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BOREL STRUCTURE IN OPERATOR ALGEBRAS

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Synopsis

Abstract measure theory is often described as the study of the two classes $C(X)$ and $\mathscr{B}(X)$ consisting, respectively, of the continuous and the Borel measurable functions on a locally compact space X. A measure μ on X is a functional on $C(X)$ and it can be extended to $\mathscr{B}(X)$, thus giving rise to various normed spaces, in particular $L^1_\mu(X)$ and $L^\infty_\mu(X)$.

This paper is concerned with certain aspects of non-commutative measure theory. In that version $C(X)$ is replaced by a non-commutative C*-algebra $\mathscr{A}$, and the Borel algebra $\mathscr{B}$ associated with $\mathscr{A}$ replaces $\mathscr{B}(X)$. Instead of $L^\infty_\mu(X)$ one must now accept any von Neumann algebra $\mathscr{M}$ for which there is a representation π of $\mathscr{A}$ into $\mathscr{M}$ such that $\pi(\mathscr{B}) = \mathscr{M}$. In a certain sense it can be said that π replaces μ.

It is shown that for each locally compact group G of automorphisms of a von Neumann algebra $\mathscr{M}$ there is an essentially unique C*-algebra $\mathscr{A}$ with Borel algebra $\mathscr{B}$ and a representation π with $\pi(\mathscr{B}) = \mathscr{M}$, such that G can be realized as a Borel group of automorphisms of $\mathscr{B}$. This generalizes classical lifting theorems by J. von Neumann and G. W. Mackey.

The problem arises (in connection with model quantum field theory) whether a group G of automorphisms of $\mathscr{M}$ can be lifted to the Borel algebra $\mathscr{B}$ of an arbitrary C*-algebra $\mathscr{A}$ with a representation π such that $\pi(\mathscr{B}) = \mathscr{M}$. This is answered in the affirmative when G is countable or has uniformly continuous action on $\mathscr{M}$.

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1. Introduction

This paper is concerned with the generalization of certain results from the theory of standard Borel spaces to the theory of (non-commutative) operator algebras. With each separable C*-algebra $\mathscr{A}$ is associated a C*-algebra $\mathscr{B}$ – the Borel algebra. In the commutative case $\mathscr{A}$ is isomorphic to the algebra of continuous functions vanishing at infinity on some separable locally compact Hausdorff space X and $\mathscr{B}$ is then the algebra of bounded Borel functions on the standard Borel space X. In the non-commutative case the Borel algebra $\mathscr{B}$ serves as an analogue of the bounded Borel functious. For example the central disintegration of representations of $\mathscr{A}$ is uniquely determined by standard measures on the spectrum of the center of $\mathscr{B}$ (see [4] and [13]).

In theoretical quantum physics the symmetries or the time evolution of a physical system is often given by a group of unitaries G on a Hilbert space H on which the observables, or rather the C*-algebra $\mathscr{A}$ they generate, have a faithful representation π. A recurrent problem is that the group G does not leave the algebra $\pi(\mathscr{A})$ invariant although it induces automorphisms of the von Neumann algebra $\pi(\mathscr{A})''$ generated by the observables. This could be explained as the effect of a wrong choice of "local algebras" (as defined in [5]), but it might also be an inherent obstacle in the model. Certainly there are many purely mathematical examples where the modular group corresponding to a cyclic and separating vector for $\pi(\mathscr{A})''$ does not give automorphisms of $\pi(\mathscr{A})$. As pointed out by E. B. Davies a natural approach (from a commutative point of view) would be to show that the automorphisms of $\pi(\mathscr{A})''$ induced by G can be lifted to a group of automorphisms of the Borel algebra $\mathscr{B}$ of $\mathscr{A}$. Then one would have a global (i.e. space free) description of G but of course at the expense of dealing now with the considerably larger algebra $\mathscr{B}$ of "measurable observables". The present paper evolved from an attempt to solve the mathematical problems involved in such a lifting.

Our point of departure is a non-commutative version of von Neumann's classical theorem on point realizations of isomorphisms between L^∞-spaces, which constitute section 3 of the paper. In section 4 we show that each separable locally compact σ-weakly continuous group of automorphisms of a von Neumann algebra on a separable Hilbert space can be lifted to a group of automorphisms of an essentially unique Borel algebra. The result is an exact analogue of a result about transformation groups of standard spaces proved by G. W. Mackey in [8]. In section 5 we show that each separable uniformly continuous group (in particular each countable group) of automorphisms of a von Neumann algebra quotient of a Borel algebra $\mathscr{B}$ can be lifted to a uniformly continuous group of automorphisms of $\mathscr{B}$. Finally, in section 6 we comment on the equivalence problem for Borel algebras.

2. Notation and preliminaries

Let $\mathscr{A}$ be a separable C*-algebra and consider $\mathscr{A}$ in its universal representation so that the enveloping von Neumann algebra $\mathscr{A}''$ is isomorphic (as a Banach space) to the second dual of $\mathscr{A}$ (see [3, § 12]). For each subset $\mathscr{S}$ of $\mathscr{A}''$ let $\mathscr{S}_{sa}$ denote the self-adjoint part of $\mathscr{S}$. Let $\mathscr{B}_{sa}$ be the smallest class of operators in $\mathscr{A}''_{sa}$ which contains $\mathscr{A}_{sa}$ and is closed under the process of taking limits of bounded monotone (increasing or decreasing) sequences from the class. The *Borel algebra* associated with $\mathscr{A}$ is the C*-algebra $\mathscr{B} = \mathscr{B}_{sa} + i\mathscr{B}_{sa}$ (see [6, p. 316] and [11, Theorem 1]). Each representation π of $\mathscr{A}$ on a Hilbert space H extends uniquely to a normal representation (again denoted by π) of $\mathscr{A}''$ onto the von Neumann algebra generated by $\pi(\mathscr{A})$. The restriction of π to $\mathscr{B}$ maps $\mathscr{B}_{sa}$ onto the monotone sequential closure of $\pi(\mathscr{A})_{sa}$ (by [10, Proposition 4.2]) and if H is separable, $\pi(\mathscr{B})$ is the von Neumann algebra generated by $\pi(\mathscr{A})$ (by [6, p. 322] or [12, Theorem 1]). The fact that the atomic representation of $\mathscr{A}$ extends to a faithful representation of $\mathscr{B}$ ([13, Corollary 3.9]) allows us to regard $\mathscr{B}$ as the non-commutative analogue of the bounded Borel functions on a standard Borel space.

3. A theorem of von Neumann

A classical result of J. von Neumann asserts that if μ_1 and μ_2 are probability measures on standard Borel spaces X_1 and X_2, respectively, such that $L^{\infty}_{\mu_1}(X_1)$ is isomorphic to $L^{\infty}_{\mu_2}(X_2)$, then the isomorphism can be lifted to a Borel isomorphism of $X_1 \setminus N_1$ onto $X_2 \setminus N_2$ where $\mu_1(N_1) = \mu_2(N_2) = 0$ (see [9]). The theorem below is the non-commutative analogue of von Neumann's result and yields a reasonably simple proof of it upon specialization to commutative C*-algebras.

Theorem 1. Let $\mathscr{A}_1$ and $\mathscr{A}_2$ be separable C*-algebras with Borel algebras $\mathscr{B}_1$ and $\mathscr{B}_2$, respectively. If for each $i = 1, 2$, π_i is a representation of $\mathscr{A}_i$ on a separable Hilbert space and ϱ is an isomorphism between the von Neumann algebras $\pi_1(\mathscr{B}_1)$ and $\pi_2(\mathscr{B}_2)$ then there are central projections e_i in $\mathscr{B}_i$ with $\pi_i(1 - e_i) = 0$ and an isomorphism λ of $e_1\mathscr{B}_1$ onto $e_2\mathscr{B}_2$ such that $\varrho\, \pi_1(x) = \pi_2\, \lambda(x)$ for each x in $e_1\mathscr{B}_1$.

Proof. Since each quotient of a separable C*-algebra $\mathscr{A}$ has a Borel algebra isomorphic to a direct summand in $\mathscr{B}$ (see section 6, Proposition 3) we may assume that the representations π_1 and π_2 are faithful on $\mathscr{A}_1$ and $\mathscr{A}_2$.

Let $\mathscr{A}_3$ (resp. $\mathscr{A}_4$) denote the separable C*-algebra generated by $\pi_1(\mathscr{A}_1)$ and $\varrho^{-1}\pi_2(\mathscr{A}_2)$ (resp. $\pi_2(\mathscr{A}_2)$ and $\varrho\,\pi_1(\mathscr{A}_1)$). Then $\varrho(\mathscr{A}_3) = \mathscr{A}_4$. Choose separable C*-algebras $\mathscr{D}_i$ in $\mathscr{B}_i$ containing $\mathscr{A}_i$ such that $\pi_i(\mathscr{D}_i) \supset \mathscr{A}_{i+2}$. There are then central projections z_i in $\mathscr{B}_i$ such that $\pi_i(1 - z_i) = 0$ and π_i is injective on $z_i\mathscr{D}_i$ (take $1 - z_i$ to be the support projection of the separable set $\ker \pi_i \cap \mathscr{D}_i$).

For each x i $\mathscr{A}_1$ define $\Phi_0(x)$ as the unique element in $z_2\mathscr{D}_2$ satisfying $\pi_2\Phi_0(x) = \varrho\pi_1(x)$. Then Φ_0 is a morphism of $\mathscr{A}_1$ into $\mathscr{B}_2$ (even an isometry). Since $\mathscr{B}_2$ is a Borel algebra there is a unique extension Φ_1 of Φ_0 to a σ-normal morphism of $\mathscr{B}_1$ into $\mathscr{B}_2$. As $\pi_2\Phi_1(x) = \varrho\pi_1(x)$ for each x in $\mathscr{A}_1$ the same is true for any x in $\mathscr{B}_1$. In the same manner we obtain a σ-normal morphism Φ_2 of $\mathscr{B}_2$ into $\mathscr{B}_1$ satisfying $\pi_1\Phi_2(x) = \varrho^{-1}\pi_2(x)$ for each x in $\mathscr{B}_2$.

Put $\Psi = \Phi_2\Phi_1$. Then for each x in $\mathscr{B}_1$

$$\pi_1\, \Psi(x) = \pi_1\, \Phi_2\Phi_1(x) = \varrho^{-1}\pi_2\, \Phi_1(x) = \varrho^{-1}\varrho\, \pi_1(x) = \pi_1(x).$$

Let (x_n) be a norm dense sequence in $\mathscr{A}_1$ and for each n let y_n be the central projection supporting $x_n - \Psi(x_n)$. Then $\pi_1(y_n) = 0$, whence $\pi_1(y) = 0$ where $y = \vee y_n$; and $(1 - y)\,(x_n - \Psi(x_n)) = 0$ for all n. Since (x_n) is a generating sequence for $\mathscr{B}_1$ we have $(1 - y)\,(x - \Psi(x)) = 0$ for all x in $\mathscr{B}_1$. Define $f_1 = \wedge \Psi^n(1 - y)$. Then

$$\pi_1(f_1) = \wedge\, \pi_1\, \Psi^n(1 - y) = \pi_1(1 - y) = 1,$$

so that $\pi_1(1 - f_1) = 0$. We have $f_1(x - \Psi(x)) = 0$ for all x in $\mathscr{B}_1$; in particular $f_1(f_1 - \Psi(f_1)) = 0$ so that $f_1 \leq \Psi(f_1)$. In the same manner we obtain a central projection f_2 in $\mathscr{B}_2$ with $\pi_2(1 - f_2) = 0$ and $f_2 \leq \Phi_1\Phi_2(f_2)$ such that $f_2(x - \Phi_1\Phi_2\,(x)) = 0$ for all x in $\mathscr{B}_2$.

Define $e_1 = f_1\, \Phi_2(f_2)$ and $e_2 = f_2\, \Phi_1(f_1)$. Then

$$\Phi_2(e_2) = \Phi_2(f_2)\, \Phi_2\Phi_1(f_1) \geq \Phi_2(f_2)\, f_1 = e_1,$$

and similarly $\Phi_1(e_1) \geq e_2$. Define λ on $e_1\mathscr{B}_1$ by $\lambda(x) = e_2\, \Phi_1(x)$ and μ on $e_2\mathscr{B}_2$ by $\mu(x) = e_1\, \Phi_2(x)$. If $x \in e_1\mathscr{B}_1$ then

$$\mu\lambda(x) = e_1\, \Phi_2(e_2\, \Phi_1(x)) = e_1\, \Phi_2(e_2)\, \Psi(x) = e_1\, \Psi(x) = e_1\, x = x,$$

and similarly $\lambda\mu(x) = x$ for each x in $e_2\mathscr{B}_2$. Thus λ is an isomorphism of $e_1\mathscr{B}_1$ onto $e_2\mathscr{B}_2$ and it is clear that $\pi_2\lambda(x) = \varrho\pi_1(x)$ for each x in $e_1\mathscr{B}_1$.

4. The Borel G-algebra associated with an automorphism group of a von Neumann algebra

Let $\mathscr{A}$ be a separable C*-algebra and G a separable locally compact group of automorphisms of $\mathscr{A}$ such that each function $g \to g(x)$, $x \in \mathscr{A}$ is continuous from G to $\mathscr{A}$. By double transposition we may extend G uniquely to a group of automorphisms of $\mathscr{A}''$. The Borel algebra $\mathscr{B}$ of $\mathscr{A}$ is a subset of $\mathscr{A}''$, and since the class of self-adjoint operators x for which $g(x) \in \mathscr{B}$ for all g in G is monotone sequentially closed and contains $\mathscr{A}_{sa}$, it contains $\mathscr{B}_{sa}$. Thus we may regard G as a group of automorphisms of $\mathscr{B}$. We shall refer to this situation by saying that $\mathscr{B}$ is a *Borel G-algebra.*

Lemma 1. If $\mathscr{B}$ is a Borel G-algebra and φ is a bounded functional on $\mathscr{A}$ then for each x in $\mathscr{B}$ the function $g \to \langle g(x), \varphi\rangle$ is Borel measurable on G. Moreover, for each ξ in $L^1(G)$ (with respect to a left Haar measure) the element $\xi(x)$ in $\mathscr{A}''$ given by

$$\langle \xi(x), \varphi\rangle = \int_G \langle g(x), \varphi\rangle\, \xi(g)dg$$

belongs to $\mathscr{B}$.

Proof. If $x \in \mathscr{A}$ then each function $g \to \langle g(x), \varphi\rangle$ is continuous on G and the element $\xi(x)$ belongs to $\mathscr{A}$. Since the class of self-adjoint operators which satisfy the two conditions in the lemma is monotone sequentially closed and contains $\mathscr{A}_{sa}$, it also contains $\mathscr{B}_{sa}$.

The following result is known (see [1, 3.2 Satz]). For completeness we indicate a proof.

Lemma 2. Let G be a separable locally compact group of automorphisms of a von Neumann algebra $\mathscr{M}$ on a separable Hilbert space H. If for each x in $\mathscr{M}$ and each pair ξ, η in H the function $g \to (g(x)\, \xi \,|\, \eta)$ is Borel measurable on G then there is a faithful normal representation ϱ of $\mathscr{M}$ on a separable Hilbert space K and a strongly continuous unitary representation $g \to u_g$ of G on K such that $\varrho(g(x)) = u_g\, \varrho(x)\, u_g^*$. In particular, G is σ-weakly continuous on $\mathscr{M}$.

Proof. Let K be the separable Hilbert space of square integrable functions ξ from G to H with the inner product

$$(\xi|\eta) = \int_G (\xi(h)|\eta(h))\, dh.$$

Define a strongly continuous unitary representation of G on K by $(u_g\xi)(h) = \xi(g^{-1}h)$. By assumption each function $g \to g(x)\xi$, $x \in \mathscr{M}$, $\xi \in H$ is weakly Borel measurable and bounded so that if $\xi \in K$ we may define $\varrho(x)\xi$ in K by $(\varrho(x)\xi)(h) = h^{-1}(x)\xi(h)$. It only requires a mildly routine argument to show that ϱ is a faithful normal representation of $\mathscr{M}$ (since H is separable it is enough to check the normality of ϱ on increasing sequences from $\mathscr{M}$, where Lebesgues monotone convergence theorem applies) and that $\varrho(g(x)) = u_g\varrho(x)u_g^*$ for all x in $\mathscr{M}$ and g in G.

A σ-normal representation π of a Borel G-algebra $\mathscr{B}$ is called *G-invariant* if the kernel of π is a G-invariant ideal in $\mathscr{B}$. The next result characterizes the G-invariant representations of $\mathscr{B}$. The equivalence of conditions (iii) and (iv) is due to H. J. Borchers in a more general setting (see [1, 2.3. Theorem]).

Proposition 1. Let $\mathscr{A}$ be a separable C*-algebra with Borel algebra $\mathscr{B}$, and G a separable locally compact group for which $\mathscr{B}$ is a Borel G-algebra. The following conditions on a representation π of $\mathscr{A}$ on a separable Hilbert space are equivalent:

(i) π extends to a G-invariant representation of $\mathscr{B}$.

(ii) The kernel of π in $\mathscr{A}$ is G-invariant and the elements in the representation of G as automorphisms of $\pi(\mathscr{A})$ are all extendable to automorphisms of $\pi(\mathscr{B})$.

(iii) The transposed action of G on the dual of $\mathscr{A}$ leaves the predual of $\pi(\mathscr{B})$ invariant (as a subset of the dual of $\mathscr{A}$) and for each φ in the predual of $\pi(\mathscr{B})$ the function $g \to g^t(\varphi)$ is norm continuous.

(iv) π is quasi-equivalent to a representation ϱ of $\mathscr{A}$ on a separable Hilbert space in which G has a strongly continuous unitary representation $g \to u_g$ such that $\varrho(g(x)) = u_g\, \varrho(x)u_g^*$.

Proof. The implications (iv) $\Rightarrow$ (iii) $\Rightarrow$ (ii) are immediate.

To prove (ii) $\Rightarrow$ (i) note first that an automorphism g of $\pi(\mathscr{A})$ is extendable to an automorphism $\bar{g}$ of $\pi(\mathscr{B})$ if and only if g is σ-weakly continuous on $\pi(\mathscr{A})$. Let $g \to \bar{g}$ denote the representation of G as automorphisms of $\pi(\mathscr{B})$. Since by assumption $\pi(g(x)) = \bar{g}(\pi(x))$ for all x in $\mathscr{A}_{sa}$ the same is true for all x in the monotone sequential closure $\mathscr{B}_{sa}$ of $\mathscr{A}_{sa}$, which shows that the kernel of π in $\mathscr{B}$ is G-invariant.

The implication (i) $\Rightarrow$ (iv) follows from Lemma 2 since G acts on $\pi(\mathscr{B})$ as a Borel group by Lemma 1.

For every topological group G of automorphisms of a C*-algebra $\mathscr{A}$ let $\mathscr{A}_c$ denote the C*-subalgebra of $\mathscr{A}$ consisting of those elements x for which the function $g \to g(x)$ is continuous.

Lemma 3. Let $\mathscr{M}$ be a von Neumann algebra on a separable Hilbert space and G a separable locally compact group of automorphisms of $\mathscr{M}$ which is σ-weakly continuous. Then there is a separable G-invariant C*-algebra of $\mathscr{M}_c$ which is weakly dense in $\mathscr{M}$.

Proof. If $x \in \mathscr{M}$ and $\xi \in L^1(G)$ then the element

$$\xi(x) = \int g(x)\ \xi(g)dg$$

(defined as a weak integral on the pre-dual of $\mathscr{M}$) belongs to $\mathscr{M}$; and if ξ runs through an approximate unit for $L^1(G)$ then $\xi(x)$ converges σ-weakly to x because G is σ-weakly continuous. However,

$$g(\xi(x)) = \int h(x)\ \xi(g^{-1}h)\ dh$$

so that

$$\left\| g(\xi(x)) - \xi(x) \right\| \leq \left\| x \right\| \int \left| \xi(g^{-1}h) - \xi(h) \right| dh \to 0$$

as g tends to the identity of G. It follows that $\xi(x) \in \mathscr{M}_c$ for each x in $\mathscr{M}$, which shows that $\mathscr{M}_c$ is weakly dense in $\mathscr{M}$.

Choose a separable weakly dense C*-algebra $\mathscr{A}_0$ of $\mathscr{M}_c$. If (g_k) is a countable dense subgroup of G let $\mathscr{A}$ be the C*-algebra generated by $\cup\, g_k(\mathscr{A}_0)$. Clearly $\mathscr{A}$ is separable and weakly dense in $\mathscr{M}$, but since $\mathscr{M}_c$ is G-invariant, $\mathscr{A} \subset \mathscr{M}_c$ so that for each x in $\mathscr{A}$ the set $\{g_k(x)\}$ is norm dense in the orbit $G(x)$; whence $G(x) \subset \mathscr{A}$ so that $\mathscr{A}$ is G-invariant.

Theorem 2. (cf. [8, Theorems 1 and 2]). Let G be a separable locally compact σ-weakly continuous group of automorphisms of a von Neumann algebra $\mathscr{M}$ on a separable Hilbert space. There is a separable C*-algebra $\mathscr{A}_1$ whose Borel algebra $\mathscr{B}_1$ is a Borel G-algebra and a G-invariant representation π_1 such that $\pi_1(\mathscr{B}_1) = \mathscr{M}$.

If $\mathscr{A}_2$, $\mathscr{B}_2$ and π_2 satisfy the same conditions then there are central G-invariant projections e_1 and e_2 with $\pi_1(1-e_1) = \pi_2(1-e_2) = 0$ and an isomorphism λ of the $e_1\mathscr{B}_1$ onto $e_2\mathscr{B}_2$ which commutes with G and satisfies $\pi_1 = \pi_2\,\lambda$.

Proof. The existence of $\mathscr{A}_1$, $\mathscr{B}_1$ and π_1 follows immediately from Lemma 3. To prove the essential uniqueness we shall repeat the constructions from the proof of Theorem 1 with minor adjustments arising from the action of G.

Note first that if $x \in \mathscr{M}_c$ and (ξ_n) is an approximate unit for $L^1(G)$ then $\| x - \xi_n(x) \| \to 0$. Thus for $i = 1, 2$, if $b_i \in \mathscr{B}_i$ with $\pi_i(b_i) = x$ then $\xi_n(b_i) \in \mathscr{B}_i$ by Lemma 1 and as in the proof of Lemma 3 we see that $\xi_n(\mathrm{b}_i) \in \mathscr{B}_{ic}$. Since $\pi_i(\xi_n(b_i)) = \xi_n(x)$ we have shown that $\pi_i(\mathscr{B}_{ic}) = \mathscr{M}_c$.

With this in mind we can choose the C*-algebras $\mathscr{D}_i$ as subalgebras of $\mathscr{B}_{ic}$. (The notation is as in the proof of Theorem 1). We may assume that the $\mathscr{D}_i$'s are G-invariant, replacing them otherwise with the C*-algebras generated by $\cup\, g_k(\mathscr{D}_i)$, where (g_k) is a countable dense subgroup of G. This implies that the support projection $1 - z_i$ of the set $\ker \pi_i \cap \mathscr{D}_i$ is G-invariant, thus $z_i\mathscr{D}_i \subset \mathscr{B}_{ic}$. Therefore the morphism Φ_1 of $\mathscr{B}_1$ into $\mathscr{B}_2$ will satisfy $\Phi_1(\mathscr{A}_1) \subset \mathscr{B}_{2c}$. Moreover, since $z_2\mathscr{D}_2$ is G-invariant, $\Phi_1(g(x)) = g(\Phi_1(x))$ for each x in $\mathscr{A}_1$ and consequently for all x in $\mathscr{B}_1$. Since a similar statement is true for Φ_2, the endomorphism $\Psi = \Phi_2\Phi_1$ of $\mathscr{B}_1$ will be G-invariant. In particular, $\| \Psi(x) - g(\Psi(x)) \| \leq \|x - g(x))\|$, which shows that $\Psi(\mathscr{B}_{1c}) \subset \mathscr{B}_{1c}$.

Since therefore $x_n - \Psi(x_n) \in \mathscr{B}_{1c}$, the central projection y_n is the limit of an increasing sequence of positive elements from $\mathscr{B}_{1c}$. (With $a_n = |x_n - \Psi(x_n)|$ and (u_j) a dense sequence of unitaries in $\mathscr{A}_1$, put $b_{nk} = \Sigma_{j=1}^k u_j^* a_n u_j$. Then $\left(\frac{1}{k} + b_{nk}\right)^{-1} b_{nk} \nearrow y_n$). Replace y_n by $y_n' = \vee\, g_k(y_n)$. Then y_n' is G-invariant. In fact, if g is a limit point of (g_k) then for each state φ of $\mathscr{A}_1$,

$$\langle g(y_n), \varphi\rangle \leq \liminf \langle g_k(y_n), \varphi\rangle \leq \langle y_n', \varphi\rangle,$$

since $g \to \langle (y_n), \varphi\rangle$ is a lower semi-continuous function on G. It follows that $g(y_n') \leq y_n'$, whence $g(y_n') = y_n'$.

We still have $\pi_1(y_n') = 0$ and the construction from the proof of Theorem 1 can now be completed without further alterations. The resulting projections e_1 and e_2 will be G-invariant and since Ψ is already G-invariant, so is the isomorphism λ from $e_1\mathscr{B}_1$ to $e_2\mathscr{B}_2$.

Simple commutative examples (e.g. rational translations in $L^\infty(\mathbf{R})$) show that even if G is ergodic on $\mathscr{M}$ it may not be possible to find a Borel G-algebra $\mathscr{B}$ and a G-invariant representation of $\mathscr{B}$ on $\mathscr{M}$ such that G is ergodic on $\mathscr{B}$ (in the strict sense). However, as the next result shows, the other possibility is excluded. If G is ergodic on $\mathscr{B}$, then it is ergodic in each G-invariant representation of $\mathscr{B}$.

Proposition 2 (cf. [8, Theorem 3]). Let $\mathscr{B}$ be a Borel G-algebra and π a G-invariant representation on a separable Hilbert space. If $x \in \mathscr{B}$ and $\pi(x)$ is G-invariant in $\pi(\mathscr{B})$ then there is a G-invariant element y in $\mathscr{B}$ such that $\pi(y) = \pi(x)$.

Proof. If $a = \pi(x)$ let $\mathscr{A}_1$ be the separable G-invariant C*–algebra generated by $\pi(\mathscr{A})$ and a. Since $a \in \pi(\mathscr{B})_c$ we have $\mathscr{A}_1 \subset \pi(\mathscr{B})_c$ so that G is pointwise norm continuous on $\mathscr{A}_1$. With $\mathscr{B}_1$ as the Borel algebra of $\mathscr{A}_1$ we see that $\mathscr{B}_1$ is a Borel G-algebra, and the identical representation of $\mathscr{A}_1$ extends to a G-invariant representation π_1 of $\mathscr{B}_1$. By Theorem 2 we can find central G-invariant projections e and e_1 with $\pi(1-e) = \pi_1(1-e_1) = 0$ and a G-invariant isomorphism λ of $e_1\mathscr{B}_1$ onto $e\mathscr{B}$ such that $\pi_1 = \pi\lambda$. Set $y = \lambda(e_2 a)$. Then y is G-invariant since $e_2 a$ is G-invariant and $\pi(y) = \pi_1(e_2 a) = \pi(x)$, as desired.

5. Lifting automorphisms from von Neumann algebras

Let again $\mathscr{A}$ be a separable C*-algebra with Borel algebra $\mathscr{B}$. If π is a representation of $\mathscr{A}$ on a separable Hilbert space and G is a group of automorphisms of the von Neumann algebra $\pi(\mathscr{B})$ is it then possible to lift G to a group of automorphisms of $\mathscr{B}$ such that $G\pi = \pi G$?

If $\mathscr{A}$ is commutative and G is separable, locally compact and σ-weakly continuous then this problem has a positive solution. We sketch the argument: The atomic part of $\pi(\mathscr{B})$ is G-invariant and corresponds to a direct summand in $\mathscr{B}$. The lifting in this case presents no problems and we assume therefore that $\pi(\mathscr{B})$ contains no minimal projections. By Theorem 2 there is a commutative separable C*-algebra $\mathscr{A}_1$ such that its Borel algebra $\mathscr{B}_1$ is a Borel G-algebra, and a G-invariant representation π_1 such that $\pi_1(\mathscr{B}_1) = \pi(\mathscr{B})$. By Theorem 1 there are projections e and e_1 with $\pi(1-e) = \pi_1(1-e) = 0$ and an isomorphism λ of $e\mathscr{B}$ onto $e_1\mathscr{B}_1$ such that $\pi = \pi_1\lambda$. We have $\mathscr{B} = \mathscr{B}(X)$ and $\mathscr{B}_1 = \mathscr{B}(X_1)$, where X and X_1 are standard Borel spaces which are uncountable (hence Borel isomorphic to $\mathbf{R}$) since $\pi(\mathscr{B})$ has no minimal projections. We may identify $1-e$ and $1-e_1$ with Borel subsets N and N_1 of X andX_1, respectively; and we may assume that they are both uncountable, replacing otherwise e and e_1 with slightly smaller projections. Then N and N_1 are Borel isomorphic so there is an isomorphism λ_0 of $(1-e)\mathscr{B}$ onto $(1-e_1)\mathscr{B}_1$. Combining λ and λ_0 we obtain an isomorphism λ_1 of $\mathscr{B}$ onto $\mathscr{B}_1$

such that $\pi = \pi_1\lambda_1$. Since $\mathscr{B}_1$ is a Borel G-algebra we can define $\bar{G} = \lambda_1^{-1}G\lambda_1$ on $\mathscr{B}$ and this gives a lifting of G from $\pi(\mathscr{B})$ to $\mathscr{B}$.

In the general case the problem remains open even when G is a σ-weakly continuous one-parameter group of automorphisms of $\pi(\mathscr{B})$. We have a positive solution, however, if G is uniformly continuous on $\pi(\mathscr{B})$.

Theorem 3. Let $\mathscr{A}$ be a separable C*-algebra with Borel algebra $\mathscr{B}$. If π is a representation of $\mathscr{A}$ on a separable Hilbert space and G is a separable uniformly continuous group of automorphisms of the von Neumann algebra $\pi(\mathscr{B})$ then G can be lifted to a uniformly continuous group of automorphisms of $\mathscr{B}$.

Proof. Choose a separable C*-algebra $\mathscr{A}_1$ in $\pi(\mathscr{B})$ which is weakly dense. We may assume that $\mathscr{A}_1$ is G-invariant, replacing it otherwise with the C*-algebra generated by $\cup\, g_k(\mathscr{A})$, where (g_k) is a countable dense subgroup of G. Then G is a uniformly continuous group of automorphisms of $\mathscr{A}_1$, hence by double transposition extends to a uniformly continuous group of automorphisms of $\mathscr{A}_1''$. By restriction we may regard G as a uniformly continuous group of automorphisms of the Borel algebra $\mathscr{B}_1$ of $\mathscr{A}_1$.

By Theorem 1 there are central projections e and e_1 with $\pi(1-e) = \pi_1(1-e_1) = 0$ and an isomorphism λ of $e\mathscr{B}$ onto $e_1\mathscr{B}_1$ such that $\pi = \pi_1\lambda$. With (g_k) a countable dense subgroup of G define $p_1 = \wedge\, g_k(e_1)$ and $p = \lambda^{-1}(p_1)$. Then p_1 is G-invariant; for if g is a limit point of (g_n) then

$$g\, g_k(e_1) = \lim g_n g_k(e_1) \geq p_1,$$

whence $g(p_1) \geq p_1$ and since this holds for all g in G we have $g(p_1) = p_1$.

For each g in G and x in $\mathscr{B}$ define

$$\bar{g}(x) = \lambda^{-1}g\lambda(px) + (1-p)\,x.$$

This gives a faithful representation of G as a uniformly continuous group of automorphisms of $\mathscr{B}$, and since for each x in $\mathscr{B}$

$$\pi\bar{g}(x) = \pi\lambda^{-1}g\lambda(px) = \pi_1 g\lambda(px) = g\pi_1\lambda(px) = g\pi(x),$$

this representation is a lifting of G.

Corollary. If π is a representation of $\mathscr{A}$ on a separable Hilbert space and G is a countable group of automorphisms of the Neumann algebra $\pi(\mathscr{B})$, then G can be lifted to a group of automorphisms of $\mathscr{B}$.

6. The isomorphism problem

One enormous simplification in the theory of standard spaces is that there are only countably many Borel isomorphism classes and that the cardinality is a complete invariant for each class. In the non-commutative situation very little is known about C*-algebras having isomorphic Borel algebras. But with uncountably many different von Neumann factors around it seems improbable that there should be only a countable number of isomorphism classes of Borel algebras.

The type I situation is completely known. If H_n is an n-dimensional Hilbert space, $1 \leq n \leq \infty$, and X_n is a standard Borel space let $\mathscr{B}(X_n, B(H_n))$ denote the algebra of bounded weakly Borel measurable functions from X_n to $B(H_n)$. If $\mathscr{A}$ is a separable C*-algebra of type I then its Borel algebra is isomorphic to an algebra $\Sigma^{\oplus} \mathscr{B}(X_n, B(H_n))$ by [11, Proposition 7] (see also [2, Theorem 4.5]). Each isomorphism class is therefore determined by a sequence of standard spaces, one from each dimension. In particular there are only countably many isomorphism classes in each dimension.

It is evidently a problem of great interest to determine when two C*-algebras have isomorphic Borel algebras. If so they have the same representations, and these representations can be decomposed in the same manner. Moreover, there is an affine Borel isomorphism between their state spaces. The isomorphism classes are much more unstable in the non-commutative theory than usual. For example, by adjoining a unit to a C*-algebra one may obtain a C*-algebra which is not Borel equivalent to the former; simply because the latter has a one-dimensional representation while the former may have none. The next result provides us with some examples of non-isomorphic C*-algebras whose Borel algebras are isomorphic.

Proposition 3. Let $\mathscr{A}$ be a separable C*-algebra and $\mathscr{I}$ a closed ideal of $\mathscr{A}$. Then the Borel algebra of $\mathscr{A}$ is isomorphic to that of $\mathscr{I} \oplus \mathscr{A}/\mathscr{I}$.

Proof. Let z be the central projection in the Borel algebra $\mathscr{B}$ of $\mathscr{A}$ obtained as the supremum of any approximate unit for $\mathscr{I}$. Then $\mathscr{I} = z\mathscr{B} \cap \mathscr{A}$ and $\mathscr{A}/\mathscr{I} = (1-z)\mathscr{A}$; and if H is the universal Hilbert space for $\mathscr{A}$ then zH and $(1-z)H$ are the universal Hilbert spaces for $\mathscr{I}$ and $\mathscr{A}/\mathscr{I}$, respectively. It follows that the Borel algebra of $\mathscr{I}$ is $z\mathscr{B}$ and the Borel algebra of $\mathscr{A}/\mathscr{I}$ is $(1-z)\mathscr{B}$, and this completes the proof.

Problem. Do all hyperfinite C*-algebras have isomorphic Borel algebras.

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