JØRGEN KALCKAR AND OLE ULFBECK

ON THE PROBLEM OF GRAVITATIONAL RADIATION

Det Kongelige Danske Videnskabernes Selskab
Matematisk-fysiske Meddelelser **39**, 6

Kommissionær: Munksgaard
København 1974

Synopsis

The problem of gravitational radiation is discussed. First, from classical electrodynamics those basic principles are isolated, the joint validity of which implies the occurrence of electromagnetic radiation, and it is investigated to what extent conclusions regarding gravitational radiation can be based solely on the same premises. Next, the consequences are explored of introducing new features, which – like the Equivalence Principle – distinguish between gravitational and electromagnetic interactions.

PRINTED IN DENMARK
BIANCO LUNOS BOGTRYKKERI A/S
ISBN 87 7304 040 1

In spite of the far reaching formal completeness of the General Theory of Relativity, the inherent conceptual and mathematical difficulties of the scheme, together with the absence of conclusive empirical evidence, has sustained the discussion of the problem of gravitational radiation, ever since the original work of Einstein* on the energy loss from a spinning rod.

In this situation it may be of interest to isolate from classical electrodynamics those basic principles, the joint validity of which implies the occurrence of electromagnetic radiation, and investigate to what extent conclusions regarding gravitational radiation can be based solely on the same premises. Having clarified this problem, one may then as a next step explore the consequences of the introduction of new features, which – like the Equivalence Principle – distinguish between gravitational and electromagnetic interactions.

The clue to the radiation problem is to be found in the limitations in the possibility of accounting for the instantaneous energy balance for a system of interacting particles solely in terms of the particle degrees of freedom. Clearly, no such limitations exist in the purely static case, and consequently, from this point of view, the impartion of energy to a static field must be regarded as a matter of convention. Quite a different situation is met with in the case of time varying charge and current distributions. Due to the retardation of physical actions, the field now represents independent degrees of freedom of the total system, which can only be ignored or eliminated at the expense of giving up the notion of instantaneous energy momentum balance.**

To illustrate this interrelationship in the case of electrodynamics, we consider two particles of charge Q — originally at rest at a relative distance

* A. Einstein, Sitzungsberichte der Preuss. Akad., Berlin, p. 688 (1916); p. 154 (1918).

** A comprehensive discussion of these problems is given in a treatise shortly to appear in Kgl. Vid. Selsk. Med.

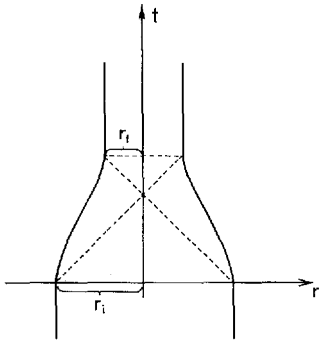

Fig. 1

$2r_i$ —, which are moved simultaneously and symmetrically towards each other to a relative distance $2r_f$ ($r_f < r_i$), where they stay at rest (see figure 1). If the process is carried out adiabatically, the external work performed equals the change in potential energy

$$W_{ad} = \frac{Q^2}{2r_f} - \frac{Q^2}{2r_i}. \tag{1}$$

If, however, the process is carried out in a finite time, the work required will in general, as a consequence of the retardation, differ from W_{ad}.

Suppose that the duration of the process, Δt, is chosen so that

$$r_i - r_f < c\Delta t \leq r_i + r_f, \tag{2}$$

which implies that the electromagnetic force on each particle due to the other one during the entire motion is given by the original static Coulomb field. In this case, the work required to overcome the electrostatic repulsion only amounts to

$$2\left(\frac{Q^2}{r_i + r_f} - \frac{Q^2}{2r_i}\right). \tag{3}$$

The very fact that this work differs from the change in potential energy (1) faces us with the choice of either giving up the customary idea of energy conservation, or recognizing the existence of some non-conservative force acting on each particle independently of the motion of the other since during the process considered no communication is possible between the particles. Within the customary mechanical framework the non-conservative character of this "damping force" is interpreted as a manifestation of an independent set of degrees of freedom with which the particles may interact and exchange energy, the damping force being just a phenomenological way of taking this interaction into account.

Reconsidering now the above process in this extended framework, we notice that the external work, W_D, required to overcome the damping force on each particle during the displacement must, for symmetry reasons, be the same for both particles and, according to its definition, independent of the motion of the other. Thus the total energy to be supplied is not given by eq. (3) but by the relation

$$W = 2\left(\frac{Q^2}{r_i + r_f} - \frac{Q^2}{2r_i} + W_D\right), \tag{4}$$

where W_D is related to the hypothetical "radiation energy" $\mathscr{E}_R$ by the requirement of energy balance

$$\mathscr{E}_R + \frac{Q^2}{2r_f} = W + \frac{Q^2}{2r_i}. \tag{5}$$

Hence:

$$2W_D - \mathscr{E}_R = Q^2 \frac{(r_i - r_f)^2}{2r_i r_f (r_i + r_f)}. \tag{6}$$

Whereas this expression is still compatible with a complete absence of radiation, corresponding to $\mathscr{E}_R = 0$, evidently, $\mathscr{E}_R$ and W_D cannot *both* vanish. Furthermore, the fact that, according to the initial conditions, $\mathscr{E}_R \geqq 0$, implies that W_D is positive definite, reflecting the irreversible character of the process of radiation emission.

Consider now in particular the case in which the equality sign in eq. (2) holds, i.e.

$$c\Delta t = r_i + r_f \tag{7}$$

and assume for simplicity that

$$\Delta r \equiv r_i - r_f \ll c\Delta t. \tag{8}$$

Then, eq. (6) may be rewritten in the suggestive form

$$2W_D - \mathscr{E}_R = 2\frac{Q^2}{c^3}\left(\frac{\Delta r}{(\Delta t)^2}\right)^2 \Delta t. \tag{9}$$

So far we cannot draw any conclusions as to the individual value of W_D and $\mathscr{E}_R$. However, since W_D, as already noticed, is independent of the motion of the other particle, it may be determined by considering another process, in which only one of the particles is displaced along the same world line as before, whereas the other is kept fixed. Denoting by E_R the energy transferred to the radiation field during this process, the energy balance now yields the relation:

$$E_R + \frac{Q^2}{r_i + r_f} = \tfrac{1}{2} W + \frac{Q^2}{2 r_i}, \tag{10}$$

where W is given by eq. (4) as before. Hence it follows that

$$E_R = W_D. \tag{11}$$

Since the role of the fixed charge in this process is purely auxiliary, we may conclude, that whenever a charge, Q, is displaced a distance Δr during a time Δt, being at rest outside this time interval, a positive net external work equal to W_D has to be performed. In view of the relation (11), it is immediately clear that eq. (6) simply expresses the amount of interference in the radiation process considered. Combined with simple invariance requirements, this fact fixes, as we shall now see, the absolute rate of radiative energy loss from the individual particle.

For the following discussion it is convenient to generalize the above experiment to include arbitrary but small displacements of the two charges. Introducing the change in dipolemoment

$$\Delta \vec{d}_1 = Q_1 \Delta \vec{x}_1, \ \Delta \vec{d}_2 = Q_2 \Delta \vec{x}_2, \tag{12}$$

the equation (9) is now easily seen to be replaced by

$$\mathscr{E}_R - E_R^{(1)} - E_R^{(2)} = - \frac{\Delta \vec{d}_1 \cdot \Delta \vec{d}_2 - 3 (\Delta \vec{d}_1 \cdot \hat{e})(\Delta \vec{d}_2 \cdot \hat{e})}{(c \Delta t)^3} \tag{13}$$

where $\hat{e}$ denotes the unit vector along the line of connection of the particles.

As far as the energy loss from the displacement of a single particle is concerned, the demand that the rate contains Q^2 as a factor, requires it for dimensional reasons to be proportional to $1/c^3$ times the square of the acceleration.* Furthermore, due to the assumption of rotational invariance, the square of the acceleration $\ddot{\vec{x}}$ can only occur in the combinations $\ddot{\vec{x}}^2$ and

* The product $(\dddot{\vec{x}} \cdot \dot{\vec{x}})$ is rejected by the demand that the rate be proportional to the square of a field of the proper dimension decreasing like $1/r$.

$(\ddot{\vec{x}} \cdot \hat{n})^2$, where $\hat{n}$ defines the direction of observation, or equivalently, in spherical components*

$$\left| \sum_{\mu} \ddot{x}_{\mu}^{*} \mathscr{D}_{\mu h}^{1}(\hat{n}) \right|^2, \qquad h = 0, \pm 1 \tag{14}$$

$\mathscr{D}_{mm'}^{j}$ denoting the well-known rotation matrices. Finally, the fact that we are dealing with a transverse vector field excludes the case $h = 0$. Since the invariants corresponding to $h = 1$ and $h = -1$ are identical, the rate of energy loss must then be of the form

$$\frac{d^2 E_R}{d\Omega dt} = \alpha \frac{1}{c^3} \left| \sum_{\mu} \ddot{d}_{\mu}^{*}(t_{\text{ret}}) \mathscr{D}_{\mu 1}^{1}(\hat{n}) \right|^2, \tag{15}$$

where $\vec{d} = Q\vec{x}$ and α is a numerical constant.

To determine the unknown constant α, we apply the general expression (15) to calculate, in the case of the experiment discussed above, the amount of interference also given by eq. (13). According to eq. (15) the total energy loss, $\mathscr{E}_R$, from the two particles amounts to

$$\mathscr{E}_R = \frac{\alpha}{c^3} \int\int d\Omega dt \left| \sum_{\mu} \{ \ddot{d}_{\mu}^{(1)*}(t_{\text{ret}}^{(1)}) + \ddot{d}_{\mu}^{(2)*}(t_{\text{ret}}^{(2)}) \} \mathscr{D}_{\mu 1}^{1}(\hat{n}) \right|^2, \tag{16}$$

where the relation between the time and angle variables is evident from figure 2. In fact, since the displacement of the individual particle is assumed to be small compared to their mutual distance $c\Delta t$, we have

$$\left.\begin{aligned} t_{\text{ret}}^{(1)} &= t - R_1/c \simeq t - R/c - \frac{\Delta t}{2} \cos\theta \\ t_{\text{ret}}^{(2)} &= t - R_2/c \simeq t - R/c + \frac{\Delta t}{2} \cos\theta . \end{aligned}\right\} \tag{17}$$

From eq. (16) we immediately get the interference term

$$\mathscr{E}_R - E_R^{(1)} - E_R^{(2)} = \frac{2\alpha}{c^3} \int\int d\Omega dt Re \{ \sum_{\mu\mu'} \ddot{d}_{\mu}^{(1)*} \ddot{d}_{\mu'}^{(2)} \mathscr{D}_{\mu 1}^{1}(\hat{n}) \mathscr{D}_{\mu' 1}^{1}{}^{*}(\hat{n}) \}, \tag{18}$$

* Since the use of spherical tensors greatly facilitates the computations in the gravitational case, we employ also here the same technique.

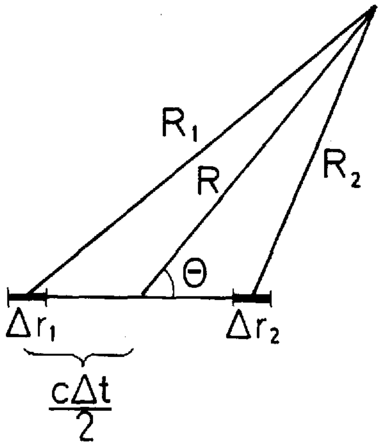

Fig. 2

where the axial symmetry around the z-axis, chosen along the line of connection between the particles, restricts the summation to the terms with $\mu = \mu'$.

Now, coupling the $\mathscr{D}$-functions, in the usual manner, by means of Clebsch-Gordan coefficients, we obtain*

$$\left.\begin{aligned} &\mathscr{E}_R - E_R^{(1)} - E_R^{(2)} = \\ &\frac{2\alpha}{c^3}\sum_{\lambda}(-)^{\lambda-1}\langle 11\ 1\ -1\,|\,\lambda 0\rangle \int\!\!\int d\Omega dt (\ddot{d}^{(1)}\ddot{d}^{(2)})_{11;\lambda 0}\mathscr{D}^{\lambda}_{00}(\hat{n}). \end{aligned}\right\} \quad (19)$$

Replacing through the relation (17) the variables t and $\cos\theta$ by $t_1 \equiv t_{\mathrm{ret}}^{(1)}$ and $t_2 \equiv t_{\mathrm{ret}}^{(2)}$, the eq. (19) takes the form

$$\left.\begin{aligned} &\mathscr{E}_R - E_R^{(1)} - E_R^{(2)} = \\ &-\frac{4\pi\alpha}{c^3}\sum_{\lambda}\langle 11\ 1\ -1\,|\,\lambda 0\rangle \int_0^{\Delta t}\!\!\int_0^{\Delta t} \frac{dt_1 dt_2}{\Delta t}(\ddot{d}^{(1)}(t_1)\ddot{d}^{(2)}(t_2))_{11;\lambda 0}\mathscr{D}^{\lambda}_{00}\left(\frac{t_2 - t_1}{\Delta t}\right). \end{aligned}\right\} \quad (20)$$

Finally, integrating by part once in each variable, we note that only the term with $\lambda = 2$ survives, and we are left with

* We are here taking advantage of a notation for the coupling of spherical tensors, so convincingly recommended for its flexibility in a recent booklet on Nuclear Structure by A. Bohr and B. Mottelson:

$$(d^{(1)}d^{(2)})_{11;\lambda\mu} \equiv \sum_{\mu'\mu''}\langle 1\mu' 1\mu''\,|\,\lambda\mu\rangle d^{(1)}_{\mu'} d^{(2)}_{\mu''}.$$

The phase conventions employed in the present paper are identical to those of the mentioned authors.

$$\left.\begin{aligned}\mathscr{E}_R - E_R^{(1)} - E_R^{(2)} &= -\frac{4\pi\alpha}{c^3}\langle 11\,1\,-1|20\rangle\frac{1}{\varDelta t}(\varDelta d^{(1)}\varDelta d^{(2)})_{11;\,20}\cdot\frac{(-3)}{(\varDelta t)^2}\\ &= 2\pi\alpha\sqrt{6}\,\frac{(\varDelta d^{(1)}\varDelta d^{(2)})_{11;\,20}}{(c\varDelta t)^3}.\end{aligned}\right\} \tag{21}$$

Comparing this result with eq. (13), rewritten in spherical components:

$$\mathscr{E}_R - E_R^{(1)} - E_R^{(2)} = \sqrt{6}\,\frac{(\varDelta d^{(1)}\varDelta d^{(2)})_{11;\,20}}{(c\varDelta t)^3}, \tag{13'}$$

we conclude that

$$\alpha = \frac{1}{2\pi}. \tag{22}$$

It needs hardly be emphasized that the argumentation has not aimed at the determination *per se* of the numerical value of the constant α, but at the elucidation of the assumptions which are crucial for such a determination. On the one hand, the existence of radiation is implied by the equations (6) and (11), which in turn rest solely on the principles of energy conservation and retardation. The answer to the further question as to the amount of radiation, emitted in a given process, demanded on the other hand explicit assumptions regarding the tensorial character of the field.

In the spirit of the above discussion, we shall commence the analysis of the gravitational case by exploring the consequences immediately to be deduced from the form of the static interaction (Newton's law), the requirement of energy balance and retardation. To exhibit most clearly the interplay of the various assumptions, it is essential to employ a sufficiently general formalism.

With the purpose of deriving relations analogous to eqs. (9) and (11), let us consider the gravitational interaction energy between two bodies which are widely separated compared to their dimensions (see figure 3). Denoting by $\vec{r}$ the vector joining two fixed points, situated inside the bodies, the interaction energy becomes*

$$\mathscr{U} = -G\int d\vec{r}_1 d\vec{r}_2\,\frac{\sigma_1(\vec{r}_1)\,\sigma_2(\vec{r}_2)}{|\vec{r}+\vec{r}_1-\vec{r}_2|}, \tag{23}$$

* The question of the interaction energy associated with the mass currents in the two bodies is not touched upon here. Accordingly, the following discussion is restricted to gravitational radiation of the "electric multipole" type.

where σ denotes the gravitational charge density, G the gravitational constant and the integration variables $\vec{r}_1$ and $\vec{r}_2$ are measured from the two fixed points mentioned. It is immediately clear, that the expansion of the rotational invariant $1/|\vec{r}+\vec{r}_1-\vec{r}_2|$ must have the form*

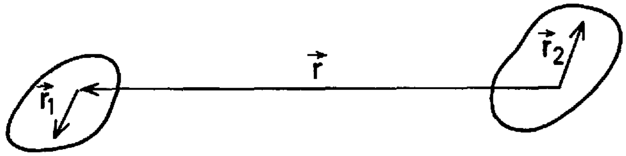

Fig. 3

$$\left.\begin{aligned} &\frac{1}{|\vec{r}+\vec{r}_1-\vec{r}_2|} = \\ &\sqrt{4\pi}\sum_{\lambda_1\lambda_2\lambda}(-)^{\lambda}A(\lambda,\lambda_1,\lambda_2)\frac{r_1^{\lambda_1}r_2^{\lambda_2}}{r^{\lambda+1}}(Y_{\lambda_1}(\hat{r}_1)Y_{\lambda_2}(\hat{r}_2)Y_{\lambda}(\hat{r}))_{(\lambda_1\lambda_2)\lambda\lambda;\,00}\end{aligned}\right\} \tag{24}$$

where the subscript again specifies the coupling scheme. For dimensional reasons the summation must be restricted to terms for which $\lambda=\lambda_1+\lambda_2$, and by considering the special case, where $\vec{r}_1$, $\vec{r}_2$ and $\vec{r}$ are all parallel, the coefficients $A(\lambda,\lambda_1,\lambda_2)$ are easily found to be

$$A(\lambda,\lambda_1,\lambda_2) = (-1)^{\lambda_1}\delta(\lambda_1+\lambda_2-\lambda)\,4\pi\sqrt{\frac{(2\lambda_1+2\lambda_2)!}{(2\lambda_1+1)!\,(2\lambda_2+1)!}}. \tag{25}$$

Introducing the gravitational charge multipole moments for the first body,

$$Q_{\lambda\mu}^{(1)} = \int d\vec{r}_1\,\sigma(\vec{r}_1)\,r_1^{\lambda}Y_{\lambda\mu}(\hat{r}_1), \tag{26}$$

and analogously for the second, the expansion of the interaction energy thus takes the form

$$\mathscr{U} = -G\sum_{\lambda_1\lambda_2\lambda}\sqrt{4\pi}(-1)^{\lambda}A(\lambda,\lambda_1,\lambda_2)\frac{(Q_{\lambda_1}^{(1)}Q_{\lambda_2}^{(2)}Y_{\lambda}(\hat{r}))_{(\lambda_1\lambda_2)\lambda\lambda;\,00}}{r^{\lambda+1}}. \tag{27}$$

Choosing the z-axis in the direction $\hat{r}$, eq. (27) reduces to

$$\mathscr{U} = -G\sum_{\lambda_1\lambda_2\lambda}A(\lambda,\lambda_1,\lambda_2)\frac{(Q_{\lambda_1}^{(1)}Q_{\lambda_2}^{(2)})_{\lambda_1\lambda_2;\,\lambda 0}}{r^{\lambda+1}}. \tag{28}$$

* See also K. Alder and Aa. Winther, Nuclear Physics A 132 (1969) 1.

Let us consider rigid motions of the two bodies, assuming for simplicity the displacement of any point of the bodies to be small, compared to their mutual distance. For an adiabatic process, the external work performed equals the change in the potential energy (28). However, if the process is carried out in a finite time, Δt, the external work, W, required, will — owing to the dependence of the interaction energy on the mutual orientation of the bodies — differ from its adiabatic value, when the retardation is taken into account. Considering, as before, the case* $c\Delta t = r$, the work W is given by

$$W = -G \sum_{\lambda_1 \lambda_2 \lambda} A(\lambda, \lambda_1, \lambda_2) \frac{[Q^{(1)}_{\lambda_1}(0) \Delta Q^{(2)}_{\lambda_2} + \Delta Q^{(1)}_{\lambda_1} Q^{(2)}_{\lambda_2}(0)]_{\lambda_1 \lambda_2; \lambda 0}}{(c\Delta t)^{\lambda+1}} + W^{(1)}_D + W^{(2)}_D, \quad (29)$$

where $Q(0)$ refers to the initial value of the multipole moment in question and ΔQ to its change. Thus, the first term in eq. (29) is obtained as the external work required to change the multipole moment of the second body in the original multipole field of the first, and vice versa, whereas $W^{(1)}_D$ and $W^{(2)}_D$ denotes the externally supplied energy to overcome the damping force. Clearly, just as in the electromagnetic case, the damping force acting on each body is independent of the motion of the other.

As a next step energy conservation is invoked in a form which leaves room for a possible gravitational "radiation energy", $\mathscr{E}_R$:

$$\mathscr{U}(t=0) + W = \mathscr{U}(t=\Delta t) + \mathscr{E}_R \quad (30)$$

or — by means of eqs. (28) and (29) —

$$\mathscr{E}_R - W^{(1)}_D - W^{(2)}_D = G \sum_{\lambda_1 \lambda_2 \lambda} A(\lambda, \lambda_1, \lambda_2) \frac{(\Delta Q^{(1)}_{\lambda_1} \Delta Q^{(2)}_{\lambda_2})_{\lambda_1 \lambda_2; \lambda 0}}{(c\Delta t)^{\lambda+1}}. \quad (31)$$

Again, considering a process in which only the first body is moved, the second being kept fixed, one concludes, in analogy to eq. (11), that $W^{(1)}_D$ equals the radiative energy loss, $E^{(1)}_R$, suffered by the first body under these circumstances. Specializing for the sake of simplicity to the case where only a single multipole moment, of order λ, is changed in each body, one obtains

* The choice of the velocity c in the present context does not amount to assuming that gravity propagates with the velocity of light, but merely that the propagation velocity does not exceed the light velocity.

in strict analogy* to eq. (13′)

$$\mathscr{E}_R - W_D^{(1)} - W_D^{(2)} = GA(2\lambda, \lambda, \lambda)\frac{(\Delta Q_\lambda^{(1)}\Delta Q_\lambda^{(2)})_{\lambda\lambda;\,2\lambda\,0}}{(c\Delta t)^{2\lambda+1}}. \tag{32}$$

Having thus obtained the interference term, we proceed to determine the general form of the radiative energy loss from a time varying multipole moment of order λ. Again, from dimensional arguments and rotational invariance (cf. also footnote on page 6), the rate can only depend on the invariants

$$\frac{G}{c^{2\lambda+1}}\left|\sum_\mu \overset{(\lambda+1)}{Q^*_{\lambda\mu}}\,\mathscr{D}^\lambda_{\mu h}(\hat{n})\right|^2,\ h = \lambda, \lambda-1, \ldots, 0, \tag{33}$$

where $\hat{n}$ defines the direction of observation and $\overset{(\lambda+1)}{Q_{\lambda\mu}}$ denotes the $(\lambda+1)$-fold time derivative of $Q_{\lambda\mu}$. Clearly, the $(\lambda+1)$ invariants (33) correspond in terms of cartesian components to the possible quadratic invariants formed of a symmetric, traceless tensor of rank λ and the vector $\hat{n}$.

Relying on the analogy to the case of electromagnetism, the idea suggests itself, that the rate of energy loss can only depend on such a combination of the invariants, which can be interpreted as a quadratic rotational invariant formed from some tensor field describing a definite spin s. Any such field can be expanded in terms of tensor spherical harmonics

$$\mathscr{D}^\lambda_{\mu h}(\hat{n})\,\varepsilon^s_h(\hat{n}) \qquad \begin{array}{l}\lambda = s, s+1, \ldots\infty\\ \mu = \lambda, \lambda-1, \ldots -\lambda;\ h = s, s-1, \ldots -s,\end{array}$$

where the $2s+1$ unit polarization tensors ε^s_h (each of which of course carries the appropriate cartesian indices) necessarily become orthogonal to each other for different values of h, if they are defined so as to transform irreducibly into each other according to the unitary representation $\mathscr{D}^s$. Thus, each of the invariants (33) has the form of the square of a tensor spherical harmonic with definite h and definite amplitude.

Although the question of the propagation velocity of gravity has been left open in the argumentation so far, nevertheless the form of Newton's

* By the comparison of (32) and (13′) it must be borne in mind that the definition of the electric dipole moment, d_μ, differs from the corresponding mass moment (26) by a factor:

$$d_\mu = \sqrt{\frac{4\pi}{3}}\int d\vec{r}\, r\varrho(\vec{r})\,Y_{1\mu}(\hat{r}).$$

Furthermore, there is an overall change of sign, due to the difference in sign of the basic interactions.

law suggests that gravity propagates with the velocity of light, and thus it is expected that only the two "helicity" amplitudes corresponding to $h = \pm s$ are present in the expansion of the free field. Hence the rate of gravitational radiative energy loss of multipole order $\lambda (\geqslant s)$ must have the general form

$$\frac{d^2 E_R}{d\Omega dt} = \alpha_\lambda \frac{G}{c^{2\lambda+1}} \left| \sum_\mu \overset{(\lambda+1)}{Q^*_{\lambda\mu}} \mathscr{D}^\lambda_{\mu h = s}(\hat{n}) \right|^2, \tag{34}$$

where α_λ is a numerical constant.

To determine this constant, we apply the general expression (34) to calculate, in the case of the experiment discussed above, the amount of interference, also given by eq. (32). According to eq. (34), the total energy loss, $\mathscr{E}_R$, from the two bodies amounts to

$$\mathscr{E}_R = \frac{\alpha_\lambda G}{c^{2\lambda+1}} \iint d\Omega dt \left| \sum_\mu \left\{ \overset{(\lambda+1)}{Q^{(1)}_{\lambda\mu}}(t^{(1)}_{\mathrm{ret}}) + \overset{(\lambda+1)}{Q^{(2)}_{\lambda\mu}}(t^{(2)}_{\mathrm{ret}}) \right\} \mathscr{D}^{\lambda *}_{\mu s}(\hat{n}) \right|^2 \tag{35}$$

where $t^{(1)}_{\mathrm{ret}}$ and $t^{(2)}_{\mathrm{ret}}$ are again given by eq. (17). Hence, by steps strictly analogous to those leading to eq. (20), one obtains the interference term

$$\left.\begin{aligned} \mathscr{E}_R - E^{(1)}_R - E^{(2)}_R &= \frac{4\pi\alpha_\lambda G}{c^{2\lambda+1}} \sum_{\lambda'} (-1)^s \langle \lambda s\ \lambda -s | \lambda' 0 \rangle \cdot \\ &\cdot \int_0^{\Delta t} \int_0^{\Delta t} \frac{dt_1 dt_2}{\Delta t} \left[\overset{(\lambda+1)}{Q^{(1)}_\lambda}(t_1)\ \overset{(\lambda+1)}{Q^{(2)}_\lambda}(t_2) \right]_{\lambda\lambda;\lambda' 0} \mathscr{D}^{\lambda'}_{00}\left(\frac{t_2 - t_1}{\Delta t}\right), \end{aligned}\right\} \tag{36}$$

where it is again understood that the z-axis is chosen along the line of connection of the two bodies.

Integrating by part λ times in each variable, we note that only the term with $\lambda' = 2\lambda$ survives, and since, moreover, the coefficient to the highest power of $\dfrac{t_2 - t_1}{\Delta t}$ in $\mathscr{D}^{2\lambda}_{00}$ is

$$\frac{(4\lambda)!}{2^{2\lambda}((2\lambda)!)^2},$$

one is left with

$$\left.\begin{array}{c}\mathscr{E}_R - E_R^{(1)} - E_R^{(2)} = \\ 4\pi\alpha_\lambda G(-)^s\langle\lambda s\,\lambda\,-s|2\lambda\,0\rangle\dfrac{(\Delta Q_\lambda^{(1)}\Delta Q_\lambda^{(2)})_{\lambda\lambda;\,2\lambda\,0}}{(c\Delta t)^{2\lambda+1}}(-)^\lambda(2\lambda)!\dfrac{(4\lambda)!}{2^{2\lambda}((2\lambda)!)^2}\cdot\end{array}\right\} \quad (37)$$

Comparing this result with eq. (32), and inserting the value for $A(2\lambda, \lambda, \lambda)$ given by eq. (25), one obtains

$$\alpha_\lambda = (-)^s\frac{2^{2\lambda}}{(2\lambda+1)\sqrt{(4\lambda)!}}\cdot\frac{1}{\langle\lambda s\,\lambda\,-s|2\lambda\,0\rangle}. \quad (38)$$

Since the Clebsch-Gordan coefficient is positive for all $s \leq \lambda$, the demand that α_λ be positive requires the rank s to be even. If the basic interaction had been repulsive, as in electrodynamics, the interference term had changed sign, and the conslusion had been that s were odd.

More specific conclusions regarding the possible spin values, s, can be drawn by noticing that the relation (32) definitely predicts the occurrence of multipole radiation of any order unless some principle forbids the change of one or more multipole moments for an isolated system. Thus, in the case of electromagnetism, where the smallest possible value of s is one, the necessary prohibition of a change in the monopole moment is expressed by the principle of charge conservation. In the case of gravity, where the empirically established equivalence between gravitational and inertial mass requires the gravitational charge density to be identified with the energy density, the conservation laws for energy and momentum prohibit the change of both monopole and dipole moments, at least in the limit where an unambiguous distinction between source and field is possible. Barring *ad hoc* assumptions to exclude the change of higher multipole moments, the spin of the gravitational field can therefore only be zero or two* Clearly, only the value $s = 2$ is immediately compatible with the fact that the energy density is a component of a four-tensor.

Returning to eq. (38) , we obtain for α_λ in the case $s = 2$

$$\alpha_\lambda = (2\lambda+1)\left\{\frac{1}{[(2\lambda+1)!!]^2}\cdot\frac{\lambda+1}{\lambda}\right\}\cdot\frac{\lambda+2}{\lambda-1}. \quad (39)$$

which only differs from the well-known result of electromagnetism by the ratio

* If, for instance, the value of s had been four, some mechanism had to be in operation to ensure conservation of the quadrupole and octupole moments appearing in eq. (32).

$$\frac{\langle \lambda 1\ \lambda-1 | 2\lambda\ 0\rangle}{\langle \lambda 2\ \lambda-2 | 2\lambda\ 0\rangle} = \frac{\lambda+2}{\lambda-1}. \tag{40}$$

For the case of special interest, $\lambda = 2$, eq. (34) then reads

$$\frac{d^2 E_R}{d\Omega dt} = \frac{2}{15}\frac{G}{c^5}\left|\sum_{\mu} \dddot{Q}^*_{2\mu}\, \mathscr{D}^2_{\mu 2}(\hat{n})\right|^2, \tag{41}$$

which, in terms of the cartesian components of the mass quadrupole moment tensor

$$Q_{ik} = \int d\vec{x}\, \sigma(\vec{x})\,[3x_i x_k - \delta_{ik}\vec{x}^2], \tag{42}$$

takes the familiar form*

$$\frac{d^2 E_R}{d\Omega dt} = \frac{G}{36\pi c^5}\left[\tfrac{1}{4}(\dddot{Q}_{ik} n_i n_k)^2 + \tfrac{1}{2}\dddot{Q}^2_{ik} - \dddot{Q}_{ik}\dddot{Q}_{kl} n_i n_l\right]. \tag{43}$$

In so far as the assumptions underlying the present analysis are intimately related to those on which the General Theory of Relativity is based, it is hardly surprising that the result (43) is identical to the one originally derived by Einstein. Accordingly, the emphasis in the above discussion has been placed on the elucidation of the interplay between those basic principles the joint validity of which implies the mentioned conclusions. In particular, it seems noteworthy that not only the existence of the gravitational radiation, but even its quantitative expression and spin character, can be so directly related to these simple premises. Of course, the price for this simplicity in the derivation has been a resignation with respect to that far reaching unification of the fundamental principles which is so remarkably achieved by the General Theory of Relativity.

* See f.i. C. Møller: Theory of Relativity (3. ed.), Oxford Univ. Press 1972. L. D. Landau and E. M. Lifshitz: The Classical Theory of Fields, Pergamon Press (3. ed.) 1971.

Acknowledgement

We take this opportunity to thank friends and colleagues at the Institutes in Copenhagen and Aarhus for many enjoyable conversations. One of us (O. U.) gratefully acknowledges the grant of a Nordita fellowship; the other one (J. K.) wishes to thank professor P. Kabir, University of Virginia, for his kind hospitality and for interesting discussions. Last but not least both of us feel it a special pleasure to express our gratitude to professor Jens Lindhard for his critical attention to our work and much friendly advice.

Jørgen Kalckar
The Niels Bohr Institute, University of Copenhagen,
DK–2100 Copenhagen Ø, Denmark

Ole Ulfbeck
NORDITA, Blegdamsvej 17,
DK–2100 Copenhagen Ø, Denmark

Indleveret til Selskabet den 13. maj 1974.
Færdig fra trykkeriet den 3. januar 1975.