth (after correction on a null set).

It remains to establish (4.13), which implies that the above f equals sgn u a.e. Suppose that the closed set $E := \{\xi \in \Sigma \mid u(\xi) = 0\}$ has measure $\sigma(E) > 0$. It is known that $\nabla u = 0$ a.e. in E because $u \in W^{1,2}(\Sigma)$. (In fact, there exists locally in $\Sigma \backslash E$ a sequence of smooth functions $u^{(j)}$ of compact support such that $u^{(j)} \to u$ in $W^{1,2}$, cf. [DL, p. 359].) Each component of ∇u (in local coordinates on Σ) is likewise of class $W^{1,2}$ because $u \in \text{dom}\,\Delta = W^{2,2}(\Sigma)$ according to [Se, p. 685] or Remark 4.4 below. Consequently, the second order partial derivatives of u (in local coordinates) are likewise null a.e. in E, and so $\tilde{f} = 0$ a.e. in E, by (4.14). It follows that $f = f_0 + f_1$ a.e. in E, and this contradicts $\sigma(E) > 0$ because $f = \pm 1$ a.e. in Σ according to Remark 4.3. (If $f_1 \equiv 0$ note that $-1 < f_0 < 1$ because we cannot have e.g. $f_0 = 1$, for then $f = 1$ a.e. on Σ, hence $u \geq 0$ a.e. on Σ, cf. Lemma 4.1, and this contradicts $u_0 = \int u \, d\sigma = 0$. And if $f_1 \not\equiv 0$, the set where $|f_0 + f_1| = 1$ is either empty or the union of at most two $(n-2)$-dimensional spheres (or single points) on Σ, hence of σ-measure 0.) □

Remark 4.4. Consider any $f \in L^\infty(\sigma)$. By standard regularity theory for elliptic operators every solution u to (4.14) (in particular the function $u = Tf$) is of class $W^{2,p}(\Sigma)$

for every finite p and hence of class $C^1(\Sigma)$ (after correction on a null set) according to the Sobolev embedding theorem, cf. e.g. [Hö1, Th. 4.5.13]. In the first place it follows e.g. from [Hö2, Theorem 17.1.1], applied in local coordinates in Σ, that (4.14) locally has solutions of class $W^{2,p}$, and hence all solutions are of this class (even globally on Σ, by compactness), the solutions of $-\Delta v - (n-1)v = 0$ being analytic.

Theorem 4.3. *Let f be a maximizing function for $\kappa(n)$, resp. $\kappa_*(n)$. In the case of $\kappa(n)$ write $u = Tf$ with T from (4.6). In the case of $\kappa_*(n)$ define $u = Tf + l$, where $l \in \mathcal{H}_1$ is uniquely determined by $\|Tf + l\|_1 = \|Tf\|_*$. In either case u is then minimizing for $c(n)$, resp. $c_*(n)$, and*

$$\operatorname{sgn} u(\xi) = f(\xi)$$

for almost every $\xi \in \Sigma$. Moreover, u is in the domain of Δ and satisfies the Euler equation (4.14). Finally, $\|u\|_1 = \kappa(n)$, resp. $\|u\|_* = \kappa_*(n)$.

Proof. Again we bring the proof for the case of $c_*(n)$, $\kappa_*(n)$, so suppose that f is maximizing for $\kappa_*(n)$. Consider any $l \in \mathcal{H}_1$ with $\|Tf + l\|_1 = \|Tf\|_*$ (cf. Definition 3), and write $u = Tf + l$. By Theorem 4.1, $\kappa_*(n) = 1/c_*(n)$, and so (4.12) holds with equality throughout. Since $u_0 = 0$ this shows that u is minimizing for $c_*(n)$, that $\|u\|_* = \kappa_*(n)$, and that $f(\xi) = \operatorname{sgn} u(\xi)$ a.e., cf. (4.13). From (4.6) follows again (4.14).

To establish the *uniqueness* (not used in the sequel) of $l \in \mathcal{H}_1$ with $\|Tf + l\|_1 = \|Tf\|_*$, suppose by contradiction that there exists $m \in \mathcal{H}_1$ with $m \neq l$ and $\|Tf + m\|_1 = \|Tf\|_*$. Write $v = Tf + m$; then v is likewise minimizing for $c_*(n)$, and $\operatorname{sgn} v = f$ a.e. With the convention $\operatorname{sgn} 0 = 0$ we infer that $\operatorname{sgn} v = \operatorname{sgn} u$ everywhere, u and v being continuous by Theorem 4.2. Consider the hemispheres

$$\Sigma_+ = \{\xi \in \Sigma \mid m(\xi) > l(\xi)\}, \quad \Sigma_- = \{\xi \in \Sigma \mid m(\xi) < l(\xi)\},$$

and their common boundary $\Sigma_0 = \{m = l\}$. Any point $\xi \in \Sigma$ at which $u(\xi) = 0$ must lie on Σ_0 because also $v(\xi) = 0$, hence $m(\xi) = l(\xi)$. It follows that u has constant sign in Σ_+ and in Σ_-, and these two signs are opposite because $u_0 = 0$. Consequently, $u = 0$ on Σ_0, and either $\operatorname{sgn} u = \operatorname{sgn}(m - l)$ throughout Σ or else $\operatorname{sgn} u = -\operatorname{sgn}(m - l)$ throughout Σ. But in either case this leads to a contradiction:

$$\pm \int_\Sigma |m - l|\, d\sigma = \int_\Sigma (m - l) \operatorname{sgn} u\, d\sigma = \int_\Sigma (m - l) f\, d\sigma = 0$$

because $f_1 = 0$ and $m - l \in \mathcal{H}_1$. □

Remark 4.5. We show in the beginning of Section 8 that the operator T from (4.6) is an integral operator on $L^2(\sigma)$ with a kernel of the form $(\xi, \eta) \mapsto \tilde{G}(\xi \cdot \eta)$, where $\tilde{G}$ is a certain continuous function on $[-1, 1]$ (finite except that $\tilde{G}(1) = +\infty$ when $n > 2$). More precisely, $\tilde{G}(t) = 1/(n-1) + G(t)$, $t \in [-1, 1]$, where $G : [-1, 1] \to\,]-\infty, +\infty]$ (apart from a negative constant factor) equals the kernel g_n constructed by Berg [Be] in his study of potential theory on the unit sphere Σ in $\mathbf{R}^n$ associated with the differential operator $\Delta + (n-1)$. When $f \in L^\infty(\Sigma)$ we thus have

$$Tf(\xi) = \int_\Sigma \tilde{G}(\xi \cdot \eta) f(\eta)\, d\sigma(\eta) \tag{4.16}$$

a.e. for $\xi \in \Sigma$; and because Tf can be taken to be continuous this holds for *every* $\xi \in \Sigma$, the function $\eta \mapsto \tilde{G}(\xi \cdot \eta)$ being integrable, cf. [Be, Theorem 3.3], and the right hand member of (4.16) being a continuous function of ξ, cf. [Be, Prop. 2.9].

We conjecture that the following theorem holds in all dimensions $n \geq 2$, but the method of proof, which is based on an idea in [HH, p. 105] for the case $n = 2$, does not seem adaptable to dimensions $n > 4$.

Theorem 4.4. *Suppose $n \leq 4$. We have*

$$\kappa(n) = \kappa_*(n), \qquad c(n) = c_*(n).$$

The maximizing functions f for $\kappa(n)$ are the same as those for $\kappa_*(n)$. The minimizing functions for $c(n)$ are precisely those minimizing functions u for $c_*(n)$ for which $\|u\|_1 = \|u\|_*$. All the stated maximizing or minimizing functions are even:

$$f(-\xi) = f(\xi), \quad u(-\xi) = u(\xi) \qquad \text{a.e. for } \xi \in \Sigma.$$

Plan of proof. The proof uses Legendre polynomials, spherical harmonics, and potential theory with respect to the operator $\Delta + (n-1)$ on Σ as developed by Berg [Be] for the purpose of studying the first surface measure of a convex body.

In the first part of the proof, given in Section 8, we establish (for $n \leq 4$) the *existence* of even, maximizing functions f for $\kappa(n)$. Any such f satisfies of course $f_1 = 0$ and is therefore *a fortiori* maximizing for $\kappa_*(n)$, and so $\kappa(n) = \kappa_*(n)$. It follows in view of Theorem 4.1 that $c(n) = c_*(n)$. The identity $\kappa(n) = \kappa_*(n)$ implies that any maximizing function for $\kappa_*(n)$ is likewise maximizing for $\kappa(n)$.

In the second part of the proof, given in Section 9, we show that *every* maximizing function for $\kappa(n)$ $(n \leq 4)$ is even.

According to Theorem 4.2, if u is minimizing for $c(n)$ and normalized so that $c(n)\|u\|_1 = 1$ then $f := \operatorname{sgn} u$ is maximizing for $\kappa(n)$, hence even. Consequently, $u = Tf$ is even, whence $\|u\|_1 = \|u\|_*$ by the end of Remark 3.1, and so u is minimizing for $c_*(n)$ as well. Conversely, any minimizing function u for $c_*(n)$ such that $\|u\|_1 = \|u\|_*$ (cf. Remark 4.1) is *a fortiori* minimizing for $c(n)$ because $c(n) = c_*(n)$. □

Corollary. *At least for $n \leq 4$ the best possible constant is the same in (3.16) and in (3.17) in Theorem 3, namely $c(n) = c_*(n)$.*

Stationary functions and values. In addition to the maximizing and normalized minimizing functions considered above we shall discuss more generally *stationary* functions. In this connection it is useful to note that, for any $f \in L^2(\sigma)$, the function $u := Tf$ from (4.6) satisfies by (4.7) the differential equation (4.14); and Tf is the *only* solution to (4.14) such that $u_1 = 0$. In fact, if $u \in \operatorname{dom}\Delta$ denotes any solution to (4.14) with $u_1 = 0$ then $u - Tf$ belongs to $\mathcal{H}_1$, the nullspace of $\Delta + (n-1)$, and hence $u - Tf = 0$ because $(u - Tf)_1 = 0$.

Definition 4.1. A $\kappa(n)$-*stationary function* is a function $f \in L^2(\sigma)$ such that the function $u := Tf$ satisfies $u(\xi) \neq 0$ σ-a.e. and $\operatorname{sgn} u = f$.

The number $\Lambda(f) = \|u\|_1$ is then called the $\kappa(n)$-*stationary value* corresponding to f. Indeed, by (4.8),

$$\Lambda(f) = \int f\, Tf\, d\sigma = \int f u\, d\sigma = \int |u|\, d\sigma = \|u\|_1. \tag{4.17}$$

Definition 4.2. A $c(n)$-*stationary* function is a function $u \in \operatorname{dom}\Delta$ with $u(\xi) \neq 0$ σ-a.e. such that $u = Tf$ holds for $f := \operatorname{sgn} u$. It follows that $u_0 = u_1 = 0$.

The number

$$\frac{\|\nabla u\|^2 - (n-1)\|u\|^2}{\|u\|_1^2} = \frac{1}{\|u\|_1} \tag{4.18}$$

is then called the $c(n)$-*stationary value* corresponding to u. Indeed, by Lemma 2,

$$\begin{aligned} \|\nabla u\|^2 - (n-1)\|u\|^2 &= \int u\big(-\Delta u - (n-1)u\big) d\sigma \\ &= \int u \tilde{f}\, d\sigma = \int u f\, d\sigma = \int |u|\, d\sigma = \|u\|_1 \end{aligned}$$

because $f - \tilde{f} \in \mathcal{H}_0 + \mathcal{H}_1$ and $u_0 = u_1 = 0$. According to Remark 4.4, $u = Tf$ is C^1-smooth. Moreover, u satisfies the Euler equation (4.14), as mentioned above.

A bijective correspondence between the class of all $\kappa(n)$-stationary functions f and the class of all $c(n)$-stationary functions u is clearly given by either of the relations

$$u = Tf, \quad f = \operatorname{sgn} u.$$

The corresponding $\kappa(n)$-stationary and $c(n)$-stationary values $\Lambda(f) = \int (Tf) f\, d\sigma$ and $\big(\|\nabla u\|^2 - (n-1)\|u\|^2\big)/\|u\|_1^2$ are the reciprocals of one another according to (4.17), (4.18).

Every maximizing function for $\kappa(n)$ is $\kappa(n)$-stationary. Every minimizing function u for $c(n)$, normalized so that $c(n)\|u\|_1 = 1$, is $c(n)$-stationary. These assertions follow immediately from Theorems 4.2 and 4.3.

5. The case $n = 2$

Using the general results in Sections 3 and 4 we shall now recover and slightly extend the results from [HHW], [HH] quoted in the Introduction.

We begin by determining on the unit circle Σ in $\mathbf{R}^2$ all those $\kappa(2)$-stationary functions f for which $f_1 = 0$, that is, the Fourier coefficients of order 1 of f considered as a 2π-periodic function of θ are both 0, whereby

$$(\cos\theta, \sin\theta) = (\xi_1, \xi_2) = \xi \in \Sigma.$$

We also determine the associated $c(n)$-stationary functions $u = Tf$, cf. (4.6), and the stationary values. In particular, this will allow us to determine $\kappa(2)$ and $c(2)$ and the associated maximizing, resp. minimizing functions.

In terms of the above coordinate θ the normalized Haar measure on the unit circle Σ is $d\sigma = (2\pi)^{-1}d\theta$, and the Laplace-Beltrami operator takes the form

$$\Delta u = \frac{d^2u}{d\theta^2}.$$

The eigenvalues of $-\Delta$ are $\lambda_k = k^2$, $k = 0, 1, 2, \dots$, and the eigenspace $\mathcal{H}_k \subset L^2(\sigma)$ has for $k \geq 1$ the two orthonormal basis vectors $\sqrt{2}\cos k\theta$, $\sqrt{2}\sin k\theta$, and for $k = 0$ the normalized basis vector 1.

Consider any $\kappa(2)$-stationary function f such that $f_1 = 0$, i.e., the Fourier series of f has the form

$$f(\theta) = a_0 + \sum_{k=2}^{\infty}(a_k \cos k\theta + b_k \sin k\theta), \tag{5.1}$$

where $|a_0| < 1$ because we cannot have $f = 1$ a.e. or $f = -1$ a.e., for that would imply $Tf = 0$ in contradiction with Definition 4.1. The associated $c(n)$-stationary function $u = Tf$ satisfies the Euler equation (4.14) from Theorem 4.2, where $\tilde{f} = f - f_0 = f - a_0$, and we have $\operatorname{sgn} u = f$ a.e.

Now consider any component U_0, resp. U_1, of the open set where $u(\theta) > 0$, resp. $u(\theta) < 0$. In U_j $(j = 0, 1)$ equation (4.14) reads

$$-u'' - u = (-1)^j - a_0, \tag{5.2}$$

and its solutions in these two intervals have the form

$$u = -(-1)^j + a_0 + (-1)^j c_j \cos(\theta - \theta_j), \tag{5.3}$$

where c_j, θ_j are constants, θ_j being the mid-point of U_j since u equals 0 at the end-points of U_j. Because $(-1)^j u(\theta_j) > 0$, we therefore have

$$c_j > 1 - (-1)^j a_0 > 0.$$

More precisely it follows from (5.3) that

$$U_j = \,]\theta_j - \rho_j,\, \theta_j + \rho_j[, \quad \cos \rho_j = \frac{1 - (-1)^j a_0}{c_j} \tag{5.4}$$

with $0 < \rho_j < \pi/2$. By differentiation of (5.3),

$$u' = -(-1)^j c_j \sin(\theta - \theta_j), \tag{5.5}$$

which has the same non-zero absolute values, but opposite signs, at the two end-points $\theta_j \pm \rho_j$ of U_j. Because u is of class C^1 everywhere, the only possibility is that an interval of type U_0 is followed immediately by an interval of type U_1, and vice versa. If U_0 is followed by U_1, let U_2 denote the interval of type U_0 following immediately after U_1, and denote by c_2, ρ_2 the numbers associated with U_2 in the same way as c_0, ρ_0 were associated with U_0 in (5.4). Then (5.5) holds at the end-points of U_1, and this in conjunction with the version of (5.4) with $j = 2$ shows that $(c_2 \cos \rho_2, c_2 \sin \rho_2) = (c_0 \cos \rho_0, c_0 \sin \rho_0)$, and consequently $(c_2, \rho_2) = (c_0, \rho_0)$. Similarly, all intervals of type U_1 have the same associated couple (c_1, ρ_1). The sum of the lengths of U_0 and U_1 must therefore divide 2π, that is,

$$2\rho_0 + 2\rho_1 = \frac{2\pi}{p} \tag{5.6}$$

for some integer $p \geq 2$ (because $\rho_0, \rho_1 < \pi/2$). We have thus shown that u and f have period $2\pi/p$ and that

$$2\pi a_0 = \int_{-\pi}^{\pi} f \, d\theta = p \int_{U_0 \cup U_1} f \, d\theta = 2p(\rho_0 - \rho_1). \tag{5.7}$$

The smoothness of u at the common end-point $\theta_0 + \rho_0 = \theta_1 - \rho_1$ of the closed intervals $\overline{U}_0$ and $\overline{U}_1$ is expressed in view of (5.5) by

$$c_0 \sin \rho_0 = c_1 \sin \rho_1.$$

From (5.6), (5.7) we get $a_0(\rho_0 + \rho_1) = \rho_0 - \rho_1$, that is

$$(1 - a_0)\rho_0 = (1 + a_0)\rho_1.$$

Combining the last two displayed equations with (5.4) leads to

$$\frac{\tan \rho_0}{\rho_0} = \frac{c_0 \sin \rho_0}{(1 - a_0)\rho_0} = \frac{c_1 \sin \rho_1}{(1 + a_0)\rho_1} = \frac{\tan \rho_1}{\rho_1},$$

hence $\rho_0 = \rho_1$, and consequently, by (5.7), (5.6), (5.4),

$$a_0 = 0, \quad \rho_0 = \rho_1 = \frac{\pi}{2p}, \quad c_0 = c_1 = 1 \Big/ \cos \frac{\pi}{2p}.$$

After a translation in the variable θ we may assume that $\theta_0 = 0$, $\theta_1 = \rho_0 + \rho_1 = \pi/p$ in (5.4). Accordingly (5.3) reads (with some integer $p \geq 2$)

$$u = (-1)^j \left(\frac{\cos(\theta - j\frac{\pi}{p})}{\cos\frac{\pi}{2p}} - 1 \right) \qquad \text{for} \qquad \left| \theta - j\frac{\pi}{p} \right| \leq \frac{\pi}{2p}, \tag{5.8}$$

valid for $j = 0, 1$, and in fact for $j = 0, 1, 2, \ldots, 2p - 1$ since u has period $2\pi/p$. We obtain

$$\begin{aligned} \|u\|_1 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} |u(\theta)|\, d\theta = 2p \frac{1}{2\pi} \int_{-\pi/(2p)}^{\pi/(2p)} \left(\frac{\cos\theta}{\cos\frac{\pi}{2p}} - 1 \right) d\theta \\ &= \left(\frac{\pi}{2p} \right)^{-1} \tan\frac{\pi}{2p} - 1. \end{aligned} \tag{5.9}$$

Conversely, the function u defined by (5.8) (with $p \geq 2$) is (π/p)-antiperiodic in the sense that $u(\theta + \pi/p) = -u(\theta)$; and hence $u_0 = u_1 = 0$. Similarly, $f := \operatorname{sgn} u = \operatorname{sgn}(\cos(p\theta))$ satisfies $a_0 := f_0 = f_1 = 0$, and so the Euler equation (5.2), or (4.14), holds. Since $u_1 = 0$ we infer from an observation in the paragraph preceding Definition 4.1 that $u = Tf$, and so u is $c(n)$-stationary and f is $\kappa(n)$-stationary, with $f_1 = 0$.

Summing up, the above analysis establishes the following result.

Theorem 5. *The $\kappa(2)$-stationary functions f of the form (5.1) are precisely the translates of the following functions $f(p, \cdot)$:*

$$f(p, \theta) = \operatorname{sgn}(\cos(p\theta)), \quad p \in \mathbf{N}, \ p \geq 2.$$

The corresponding $c(2)$-stationary functions $u = Tf$ are the translates of the functions $u(p, \cdot)$ given by

$$u(p, \theta) = \frac{\cos\theta}{\cos\frac{\pi}{2p}} - 1 \qquad \textit{for} \quad \theta \in \left[-\frac{\pi}{2p}, \frac{\pi}{2p} \right],$$

continued so as to be (π/p)-antiperiodic: $u(\theta + \pi/p) = -u(\theta)$. The $\kappa(2)$-stationary values are

$$\Lambda\big(f(p, \cdot)\big) = \|u(p, \cdot)\|_1 = \frac{2p}{\pi} \tan\frac{\pi}{2p} - 1.$$

Since the function $\rho \mapsto \rho^{-1} \tan\rho$ occurring in (5.9) is increasing for $0 < \rho < \pi/2$, the $\kappa(2)$-stationary value $\Lambda\big(f(p, \cdot)\big)$ is biggest when p is smallest, i.e. for $p = 2$. By Theorem 4.4 (valid in the present case $n = 2$) the *maximizing* functions $f(\theta)$ for $\kappa(2)$ have period π and are therefore in particular of the form (5.1). Consequently, we have the following corollary of the above theorem, containing the result obtained by Hall and Hayman [HH] quoted in (1.2) in the Introduction:

Corollary. $\kappa(2) = \kappa_*(2) = 4/\pi - 1$. *The maximizing functions* f *for* $\kappa(2)$ *are the translates of the function* $f(\theta) = \operatorname{sgn}(\cos(2\theta))$, *and the minimizing functions* u *for* $c(2)$, *normalized so that* $\|u\|_1 = \kappa(2)$, *are the translates of the function*

$$u(\theta) = \sqrt{2}\cos\theta - 1 \quad \textit{for } \theta \in [-\pi/4, \pi/4],$$

continued so as to be $(\pi/2)$*-antiperiodic.*

Remark 5.1. For the $c(2)$-stationary function $u = u(p, \cdot)$ defined in Theorem 5 the set $\{\xi \in \Sigma \mid u(\xi) \neq 0\}$ has $2p$ connectivity components (arcs). Note that u is an even, resp. odd, function on the circle Σ if p is even, resp. odd.

Remark 5.2. Returning to the geometric interpretation given in Theorem 3 consider convex domains K in $\mathbf{R}^2$ with area $A = A(K)$, perimeter $L = L(K)$, and barycentric asymmetry $\beta = \beta(K) = A(K \setminus B)/A(K)$, where B denotes the disc of equal area $A(B) = A(K)$ centred at the barycentre of K, cf. (2.2). In view of Theorem 3, Remark 3.5, and the inequality $(1 + D)^2 \geq 1 + 2D$, we thus obtain

$$\frac{L^2}{4\pi A} = (1 + D)^2 \geq 1 + \frac{\pi}{4 - \pi}\beta^2 + O(\beta^3) \tag{5.10}$$

as $\beta \to 0$, and $c(2) = 1/\kappa(2) = \pi/(4 - \pi)$ is the best possible constant here. This strengthening of the estimate (1.3) in the Introduction is essentially what was proved in [HHW, p. 109–113], [HH], though with the Steiner disc of K in place of B in the above definition of β. As to $\pi/(4 - \pi)$ being best possible in (5.10), it suffices to produce a family of planar convex domains K_t such that $\beta(K_t) \to 0$ as $t \to 0$ while the equality sign prevails in (5.10). (Similarly for the weaker estimate with β replaced by α.) Such a family (K_t) can be obtained in polar coordinates (r, θ) in the form

$$0 \leq r \leq 1 + tu(\theta)$$

in terms of the solution u from the above Corollary because u is even (as a function on the unit circle) and C^1-smooth. For infinitesimal $t \neq 0$ the C^1-smooth boundary ∂K_t consists of 4 nearly quartercircles, two of which have radius slightly smaller than 1 and separate the other two which have radius slightly bigger than 1, all four circular arcs having their end-points on the unit circle. Also this geometric interpretation is given in [HHW, p. 113], based on an elegant, heuristic argument involving the classical isoperimetric property of circular arcs.

6. The case $n \geq 3$. Examples, estimates, and a conjecture

Our starting point is, for each dimension n, a detailed study of altogether $n - 1$ even $\kappa(n)$-stationary functions and their corresponding $c(n)$-stationary functions (Lemma 6.1, Lemma 6.2).

Lemma 6.1. *For any dimension $n \geq 2$ and any integer $m = 1, 2, \ldots, [n/2]$ (the biggest integer $\leq n/2$) there is precisely one constant α (necessarily with $0 < \alpha < 1$) such that the following even function $f = f(n, m)$ on the unit sphere Σ in $\mathbf{R}^n$ is $\kappa(n)$-stationary (Definition 4.1):*

$$f(\xi) = f(n, m; \xi) = \operatorname{sgn}(z - \alpha), \qquad z = \xi_1^2 + \ldots + \xi_m^2, \tag{6.1}$$

for $\xi = (\xi_1, \ldots, \xi_n) \in \Sigma$. This constant $\alpha = \alpha(n, m)$ is the unique root with $0 < \alpha < 1$ in the transcendental equation (6.10), (6.11) below, involving the hypergeometric functions U, V from (6.8) and B from (6.4). The $c(n)$-stationary function $u = u(n, m) = Tf$ corresponding to f (cf. (4.6) and Definition 4.2) likewise depends only on z from (6.1), and u is given by (6.9). The $\kappa(n)$-stationary value $\Lambda(f) = \|u\|_1$, cf. (4.17), is given by (6.17).

The limitation $m \leq [n/2]$ is only apparent in view of the isometry of Σ taking ξ into $(\xi_{m+1}, \ldots, \xi_n, \xi_1, \ldots, \xi_m)$. Note that, for any constant α, we have $z - \alpha \neq 0$ σ-a.e. on Σ. Clearly, z and hence f and u are *even* functions of ξ. In the particular case $n = 2m$ we have $\alpha = \alpha(2m, m) = 1/2$, see below.

Proof. We use the following parametric representation of Σ consistent with the definition of z in (6.1):

$$\begin{aligned} &\xi = \left(\sqrt{z}\,\eta,\ \sqrt{1-z}\,\zeta\right), \qquad z \in [0, 1], \\ &\eta = (\eta_1, \ldots, \eta_m) \in \Sigma_m, \quad \zeta = (\zeta_1, \ldots, \zeta_{n-m}) \in \Sigma_{n-m}, \end{aligned} \tag{6.2}$$

where e.g. Σ_m denotes the unit sphere in $\mathbf{R}^m$.

Suppose first that the function f in (6.1) is $\kappa(n)$-stationary for a certain α, and denote by $u = Tf$, cf. (4.6), the corresponding $c(n)$-stationary function, which is C^1-smooth. Clearly $0 < \alpha < 1$, for otherwise $f = 1$ or $f = -1$, hence $u = 0$. To see that u depends only on z, note that $f(\xi)$ is invariant under isometries of Σ leaving z invariant, and so is therefore each term $f_k(\xi)$ in the expansion $f = \sum f_k$. It follows that each f_k depends only on z, and so does therefore $u = \sum_{k=2}^{\infty} f_k/(\lambda_k - \lambda_1)$.

Because u is analytic in the two open subsets of Σ where $f = \operatorname{sgn} u = \pm 1$, that is $z \gtrless \alpha$, we see by using the parametric representation (6.2) (writing $z = t^2$ or $1 - z = t^2$ and noting that e.g. $\left(t\,\eta,\ \sqrt{1-t^2}\,\zeta\right)$ remains unchanged if t is replaced by $-t$ and η by $-\eta$) that u as a function of t extends in a neigbourhood of $0 \in \mathbf{C}$ to a function which is even and holomorphic, and hence u as a function of z extends holomorphically near $z = 0$ and $z = 1$.

In terms of the normalized surface measures σ_m, σ_{n-m}, and σ on Σ_m, Σ_{n-m}, and Σ, one finds from (6.2) that

$$d\sigma(\xi) = B(1)^{-1} B'(z)\, dz\, d\sigma_m(\eta)\, d\sigma_{n-m}(\zeta), \tag{6.3}$$

where

$$B(z) = \int_0^z x^{m/2-1}(1-x)^{n/2-m/2-1}\,dx \tag{6.4}$$

is an incomplete betafunction. The Laplace-Beltrami operator Δ on Σ is

$$\Delta = 4z(1-z)\frac{\partial^2}{\partial z^2} + 2(m-nz)\frac{\partial}{\partial z} + \frac{1}{z}\Delta_m + \frac{1}{1-z}\Delta_{n-m}, \tag{6.5}$$

where e.g. Δ_m denotes the Laplace-Beltrami operator on Σ_m with variable point η as in (6.2). In (6.3) and (6.5) there is the limitation $0 < z < 1$.

The Euler equation (4.14) for u as a function of z now reads in view of (6.5)

$$-4z(1-z)\frac{d^2u}{dz^2} - 2(m-nz)\frac{du}{dz} - (n-1)u = f(z) - f_0, \tag{6.6}$$

where $f_0 = \int_\Sigma f\,d\sigma$; this is because f is even and so $f_1 = 0$. The corresponding homogeneous equation (after division by -4) is the standard differential equation for the hypergeometric function $u = F(a, b; c; z)$, $|z| < 1$, with parameters

$$a = -1/2, \quad b = n/2 - 1/2, \quad c = m/2. \tag{6.7}$$

A second solution is known to be $F(a, b; a+b+1-c; 1-z)$, cf. e.g. [E1, (5), p. 105]. In the present situation we thus have in the interval $0 < z < 1$ the linearly independent solutions to the homogeneous equation:

$$\begin{aligned} U(z) &= F(-1/2,\ n/2-1/2;\ m/2;\ z), \\ V(z) &= F(-1/2,\ n/2-1/2;\ n/2-m/2;\ 1-z). \end{aligned} \tag{6.8}$$

Because none of the parameters a, b, c in (6.7) is an integer ≤ 0, U is not a polynomial, and $U(z)$ as a function of a complex variable z is holomorphic for $|z| < 1$ (in particular for $0 \leq z < 1$), but not extendable holomorphically near $z = 1$. (Indeed, the power series of $U(z)$ has radius of convergence 1, and the only possible finite singularities of a solution to (6.6) are at $z = 0$ or 1.) Similarly $V(z)$ is holomorphic for $|1-z| < 1$, but not at $z = 0$.

The mean-value of f from (6.1) over Σ is according to (6.3), (6.4)

$$f_0 = 1 - 2B(\alpha)/B(1).$$

In the intervals of constancy $z \lessgtr \alpha$ for the right hand member of (6.6) this equation has the following constant solutions $\gtrless 0$:

$$\frac{f_0+1}{n-1} = \frac{2}{n-1}\frac{B(1)-B(\alpha)}{B(1)} \quad \text{for } z < \alpha, \qquad \frac{f_0-1}{n-1} = -\frac{2}{n-1}\frac{B(\alpha)}{B(1)} \quad \text{for } z > \alpha.$$

Invoking the solutions (6.8) to the homogeneous equation for $z \lessgtr \alpha$ and taking into account the singularity of U at 1 and of V at 0 we find from $f = \operatorname{sgn} u$ (with $u = u(z)$ continuous for $0 \le z \le 1$) that $u(\alpha) = 0$, $U(\alpha) \ne 0$, $V(\alpha) \ne 0$, and hence we must have

$$u(z) = \begin{cases} -\dfrac{2}{n-1}\dfrac{B(1)-B(\alpha)}{B(1)}\left(\dfrac{U(z)}{U(\alpha)} - 1\right), & 0 \le z \le \alpha \\[2ex] \dfrac{2}{n-1}\dfrac{B(\alpha)}{B(1)}\left(\dfrac{V(z)}{V(\alpha)} - 1\right), & \alpha \le z \le 1. \end{cases} \tag{6.9}$$

The smoothness of $u(z)$ at $z = \alpha$ leads to the following equation serving to determine α:

$$\Phi(\alpha) = 0, \tag{6.10}$$

where

$$\Phi(z) = B(z)U(z)V'(z) + (B(1) - B(z))U'(z)V(z). \tag{6.11}$$

Note that

$$U(0) = V(1) = 1, \quad U'(z) < 0, \quad V'(z) > 0, \tag{6.12}$$

e.g. by use of the hypergeometric series for U and V from (6.8). For example,

$$V(z) = 1 + \sum_{k=1}^{\infty} \frac{(-1/2)_k\,(n/2 - 1/2)_k}{(n/2 - m/2)_k\,k!}(1-z)^k \tag{6.13}$$

for $0 < z \le 1$, using the notation

$$(a)_k = a(a+1)\cdots(a+k-1), \qquad k = 1, 2, \dots .$$

For $m = 1$ this gives $V(z) = \sqrt{z}$, and for arbitrary $m \ge 1$ we therefore find for $0 < z < 1$ by comparison of the (negative) terms in (6.13)

$$V(z) \le -\frac{m-1}{n-m} + \frac{n-1}{n-m}z^{1/2}, \quad V'(z) \ge \frac{n-1}{n-m}\frac{1}{2}z^{-1/2} \tag{6.14}$$

with equality for $m = 1$. Similarly for $0 < z < 1$

$$U(z) \le -\frac{n-m-1}{m} + \frac{n-1}{m}(1-z)^{\frac{1}{2}}, \quad -U'(z) \ge \frac{n-1}{2m}(1-z)^{-1/2} \tag{6.15}$$

with equality for $n - m = 1$.

Using (6.12), (6.14), (6.15), and the behaviour of $B(z)$ for z near 0 or 1, it is easy to check that $\Phi(z)$ from (6.11) satisfies

$$\Phi(z) > 0 \;\text{ for } z \text{ near } 0, \qquad \Phi(z) < 0 \;\text{ for } z \text{ near } 1,$$

and so (6.10) *has at least one solution* in the interval $]0,1[$. (Consider separately the cases $m=1$ and $2 \leq m \leq [n/2]$.)

Conversely, let $\alpha \in]0,1[$ denote a solution of (6.10), and define u by (6.9) in terms of z from (6.1), noting that $U(\alpha)$ and $V(\alpha)$ are non-zero and have the same sign. Indeed, if e.g. $U(\alpha) \geq 0$ and $V(\alpha) \leq 0$ then both terms on the right of (6.11) would be ≥ 0 at α in view of (6.12), and hence $U(\alpha) = V(\alpha) = 0$ by (6.10), but that contradicts U and V being linearly independent solutions to the homogeneous equation corresponding to (6.6), whence their Wronskian $UV' - U'V$ does not take the value 0. Reversing our steps we see that u satisfies the Euler equation (4.14), namely (6.6) for $0 \leq z \leq 1$. As observed in the paragraph preceding Definition 4.1 it follows that $u = Tf$ because $u_1 = 0$; in fact, u is an even function on Σ since u depends on z only. To prove that f is $\kappa(n)$-stationary it remains to show that $\operatorname{sgn} u = f$, and this follows from (6.9) and (6.12) because

$$U(\alpha) > 0, \quad V(\alpha) > 0. \tag{6.16}$$

In fact, we have just seen that the only alternative here would be that $U(\alpha) < 0$ and $V(\alpha) < 0$, but then it would follow from (6.9) that $\operatorname{sgn} u = -f$, and we would be led to the contradiction

$$-\|u\|_1 = \int_\Sigma uf\,d\sigma = \int (Tf)f\,d\sigma = \sum_{k=2}^{\infty} \frac{\|f_k\|^2}{\lambda_k - \lambda_1} \geq 0.$$

Because $u = Tf$ we have $u_0 = \int u\,d\sigma = 0$ and hence from (4.17) and (6.9), (6.3)

$$\begin{aligned} \Lambda(f) = \|u\|_1 &= -2\int_{\{z<\alpha\}} u\,d\sigma \\ &= \frac{4}{n-1}\frac{B(1)-B(\alpha)}{B(1)^2}\left(\frac{-4}{n-1}\alpha^{m/2}(1-\alpha)^{n/2-m/2}\frac{U'(\alpha)}{U(\alpha)} - B(\alpha)\right), \end{aligned} \tag{6.17}$$

where we have used that $\int_0^\alpha z^{m/2-1}(1-z)^{n/2-m/2-1}U(z)\,dz$ can be expressed in terms of $U'(\alpha)$ in view of the homogeneous equation corresponding to (6.6) (e.g. in divergence form) applied to U. There is a similar expression for $\|u\|_1 = 2\int_{\{z>\alpha\}} u\,d\sigma$ containing $V'(\alpha)/V(\alpha)$, and the two expressions are equal on account of (6.10), (6.11).

The *uniqueness* of the solution $\alpha \in]0,1[$ of (6.10) will be established first for $m=1$, where we show that the function Φ from (6.11) is *strictly decreasing* in $]0,1[$.

For $n=2$, $m=1$, we write $z = \sin^2\theta$, $0 < \theta < \pi/2$, and obtain $B = 2\theta$, $U = \cos\theta$, $V = \sin\theta$ (cf. above), and hence

$$\Phi = \theta/\tan\theta - (\pi/2-\theta)/\tan(\pi/2-\theta),$$

which is strictly decreasing and has the unique zero $\theta = \pi/4$, i.e., $z = \alpha = 1/2$. Inserting in (6.17) leads to $\|u\|_1 = 4/\pi - 1$, cf. Section 5.

For $n \geq 3$ and $m = 1$ we have from (6.13) and from the power series of $U(z)$:

$$V(z) = z^{1/2}, \quad U(z) = 1 - \frac{1}{2} z^{1/2} \int_0^z x^{-3/2}\big((1-x)^{1/2-n/2} - 1\big)\,dx. \tag{6.18}$$

(As functions of t, with $t^2 = z$, the functions $V = t$ and $-U$ are the Legendre functions in dimension n of degree 1 and of the first and second kind, respectively.) The Wronskian of U and V as fuctions of $z \in \,]0, 1[$ is

$$U(z)V'(z) - U'(z)V(z) = \frac{1}{2} z^{-1/2}(1-z)^{1/2-n/2},$$

and we therefore obtain from (6.11), (6.18) for $0 < z < 1$

$$2\Phi(z) = z^{-1/2}B(1)U(z) - z^{-1/2}(1-z)^{1/2-n/2}(B(1) - B(z)),$$

$$4z(1-z)B'(z)\Phi'(z) = -\frac{B(z)}{z} + 2B'(z) - (n-1)\frac{B(1) - B(z)}{1-z} < 0$$

because (6.4) in the present case $m = 1$, $n \geq 3$, implies

$$B(z) > \int_0^z x^{-1/2}(1-z)^{n/2-3/2}\,dx = 2z^{1/2}(1-z)^{n/2-3/2} = 2zB'(z).$$

In the remaining case, where $m \geq 2$ and $n - m \geq 2$, it follows from (6.16) that we shall work only in the interval

$$J = \big\{z \in \,]0, 1[\;\big|\; U(z) > 0, V(z) > 0\big\},$$

and here we consider the function $\Psi = \Phi/(UV)$, that is by (6.11)

$$\Psi(z) = B(z)\frac{V'(z)}{V(z)} + (B(1) - B(z))\frac{U'(z)}{U(z)}.$$

To establish the uniqueness of the zero $\alpha \in J$ of Φ, or equivalently of Ψ, we will show that $z(1-z)B'(z)\Psi(z)$ is *strictly decreasing* in J (when m, $n - m \geq 2$). Writing for brevity B for $B(z)$, etc., we obtain by differentiating Ψ and eliminating U'' and V'' by use of the hypergeometric differential equation satisfied by U and V:

$$-\frac{\big(z(1-z)B'\Psi\big)'}{z(1-z)B'} = B\left(\frac{V'}{V}\right)^2 - B'\frac{V'}{V} + \frac{n-1}{4z(1-z)}(B(1) - B)$$
$$+ (B(1) - B)\left(\frac{U'}{U}\right)^2 + B'\frac{U'}{U} + \frac{n-1}{4z(1-z)}B.$$

The latter line arises from the former (after the equality sign) by interchanging m and $n-m$, z and $1-z$, $B(z)$ and $B(1)-B(1-z)$, cf. (6.4), (6.8). It therefore suffices to prove that the former sum is > 0, and we do that by showing that the discriminant of this quadratic polynomial in V'/V (> 0) is negative for all $z \in]0,1[$, or equivalently that

$$D(z) := z(1-z)B'(z)^2 - (n-1)B(z)(B(1)-B(z)) < 0 \tag{6.19}$$

for $0 < z < 1$. Note that $B'(0) = B'(1) = 0$ because $m,\ n-m \geq 2$, cf. (6.4). Hence $D(0) = D(1) = 0$, and it suffices to prove that D is *strictly convex* as a function of B, or equivalently that

$$D'(z)/B'(z) \ \text{ is } \textit{strictly increasing}. \tag{6.20}$$

For convenience write

$$m = p+2, \quad n-m = q+2, \quad n = p+q+4,$$

whereby $p \geq 0$, $q \geq 0$. By differentiating (6.19) and expressing B'' in terms of B' one finds after reduction (for $0 < z < 1$), writing $w = 1-z$:

$$D'(z)/B'(z) = \big((p+1)w - (q+1)z\big)B'(z) - (p+q+3)\big(B(1) - 2B(z)\big),$$

$$2z(1-z)(D'/B')'/B' = (pw-qz)^2 + pw^2 + qz^2 + (p+q+8)zw \geq 8zw > 0.$$

This establishes (6.20) and thus completes the proof of the lemma. □

The case n even and $m = n/2$ (≥ 1). This is the simplest and most interesting case. Here $m = n-m$ and hence $V(z) = U(1-z)$, $B(\frac{1}{2}) = \frac{1}{2}B(1)$, cf. (6.8) and (6.4). Consequently, (6.10) holds with $\alpha = \alpha(2m,m) = \frac{1}{2}$ and we may therefore write

$$f(2m,m;\,\xi) = \operatorname{sgn}\Big(2\sum_{i=1}^{m}\xi_i^2 - 1\Big) = \operatorname{sgn}\Big(\sum_{i=1}^{m}\xi_i^2 - \sum_{i=1}^{m}\xi_{m+i}^2\Big). \tag{6.21}$$

Moreover, (6.17) leads to

$$(2m-1)\|u\|_1 = \frac{2^{1-m}}{B(\frac{m}{2},\frac{m}{2})}\frac{-4}{2m-1}\frac{U'(\frac{1}{2})}{U(\frac{1}{2})} - 1,$$

where $U(z) = F(-\frac{1}{2},\, m-\frac{1}{2};\, \frac{m}{2};\, z)$. Applying (50), (20) in [E1, §2.8] we obtain

$$-\frac{U'(\frac{1}{2})}{U(\frac{1}{2})} = \frac{2m-1}{4}\frac{\Gamma(\frac{1}{4})\Gamma(\frac{2m+1}{4})}{\Gamma(\frac{3}{4})\Gamma(\frac{2m+3}{4})},$$

and

$$\Lambda\big(f(2m,m)\big) = \|u\|_1 = \frac{1}{2m-1}\left(\frac{\Gamma(\frac{m+1}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{m}{2})}\frac{\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})}\frac{\Gamma(\frac{m}{2}+\frac{1}{4})}{\Gamma(\frac{m}{2}+\frac{3}{4})} - 1\right), \tag{6.22}$$

where we have used that $2^{1-m}\Gamma(m)/\big(\Gamma(\frac{m}{2})\big)^2 = \Gamma(\frac{m+1}{2})/\big(\Gamma(\frac{1}{2})\Gamma(\frac{m}{2})\big)$ according to Legendre's identity. For $n=2$, $m=1$, we thus recover once more the $\kappa(2)$-stationary value $4/\pi - 1$ from Section 5.

Using [E1, (4), p. 47] (or Stirling's formula) we find from (6.22) the asymptotic formula

$$\Lambda(f(2m,m)) = \frac{1}{2m-1}\left(\frac{1}{\sqrt{\pi}}\frac{\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})} - 1 + O(1/m^2)\right),$$

and so

$$\lim_{\substack{n\to\infty \\ n \text{ even}}} (n+1)\Lambda\big(f(n,\, n/2)\big) = \frac{1}{\sqrt{\pi}}\frac{\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})} - 1 \approx 0.6692. \tag{6.23}$$

It can also be shown by Stirling's formula that the sequence $(2m+1)\Lambda\big(f(2m,m)\big)$ is strictly *decreasing* (from $3(\frac{4}{\pi}-1) \approx 0.8197$ to the above limit).

The case n odd and $m = [n/2]$ (≥ 1). This is the main case for odd n.

For $n=3$, $m=1$, we write $z = t^2$, $0<t<1$, and obtain from (6.4) $B = 2t$ and from (6.18), (6.11):

$$V = t, \quad U = 1 - \frac{t}{2}\log\frac{1+t}{1-t}, \quad \Phi = \frac{1}{1+t} - \frac{1}{2}\log\frac{1+t}{1-t}.$$

One finds that the strictly decreasing function Φ equals 0 for $t = \tau \approx 0.5644$, i.e. $z = \alpha = \tau^2 \approx 0.3185$; and (6.17) gives $\Lambda(f(3,1)) = \|u\|_1 = (1-\tau)^2 \approx 0.1898$.

In the next example we exclude the case $n = 2m$ in which the two examples would be the same: $f(2m,m) = g(2m,m)$, cf. (6.21), (6.24). See however Remark 6.1 below.

Lemma 6.2. *For any dimension $n \geq 2$ and any integer m with $1 \leq m < n/2$ the following even function $g = g(n,m)$ on Σ is $\kappa(n)$-stationary (Definition 4.1):*

$$g(\xi) = g(n,m;\xi) = \operatorname{sgn} v, \qquad v = \sum_{i=1}^{m}\xi_i^2 - \sum_{i=1}^{m}\xi_{m+i}^2. \tag{6.24}$$

For even $k = 0, 2, 4, \ldots$ the projection g_k of g on $\mathcal{H}_k$ is given by (6.37) (see also (6.34) and (6.41)–(6.45)), and $\|g_k\|^2$ is given by (6.46). The $\kappa(n)$-stationary value corresponding to g can thus be computed by

$$\Lambda(g) = \sum_{\substack{k \text{ even} \\ k\geq 2}} \frac{\|g_k\|^2}{\lambda_k - \lambda_1}. \tag{6.25}$$

Proof. Note that $v \neq 0$ σ-a.e. on Σ. In order to show that g is $\kappa(n)$-stationary write $u = Tg$, cf. (4.6). Consider the isometry J of Σ by which ξ_i and ξ_{m+i} are interchanged for $\xi \in \Sigma$ and $i = 1, \dots, m$. Then $g(J\xi) = -g(\xi)$, and hence

$$g_0 = \int_\Sigma g \, d\sigma = 0. \tag{6.26}$$

It also follows that $g_k(J\xi) = -g_k(\xi)$ and hence $u(J\xi) = -u(\xi)$. In particular, $u(\xi) = 0$ if $J\xi = \xi$, and more generally if $\sum_{i=1}^m \xi_i^2 = \sum_{i=1}^m \xi_{m+i}^2$ because g and hence g_k and u are invariant under any isometry of Σ involving only $\xi_1, \dots, \xi_m$ or only $\xi_{m+1}, \dots, \xi_{2m}$. This also shows that g_k as well as u only depends on $\xi_1^2 + \dots + \xi_m^2$ and $\xi_{m+1}^2 + \dots + \xi_{2m}^2$.

We proceed to show that

$$u > 0 \text{ in } \{v > 0\}, \quad u < 0 \text{ in } \{v < 0\} \tag{6.27}$$

with v from (6.24), hence $\operatorname{sgn} u = \operatorname{sgn} v = g$. Consider any connectivity component V of $\{v > 0\}$, say. Note that v is of class $\mathcal{H}_2(\Sigma)$ because $\sum_{i=1}^m (x_i^2 - x_{m+i}^2)$ is harmonic in $\mathbf{R}^n$. In particular,

$$\Delta v + \lambda_2 v = 0 \text{ in } V, \quad v \to 0 \text{ at } \partial V,$$

and since $v > 0$ in V, λ_2 must be the first non-zero eigenvalue of $-\Delta$ considered in V, with the requirement of zero boundary values, cf. [CH, Chap. 6.6]. Choose a connected open proper subset W of Σ with smooth boundary ∂W so that $W \supset \overline{V}$, but also so that the first non-zero eigenvalue λ ($< \lambda_2$) of $-\Delta$ considered in W still exceeds $\lambda_1 = n - 1$, cf. [Co]. Let w denote a corresponding eigenfunction of $-\Delta$ in W:

$$\Delta w + \lambda w = 0 \text{ in } W, \quad w \to 0 \text{ at } \partial W.$$

It is known that $w(\xi) \neq 0$ for $\xi \in W$, and we may hence assume that $w > 0$ in W. Then

$$-\Delta w - (n-1)w = (\lambda - (n-1))w > 0 \quad \text{in } W.$$

This shows that the C^2-smooth function w in W is spherically superharmonic in the sense of Berg [Be, p. 48–49], or equivalently, by [He, Prop. 34.1], superharmonic in the sense of Hervé [He, Chap. 7] applied to the elliptic operator $\Delta + (n-1)$ (expressed in local coordinates), with reference to the axiomatic potential theory of Brelot [Br].

By (4.11), $u = Tg$ satisfies the Euler equation

$$-\Delta u - (n-1)u = \tilde{g} = g \quad \text{in } \Sigma \tag{6.28}$$

because $g_0 = g_1 = 0$, by (6.26) together with the fact that g is even and so

$$g_k = 0 \quad \text{for odd } k. \tag{6.29}$$

Since $g > 0$ in V, u is spherically superharmonic in V, by (6.28). In the beginning of the proof we saw that $u = 0$ on $\{v = 0\}$, cf. (6.24), hence on ∂V. Since the spherically superharmonic function $w > 0$ is bounded away from 0 on $\overline{V} \subset W$ we conclude from a well-known boundary minimum principle that $u \geq 0$ in V, and in fact $u > 0$ in V, see [Br, p. 33]. This proves that $u > 0$ in $\{g > 0\}$, and similarly the latter assertion in (6.27). Consequently, $\operatorname{sgn} u = g$ σ-a.e. in Σ, and g is indeed $\kappa(n)$-stationary.

In the sequel we use the parametric representation (6.2) of $\Sigma = \Sigma_n$, but now we replace m by $2m$ $(< n)$ in the notation for z, η, ζ. This leads to

$$\begin{aligned} &\xi = \left(\sqrt{z}\,\eta, \sqrt{1-z}\,\zeta\right), \qquad z = \xi_1^2 + \ldots + \xi_{2m}^2 \in [0, 1], \\ &\eta = (\eta_1, \ldots, \eta_{2m}) \in \Sigma_{2m}, \quad \zeta = (\zeta_1, \ldots, \zeta_{n-2m}) \in \Sigma_{n-2m}, \end{aligned} \tag{6.30}$$

and so by (6.3), (6.5) for $0 < z < 1$ (with B denoting the betafunction)

$$d\sigma(\xi) = \frac{1}{B(m, n/2 - m)} z^{m-1}(1-z)^{n/2-m-1} dz\, d\sigma_{2m}(\eta)\, d\sigma_{n-2m}(\zeta), \tag{6.31}$$

$$\Delta = 4z(1-z)\frac{\partial^2}{\partial z^2} + (4m - 2nz)\frac{\partial}{\partial z} + \frac{1}{z}\Delta_{2m} + \frac{1}{1-z}\Delta_{n-2m}. \tag{6.32}$$

For any *even* integer $k \geq 2$ (cf. (6.26), (6.29)) let $\mathcal{H}_k^*(\Sigma_n)$ denote the subspace of $\mathcal{H}_k(\Sigma_n)$ consisting of all functions $R \in \mathcal{H}_k(\Sigma_n)$ for which $R(\xi)$ only depends on $(\xi_1, \ldots, \xi_{2m})$, that is on (z, η) in (6.30). Adapting the procedure leading to the definition of the associated Legendre functions in Müller [M] we proceed to determine for each even integer $k \geq 2$ a family of functions $z \mapsto A_{k,j}(z)$, j even, $0 \leq j \leq k$, such that the functions

$$R(\xi) = A_{k,j}(z) S_j(\eta), \qquad S_j \in \mathcal{H}_j(\Sigma_{2m}), \tag{6.33}$$

belong to $\mathcal{H}_k(\Sigma_n)$ and hence to $\mathcal{H}_k^*(\Sigma_n)$, and that they together span $\mathcal{H}_k^*(\Sigma_n)$. In view of (6.31) any two functions of the form (6.33) corresponding to distinct values of j are orthogonal to one another in $L^2(\Sigma_n)$ because the respective $S_j \in \mathcal{H}_j(\Sigma_{2m})$ are mutually orthogonal in $L^2(\Sigma_{2m})$. From the requirement $\Delta R + \lambda_k R = 0$ one finds by separation of the variables z, η and using (6.32) a differential equation for $A_{k,j}$ solved by

$$\begin{aligned} A_{k,j}(z) &= z^{j/2}\binom{k/2 + j/2 + m - 1}{k/2 - j/2} F\left(-\tfrac{k-j}{2}, \tfrac{k+j+n-2}{2}; j+m; z\right) \\ &= (-1)^h z^{j/2} P_h^{(\alpha,\beta)}(t), \qquad z = \frac{1+t}{2}, \end{aligned} \tag{6.34}$$

in terms of the Jacobi polynomial $P_h^{(\alpha,\beta)}$ in Szegö's notation, cf. [E2, §10.8], whereby

$$\alpha = \frac{n}{2} - m - 1, \quad \beta = j + m - 1, \quad h = \frac{k-j}{2}. \tag{6.35}$$

The fact that the functions (6.33) span the whole of $\mathcal{H}_k^*(\Sigma_n)$ follows from the dimension relation

$$\dim \mathcal{H}_k^*(\Sigma_n) = \sum_{\substack{j \text{ even} \\ 0 \le j \le k}} \dim \mathcal{H}_j(\Sigma_{2m}),$$

which we shall establish by extending the proof given in [M, p. 25] for the particular case $n - 2m = 1$. First we note that the extensions to $\mathbf{R}^n$ of the functions in $\mathcal{H}_k^*(\Sigma_n)$ by homogeneity of degree k are precisely the harmonic polynomials H on $\mathbf{R}^n$ of the form

$$H(x) = \sum_{\substack{j \text{ even} \\ 0 \le j \le k}} H_{k-j}(x_1, \dots, x_{2m})(x_{2m+1}^2 + \cdots + x_n^2)^{j/2}$$

with H_{k-j} a homogeneous polynomial on $\mathbf{R}^{2m}$ of degree $k - j$. From the requirement that the Laplacian of H be 0 one easily finds that H_k can be prescribed arbitrarily, and that $H_{k-1}, \dots, H_0$ are uniquely determined from H_k. This implies that $\dim \mathcal{H}_k^*(\Sigma_n) = M(2m, k)$, where $M(q, r)$ denotes the dimension of the space of all homogeneous polynomials of degree r in q variables. As shown in [M, p. 3],

$$\dim \mathcal{H}_j(\Sigma_{2m}) = M(2m - 1, j) + M(2m - 1, j - 1),$$

and since $M(2m, k) = \sum_{i=0}^{k} M(2m-1, i)$ we obtain the stated expression for $\dim \mathcal{H}_k^*(\Sigma_n)$.

For $z > 0$ write

$$s = \sum_{i=1}^{m} \eta_i^2 - \sum_{i=1}^{m} \eta_{m+i}^2 = \frac{v}{z} \tag{6.36}$$

with v from (6.24). Then $s \neq 0$ σ-a.e. on Σ, and $g(\xi) = \operatorname{sgn} v = \operatorname{sgn} s$. As shown in the first paragraph of the proof g_k depends only on $\xi_1^2 + \ldots + \xi_m^2 = \frac{1}{2}z(1 + s)$ and $\xi_{m+1}^2 + \ldots + \xi_{2m}^2 = \frac{1}{2}z(1 - s)$, in other words on (z, s). In particular, $g_k(\xi)$ depends only on (z, η) in view of (6.36), and so g_k belongs to $\mathcal{H}_k^*(\Sigma_n)$ as defined in the paragraph containing (6.33). Accordingly, g_k has (for even $k \ge 2$) a unique representation of the form

$$g_k(\xi) = \sum_{\substack{j \text{ even} \\ 0 \le j \le k}} c_{k,j} A_{k,j}(z) S_j(\eta), \tag{6.37}$$

where the constants $c_{k,j}$ and the normalized functions $S_j \in \mathcal{H}_j(\Sigma_{2m})$ are to be determined. From (6.31) we obtain since $\int S_j^2 \, d\sigma_{2m} = 1$

$$\begin{aligned} a_{k,j} &:= \int_\Sigma \big(A_{k,j}(z) S_j(\eta)\big)^2 \, d\sigma(\xi) \\ &= \frac{1}{B(m, n/2 - m)} \int_0^1 \big(A_{k,j}(z)\big)^2 z^{m-1} (1 - z)^{n/2-m-1} dz. \end{aligned} \tag{6.38}$$

Inserting (6.34), (6.35) and writing $h = (k - j)/2$ we obtain (cf. [E2, §10.8])

$$\begin{aligned} a_{k,j} &= \frac{2^{-\alpha-\beta-1}}{B(m, n/2 - m)} \int_{-1}^{1} \big(P_h^{(\alpha,\beta)}(t)\big)^2 (1-t)^\alpha (1+t)^\beta dt \\ &= \frac{\Gamma(n/2)\Gamma(h + n/2 - m)\Gamma(k - h + m)}{\Gamma(n/2 - m)(m-1)!\, h!\, (k + n/2 - 1)\Gamma(k - h + n/2 - 1)}. \end{aligned} \tag{6.39}$$

Next we determine $S_j(\eta)$. Because $g_k(\xi)$ depends only on (z, s), as mentioned after (6.36), it follows by the uniqueness of the representation (6.37) that S_j depends only on s and so is a polynomial in s of degree $j/2$. Applying (6.5) with n, m, ξ, z replaced by $2m, m, \eta, \frac{1}{2}(1+s)$, respectively, to $S_j(\eta)$ as a function of s, we obtain, noting that S_j is an eigenfunction to $-\Delta_{2m}$ corresponding to its jth eigenvalue $j(j + 2m - 2)$:

$$(1 - s^2)\frac{d^2 S_j}{ds^2} - ms\frac{dS_j}{ds} + \frac{1}{4}j(j + 2m - 2)S_j = 0 \tag{6.40}$$

with the normalized solution

$$S_j(\eta) = b_j^{-\frac{1}{2}} P_{j/2}(m+1, s), \tag{6.41}$$

where $P_{j/2}(m+1, s)$ denotes the (generalized) Legendre polynomial in dimension $m+1$ and of degree $j/2$, cf. [M, pp. 17, 21], and where

$$\begin{aligned} b_j &= \frac{1}{B(1/2, m/2)} \int_{-1}^{1} \big(P_{j/2}(m+1, s)\big)^2 (1 - s^2)^{m/2-1} ds \\ &= \frac{1}{N(m+1, j/2)} = \frac{(j/2)!\,(m-1)!}{(j + m - 1)(j/2 + m - 2)!} \end{aligned} \tag{6.42}$$

with the notation $N(q, r) = \dim \mathcal{H}_r(\Sigma_q)$. Here we have used [M, p. 1], [M, Lemma 10], and [M, (11), p. 4].

From (6.37), (6.38), (6.31), (6.41) we obtain (always for even $k \geq 2$)

$$c_{k,j} = \frac{1}{a_{k,j}} \int_\Sigma g(\xi) A_{k,j}(z) S_j(\eta)\, d\sigma(\xi) = \frac{p_{k,j} q_{k,j}}{a_{k,j} b_j^{1/2}}, \tag{6.43}$$

where

$$p_{k,j} = \frac{1}{B(m, n/2 - m)} \int_0^1 A_{k,j}(z) z^{m-1} (1 - z)^{n/2-m-1} dz,$$

$$q_{k,j} = \frac{1}{B(1/2, m/2)} \int_{-1}^{1} (\operatorname{sgn} s) P_{j/2}(m+1, s)(1 - s^2)^{m/2-1} ds.$$

Inserting the expansion of the hypergeometric polynomial $z^{-j/2}A_{k,j}(z)$, cf. (6.34), in powers of z and integrating term by term leads after some calculation to

$$p_{k,j} = \binom{k/2-1}{k/2-j/2} \frac{\Gamma(j/2+m)\Gamma(k/2-j/2+n/2-m)}{B(m,n/2-m)\Gamma(k/2+n/2)} \tag{6.44}$$

by use of the Pfaff-Saalschütz identity, cf. [E1, p. 188].

From (6.40) with S_j replaced by $P_{j/2}(m+1,\cdot)$, cf. (6.41), we get after integrating from 0 to 1

$$\begin{aligned} B(1/2,m/2)\, q_{k,j} &= \frac{2}{j/2\,(j/2+m-1)} P'_{j/2}(m+1,0) \\ &= \frac{2}{m} P_{j/2-1}(m+3,0) \end{aligned}$$

according to [Be, p. 32, line 1]. Expressing $P_{j/2-1}(m+3,\cdot)$ by Gegenbauer functions and applying [E2, §10.9, (19)] leads to

$$q_{k,j} = (-1)^i \frac{\Gamma(i+m/2+1/2)}{i!\sqrt{\pi}\,\Gamma(m/2+1)} \binom{2i+m}{2i}^{-1}, \qquad i=(j-2)/4, \tag{6.45}$$

for $j/2$ *odd*, while clearly $q_{k,j}=0$ for $j/2$ even.

From (6.37), (6.38), (6.43) we obtain for even $k \geq 2$, cf. (6.29),

$$\|g_k\|^2 = \sum_{\substack{j \text{ even} \\ 0\le j\le k}} c_{k,j}^2 a_{k,j} = \sum_{\substack{j/2 \text{ odd} \\ 0\le j\le k}} \frac{p_{k,j}^2 q_{k,j}^2}{a_{k,j} b_j} \tag{6.46}$$

in which (6.39), (6.42), (6.44), and (6.45) can be inserted. □

Remark 6.1. The formulae obtained in Lemma 6.2 and its proof (notably (6.25), (6.37), (6.39), and (6.42)–(6.46)) hold also for $n = 2m$ (the case where $f(n,m) = g(n,m)$, as observed just before Lemma 6.2). One merely has to interpret various undefined expressions in the obvious way, whereby $c_{k,j} = a_{k,j} = p_{k,j} = 0$ for $j < k$, while $a_{k,k} = p_{k,k} = 1$.

Theorem 6. *For any dimension n we have*

$$1 > (n+1)\kappa(n) \geq (n+1)\kappa_*(n) > \frac{1}{\sqrt{\pi}} \frac{\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})} - 1 \ (\approx 0.6692).$$

Proof. The first inequality is contained in Remark 4.2 and the second is trivial. For *even* $n = 2m$ the third inequality follows from (6.23) and the subsequent lines since

$\kappa_*(2m) \ge \Lambda\big(f(2m,m)\big)$ in the notation of Lemma 6.1. For *odd* $n = 2m+1$ use the fact that $\kappa_*(2m+1) \ge \Lambda(g)$ with $g = g(n,m)$ from Lemma 6.2. Taking $k = 2$ in (6.46), and inserting (6.39), (6.42), (6.44), (6.45), leads in view of (6.25) (where $\lambda_2 - \lambda_1 = n+1$) to

$$\begin{aligned}(n+1)\kappa_*(n) &\ge (n+1)\Lambda(g) \ge \|g_2\|^2 \ge \frac{p_{2,2}^2 q_{2,2}^2}{a_{2,2}b_2} = \frac{4}{\pi m}\frac{2m+3}{2m+1}\left(\frac{\Gamma(\frac{m+1}{2})}{\Gamma(\frac{m}{2})}\right)^2\\ &\ge \frac{2}{\pi}\frac{2m+3}{2m+1}\exp\Big(-\frac{1}{2m}\Big) > \frac{2}{\pi} \quad (\approx 0.6366),\end{aligned}$$

using Stirling's formula. In order to obtain the slightly sharper lower estimate stated in the theorem one must take also the terms in (6.25) with $k > 2$ into account. In the notation explained after (6.13) one obtains for $k = 4i+2$, $i = 0, 1, 2, \dots$,

$$\lim_{\substack{n\to\infty\\ n\text{ odd}}} \frac{n+1}{\lambda_k - \lambda_1}\|g_k\|^2 = \frac{1}{\pi}\left(\frac{1}{i+\frac14} - \frac{1}{i+\frac12}\right)\frac{(\frac12)_i}{i!}, \tag{6.47}$$

while the limit is 0 for other values of k. This is because the terms with $j < k$ in (6.46) contribute with 0 to the stated limit, while the term with $j = k$ equals 0 unless k has the stated form $k = 4i+2$, cf. (6.45). It follows now by (6.25) that

$$\begin{aligned}\liminf_{\substack{n\to\infty\\ n\text{ odd}}} (n+1)\,\kappa_*(n) &\ge \sum_{i=0}^{\infty} \lim_{\substack{n\to\infty\\ n\text{ odd}}} \frac{n+1}{\lambda_k - \lambda_1}\|g_k\|^2 \ge \frac{1}{\pi}\sum_{i=0}^{\infty}\left(\frac{4(\frac14)_i}{(\frac54)_i} - \frac{2(\frac12)_i}{(\frac32)_i}\right)\frac{(\frac12)_i}{i!}\\ &= \frac{4}{\pi}F(\tfrac14,\tfrac12;\tfrac54;1) - \frac{2}{\pi}F(\tfrac12,\tfrac12;\tfrac32;1) = \frac{1}{\sqrt{\pi}}\frac{\Gamma(\frac14)}{\Gamma(\frac34)} - 1\end{aligned}$$

according to [E1, (14), p. 61]. By the way, the same holds for *even* $n \to \infty$ (cf. Remark 6.1 above), and this leads to an alternative proof of (6.23) and of the third inequality in the present theorem for even n because it can be shown that $(n+1)\|g_k\|^2/(\lambda_k - \lambda_1)$ is a strictly decreasing function of even n for fixed $k = 4i+2$ (now of course with $g = g(n, n/2)$). Unfortunately, when $i \ge 1$ (i.e., $k \ge 6$), the corresponding function of odd n is *not* decreasing, and its values for large n are smaller than the limit in (6.47). The completion of the proof of the theorem for odd n therefore requires a further analysis which we omit here. □

Comparison of stationary values. For any measurable function $f : \Sigma \to [-1, 1]$ we have from Remark 4.2 (or the above theorem)

$$\Lambda(f) \le \kappa(n) < \frac{1}{n+1}.$$

We begin by comparing the particular $\kappa(n)$-stationary values $\Lambda(f(n,m))$ and $\Lambda(g(n,m))$ from Lemmas 6.1 and 6.2 above. In Table 1 below we list for a few pairs n, m with $1 \le m \le [n/2]$ the root $\alpha = \alpha(n,m)$ of the transcendental equation $\Phi(\alpha) = 0$ with Φ from (6.11) in Lemma 6.1, and the corresponding $\kappa(n)$-stationary value $\Lambda(f(n,m))$ given by (6.17). In view of Theorem 6 we also list the values of $(n+1)\Lambda(f(n,m))$ (< 1). In the last column we list the analogous products $(n+1)\Lambda(g(n,m))$ from Lemma 6.2 (cf. Remark 6.1). Entries followed by a double, resp. single, asterisk pertain to the main case $n = 2m$, resp. $n = 2m+1$, in the example in Lemma 6.1.

n	m	$\alpha(n,m)$	$\Lambda(f(n,m))$	$(n+1)\times$ $\Lambda(f(n,m))$	$(n+1)\times$ $\Lambda(g(n,m))$
2	1**	0.5000**	0.2732**	0.8197**	0.8197
3	1*	0.3185*	0.1898*	0.7591*	0.7051
4	1	0.2323	0.1424	0.7119	0.6487
4	2**	0.5000**	0.1530**	0.7649**	0.7649
5	1	0.1825	0.1134	0.6804	0.6152
5	2*	0.3933*	0.1241*	0.7445*	0.7252
6	1	0.1502	0.0941	0.6584	0.5930
6	2	0.3237	0.1037	0.7256	0.6989
6	3**	0.5000**	0.1056**	0.7390**	0.7390
20	10**	0.5000**	0.0330**	0.6931**	0.6931
21	10*	0.4757*	0.0315*	0.6920*	0.6910
50	25**	0.5000**	0.0133**	0.6791**	0.6791
51	25*	0.4901*	0.0131*	0.6789*	0.6787

Table 1

By comparison of the last two columns in Table 1 it seems that

$$\Lambda(f(n,m)) > \Lambda(g(n,m)) \quad \text{when } n > 2m. \tag{6.48}$$

(Recall that $f(2m,m) = g(2m,m)$.) The table furthermore seems to indicate that, for each dimension n (≥ 4), the biggest among the $\kappa(n)$-stationary values $\Lambda(f(n,m))$ is the one for which $m = [n/2]$, the main case discussed after the proof of Lemma 6.1.

For each dimension n there are infinitely many equivalence classes (modulo isometry of Σ) of $\kappa(n)$-stationary functions, cf. Remark 6.2 below, and it seems difficult to classify them, except perhaps for $n = 2$, where at least we have found in Theorem 5 all $\kappa(2)$-stationary functions f such that $f_1 = 0$ (probably there are no others). For each integer $p \geq 2$ we found that there is precisely one equivalence class of $\kappa(2)$-stationary functions f with $f_1 = 0$ such that the set

$$\{\xi \in \Sigma \mid u(\xi) \neq 0\}, \quad \text{where } u = Tf, \tag{6.49}$$

has precisely $2p$ connectivity components; and this is the class of all translates of the function $f(p, \theta) = \text{sgn}(\cos(p\theta))$. The maximizing functions for $\kappa(2)$, i.e. the translates of $f(2, \cdot)$, thus have the smallest possible number of components of the set in (6.49) above, namely 4.

Remark 6.2. For any dimension $n \geq 2$ and any integer $p \geq 2$ it can be shown by the method from Lemma 6.2 that the function $\text{sgn}(\cos(p\theta))$ is $\kappa(n)$-stationary, writing $(\cos\theta, \sin\theta) = (\xi_1, \xi_2)$, $\xi = (\xi_1, \ldots, \xi_n)$ (cf. Theorem 5 in case $n = 2$, and Lemma 6.2 with $m = 1$ in case $p = 2$). But in dimension $n \geq 3$ there seem to be infinitely many other (equivalence classes of) $\kappa(n)$-stationary functions, among which certain functions depending only on z from Lemma 6.1. Yet another example (for $n \geq 4$) is $f(\xi) = \text{sgn}(\xi_1\xi_2\xi_3\xi_4)$, etc.

Conjecture. *For any dimension $n \geq 2$ the particular $\kappa(n)$-stationary function $f = f(n, [n/2])$ from Lemma 6.1 is maximizing for $\kappa(n)$; and f and $-f$ are the only maximizing functions for $\kappa(n)$ up to isometry of Σ.*

For $n = 2$ this conjecture is true by Theorem 5, but the case $n > 2$ remains open. Recall that the isometry $\xi \mapsto (\xi_{m+1}, \ldots, \xi_n, \xi_1, \ldots, \xi_m)$ transforms $f(n, m)$ into $-f(n, n - m)$. In particular, the function $f(n, [n/2])$ from the conjecture is equivalent (under isometry of Σ) to $-f(n, [n/2])$ if n is even, and to $-f(n, [n/2] + 1)$ if n is odd.

The conjecture, if confirmed, implies in view of (6.23) that the lower estimate in Theorem 6 above is best possible in the limit as $n \to \infty$.

In order to somehow support the conjecture we first observe that, for any *even* continuous function $u \not\equiv 0$ on Σ with mean-value $u_0 = 0$ (thus in particular for any minimizing function u for $c(n)$, at least if $n \leq 4$, cf. Theorem 4.4), the open set $\{u \neq 0\}$, cf. (6.49), has at least 4 connectivity components if $n = 2$; at least 3 components if $n = 3$ (this uses the Jordan curve theorem); and of course at least 2 components if $n \geq 4$.

This minimal number of components of the set $\{u \neq 0\}$ is attained by the $c(n)$-stationary function $u = Tf = Tf(n, [n/2])$ corresponding to the $\kappa(n)$-stationary function $f = f(n, [n/2])$ entering in the conjecture. More generally it is attained by the $c(n)$-stationary function $u = Tf(n, m)$ from Lemma 6.1 except if $n \geq 4$, $m = 1$ (in which case there are 3 components instead of 2). This follows easily from the parametric representation (6.2) of $\Sigma = \Sigma_n$ because the unit sphere Σ_m in $\mathbf{R}^m$ is connected when $m \geq 2$, but has 2 components when $m = 1$.

For the $c(n)$-stationary function $u = Tg(n, m)$ from Lemma 6.2 the number of components of $\{u \neq 0\}$ is minimal except for $n \geq 3$, $m = 1$ (in which case there are 4 components instead of 3 or 2).

For any dimension $n > 2$ it seems plausible (in view of Theorem 5 and the observation just before Remark 6.2) that any maximizing function f for $\kappa(n)$ must lead to the smallest possible number of components of the set $\{Tf \neq 0\}$; and further that the *only* $\kappa(n)$-stationary functions f for which $\{Tf \neq 0\}$ has this minimal number of components are (up to isometry of Σ) the functions $\pm f(n, m)$ and $g(n, m)$ from Lemma 6.1 and Lemma 6.2, with (n, m) as stated above; here we appeal to the high degree of symmetry of these functions. Finally, Table 1 above indicates that it is $f(n, [n/2])$ which is maximizing for $\kappa(n)$. However, no proof of the conjecture is in sight.

All I can show is that, for even $n = 2m$, we have $\Lambda(f(n, m)) \geq \Lambda(f)$ for any measurable function f depending only on z from (6.1) and taking values in $[-1, 1]$; and the sign of equality holds only for $f = \pm f(n, m) = \pm \operatorname{sgn} z$.

7. Completion of the proof of Lemma 3.1

Proof of the expression for D. We may assume that K is normalized, and we begin by estimating D from below. Because $\sqrt{1+t} \geq 1 + \frac{t}{2}(1 - \frac{t}{4})$ for $t \geq 0$, the integrand on the right of (3.1) is no less than

$$\begin{aligned}&(1+u)^{n-1} + \tfrac{1}{2}(1+u)^{n-3}|\nabla u|^2\big(1 - \tfrac{1}{4}(1+u)^{-2}|\nabla u|^2\big)\\ \geq &(1+u)^{n-1} + \tfrac{1}{2}\big(1 + O(d + \|\nabla u\|_\infty^2)\big)\,|\nabla u|^2\end{aligned}$$

since $(1+u)^{-2} \leq (1-d)^{-2} \leq 4$ if $d \leq \frac{1}{2}$; and $(1+u)^{n-3} \geq 1 - |n-3|\,|u| = 1 + O(d)$. Inserting this estimate in (3.1), and using the relation

$$\int_\Sigma (1+u)^{n-1}\, d\sigma = 1 - \tfrac{n-1}{2}\big(1 + O(d)\big)\|u\|^2 \tag{7.1}$$

(cf. the proof of [F1, (14)]), leads to

$$\begin{aligned}D &\geq \tfrac{1}{2}\|\nabla u\|^2 - \tfrac{n-1}{2}\|u\|^2 + O\big(d + \|\nabla u\|_\infty^2\big)\big(\|\nabla u\|^2 + \|u\|^2\big)\\ &\geq \tfrac{1}{2}\big(\|\nabla u\|^2 - (n-1)\|u\|^2\big)\big(1 + O\big(d + \|\nabla u\|_\infty^2\big)\big),\end{aligned}$$

the desired lower estimate. Here we have used that

$$\|\nabla u\|^2 + \|u\|^2 \leq 2\big(\|\nabla u\|^2 - (n-1)\|u\|^2\big) \tag{7.2}$$

for d sufficiently small. In fact, by Lemma 2,

$$\begin{aligned}
\|\nabla u\|^2 - (n-1)\|u\|^2 &= \sum_{k=2}^{\infty}(\lambda_k - \lambda_1)\|u_k\|^2 - \lambda_1\|u_0\|^2 \\
&\ge \frac{n+1}{2n+1}\sum_{k=2}^{\infty}(\lambda_k + 1)\|u_k\|^2 - \lambda_1\|u_0\|^2 \\
&\ge \left(\frac{n+1}{2n+1} + O(d^2)\right)\sum_{k=0}^{\infty}(\lambda_k + 1)\|u_k\|^2 \\
&\ge \tfrac{1}{2}\big(\|\nabla u\|^2 + \|u\|^2\big)
\end{aligned}$$

for small enough d since $\|u_0\|$, $\|u_1\| = O(d)\|u\|$ by (3.5), and $(n+1)/(2n+1) > 1/2$.

For the easier estimate of D from above we use that $\sqrt{1+t} \le 1 + \frac{1}{2}t$ for $t \ge 0$, and hence by (3.1), (7.1), (7.2)

$$\begin{aligned}
1 + D &\le \int_{\Sigma}\big((1+u)^{n-1} + \tfrac{1}{2}(1+u)^{n-3}|\nabla u|^2\big)\,d\sigma \\
&\le 1 + \tfrac{1}{2}\big(\|\nabla u\|^2 - (n-1)\|u\|^2\big) + O(d)\big(\|\nabla u\|^2 + \|u\|^2\big) \\
&\le 1 + \tfrac{1}{2}\big(1 + O(d)\big)\big(\|\nabla u\|^2 - (n-1)\|u\|^2\big),
\end{aligned}$$

noting that $(1+u)^{n-3} = 1 + O(d)$ since $|u| \le d$. □

Proof of the estimate of $|F|$. We may assume that K is normalized, hence $v = 1$. Consider any point $x \in F$. Since $K \subset B(0, 1+d)$ we have for small d

$$|x| < 1 + d < \sqrt{2},$$

for otherwise $B(x,1) \setminus K$ would contain more than half of $B(x,1)$, and so $\alpha > \frac{1}{2}$, contradicting (for small d) $\alpha \le \beta = O(d)$, a consequence e.g. of the second relation in the lemma. – Since $K \supset B(0, 1-d)$ we have

$$B(0,1) \setminus B(x,1) \subset \big(K \setminus B(x,1)\big) \cup \big(B(0,1) \setminus B(0,1-d)\big).$$

Because the $(n-2)$-sphere $\partial B(0,1) \cap \partial B(x,1)$ has radius $\sqrt{1 - |x/2|^2} > \sqrt{1/2}$, we obtain (for $x \in F$)

$$\begin{aligned}
\omega_{n-1}\sqrt{2}^{\,1-n}|x| &\le V\big(B(0,1) \setminus B(x,1)\big) \\
&\le V\big(K \setminus B(x,1)\big) + \omega_n(1 - (1-d)^n) \\
&\le \omega_n(\beta + nd) = O(d),
\end{aligned}$$

again by the second relation in the lemma. This shows that indeed $|F| = O(d)$. □

Proof of the expression for α. Again we assume that K is normalized. Consider an arbitrary point $x \in B(0,1)$. In polar coordinates R, ξ the sphere $\partial B(x,1)$ is given by an equation of the form

$$R = 1 + v(\xi), \qquad \xi \in \Sigma.$$

Elementary estimates show that

$$0 \le x\cdot\xi - v(\xi) \le 2|x|^2, \tag{7.3}$$

and hence

$$\|l_x - v\|_1 \le 2|x|^2, \qquad l_x(\xi) := x\cdot\xi. \tag{7.4}$$

We now obtain

$$\begin{aligned}\frac{2}{\omega_n} V\big(K \setminus B(x,1)\big) &= \frac{1}{\omega_n} V\big(K \setminus B(x,1)\big) + \frac{1}{\omega_n} V\big(B(x,1) \setminus K\big) \\ &= \int_\Sigma |(1+u)^n - (1+v)^n| \, d\sigma \\ &= \int_\Sigma |u - v| \sum_{j=1}^{n} (1+u)^{n-j}(1+v)^{j-1} \, d\sigma \\ &= n \, \|u - v\|_1 \big(1 + O(d + |x|)\big) \end{aligned} \tag{7.5}$$

because $|u(\xi)| \le d$ and $|v(\xi)| \le |x| + 2|x|^2 \le 3|x| < 3$ by (7.3) together with $x \in B(0,1)$.

From the former estimate (3.5) of $\|u_1\|_\infty$ together with $\|u\|^2 = \|u - u_1\|^2 + \|u_1\|^2$ we easily obtain $\|u_1\|_\infty = O(\|u - u_1\|^2)$. From (7.4) and the fact that $|x| = \|l_x\|_\infty$ is a constant times $\|l_x\|$ we therefore get

$$\begin{aligned}\|u_1\|_1 + \|l_x - v\|_1 &= O\big(\|u - u_1\|^2 + \|l_x\|^2\big) \\ &= O\big(\|u - u_1 - l_x\|^2\big) \\ &= O\big(\|u - u_1 - l_x\|_1 \, \|u - u_1 - l_x\|_\infty\big) \\ &= \|u - u_1 - l_x\|_1 \, O(d + |x|). \end{aligned} \tag{7.6}$$

For the second equation here we have used that $u - u_1$ is orthogonal to $\mathcal{H}_1$ in $L^2(\sigma)$, in particular to l_x; and for the last relation that $\|u\|_\infty = d$ and $\|u_1\|_\infty = O(\|u\|^2) = O(d^2)$ by (3.5). By the triangle inequality (7.6) leads to

$$\|u - v\|_1 = \|u - u_1 - l_x\|_1 \big(1 + O(d + |x|)\big). \tag{7.7}$$

In order first to prove that $\alpha \geq \frac{n}{2}\|u\|_*(1 + O(d))$ take x in F (defined in Lemma 3.1), whereby $|x| \leq |F| = O(d)$ as shown above, in particular $x \in B(0, 1)$ for small d. Combining (7.5) with (7.7) after inserting $|x| = O(d)$ in both we get by (2.1)

$$\begin{aligned}\alpha = \frac{1}{\omega_n} V\big(K \setminus B(x, 1)\big) &= \frac{n}{2}\|u - u_1 - l_x\|_1(1 + O(d)) \\ &\geq \frac{n}{2}\|u\|_*(1 + O(d))\end{aligned} \tag{7.8}$$

in view of Definition 3, noting that $u_1 + l_x \in \mathcal{H}_1$.

To prove the opposite inequality we apply (3.6) to $u - u_1$ (orthogonal to $\mathcal{H}_1$). We thus find $l = l_x \in \mathcal{H}_1$ (cf. (7.4)) such that

$$\|u\|_* = \|u - u_1\|_* = \|u - u_1 - l_x\|_1 \tag{7.9}$$

and $|x| = O\big(\|l_x\|_1\big) = O\big(\|u\|_*\big) = O(d)$. In (7.8) the first equality sign must now be replaced by $\leq$ while the sign $\geq$ can be replaced by $=$ according to (7.9). □

8. The first part of the proof of Theorem 4.4

The projection f_k of a function $f \in L^2(\sigma)$ onto the kth eigenspace $\mathcal{H}_k$ of $-\Delta$ is given by

$$f_k(\xi) = \int F_k(\xi, \eta) f(y)\, d\sigma(y)$$

in terms of the reproducing kernel F_k for $\mathcal{H}_k$ determined by

$$F_k(\xi, \eta) = N(n, k) P_k(n, \xi \cdot \eta), \tag{8.1}$$

where

$$N(k) = N(n, k) = \frac{(2k + n - 2)\,(k + n - 3)!}{(n - 2)!\, k!}$$

is the dimension of $\mathcal{H}_k$, and $P_k(n, t) = P_k(t)$, $k = 0, 1, 2, \ldots$, is the kth (generalized) Legendre polynomial in dimension n, given by Rodriques' formula (cf. [M])

$$P_k(n, t) = \left(-\frac{1}{2}\right)^k \frac{\Gamma(\frac{n-1}{2})}{\Gamma(k + \frac{n-1}{2})} (1 - t^2)^{\frac{3-n}{2}} \left(\frac{d}{dt}\right)^k (1 - t^2)^{k + \frac{n-3}{2}}.$$

The polynomials P_k are mutually orthogonal with respect to the measure with density $(1 - t^2)^{\frac{n-3}{2}} dt$ w.r.t. Lebesgue measure on $[-1, 1]$.

In view of (8.1) the $L^2(\sigma)$-norm of f_k is determined by

$$\|f_k\|^2 = \int f f_k \, d\sigma = N(k) \iint P_k(\xi\cdot\eta) f(\xi) f(\eta) \, d\sigma(\xi) \, d\sigma(\eta). \tag{8.2}$$

Recall that $F_k(\xi, \cdot) = F_k(\cdot, \xi)$ is in $\mathcal{H}_k$ for every $\xi \in \Sigma$, and that $f(-\xi) = (-1)^k f(\xi)$ for $f \in \mathcal{H}_k$. With the notation (4.8) we thus have from (8.2) for $f \in L^2(\sigma)$

$$\Lambda(f) = \sum_{k=2}^{\infty} \frac{\|f_k\|^2}{\lambda_k - \lambda_1} = \iint \tilde{G}(\xi\cdot\eta) f(\xi) f(\eta) \, d\sigma(\xi) \, d\sigma(\eta), \tag{8.3}$$

$\lambda_k = k(k+n-2)$ being the kth eigenvalue of $-\Delta$, hence $\lambda_k - \lambda_1 = (k-1)(k+n-1)$; and

$$\tilde{G}(t) = \tilde{G}(n,t) = \sum_{k=2}^{\infty} \frac{N(n,k)}{(k-1)(k+n-1)} P_k(n,t) \tag{8.4}$$

converges in $L^2\big((1-t^2)^{\frac{n-3}{2}} dt\big)$ when $n \le 4$, as noted in [Be, p. 25–26]. In view of the Funk-Hecke formula (cf. p. 20 in [M] or [Be]), this amounts to the kernel $(\xi, \eta) \mapsto \tilde{G}(\xi\cdot\eta)$ being of Hilbert-Schmidt class and representable within $L^2\big(d\sigma(\xi) d\sigma(\eta)\big)$ as in (8.4), now with t replaced by $\xi\cdot\eta$; and so the term by term integration in (8.3) is justified. Moreover, T from (4.6) is the integral operator with the kernel $\tilde{G}(\xi\cdot\eta)$, as expressed in (4.16) in Remark 4.5. In the present case $n \le 4$ this appears from the polarized form of (4.8). (For general n one may apply [Be, p. 20–23] to verify (4.16), and hence (8.3) above, by checking that both members of (4.16) have the same formal expansion in spherical harmonics, on account of (8.4), now as a formal expansion. Thus it is not at the present stage that the limitation $n \le 4$ in our proof of Theorem 4.4 is essential.)

In [Be] the function $\tilde{G}$, or rather the 'full' sum

$$\begin{aligned} G(t) = G(n,t) &= \sum_{k \ne 1} \frac{N(n,k)}{(k-1)(k+n-1)} P_k(n,t) \\ &= -\frac{1}{n-1} + \tilde{G}(n,t), \end{aligned} \tag{8.5}$$

is determined explicitly by recursion with respect to the dimension n. (In the notation in [Be] our $G(n,t)$ is expressed as $-\frac{1}{n-1}\frac{\|\omega_n\|}{\|\omega_{n-1}\|} g_n(t)$, where $\|\omega_n\|$ denotes the surface area of the unit sphere in $\mathbf{R}^n$.)

Note that $G(n,\cdot)$ and $\tilde{G}(n,\cdot)$ are analytic in $[-1,1[$, and they approach $+\infty$ as $t \to 1$ (except for $n=2$, where the limits are finite).

In the sequel we shall also need the even part $\tilde{H}(t)$ and the odd part $\tilde{J}(t)$ of $\tilde{G}(t) = \tilde{G}(n,t)$:

$$\tilde{H}(t) = \frac{1}{2}\big(\tilde{G}(t) + \tilde{G}(-t)\big) = \sum_{k \text{ even } \ge 2} \frac{N(k)}{(k-1)(k+n-1)} P_k(t), \tag{8.6}$$

$$\tilde{J}(t) = \frac{1}{2}\big(\tilde{G}(t) - \tilde{G}(-t)\big) = \sum_{k \text{ odd} \geq 3} \frac{N(k)}{(k-1)(k+n-1)} P_k(t). \tag{8.7}$$

(The even part H of G itself equals $\frac{1}{2}\|\omega_n\|/\|\omega_{n-1}\|$ times the Legendre function of the second kind of degree 1 in dimension n.)

Consider now any maximizing function f for $\kappa(n)$. Inspired by a construction in [HH, p. 105] we associate with each point $a \in \Sigma$ the *even* function $f(a,\cdot) \in L^\infty(\sigma)$ which coincides with f on the hemisphere $\{\xi \in \Sigma \mid a\cdot\xi > 0\}$:

$$f(a,\xi) = \chi(a\cdot\xi) f(\xi) + \chi(-a\cdot\xi) f(-\xi), \qquad \xi \in \Sigma, \tag{8.8}$$

where $\chi(t) = 1$ for $t > 0$, $\chi(t) = 0$ for $t < 0$, and so $\chi(-t) = 1 - \chi(t)$ for all $t \neq 0$. It is our aim to show that $f(a,\cdot)$ is maximizing for $\kappa(n)$ (and hence for $\kappa_*(n)$) for every $a \in \Sigma$. (This is trivial if f is itself even, hence $f(a,\cdot) = f$.)

Inserting (8.8) in place of f in (8.2) we easily obtain for the projection $f_k(a,\cdot)$ of $f(a,\cdot)$ on $\mathcal{H}_k$

$$\|f_k(a,\cdot)\|^2 = 4N(k) \iint P_k(\xi\cdot\eta)\chi(a\cdot\xi)\chi(a\cdot\eta) f(\xi) f(\eta)\, d\sigma(\xi)\, d\sigma(\eta)$$

for *even* k, while $\|f_k(a,\cdot)\| = 0$ for odd k. From this we derive similarly to (8.3), using (8.6),

$$\Lambda(f(a,\cdot)) = 4 \iint \tilde{H}(\xi\cdot\eta)\chi(a\cdot\xi)\chi(a\cdot\eta) f(\xi) f(\eta)\, d\sigma(\xi)\, d\sigma(\eta). \tag{8.9}$$

Because $\tilde{G}(t)$ and hence $\tilde{H}(t)$ are lower bounded (see the short paragraph between (8.5) and (8.6)), Fubini's theorem applies to (8.9), and since

$$4 \int \chi(a\cdot\xi)\chi(a\cdot\eta)\, d\sigma(a) = 1 + \frac{2}{\pi} \arcsin(\xi\cdot\eta),$$

we obtain from (8.3) together with $\tilde{G} = \tilde{H} + \tilde{J}$ (cf. (8.6), (8.7))

$$\int \Lambda(f(a,\cdot))\, d\sigma(a) - \Lambda(f) =$$

$$\iint \left(\frac{2}{\pi} \arcsin(\xi\cdot\eta)\ \tilde{H}(\xi\cdot\eta) - \tilde{J}(\xi\cdot\eta)\right) f(\xi) f(\eta)\, d\sigma(\xi)\, d\sigma(\eta). \tag{8.10}$$

Each $f(a,\cdot)$ is even and takes a.e. the values 1 and -1 only, hence $\Lambda(f(a,\cdot)) \leq \kappa_*(n) \leq \kappa(n)$. For $n \leq 4$ we proceed to show that the right hand member of (8.10) is ≥ 0, and since $\Lambda(f) = \kappa(n)$ by hypothesis, this will imply that $f(a,\cdot)$ is maximizing for $\kappa(n)$, i.e., $\Lambda(f(a,\cdot)) = \kappa(n) = \kappa_*(n)$, for almost every $a \in \Sigma$, and indeed for *every* a

because the right hand member of (8.9) is a continuous function of a by the dominated convergence theorem and the fact that the kernel $(\xi, \eta) \mapsto \tilde{H}(\xi\cdot\eta)$ is integrable with respect to $d\sigma(\xi)\, d\sigma(\eta)$. (This property of integrability follows easily from [Be, Prop. 2.7] because G and hence $\tilde{H}$ are integrable w.r.t. the measure $(1-t^2)^{\frac{n-3}{2}} dt$ in view of [Be, Theorem 3.3].)

According to eq. (8.10) and the above comments to it, the first part of the proof of Theorem 4.4 will be achieved if we can show that the kernel

$$\frac{2}{\pi} \arcsin(\xi\cdot\eta)\, \tilde{H}(\xi\cdot\eta) - \tilde{J}(\xi\cdot\eta) \tag{8.11}$$

on $\Sigma \times \Sigma$ is *positive semidefinite*. Because the kernel (8.11) depends on $\xi\cdot\eta$ only, this amounts, by the Funk-Hecke formula [M, p. 20], to the corresponding function of t being positive semi-definite in the sense that

$$\int_{-1}^{1} \left(\frac{2}{\pi} \arcsin t\, \tilde{H}(t) - \tilde{J}(t)\right) P_k(n,t)\,(1-t^2)^{\frac{n-3}{2}} dt \geq 0 \tag{8.12}$$

for $k = 0, 1, 2, \dots$. The inequality (8.12) is trivially fulfilled for even values of k because the integrand then is an odd function of t. Moreover, (8.12) holds for $k = 1$ because $\int \tilde{J}(t) P_1(n,t)\,(1-t^2)^{\frac{n-3}{2}} dt = 0$ by (8.7) (no term with $k = 1$) and because the even function $\arcsin t\; P_1(n,t) = t \arcsin t$ has a power series expansion for $-1 \leq t \leq 1$ with exclusively non-negative coefficients. Note at this point that (cf. [Be, p. 19])

$$nt^2 = P_0(n,t) + (n-1) P_2(n,t)$$

is positive semidefinite (i.e., the kernel $n(\xi\cdot\eta)^2$ is positive semidefinite), and that any pointwise product of positive semi-definite kernels or functions is positive semi-definite. Finally, $\tilde{H}$ is positive semi-definite in view of (8.6).

Thus it remains to verify (8.12) for *odd* $k \geq 3$. We distinguish the cases n even (= 2 or 4) and n odd (= 3).

The case of even dimension n. 1° In the known case $n = 2$, cf. [HH], we have from [Be, p. 35] in view of (8.5) and subsequent lines

$$\tilde{G}(t) = 1 + G(t) = 1 - \left(\frac{\pi}{2} + \arcsin t\right)\sqrt{1-t^2} + \frac{1}{2}t,$$

and hence

$$\frac{2}{\pi} \arcsin t\, \tilde{H}(t) - \tilde{J}(t) = \frac{2}{\pi} \arcsin t - \frac{1}{2}t.$$

Since $P_k(2,t) = T_k(t)$ = the kth Čebyšev polynomial, the left hand member of (8.12) becomes (for odd $k \geq 3$) after elementary evaluation

$$\int_{-1}^{1} \left(\frac{2}{\pi} \arcsin t - \frac{1}{2}t\right) T_k(t)\,(1-t^2)^{-\frac{1}{2}} dt = \frac{4}{k^2\pi} > 0.$$

2° For $n = 4$ we similarly obtain from [Be]

$$\tilde{G}(t) = \frac{1}{3} - \frac{1}{2}\left(\frac{\pi}{2} + \arcsin t\right)(1-2t^2)(1-t^2)^{-\frac{1}{2}} - \frac{1}{4}t,$$

$$\frac{2}{\pi}\arcsin t\, \tilde{H}(t) - \tilde{J}(t) = \frac{2}{3\pi}\arcsin t + \frac{1}{4}t.$$

It is known that

$$P_k(4,t) = \frac{1}{(k+1)^2}\frac{d}{dt}T_{k+1}(t),$$

cf. [M, Lemma 13], and the left hand member of (8.12) becomes (for odd $k \geq 3$) after evaluations involving the substitution $t = \cos\theta$ and a partial integration

$$\int_{-1}^{1}\left(\frac{2}{3\pi}\arcsin t + \frac{1}{4}t\right) P_k(4,t)\,(1-t^2)^{\frac{1}{2}}\,dt = \frac{8}{3\pi}\frac{1}{k^2(k+2)^2} > 0.$$

3° For $n = 6$ one obtains from [Be]

$$\frac{2}{\pi}\arcsin t\, \tilde{H}(t) - \tilde{J}(t) = \frac{2}{5\pi}\arcsin t - \frac{1}{8}\frac{t}{1-t^2} + \frac{2}{3}t,$$

but now (8.12) breaks down already for $k = 3$. It is the middle term $-\frac{1}{8}t/(1-t^2)$ which tends to $-\infty$ as $t \to 1-$, and thus causes the kernel (8.11) to approach $-\infty$ on the diagonal ($\xi = \eta$), showing that the kernel (8.11) cannot be positive semidefinite.

The case $n = 3$. Using again (8.5) we obtain from [Be, p. 35]

$$\tilde{G}(t) = -\frac{1}{2} - t\log(1-t) - \left(\frac{4}{3} - \log 2\right)t,$$

$$\frac{2}{\pi}\arcsin t\, \tilde{H}(t) - \tilde{J}(t) =$$
$$\frac{1}{\pi}\arcsin t\left(-1 + t\log\frac{1+t}{1-t}\right) + \frac{1}{2}t\log(1-t^2) + \left(\frac{4}{3} - \log 2\right)t. \tag{8.13}$$

The presence of both arcsin and log makes this case more complicated than the above case of even dimension n, and the estimates become quite delicate, as we shall see. The polynomials $P_k(n,t)$ are now the classical Legendre polynomials $P_k(t)$, and the density $(1-t^2)^{\frac{n-3}{2}}$ equals 1. Recall that P_k satisfies the differential equation

$$\left((1-t^2)P_k'(t)\right)' + k(k+1)P_k(t) = 0 \tag{8.14}$$

and is the only solution regular at $t = 1$ with the value $P_k(1) = 1$. Also recall for $k \geq 1$ the recursion formula (cf. e.g. [Be, p. 32])

$$(k+1)P_{k+1}(t) - (2k+1)tP_k(t) + kP_{k-1}(t) = 0. \tag{8.15}$$

From (8.14) follows for $k \geq 1$ by integration

$$\int_{-1}^{t} P_k(s)\,ds = -\frac{1}{k(k+1)}(1-t^2)P_k'(t). \tag{8.16}$$

For *even* $k \geq 2$ we have by partial integration since $P_k(-1) = P_k(1) = 1$

$$\int_{-1}^{1} tP_k'(t)\,dt = 2 - \int_{-1}^{1} P_k(t)\,dt = 2. \tag{8.17}$$

From (8.15), (8.16) we obtain for any $k \geq 2$

$$(2k+1)\int_{-1}^{t} sP_k(s)\,ds = (1-t^2)\left(\frac{-1}{k+2}P_{k+1}'(t) + \frac{-1}{k-1}P_{k-1}'(t)\right) \tag{8.18}$$

and hence for *odd* $k \geq 3$ after elementary evaluations, using (8.17),

$$\left(k+\frac{1}{2}\right)\int_{-1}^{1} \frac{1}{2}\log(1-t^2)\; t\; P_k(t)\,dt = \frac{-1}{k+2} + \frac{-1}{k-1}. \tag{8.19}$$

In view of (8.16) we get for odd k

$$\int_{-1}^{1} \frac{1}{\pi}\arcsin t\; P_k(t)\,dt = \frac{1}{k(k+1)}\frac{1}{\pi}\int_{-1}^{1} \frac{t}{\sqrt{1-t^2}}\, P_k(t)\,dt$$

and hence in view of (8.15), again for odd k,

$$\left(k+\frac{1}{2}\right)\int_{-1}^{1} \frac{1}{\pi}\arcsin t\; P_k(t)\,dt = \frac{p_{k+1}}{2k} + \frac{p_{k-1}}{2k+2}, \tag{8.20}$$

where for even k

$$p_k := \frac{1}{\pi}\int_{-1}^{1} \frac{P_k(t)}{\sqrt{1-t^2}}\,dt = \binom{-1/2}{k/2}^2, \tag{8.21}$$

as it follows from (8.26) below by integration. We therefore obtain for even k by partial integrations

$$\frac{1}{\pi}\int_{-1}^{1} \arcsin t\; P_k'(t)\,dt = 1 - p_k, \tag{8.22}$$

$$\frac{1}{\pi}\int_{-1}^{1}\log\frac{1+t}{1-t}\,\sqrt{1-t^2}\,P_k'(t)\,dt = 2q_k - 2p_k, \tag{8.23}$$

where (for even k)

$$q_k = \frac{1}{2\pi}\int_{-1}^{1}\log\frac{1+t}{1-t}\,\frac{tP_k(t)}{\sqrt{1-t^2}}\,dt. \tag{8.24}$$

Summing up, we find for odd $k \geq 3$ by partial integration, using (8.18), (8.22), (8.23),

$$\left(k+\frac{1}{2}\right)\int_{-1}^{1}\frac{1}{\pi}\arcsin t\,\log\frac{1+t}{1-t}\,tP_k(t)\,dt =$$
$$\frac{1}{k+2}+\frac{1}{k-1}+\frac{q_{k+1}-2p_{k+1}}{k+2}+\frac{q_{k-1}-2p_{k-1}}{k-1}.$$

Combining this with (8.13), (8.20), (8.19) we evaluate the left hand member of (8.12) as follows for odd $k \geq 3$ in dimension $n = 3$:

$$\left(k+\frac{1}{2}\right)\int_{-1}^{1}\left(\frac{2}{\pi}\arcsin t\,\tilde{H}(t)-\tilde{J}(t)\right)P_k(t)\,dt =$$
$$\frac{1}{k+2}\left[q_{k+1}-\left(\frac{5}{2}+\frac{1}{k}\right)p_{k+1}\right]+\frac{1}{k-1}\left[q_{k-1}-\left(\frac{5}{2}-\frac{1}{k+1}\right)p_{k-1}\right], \tag{8.25}$$

noting that t is orthogonal to $P_k(t)$.

To prove that the right hand member of (8.25) is positive for odd $k \geq 3$ we evaluate q_k from (8.24) for *even* k, using the known identity, cf. [E1, p. 176],

$$P_k(\cos\theta) = \sum_{j=0}^{k}\gamma_j\gamma_{k-j}\cos\big((k-2j)\theta\big), \tag{8.26}$$

$$\gamma_j = \frac{1}{2}\frac{3}{4}\cdots\frac{2j-1}{2j} = (-1)^j\binom{-1/2}{j}. \tag{8.27}$$

For *even* $h \in [-k, k]$ one easily finds by partial integration, invoking the Dirichlet kernel,

$$\frac{1}{\pi}\int_0^{\pi}\log\left(\cot^2\frac{\theta}{2}\right)\cos\theta\,\cos(h\theta)\,d\theta = \frac{1}{|h+1|}+\frac{1}{|h-1|}.$$

In view of (8.26), (8.27) this leads after some calculation to the following evaluation of q_k from (8.24) (k being even)

$$q_k = \binom{-1/2}{k/2}^2 + \sum_{j=1}^{k/2}\frac{4j}{4j^2-1}\binom{-1/2}{k/2+j}\binom{-1/2}{k/2-j}. \tag{8.28}$$

It is also elementary to verify the inequality

$$\gamma_{k/2+j}\,\gamma_{k/2-j} \geq \left(\gamma_{k/2}\right)^2 = \binom{-1/2}{k/2}^2 = p_k,$$

cf. (8.27), (8.21), and we thus deduce from (8.28) that, for even k,

$$q_k \geq p_k\left(1 + \sum_{j=1}^{k/2} \frac{4j}{4j^2-1}\right) \geq \frac{43}{15}p_k$$

if $k \geq 4$, while $q_2 \geq \frac{7}{3}p_2$. It follows that both terms on the right of (8.25) are indeed positive for odd $k \geq 3$:

$$q_{k+1} - \left(\frac{5}{2} + \frac{1}{k}\right)p_{k+1} \geq \left(\frac{43}{15} - \frac{5}{2} - \frac{1}{3}\right)p_{k+1} = \frac{1}{30}\,p_{k+1} > 0,$$

$$q_{k-1} - \left(\frac{5}{2} - \frac{1}{k+1}\right)p_{k-1} \geq \left(\frac{43}{15} - \frac{5}{2}\right)p_{k-1} = \frac{11}{30}\,p_{k-1} > 0$$

if $k \geq 5$, while for $k = 3$:

$$q_{k-1} - \left(\frac{5}{2} - \frac{1}{k+1}\right)p_{k-1} \geq \left(\frac{7}{3} - \frac{5}{2} + \frac{1}{4}\right)p_2 = \frac{1}{12}\,p_2 > 0.$$

In view of the text following (8.12) this completes the first part of the proof of Theorem 4.4 in dimension $n = 3$, and altogether for $n \leq 4$. □

For $n = 5$ the proof breaks down (for the same reason as in the case $n = 6$ above) since $\frac{2}{\pi}\arcsin t\ \tilde{H}(t) - \tilde{J}(t) \to -\infty$ as $t \to 1-$, and so the kernel (8.13) (with $t = \xi\cdot\eta$) is not positive semidefinite.

9. The second part of the proof of Theorem 4.4

Let $n \leq 4$, and suppose by contradiction that there exists a maximizing function f for $\kappa(n)$ which is *not* an even function. The associated minimizing function $u = Tf$ for $c(n)$ is C^1-smooth, and $u \neq 0$ a.e., $\operatorname{sgn} u = f$ a.e. (cf. Theorems 4.2 and 4.3). Because f is not an even function (after correction on a null set), the open sets $\{u > 0\}$ and $\{\check{u} < 0\}$ meet (we write $\check{u}(\xi) = u(-\xi)$), and so do therefore the larger open sets

$$\operatorname{int}\{u \geq 0\}, \quad \operatorname{int}\{\check{u} \leq 0\}.$$

Consider two components U and V of $\operatorname{int}\{u \geq 0\}$ and of $\operatorname{int}\{\check{u} \leq 0\}$, respectively, such that $U \cap V \neq \emptyset$. Clearly $u = 0$ on ∂U and $\check{u} = 0$ on ∂V.

For any $a \in \Sigma$ write Σ_a, Σ_a^+, Σ_a^- for the set of $\xi \in \Sigma$ such that $a{\cdot}\xi = 0$, $a{\cdot}\xi > 0$, $a{\cdot}\xi < 0$, respectively. As in (8.8) define

$$f(a,\cdot) = f \text{ in } \Sigma_a^+, \quad f(a,\cdot) = \check{f} \text{ in } \Sigma_a^-, \tag{9.1}$$

and recall from Section 8 that $f(a,\cdot)$ is again maximizing for $\kappa(n)$ because $n \leq 4$ (see the lines following (8.10)). Applying the operator T from (4.6) we know as above that $u(a,\cdot) := Tf(a,\cdot)$ is minimizing for $c(n)$ and hence C^1-smooth; that $u(a,\cdot) \neq 0$ a.e.; and that $\operatorname{sgn} u(a,\cdot) = f(a,\cdot)$ a.e. Also, $f(a,\cdot)$ is an even function, and so is therefore $u(a,\cdot)$.

The case $n = 2$. On the unit circle Σ in $\mathbf{R}^2 = \mathbf{C}$ we choose a point a so that $-ia \in U \cap V$, and further that the circular distance between $-ia$ and the first point b of ∂U following $-ia$ (in the standard orientation of Σ) is an irrational multiple of π. Note that $u > 0$ a.e. in a neighbourhood of $-ia$ ($\in U$) and $u < 0$ a.e. in a neighbourhood of ia ($\in -V$). Hence $-ia$, b, and ia follow each other in this order. In view of (9.1) $u(a,\xi)$ changes sign when ξ passes $-ia$ ($\in \Sigma_a$), while $u(a,\xi)$, like $u(\xi)$, takes both signs in any neighbourhood of b ($\in \Sigma_a^+$). It follows that the open arc from $-ia$ to b is a component of $\operatorname{int}\{u(a,\cdot) \geq 0\}$ of length $\operatorname{dist}(-ia, b) \notin \mathbf{Q}\pi$, in contradiction with Theorem 5 or its corollary applied to the even minimizing function $u(a,\cdot)$ for $c(2)$.

The case $n = 3$ *or* 4. First a general observation concerning the even, minimizing function $u(a,\cdot) = Tf(a,\cdot)$ for $c(n)$. If $u(a,\cdot) \geq 0$ (resp. ≤ 0) in some open set $E \subset \Sigma$ then actually $u(a,\cdot) > 0$ (resp. < 0) everywhere in E. In fact, the Euler equation (4.14) for $u(a,\xi)$, viewed in E, reads (in the former case)

$$-\Delta u(a,\cdot) - (n-1)u(a,\cdot) = 1 - \int f(a,\cdot)\,d\sigma > 0. \tag{9.2}$$

(Note that $\int f(a,\cdot)\,d\sigma = 1$ would imply $f(a,\cdot) = 1$ a.e. on Σ, hence $u(a,\cdot) \geq 0$ on Σ, and since $\int u(a,\cdot)\,d\sigma = 0$ this would lead to $u(a,\cdot) \equiv 0$.) It follows from (9.2) that $u(a,\cdot)$ is spherically superharmonic in E, cf. page 40, and if $u(a,\cdot)$ equals 0 at some point of E then also in a neighbourhood, cf. [Br, p. 33], in contradiction with (9.2).

First step. We show that the two maximal domains U and V chosen in the beginning of this section must be equal:

$$U = V.$$

Because $U \cap V \neq 0$ this amounts to proving that $V \cap \partial U = \emptyset$ and (similarly) $U \cap \partial V = \emptyset$. Suppose there is a point

$$\xi^* \in V \cap \partial U. \tag{9.3}$$

Clearly $\Sigma \setminus U$ has no isolated points, and neither has ∂U because Σ has dimension $n - 1 \geq 2$ and so $N \setminus \{\xi\}$ is connected for any connected open neighbourhood N of a point $\xi \in N$.

There exists therefore a closed solid cone of revolution Q in $\mathbf{R}^n$ with opening angle $< \pi/4$ and with vertex at ξ^* such that ξ^* is a limit point of $(\operatorname{int}(Q \cap \Sigma)) \cap \partial U$, hence also of $Q \cap (\partial U) \setminus \{\xi^*\}$ and of $Q \cap U$ (interiors and boundaries of subsets of Σ being taken relatively to Σ). In particular, $Q \setminus \{\xi^*\}$ meets the tangent space to Σ at ξ^*, hence does not meet the line $\mathbf{R}\xi^*$ passing through 0 and ξ^*. By the separation theorem there exists therefore a point $a \in \Sigma$ such that $\xi^* \in \Sigma_a$ and

$$Q \cap \Sigma \setminus \{\xi^*\} \subset \Sigma_a^+. \tag{9.4}$$

Now $f(a, \xi) = f(\xi) = 1$ a.e. in $U \cap \Sigma_a^+$, cf. (9.1), and hence $u(a, \xi) > 0$ for every $\xi \in U \cap \Sigma_a^+$ by the above general observation. Similarly, $u(a, \xi) < 0$ for every $\xi \in V \cap \Sigma_a^-$ because $f(a, \xi) = f(-\xi) = -1$ a.e. for $\xi \in V \cap \Sigma_a^-$. By continuity we obtain

$$u(a, \xi^*) = 0 \tag{9.5}$$

because $\xi^* \in V \cap \Sigma_a$, cf. (9.3), is a limit point of $V \cap \Sigma_a^-$ and also of $Q \cap U \subset U \cap \Sigma_a^+$ (noting that $U \subset \Sigma \setminus \{\xi^*\}$ by (9.3), and so $Q \cap U \subset \Sigma_a^+$ by (9.4)). It follows that

$$\nabla u(a, \xi^*) \neq 0 \tag{9.6}$$

according to Lemma 9 below applied to the C^1-smooth function $-u(a, \cdot)$ (which we have just shown is positive in $V \cap \Sigma_a^-$). In fact, the Euler equation for the even, minimizing function $u(a, \cdot)$ for $c(n)$, considered in $V \cap \Sigma_a^-$, has the (constant) right hand member $-1 - \int f(a, \cdot)\, d\sigma < 0$, cf. the argument following (9.2). Moreover, since $\xi^* \in V \cap \Sigma_a$ there is a small closed cap K such that $\xi^* \in \partial K$ and $\operatorname{int} K \subset V \cap \Sigma_a^-$, hence $-u(a, \cdot) \geq 0$ on ∂K.

By the implicit function theorem it follows from (9.5), (9.6) that the zero set $\{u(a, \cdot) = 0\}$ is (in a neighbourhood of ξ^*) an $(n-2)$-dimensional submanifold of Σ. This manifold does not meet $V \cap \Sigma_a^-$ (where $u(a, \cdot) < 0$), and is therefore *tangential* to Σ_a at ξ^*. Near each point of $Q \cap (\partial U) \setminus \{\xi^*\}$ the function u takes both positive and negative values, and hence so does $u(a, \cdot)$ by (9.1) because $Q \cap (\partial U) \setminus \{\xi^*\} \subset \Sigma_a^+$ by (9.4). It follows that $u(a, \cdot) = 0$ in the set $Q \cap (\partial U) \setminus \{\xi^*\}$ for which ξ^* is a limit point by the definition of Q above; and this set $Q \cap (\partial U) \setminus \{\xi^*\}$ is *non-tangential* to Σ_a at ξ^*, by (9.4). We have thus arrived at a contradiction which shows that our hypothesis of the existence of a point ξ^* as in (9.3) is false, and so actually $U = V$, as asserted.

Second step. The Euler equation (4.14) for u itself, considered in $-V$ where $u \leq 0$, reads

$$-\Delta u - (n-1)u = -1 - f_0 - f_1. \tag{9.7}$$

Here $1 + f_0 > 0$ (cf. the argument following (9.2)), and the set

$$A := \{\xi \in \Sigma \mid f_1(\xi) \leq -1 - f_0\} \tag{9.8}$$

is therefore a closed cap of spherical radius $< \pi/2$ (except that $A = \emptyset$ if $f_1 \equiv 0$). Anyway, A cannot contain $-V$, for then the analytic function u in $-V$ would be spherically superharmonic in the sense of Berg in view of (9.7), (9.8), cf. [Be, Theorem 4.9]; and since $u = 0$ on $\partial(-V)$ and there exists a spherically superharmonic function > 0 on a cap containing A (cf. the proof of Lemma 9 below) it would follow from the boundary minimum principle [Br, p. 33] that $u \geq 0$ in $-V$, hence $u \equiv 0$ in $-V$, in contradiction with $u \neq 0$ a.e. on Σ by Theorem 4.2.

We have thus proved that the connected set $\complement A$ meets $-V$, hence also $\partial(-V)$ (complements and boundaries being taken relative to Σ); for otherwise $\complement A \subset -V$, hence $\complement(-A) \subset V$, and so $\complement(A \cup (-A)) \subset V \cap (-V)$, showing that $\sigma(V \cap (-V)) > 0$, in contradiction with $u \leq 0$ in $-V$, $u \geq 0$ in $V = U$, and $u \neq 0$ a.e. on Σ, by Theorem 4.2.

Accordingly we may choose a point $\eta \in (\partial(-V)) \setminus A$ and next a point $b \in -V$ so that $2\,\mathrm{dist}(b, \eta) < \mathrm{dist}(\eta, A)$ $(\leq \pi)$, where dist refers to the geodesic distance on Σ (and where $\mathrm{dist}(\eta, A) := \pi$ if $A = \emptyset$). Fix a point $\eta^* \in \partial(-V)$ nearest to b. The closed cap B in Σ centred at b and such that $\eta^* \in \partial B$ has then spherical radius $< \pi/2$ and does not meet A because

$$\mathrm{dist}(b, \eta^*) \leq \mathrm{dist}(b, \eta) < \mathrm{dist}(\eta, A) - \mathrm{dist}(b, \eta) \leq \mathrm{dist}(b, A) \leq \pi.$$

From $b \in -V$ and $(\mathrm{int}\, B) \cap \partial(-V) = \emptyset$ (by the definition of η^*) follows

$$\mathrm{int}\, B \subset -V. \tag{9.9}$$

Since $B \subset \Sigma \setminus A$ there is, in view of (9.8), a constant $\alpha > 0$ such that $f_1(\xi) > -1 - f_0 + \alpha$ for $\xi \in B$, and so the right hand member of (9.7) is $< -\alpha$ in $\mathrm{int}\, B$. We have $-u \geq 0$ in $-V$, in particular in B, by (9.9), while $u = 0$ on $\partial(-V)$, in particular $u(\eta^*) = 0$. It follows by Lemma 9 applied to $-u$ and the point $\eta^* \in \partial B$ that $\nabla u(\eta^*) \neq 0$. Writing $\zeta^* = -\eta^*$ we thus have

$$\check{u}(\zeta^*) = 0, \quad \nabla\check{u}(\zeta^*) \neq 0. \tag{9.10}$$

By the implicit function theorem ζ^* has a connected open neighbourhood N in Σ such that $N_0 := N \cap \{\check{u} = 0\}$ is an $(n-2)$-dimensional submanifold of N $(\subset \Sigma)$, separating N into the connected open sets $N_+ := N \cap \{\check{u} > 0\}$, $N_- := N \cap \{\check{u} < 0\}$. Because $\check{u} \leq 0$ in V while $\check{u} = 0$ on ∂V we actually have $\check{u} < 0$ in $N \cap V$, and indeed $N \cap V = N_-$, whence $N \cap \partial V = N_0$. It follows that $\mathrm{int}\{\check{u} \geq 0\}$ has a (unique) component W such that $N \cap W = N_+$ and hence

$$N \cap \partial W = N_0 = N \cap \partial U \tag{9.11}$$

(recall that $U = V$, as shown in the first step in the proof); and within N this submanifold N_0 separates U from W.

Because the cap $-B$ has $\zeta^* = -\eta^*$ on its boundary, there is a (unique) point $a \in \Sigma$ such that $a \cdot \zeta^* = 0$ and

$$\mathrm{int}(-B) \subset \Sigma_a^+. \tag{9.12}$$

Hence Σ_a and $\partial(-B)$ have the same tangent space at ζ^*, and so have $N_0 = N \cap \partial V$ (smooth) and $\partial(-B)$ because $\zeta^* \in N_0$ and $\mathrm{int}(-B) \subset V$, by (9.9).

Now consider the maximizing function $f(a, \cdot)$ for $\kappa(n)$ with the above $a \in \Sigma$, cf. (9.1), and the corresponding minimizing function $u(a, \cdot) = Tf(a, \cdot)$ for $c(n)$, cf. Theorem 4.3. By (9.1) we obtain

$$u(a, \cdot) \geq 0 \quad \text{in } U \cap \Sigma_a^+ \text{ and in } W \cap \Sigma_a^- \tag{9.13}$$

because $f(a, \xi) = f(\xi) = 1$ a.e. in the former set and $f(a, \xi) = \check{f}(\xi) = 1$ a.e. in the latter set where $\check{u} \geq 0$. Also by (9.1), (9.11), and by continuity,

$$u(a, \cdot) = 0 \quad \text{on } Z := N_0 \setminus \Sigma_a. \tag{9.14}$$

In fact, for given $\xi \in Z$ we have $u(a, \xi) \geq 0$, by passing to the limit in (9.13) (consider separately the cases $\xi \in \Sigma_a^+$ and $\xi \in \Sigma_a^-$). To see that also $u(a, \xi) \leq 0$, note that, if $\xi \in \Sigma_a^+$, every neighbourhood of ξ ($\in \partial U$) contains points $\xi' \in \Sigma_a^+$ with $u(\xi') < 0$ (by the maximality of U), hence $u(a, \xi') < 0$; and if $\xi \in \Sigma_a^-$, every neighbourhood of ξ ($\in \partial U$) contains points $\xi' \in U \cap \Sigma_a^-$ such that $\check{u}(\xi') < 0$ (because $\check{u} < 0$ a.e. in $U = V$) and so $u(a, \xi') < 0$.

If the point ζ^* ($\in \Sigma_a$) is a limit point of this set Z then $u(a, \zeta^*) = 0$, and hence $\nabla u(a, \zeta^*) \neq 0$ according to Lemma 9 and the Euler equation (9.2) considered in the cap

$$\mathrm{int}(-B) \subset U \cap \Sigma_a^+ \tag{9.15}$$

in which $u(a, \cdot) \geq 0$, cf. (9.9), (9.12), (9.13), recalling that $U = V$ as shown in the first step in the proof. By the implicit function theorem, however, this conclusion $\nabla u(a, \zeta^*) \neq 0$ contradicts (9.13) according to which $u(a, \cdot) \geq 0$ near ζ^*, e.g. on the geodesic on Σ passing through ζ^* and perpendicular to Σ_a, hence also to N_0 as noted above after (9.12).

Third step. We are thus left with the only possibility that there is an open cap C contained in N, centred at ζ^*, and not meeting Z from (9.14), whence $C \cap N_0 \subset C \cap \Sigma_a$. Actually,

$$C \cap N_0 = C \cap \Sigma_a \tag{9.16}$$

because $C \setminus N_0$ is disconnected, being the union of the non-void disjoint open sets $C \cap N_+$ and $C \cap N_-$, cf. the text between (9.10) and (9.11). Since $\zeta^* \in \partial(-B)$, C meets $\mathrm{int}(-B)$ and hence also $U \cap \Sigma_a^+$, by (9.15). Thus $C \cap \Sigma_a^+$ meets U, but not ∂U in view of (9.11) and (9.16). Because $C \cap \Sigma_a^+$ is connected it follows that

$$C \cap \Sigma_a^+ \subset U, \quad C \cap \Sigma_a^- \subset W, \tag{9.17}$$

the latter relation by a similar argument involving the lines following (9.11).

From (9.13), (9.17) we infer that $u(a,\cdot) \geq 0$ in $C \cap \Sigma_a^+$ and in $C \cap \Sigma_a^-$, hence in all of C, by continuity. By the general observation in the beginning of the proof for $n \geq 3$ it follows that $u(a,\cdot) > 0$ in C, in particular

$$u(a, \zeta^*) > 0. \tag{9.18}$$

Since $\zeta^* \in \Sigma_a$, that is $a \cdot \zeta^* = 0$, we have $a \in \Sigma_{\zeta^*}$. Being a sphere of dimension $n-2 \geq 1$, Σ_{ζ^*} contains points $c \neq a$ arbitrarily close to a. For any point c of $\Sigma_{\zeta^*} \setminus \{a, -a\}$ we have $u(c,\cdot) = Tf(c,\cdot) = 0$ on $C \cap \Sigma_a \cap \Sigma_c^+$. In the first place, $u(c,\cdot) \geq 0$ in $C \cap \Sigma_a^+ \cap \Sigma_c^+$ by (9.1) with c in place of a because $u > 0$ a.e. in this open subset of U, cf. (9.17). And secondly we have $u(\xi) < 0$ and hence $u(c,\xi) < 0$ for suitable points $\xi \in C \cap \Sigma_a^- \cap \Sigma_c^+$ arbitrarily close to any given point of $C \cap \Sigma_a \cap \Sigma_c^+ = C \cap (\partial U) \cap \Sigma_c^+$, by the maximality of U, cf. (9.11), (9.16). It follows that

$$u(c, \zeta^*) = 0 \quad \text{for } c \in \Sigma_{\zeta^*} \setminus \{a, -a\} \tag{9.19}$$

because $\zeta^* \in \Sigma_a \cap \Sigma_c$ is a limit point of $C \cap \Sigma_a \cap \Sigma_c^+$. By the dominated convergence theorem we have, using again (9.1),

$$\lim_{c \to a} f(c,\cdot) = f(a,\cdot) \tag{9.20}$$

in the weak* topology on $L^\infty(\sigma)$ as the dual of $L^1(\sigma)$.

As mentioned in Remark 4.5 (see also Section 8 after (8.4)), T is an integral operator with the kernel $(\zeta, \xi) \mapsto \tilde{G}(\zeta \cdot \xi)$, $\tilde{G}(t)$ being defined in (8.4). It follows from [Be, Theorem3.3] that G and hence $\tilde{G}$ are integrable over $[-1, 1]$ w.r.t. the measure $(1-t^2)^{\frac{n-3}{2}}dt$. For fixed $\zeta \in \Sigma$ the function $\xi \mapsto \tilde{G}(\zeta \cdot \xi)$ is therefore integrable w.r.t. σ by virtue of [Be, Prop. 2.7]. In view of (4.16) and (9.19), (9.20) we therefore obtain for $c \to a$ through $\Sigma_{\zeta^*} \setminus \{a, -a\}$:

$$0 = u(c, \zeta^*) = [Tf(c,\cdot)](\zeta^*) = \int \tilde{G}(\zeta^* \cdot \xi) f(c, \xi)\, d\sigma(\xi)$$

$$\to \int \tilde{G}(\zeta^* \cdot \xi) f(a, \xi)\, d\sigma(\xi) = u(a, \zeta^*)$$

in contradiction with (9.18). When Lemma 9 below has been established, this completes the second part of the proof of Theorem 4.4. □

Lemma 9. *Let $B = \{\xi \in \Sigma \mid b \cdot \xi \geq \cos\rho\}$ denote a closed cap in Σ with centre $b \in \Sigma$ and spherical radius $\rho < \pi/2$. Let $u : B \to \mathbf{R}$ be continuous in B and C^2-smooth in* int B*, and suppose that u satisfies*

$$-\Delta u - (n-1)u \geq \alpha \quad \textit{in } \operatorname{int} B$$

for some constant $\alpha \geq 0$. If $u \geq 0$ on ∂B then

$$u(\xi) \geq \frac{\alpha}{n-1}\left(\frac{b \cdot \xi}{\cos\rho} - 1\right) \geq 0 \quad \textit{for } \xi \in B. \tag{9.21}$$

In the case $\alpha > 0$ it follows that $\nabla u(\xi^*) \neq 0$ for any $\xi^* \in \partial B$ at which $u(\xi^*) = 0$ and $\nabla u(\xi^*)$ exists in the classical sense.

Proof. The function

$$v(\xi) = \frac{1}{n-1}\left(\frac{b\cdot\xi}{\cos\rho} - 1\right), \qquad \xi \in B,$$

satisfies $v = 0$ on ∂B and

$$-\Delta v - (n-1)v = 1 \quad \text{in } \operatorname{int} B$$

because the function $\xi \mapsto b\cdot\xi$ is in $\mathcal{H}_1$ and $\lambda_1 = n - 1$. Thus u and v are spherically superharmonic in int B (cf. page 40). Replacing ρ by a bigger number, again $< \pi/2$, leads similarly to a spherically superharmonic function $\bar{v} > 0$ in an open cap containing B. Since $\bar{v}$ is bounded away from 0 on B we infer from the boundary minimum principle [Br, p. 33] that indeed $u \geq 0$ in int B and hence in B. (Alternatively, argue as in [Be, pp. 49–50].) Applying the above to $u - \alpha v$ in place of u leads to (9.21). As to the last assertion of the lemma, note that the inner normal derivative of v (as a function in B) is > 0 at any point $\xi^* \in \partial B$, and it follows by (9.21) that the inner lower normal derivative of u at ξ^* is > 0 when $u(\xi^*) = 0$. □

References

[Be] Berg, C., *Corps convexes et potentiels sphériques*, Mat. Fys. Medd. Dan. Vid. Selsk. **37** (1969), no. 6.

[Bo] Bonnesen, T., *Über das isoperimetrische Defizit ebener Figuren*, Math. Ann. **91** (1924), 252–268.

[Br] Brelot, M., *Axiomatique des fonctions harmoniques*, Les presses de l'Université de Montréal, Montreal, 1966.

[Co] Courant, R., *Über die Eigenwerte bei den Differentialgleichungen der mathematischen Physik*, Math. Z. **7** (1920), 1–57.

[CH] Courant, R., and D. Hilbert, *Methoden der mathematischen Physik*, Springer-Verlag, Berlin, 1931.

[DL] Deny, J., and J. L. Lions, *Les espaces du type de Beppo Levi*, Ann. Institut Fourier Grenoble **5** (1953–54), 305–370.

[E1] Erdélyi et al., *Higher Transcendental Functions I*, McGraw-Hill, New York-Toronto-London, 1954.

[E2] Erdélyi et al., *Higher Transcendental Functions II*, McGraw-Hill, New York-Toronto-London, 1954.

[F1] Fuglede, B., *Stability in the isoperimetric problem for convex or nearly spherical domains in* $\mathbf{R}^n$, Trans. Amer. Math. Soc. **314** (1989), 619–638.

[F2] Fuglede, B., *Bonnesen's inequality for the isoperimetric deficiency of closed curves in the plane*, Geometriae Dedicata **38** (1991), 283–300.

[F3] Fuglede, B., *Lower estimate of the isoperimetric deficit of convex domains in* $\mathbf{R}^n$ *in terms of asymmetry*, Geometriae Dedicata **47** (1993), 41–48.

[F4] Fuglede, B., *Capacity, convexity, and isoperimetric inequalities*, Expo. Math. **11** (1993), 455–464.

[F5] Fuglede, B., *Poincaré inequalities in L^1-norm for the sphere and a strong isoperimetric inequality in $\mathbf{R}^n$*, In: "Classical and Modern Potential Theory and Applications", Eds. K. GowriSankaran et al., Kluwer Acad. Publ., Dordrecht-Boston-London 1994, 165–183.

[G] Groemer, H., *Stability of geometric inequalities*, Chapter 1.4 in "Handbook of Convex Geometry I", Ed. P. M. Gruber and J. M. Wills, Elsevier Science Publishers, 1993.

[GS] Groemer, H., and R. Schneider, *Stability estimates for some geometric inequalities*, Bull. London Math. Soc. **23** (1991), 67–74.

[H] Hall, R. R., *A quantitative isoperimetric inequality in n-dimensional space*, J. reine angew. Math. **428** (1992), 161–176.

[HH] Hall, R. R., and W. K. Hayman, *A problem in the theory of subordination*, J. d'Analyse Math. **60** (1993), 99–111.

[HHW] Hall, R. R., W. K. Hayman, and A. W. Weitsman, *On asymmetry and capacity*, J. d'Analyse Math. **56** (1991), 87–123.

[He] Hervé, R.-M., *Recherches axiomatiques sur la théorie des fonctions surharmoniques et du potentiel*, Ann. Inst. Fourier Grenoble **12** (1962), 415–471.

[Hö1] Hörmander, L., *The Analysis of Linear Partial Differential Operators I*, Springer-Verlag, Berlin, 1983, 1990.

[Hö2] Hörmander, L., *The Analysis of Linear Partial Differential Operators III*, Springer-Verlag, Berlin, 1985.

[M] Müller, C., *Spherical Harmonics*, Lecture Notes in Mathematics, No. 17, Springer-Verlag, Berlin-Heidelberg-NewYork, 1966.

[PS] Pólya, G., and G. Szegö, *Isoperimetric Inequalities in Mathematical Physics*, Annals of Mathematics Studies No. 27, Princeton, 1951.

[R] Rellich, F., *Ein Satz über mittlere Konvergenz*, Nach. Akad. Wiss. Göttingen Math.-Phys. Kl. (1930), 30–35.

[Sc] Schneider, R., *Stability in the Alexandrov-Fenchel-Jessen theorem*, Mathematika **36** (1989), 50–59.

[Se] Seeley, R.T., *Singular integrals on compact manifolds*, Amer. J. Math. **81** (1959), 658–690.

[Sp] Spivak, M., *A Comprehensive Introduction to Differential Geometry IV*, Publish or Perish, Boston, 1975.

Submitted to the Academy July 1994.
Published June 1995.