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ON TORUS MAPS AND ALMOST PERIODIC MOVEMENTS

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Notations

$S^n$: the $n$-sphere.
$E^n$: the $n$-cell.
$T^n = S^1 \times \ldots \times S^1$ ($n$ factors): the $n$-dimensional torus.
$\sim$ homeomorphic.
$\simeq$ homotopic, bijective equivalent, isomorphic.
$f_\#$, $f^\#$ maps induced by $f$ in homology, cohomology.
$f_*$, $f^*$ maps induced by $f$ in homotopy, cohomotopy.

Synopsis

In this paper I show that the cohomotopy groups of the $n$-dimensional torus $T^n$ usually are direct sums of homotopy groups of spheres. Further, I investigate homotopy classification problems of continuous maps from $T^n$ into other topological spaces – especially spaces with "nice" homotopy groups in the lower dimensions. The results are applied to some "reducibility problems" for torus maps and almost periodic movements – in particular I find conditions for almost periodic movements being almost periodically homotopic to periodic movements.


Introduction

In 1954 H. Tornehave [4] investigated the following problem: In which topological spaces $X$ is every almost periodic movement $x = f(t)$ ($t \in \mathbb{R} = [-\infty, \infty]$, $x \in X$) almost periodically homotopic to a periodic movement?

A(almost) p(eriodic) homotopy between two a.p. movements $f_0(t)$ and $f_1(t)$ means that there exists a uniformly continuous family $f(t, \nu)$, $\nu \in [0,1]$, of almost periodic movements starting with $f_0(t)$ and ending with $f_1(t)$.

Let $\mathcal{C}$ denote the class of metric spaces $X$ which are "continuously locally arcwise connected" (see p. 21). Because every CW complex is "continuously locally arcwise connected", $\mathcal{C}$ includes the class $\mathcal{C}'$ of metrizable CW complexes. $\mathcal{C}'$ includes the class $\mathcal{C}''$ of locally compact polyhedrons. Remark: In this paper I only look at continuous maps between topological spaces, though I do not explicitly write continuous everywhere. Nor do I everywhere write that I assume my spaces different from the empty set.

A theorem ([4] p. 28) states that every a.p. movement $\tilde{f}(t)$ in a space $X \in \mathcal{C}$ corresponds to some rationally independent real numbers $(\beta_1, \ldots, \beta_n)$ and a continuous torus map $f: \mathbb{R}^n \to X$ ($f$ is periodic in all the variables with the period $2\pi$) in such a way that $\tilde{f}(t)$ is almost periodically homotopic to the almost periodic movement $f(\beta_1 t, \ldots, \beta_n t)$.

A small correction and generalization of Theorem 13 in [4] gives:

Every almost periodic movement in $X \in \mathcal{C}$ is almost periodically homotopic to a periodic movement if and only if for every continuous torus map $f$ (of any dimension) into $X$ there exists a number $N \in \mathbb{N}$ so that $f_k (xN)$ is homotopic to a torus map into a closed curve in $X$, where $(xN) (u_1, \ldots, u_n) = (Nu_1, \ldots, Nu_n)$.

Because of this theorem it is natural to look at the following problem: For which $X$ is every torus map (of any dimension) into $X$ homotopic to a torus map into a closed curve in $X$? H. Tornehave had some intuitive ideas.
of how to solve the new problem: he thought it was a necessary (and if all the homotopy groups \( \pi_i(X,x_0) \) are trivial for \( i > 1 \), sufficient) condition that for all \( x_0 \in X \) every abelian subgroup of the fundamental group \( \pi_1(X,x_0) \) is cyclic. (This is not quite correct). In this paper I shall find a partial solution of a more general problem.

**Definition 1.** Let \( X \) be a topological space. An \( n \)-dimensional torus map \( f: T^n \to X \) is called \( m \)-reducible iff it is homotopic to a continuous map \( g \) through \( T^m \), i.e. \( T^n \to T^m \to X \), where \( 1 \leq m < n \) are the only interesting cases.

**Definition 2.** The space \( X \) is called \( n \)-dimensionally \( m \)-reducible iff every \( n \)-dimensional torus map into \( X \) is \( m \)-reducible.

**Definition 3.** The space \( X \) is called \( m \)-reducible iff every torus map into \( X \) (of any dimension) is \( m \)-reducible.

**Definition 4.** An a.p. movement \( x = \tilde{f}(t) \) in \( X \) is said to be of dimension \( \leq n \) iff it is a.p. homotopic to some \( x = f(\beta_1 t, \ldots, \beta_n t) \), where \( (\beta_1, \ldots, \beta_n) \) are rationally independent real numbers and \( f: T^n \to X \) is an \( n \)-dimensional torus map.

**Definition 5.** The a.p. movement \( x = \tilde{f}(t) \) in \( X \) of dimension \( \leq n \) is called a.p. \( m \)-reducible iff it is a.p. homotopic to an a.p. movement of dimension \( \leq m \).

**Definition 6.** The space \( X \) is called a.p. \( n \)-dimensionally \( m \)-reducible iff every a.p. movement in \( X \) of dimension \( \leq n \) is a.p. \( m \)-reducible.

**Definition 7.** The space \( X \) is called a.p. \( m \)-reducible iff every a.p. movement in \( X \) is a.p. \( m \)-reducible.

We have the following theorem:

**Theorem 16.** Let the almost periodic movement \( x = \tilde{f}(t) \) in \( X \in \mathcal{C} \) be a.p. homotopic to \( f(\beta_1 t, \ldots, \beta_n t) \), where \( f: T^n \to X \), \( (\beta_1, \ldots, \beta_n) \) rationally independent. Then

\[
\tilde{f}(t) \text{ is a.p. } m \text{-reducible iff } f \circ (\times N) \text{ for some } N \text{ is } m \text{-reducible.}
\]

I shall further study the following problems:
1) When is a given \( n \)-dimensional torus map \( m \)-reducible?
1a) When is a given a.p. movement \( \tilde{f}(t) \) of dimension \( \leq n \), a.p. \( m \)-reducible?
2) Which spaces \( X \) are \( n \)-dimensionally \( m \)-reducible?
2a) Which spaces \( X \) are a.p. \( n \)-dimensionally \( m \)-reducible?
3) Which spaces $X$ are $m$-reducible?

3a) Which spaces $X$ are a.p. $m$-reducible?

The case $m = 1$ is of course of special interest.

My way through the problems is the following:

First I use obstruction theory in dealing with extension problems and homotopy classification problems of continuous functions from subspaces of $T^n$ into topological spaces. I shall not go beyond the primary obstruction because otherwise the problems get too complicated to be of use for my original problems.

Next I look at the special case $X = S^p$, the $p$-dimensional sphere. The set of homotopy classes of torus maps $T^n \to S^p$, $\pi^p(T^n)$, is called the $p$-dimensional cohomotopy set. For some $p$ and $n$ the set $\pi^p(T^n)$ is an abelian group which is called the $p$-dimensional cohomotopy group of $T^n$. I compute the cohomotopy groups by means of the homotopy groups of the spheres $\pi_i(S^m,s_0)$. Unfortunately, most of these homotopy groups are not yet known.

The homotopy groups of $T^n$, on the other hand, are very simple: the fundamental group is free abelian of rank $n$: $\pi_1(T^n,t_0) \cong \mathbb{Z}^n$, and all the higher homotopy groups $\pi_i(T^n,t_0)$ ($i > 1$) are zero. – In general we know that $\pi_i(X,x_0)$ ($i > 1$) is an abelian group, while $\pi_1(X,x_0)$ is a group, but not always abelian.

The results obtained on the homotopy classification of torus maps into $X$, are applied to the torus reducibility problems and this leads to results on the a.p. reducibility problems. For some special topological spaces $X$ this gives simple results, but for further work on almost periodic movements my method does not seem fruitful because the homotopy classification problems soon become extremely complicated and the torus reducibility problems turn out to be more complicated than the a.p. reducibility problems.

I shall state the principal results in a form independent of the choice of basis point. The corresponding theorems in the paper will be stated only for a fixed basis point.

**Theorem 18'.** Let $x = \tilde{f}(t)$ be an almost periodic movement in $X \in \mathcal{C}$ corresponding to the torus map $f: T^n \to X$. If $\pi_2(X,x_0) = \ldots = \pi_m(X,x_0) = 0$ for all $x_0 \in X$, then a necessary and sufficient condition for $\tilde{f}$ to be a.p. $m$-reducible ($1 \leq m < n$) is that $f_*\pi_1(T^n,t_0)$, which is a finitely generated abelian subgroup of $\pi_1(X,f(t_0))$, has rank $\leq m$.

**Theorem 19'.** Let $X \in \mathcal{C}$ and let $\pi_i(X,x_0) = 0$ for all $x_0 \in X$, $i > 1$. Then a necessary and sufficient condition for $X$ to be a.p. $m$-reducible ($m \geq 1$) is that for all $x_0 \in X$ every abelian subgroup of $\pi_1(X,x_0)$ has rank $\leq m$. 
Remark: If $\pi_1(X, x_0)$ is itself abelian, the above condition on $\pi_1(X, x_0)$ is equivalent to: rank $\pi_1(X, x_0) \leq m$.

Theorem 20'. A necessary and sufficient condition for $X$ to be a.p. 1-reducible is that, for all $x_0 \in X$, every abelian subgroup $G$ of $\pi_1(X, x_0)$ has rank $\leq 1$ and that, for all $n > 1$, the condition that $f_*: \pi_1(T^n, t_0) \to \pi_1(X, x_0)$ are conjugate implies that for some natural number $N$ the maps $f^*(N)$, $g^*(N)$ are homotopic.

Theorem 21'. Let $X \in \mathcal{C}$ be an $H$-space (for instance a topological group) with $\pi_1(X, x_0) = \ldots = \pi_{p-1}(X, x_0) = 0$ for all $x_0 \in X$. Then a necessary condition for $X$ to be a.p. $m$-reducible is that for all $x_0 \in X$ the rank of $\pi_p(X, x_0)$ is $\leq \left(\begin{array}{c} m \\ p \end{array}\right)$.

Theorem 22'. Let $X \in \mathcal{C}$ and let for all $x_0 \in X$

$$\pi_1(X, x_0) = \ldots = \pi_{p-1}(X, x_0) = \pi_{p+1}(X, x_0) = \ldots = \pi_m(X, x_0) = 0;$$

then a necessary condition for $X$ to be a.p. $m$-reducible is that for all $x_0 \in X$ the rank of $\pi_p(X, x_0)$ is $\leq \left(\begin{array}{c} m \\ p \end{array}\right)$ if $p > 1$, and that every abelian subgroup of $\pi_1(X, x_0)$ has rank $\leq m$, if $p = 1$.

Chapter 1

The Functions $F$, $G$, and $G'$

Definition. Let $X$ be a topological space with $x_0 \in X$. Let $n \in \mathbb{N}$ and let $\pi$ be a group (abelian for $n > 1$). I call $X$ an $m$-space of type $(\pi, n)$ ($m \in \mathbb{N} \cup \{+\infty\}, m \geq n$) when $X$ is path-connected and its homotopy groups in dimensions $\leq m$, except $\pi_n(X, x_0)$, are zero, while $\pi_n(X, x_0)$ is isomorphic to $\pi$.

A usual $(\pi, n)$ space is then the same as my $\infty$-space of type $(\pi, n)$.

Examples: $T^n$ is an $\infty$-space of type $(\mathbb{Z}^n, 1)$.

$S^n$ is an $n$-space of type $(\mathbb{Z}, n)$.

For all $n \in \mathbb{N}$ and all groups $\pi$ (abelian if $n > 1$) there exists a topological space $X$ of type $(\pi, n)$ ([3] p. 426).

We shall now look at maps $T^n \to X$ where $X$ is path-connected and $x_0 \in X$. We know that $f$ is homotopic to a map $f': (T^n, t_0) \to (X, x_0)$. Because we are interested only in the homotopy classes of maps $T^n \to X$ we shall always choose representatives $g$ of the homotopy classes for which $g(t_0) = x_0$, but we do not write this explicitly everywhere.

If we denote the homotopy classes of maps equivalent to $f$: $T^n \to X$ by $[f]$ and the set of these classes by $[T^n, X]$, then we know that a function
\[ F_n : [T^n, X] \to \text{Hom}(\pi_1(T^n, t_0), \pi_1(X, x_0)) \] \\
where \( \sim \) means conjugate, (if \( \pi_1(X, x_0) \) is abelian, then this group is isomorphic to \( \pi_1(X, x_0)^n \)) is defined by:

\[ F_n[f] = \{ f_* \} \text{ (the conjugacy class of } f_*. \) \]

If we denote the homotopy class relative to \( t_0 \) of maps \( (T^n, t_0) \to (X, x_0) \) equivalent to \( f \) by \([f]_t \) and the set of these maps by \([T^n, t_0; X, x_0]_t\), then we know that \( F_n'[f]_t = f_* \) defines a map

\[ F_n' : [T^n, t_0; X, x_0]_t \to \text{Hom}(\pi_1(T^n, t_0), \pi_1(X, x_0)). \]

If \( X \) is an \( H \)-space, we know that \([T^n, X] \) and \([T^n, t_0; X, x_0]_t \) are groups and that \( \pi_1(X, x_0) \) is abelian. In this case it is easy to see that \( F_n \) and \( F_n' \) are homomorphisms.

As a cell complex \( T^n = S_1 \times \ldots \times S_n \) consists of

- 1 0-dimensional cell \( t_0 \),
- \( n \) 1-dimensional cells \( S_1^{1} \) (circles), \ldots ,
- \( (\begin{array}{c} n \\ p \end{array}) \) \( p \)-dimensional cells \( T_1^p = S_1^{1} \times \ldots \times S_1^{1} \), \ldots ,

and 1 \( n \)-dimensional cell \( T^n \).

Let \( L \) be a subcomplex of \( T^n \), and let \( i_* : \pi_1(L, t_0) \to \pi_1(T^n, t_0) \) be induced by the inclusion map \( i : L \to T^n \). Because \( \pi_1(T^n, t_0) \cong \pi_1(S_1^1, t_0) \oplus \ldots \oplus \pi_1(S_n^1, t_0) \cong \mathbb{Z}^n \) we have that \( i_* \pi_1(L, t_0) \) is a direct summand in \( \pi_1(T^n, t_0) \) with rank equal to the number \( r \) of 1-cells in \( L \). We also have \( \pi_1(L, t_0) \cong \mathbb{Z}a_1 \ast \ldots \ast \mathbb{Z}a_r \), where \( \ast \) denotes free product and \( \sim \) means that two generators \( a_i \) and \( a_j \) commute when the corresponding 1-cells in \( L \) are sides of a 2-cell in \( L \).
A homomorphism \( h \in \text{Hom}(\pi_1(L,t_0), \pi_1(X,x_0)) \) is said to be extensible over \( T^n \) iff there exists an \( \tilde{h} \in \text{Hom}(\pi_1(T^n,t_0), \pi_1(X,x_0)) \) with \( \tilde{h}i_* = h \).

The statements in the following theorem are proved in [2], VI.

**Theorem 1.** Let \( X \) be an \( m \)-space of type \( (\pi,1) \). Then \( F_n \) and \( F_n' \) are surjective if \( m \geq n-1 \), and bijective if \( m \geq n \). Let \( L \neq \emptyset \) be a subcomplex of \( T^n \) and let \( g : L \to X \) be a given map. Then the following 3 statements are equivalent for \( m \geq n-1 \):

(i) \( g \) is continuously extensible over \( T^n \);
(ii) \( g_* \in \text{Hom}(\pi_1(L,t_0), \pi_1(X,x_0)) \) is extensible over \( T^n \);
(iii) \( g_*(\pi_1(L,t_0)) \) abelian.

If \( g_* \) is extensible to \( h \in \text{Hom}(\pi_1(T^n,t_0), \pi_1(X,x_0)) \), then we can find an extension \( f \) of \( g \) with \( f_* = h \).

**Examples.** Let \( P^m \) denote the \( m \)-dimensional real projective space. Because \( \pi_i(P^m,p_0) \cong \mathbb{Z}_2 \) for \( m \geq 2 \) and \( \pi_i(P^m,p_0) = 0 \) for \( 1 < i < m \), while \( \pi_m(P^m,p_0) \cong \mathbb{Z} \), we have the bijections

\[
[T^n,P^m] \cong \mathbb{Z}_2^n \cong [T^n,t_0; P^m,p_0]_{t_0} \text{ for } n < m.
\]

We also have

\[
[T^n,S^m] \cong 0 \cong [T^n,t_0; S^m,s_0]_{t_0} \text{ for } n < m,
\]

and

\[
[T^n,T^m] \cong \mathbb{Z}^{m-n} \cong [T^n,t_0; T^n,t_0]_{t_0},
\]

where the two last bijections are group isomorphisms.

**Remark.** It is easy to see that \( F_n \) surjective (injective) implies \( F_m \) surjective (injective) if \( m \leq n \).

Let \( (T^n)^m \) denote the \( m \)-dimensional skeleton of \( T^n \) as a cell-complex. The absolute and relative (with respect to subcomplexes) homology- and cohomology groups of \( T^n \) and \( (T^n)^m \) are easily computed: With coefficients in an abelian group \( G \) we have

\[
H_p(T^n; G) \cong G^\binom{n}{p} \cong H^p(T^n; G),
\]

\[
H_p(T^n,L; G) \cong G^\binom{n}{p} - \iota(p) \cong H^p(T^n,L; G),
\]

\[
H_p((T^n)^m,L'; G) \cong G^\binom{n}{p} - \iota'(p) \cong H^p((T^n)^m,L'; G),
\]

where \( \iota(p) \) and \( \iota'(p) \) denote the number of \( p \)-cells in \( L \) and \( L' \). As generators for the homology groups we can take the elements \( \{\sigma_i \otimes g\}, g \in G \), where
σ_i is a p-cell of T^m, T^m\mid L, (T^m)^m\mid L' respectively corresponding to a p-dimensional torus T_p^m in T^n. As generators for the cohomology groups we can take the elements \{f^p_i\}, where f^p_i is the homomorphism determined by f^p_i(σ_i) = g and f^p_i(σ_j) = 0, j ≠ i.

Further, it is easy to see that H^p(T^n, \mathbb{Z}) as an algebra is the exterior algebra over \mathbb{Z} with n generators in dimension 1 corresponding to the generators of H^1(S^1).

The homomorphism

\[ h: H^p((T^n)^m, L'; G) \to \text{Hom}(H_p((T^n)^m, L'; G)) \]

defined by \( h\{f\}\{c\} = f(c) \) is an isomorphism. Generators in \text{Hom}(H_p((T^n)^m, L'; G)) corresponding to the \{f^p_i\} are the \( f^p_i \) themselves.

Dividing the p-cells σ_i into those in (T^n)^m\mid L and those in L I find that

\[
H_p((T^n)^m; G) \cong H_p((T^n)^m, L'; G) \oplus H_p(L', G)
\]

\[
H^p((T^n)^m; G) \cong H^p((T^n)^m, L'; G) \oplus H^p(L', G).
\]

Thus the homomorphisms

\[
i^\# \in \text{Hom}(H^p((T^n)^m; G); H^p(L', G)) \quad \text{and} \quad j^\# \in \text{Hom}(H^p((T^n)^m, L'; G), H^p((T^n)^m; G))
\]

induced by the inclusion maps \( i: L' \to (T^n)^m, j: (T^n)^m \to ((T^n)^m, L') \) are injective, and so are the corresponding homomorphisms between the homology groups \( i^\# \) and \( j^\# \).

Now, let \( g: L' \to X \) be a given map. Suppose \( f: (T^n)^m \to X \) is an extension of \( g \) and denote by \([f]_{L'}\) the homotopy class relative to \( L' \) of maps \((T^n)^m \to X\) equivalent to \( f \). By \([f]_{L'}\), we denote the set of these maps. Then there is a well-defined function

\[
G'_{n,m,p}: [(T^n)^m, L', g; X]_{L'} \to \text{Hom}(H_p((T^n)^m, L'; H_p(X))
\]

defined by \( G'_{n,m,p}[f]_{L'} = i^\#_p \circ H_p((T^n)^m, L') \). We observe that

\[
\text{Hom}(H_p((T^n)^m, L'); H_p(X)) \cong \begin{cases} 
(H_p(X))^{(n)_{p,\infty}} & \text{if } p \leq m \\
0 & \text{if } p > m.
\end{cases}
\]

Even if \( X \) is an H-space, \( G' \) is not in general a homomorphism.

If \( X \) is a p-space of type \((\pi^p, p) \) (\( \pi^p \) abelian), we have the Hurewicz isomorphism \( \kappa: \pi^p \to H_p(X) \) and \( G'_{n,m,p} \) corresponds to
\[ G_{n,m,p} : [T^n,L',g;X]_{L'} \to \text{Hom}(H_p((T^n)^m,L'), \pi_p) \]

where \( G_{n,m,p}[f]_{L'} = \pi^{-1} \circ f \#_p \mid H_p((T^n)^m,L') \). We know from the theory of obstruction that \( \pi_1(X,x_0) = \ldots = \pi_{n-1}(X,x_0) = 0 \Rightarrow f|((T^n)^{p-1}) \cong 0 \), hence we can choose a map \( f' : (T^n)^m \to X \) with \( f' \cong f \) and \( f'|((T^n)^{p-1}) = x_0 \). If \( \sigma \) denotes a \( p \)-cell in \( (T^n)^m \setminus L' \), then \( f'(\partial \sigma) = x_0 \) and \( f'|\sigma \) represents an element of \( \pi_p \cong [S^p,X] \). Because \( \pi([f'|\sigma]) = f''_p(\sigma) = f''_p(\sigma) \) we have:

\[ G_{n,m,p}[f]_{L'}(\sigma) = [f'|\sigma]. \]

If \( X \not\cong \emptyset \) is also an \( H \)-space, we know that \( \pi_1(X,x_0) \) is abelian and that the group structure in \( \pi_p(X,x_0) \) is defined by the multiplication map in \( X \), so that in this case the functions \( G \) and \( G' \) are homomorphisms.

**Remark.** \( G'_{n} : [T^n,X] \to \text{Hom}(H_p(T^n), H_p(X)) \) surjective (injective) implies that \( G'_{m} : [T^m,X] \to \text{Hom}(H_p(T^m), H_p(X)) \) is surjective (injective) for \( m \leq n \).

From the above remarks and the results in [2], VI we get:

**Theorem 2.** Let \( X \) be an \( (m-1) \)-space of type \( (\pi_p,p) \) (\( \pi_p \) abelian). Then we have \( [(T^n)^m,L',g;X]_{L'} = \emptyset \) and the functions \( G_{n,m,p} \) and \( G'_{n,m,p} \) are surjective. Let \( X \) be an \( m \)-space of type \( (\pi_p,p) \) (\( \pi_p \) abelian). Then we have \( [(T^n)^m,L',g;X]_{L'} = \emptyset \) and \( G_{n,m,p} \) and \( G'_{n,m,p} \) are bijective.

Thus

\[ [(T^n)^m,L',g;X]_{L'} \cong \pi_p^{(n)-\ell(p)}, \]

and, in particular

\[ [T^n,L',g;X]_{L} \cong \pi_p^{(n)-\ell(p)}, \]

\[ [T^n,X] \cong [T^n,l_0;X,x_0]_{L} \cong \pi_p^{(n)}. \]

**Remark.** Let \( [(T^n)^m,L',g;X] \) denote the set of homotopy classes of maps \( f : (T^n)^m \to X \) for which \( f|L' = g \). Then \( I[f]_{L'} = [f] \) defines a map \( [(T^n)^m,L',g;X]_{L'} \to [(T^n)^m,L',g;X] \), which is surjective. The map

\[ \tilde{G}'_{n,m,p} : [(T^n)^m,L',g;X] \to \text{Hom}(H_p((T^n)^m,L'), H_p(X)), \]

where \( \tilde{G}'_{n,m,p}[f] = f\# p \mid H_p((T^n)^m,L') \) is also well defined. From this we get \( (G_{n,m,p} \circ I) : G_{n,m,p} \) surjective (injective) \( \Rightarrow \tilde{G}'_{n,m,p} \) surjective (injective).
Examples. \( [(T^n)_m, S^m] \cong \mathbb{Z}^{(m)}_n, [T^n, S^n] \cong \mathbb{Z} \).

(Here \( \cong \) denotes bijection.)

Now, let \( X \) be an \( H \)-space and a \( p \)-space of type \( (\pi_p, p) \) (then \( \pi_1 \) is automatically abelian). That \( X \) is an \( H \)-space means that we have a fixed point \( x_0 \in X \), a continuous multiplication \( \mu: (X \times X, x_0 \times x_0) \to (X, x_0) \) for which the constant map \( X \to x_0 \) is a homotopy identity, i.e. \( \mu \circ (c, 1_x) \cong \mu \circ (1_x, c) \cong 1_x \) relative to \( x_0 \). From the above theorem we know that for \( n \geq p \)

\[ [(T^n)^{p-1}, X] = 0 \text{ and } [(T^n)^p, X] \cong \text{Hom}(H_p((T^n)^p), \pi_p) \cong \text{Hom}(H_p(T^n), \pi_p). \]

Hence every \( h \in \text{Hom}(H_p(T^n), \pi_p) \) corresponds to a continuous map \( (T^n)^p \to X \) with \( f((T^n)^{p-1}) = x_0 \) and \( f\#_{p} = \varepsilon h \). Let \( f_i \) denote the map

\[ T^n \xrightarrow{\text{proj}} T_i^p \xrightarrow{f_i} X. \]

Let \( \tilde{f} \) denote "one of the products of the \( f_i \)'s", for instance

\[ \mu \circ (1_x \times \mu) \circ \ldots \circ (1_x \times \ldots \times 1_x \times \mu) \circ (f_1^{(p)}), \ldots, f_{\ell}) \).

Then it is easy to see that \( \tilde{f} \) is a continuous map \( T^n \to X \) with \( \tilde{f}(T^n)^p \xrightarrow{\varepsilon} f, \)

i.e. \( f\#_{p} = f\#_{p} = \varepsilon h \). Thus we have \( (n < p \text{ trivial}) : \)

**Theorem 3.** Let \( X \) be an \( H \)-space and a \( p \)-space of type \( (\pi_p, p) \). Then for all \( n \in \mathbb{N} \), \( G \) and \( G' \) are surjective homomorphisms; \( G: [T^n, X] \to \text{Hom}(H_p(T^n), \pi_p) \cong \pi_p^{(n)} \), where \( G([f]) = \varepsilon^{-1}f\#_{p} \), \( G': [T^n, X] \to \text{Hom}(H_p(T^n), H_p(X)) \cong \pi_p^{(n)} \), where \( G'([f]) = f\#_{p} \).

For \( p = 1 \) we have the Hurewicz isomorphism \( \varepsilon_{T^n}: \pi_1(T^n, l_0) \to H_1(T^n) \) and \( G([f]) = f_* \circ \varepsilon_{T^n}^{-1} \) so that \( F_n \) and \( F_n' \) are surjective.

**Chapter 2**

**The Cohomotopy Groups of \( T^n \)**

**Definition.** Let \( X \) be a topological space and \( A \) a subspace of \( X \). The \( m \)'th (relative) cohomotopy set \( \pi^m(X, A) \) of \( (X, A) \) \((m \in \mathbb{N})\) is defined to be \([X, A; S^m, S_0]_A\) and the \( m \)'th (absolute) cohomotopy set \( \pi^m(X) \) of \( X \) is defined to be \([X, S^m]\).
A pair \((X,A)\) is called \(n\)-coconnected if it satisfies the condition
\[H^q(X,A; G) = 0\] for every \(q \geq n\) and every coefficient group \(G\).

A cellular pair \((X,A)\) is a pair of a finite cell complex \(X\) and a sub-complex \(A\).

We need the following theorem ([2], VII, Theorem 5.2).

**Theorem 4.** If \((X,A)\) is a \((2m-1)\) coconnected cellular pair, we can define a certain abelian group structure \(+\) in \(\pi^m(X,A)\) with the class of the constant map as the neutral element. In this case \(\pi^m(X,A)\) is called the \(m'\)th cohomotopy group of \((X,A)\).

**Remarks.** 1) Every map \(f: (X,A) \to (Y,B)\) induces a transformation \(f^*: \pi^m(Y,B) \to \pi^m(X,A)\). If both \((X,A)\) and \((Y,B)\) are \((2m-1)\)-coconnected cellular pairs, then \(f^*\) is a homomorphism ([2], VII, prop. 5.4).

2) When the cohomotopy group structure is defined in \(\pi^m(S^n,s_0)\) (thus \(n \leq 2m-2\)), then the bijection \(\pi^m(S^n,s_0) \cong \pi_n(S^m,s_0)\) is an isomorphism ([2], VII, prop. 12.1).

We now try to compute the cohomotopy groups of \(T^n\) by means of some exact sequences for the pairs \([(T^n)^m, (T^n)^{m-1}]\).

If \(i \geq \frac{m}{2}(+1)\), where for \(a\) real \(a = \min_{p \in \mathbb{Z}} \{p|p \geq a\}\), then \(\pi^i((T^n)^m)\) and \(\pi^i((T^n)^m, (T^n)^{m-1})\) have the cohomotopy group structure.

From Theorem 2 we have the bijections \((p \geq m)\)

\[
\pi^p\left((T^n)^m, (T^n)^{m-1}\right) \xrightarrow{\sim} \pi^p\left((T^n)^m\right) \cong \begin{cases} 0 & p > m \\ \mathbb{Z}_{(m)} & p = m, \end{cases}
\]

where \(i^*\) is a homomorphism for \(p \geq \frac{m}{2}(+1)\), i.e. \(p > 1\).

If \(X,Y\) are spaces with basis point, \(X \vee Y\) denotes their one point union.

If \(p\) denotes the map

\[
(T^n)^m \to (T^n)^{m-1} \sim S^m_1 \vee \ldots \vee S^m_m
\]

then

\[
p^*: \pi^i(S^m_1 \vee \ldots \vee S^m_m, s_0) \to \pi^i((T^n)^m, (T^n)^{m-1})
\]

is a bijection and for \(i \geq \frac{m}{2}(+1)\) an isomorphism. It is easy to see that there is a bijection

\[
\pi^i(S^m_1 \vee \ldots \vee S^m_m, s_0) \cong \pi^i(S^m, s_0)_{(m)}.
\]

Let \(i'\) denote the inclusion \(S^m_1 \vee \ldots \vee S^m_r \to S^m_1 \vee \ldots \vee S^m_r \vee \ldots \vee S^m_q\) and
$p'$ the projection $S^n_1 \vee \ldots \vee S^n_{q-1} \rightarrow S^n_1 \vee \ldots \vee S^n_{q}$ when $i \geq \frac{m}{2}(+1)$ we get the following split exact sequences of abelian groups and homomorphisms because $\pi^i$ is a functor:

$$0 \rightarrow \pi^i(S^n_1 \vee \ldots \vee S^n_{q-1}, S_0) \xrightarrow{p'^*} \pi^i(S^n_1 \vee \ldots \vee S^n_{q}, S_0) \xrightarrow{i'^*} \pi^i(S^n_q, S_0) \rightarrow 0.$$ 

An induction thus gives us the isomorphism

$$\pi^i(S^n_1 \vee \ldots \vee S^n_{q}, S_0) \cong \pi^i(S^n_1, S_0) \oplus \ldots \oplus \pi^i(S^n_q, S_0).$$

Because of the isomorphism $\pi^i(S^n, S_0) \cong \pi_i(S^n, S_0)$ we have that the bijection

$$\pi^i((T^n)^m, (T^n)^{m-1}) \cong \pi_m(S^n, S_0) \left(\begin{array}{c} m \\frac{m}{2} \end{array}\right)$$

is an isomorphism for $i \geq \frac{m}{2}(+1)$.

We have the following long exact cohomotopy sequences ([2], VII, 6.-9.) of abelian groups and homomorphisms except the first set and the first map in the first sequence and the two first sets and the two first maps in the second sequence.

1) $m = 2q \geq 2$.

$$\pi^q((T^n)^{2q-1}) \xrightarrow{\delta^q}$$

$$\rightarrow \pi^{q+1}((T^n)^{2q}, (T^n)^{2q-1}) \xrightarrow{j_{2q+1}} \pi^{q+1}((T^n)^{2q}) \xrightarrow{i_{2q+1}} \pi^{q+1}((T^n)^{2q-1}) \xrightarrow{\delta_{2q+1}} \ldots$$

$$\rightarrow \pi^{2q-1}((T^n)^{2q}, (T^n)^{2q-1}) \xrightarrow{j_{2q-1}} \pi^{2q-1}((T^n)^{2q}) \xrightarrow{i_{2q-1}} \pi^{2q-1}((T^n)^{2q-1}) \xrightarrow{\delta_{2q-1}}$$

$$\rightarrow \pi^{2q}((T^n)^{2q}, (T^n)^{2q}) \xrightarrow{j_{2q}} \pi^{2q}((T^n)^{2q}) \xrightarrow{i_{2q}} 0.$$ 

2) $m = 2q-1 \geq 3$.

$$\pi^q((T^n)^{2q-1}, (T^n)^{2q-2}) \xrightarrow{j_{2q-1}} \pi^q((T^n)^{2q-1}) \xrightarrow{i_{2q-1}} \pi^q((T^n)^{2q-2}) \xrightarrow{\delta^q}$$

$$\rightarrow \pi^{q+1}((T^n)^{2q-1}, (T^n)^{2q-2}) \xrightarrow{j_{2q+1}} \pi^{q+1}((T^n)^{2q-1}) \xrightarrow{i_{2q+1}} \pi^{q+1}((T^n)^{2q-2}) \xrightarrow{\delta_{2q+1}} \ldots$$

$$\rightarrow \pi^{2q-2}((T^n)^{2q-1}, (T^n)^{2q-2}) \xrightarrow{j_{2q-2}} \pi^{2q-2}((T^n)^{2q-1}) \xrightarrow{i_{2q-2}} \pi^{2q-2}((T^n)^{2q-2}) \xrightarrow{\delta_{2q-2}}$$

$$\rightarrow \pi^{2q-1}((T^n)^{2q-1}, (T^n)^{2q-2}) \xrightarrow{j_{2q-1}} \pi^{2q-1}((T^n)^{2q-1}) \xrightarrow{i_{2q-1}} 0.$$ 

All maps $i, j$ are induced by inclusions, and the $\delta$'s are connecting "homomorphisms".
We know that every finitely generated abelian group $G$ is isomorphic to a direct sum of cyclic groups. If $(b_1, \ldots, b_n)$ are generators of the cyclic groups in such a decomposition, then they are weakly linearly independent in the sense that $p_1 b_1 + \ldots + p_n b_n = 0$ implies $p_1 b_1 = \ldots = p_n b_n = 0$. We shall call $(b_1, \ldots, b_n)$ a basis of $G$.

We now choose basis elements for the finitely generated abelian groups $\pi_i(S^i, s_0)$ represented by $g_{j,i}^x$. A generator of $\pi_i(T^y)$ is represented by the projection $p_j: T^j \to T^j/(T^j)^{j-1} \sim S^j$. We then look at the elements of $\pi_i((T^n)^m)$ represented by

$$(T^n)^m \xrightarrow{\text{proj}_{m,j}} T_{\beta}^j \xrightarrow{p_j} S_i \xrightarrow{g_{j,i}^x} S^j; \quad j \leq m, \beta \in \{1, \ldots, (n)\}.$$ 

By means of induction on $m$ we prove:

For all $i, n, m$ in $\mathbb{N}$ with $n \geq m$ such that $\pi^i((T^n)^m)$ is a cohomotopy group, i.e. $i \geq \frac{n^2}{2} + 1$, the following short exact sequence is split exact,

$$0 \to j^* \pi^i((T^n)^m), \quad (T^n)^{m-1} \to \pi^i((T^n)^m) \xrightarrow{i^*} \pi^i((T^n)^{m-1}) \to 0,$$ 

and a basis for $\pi^i((T^n)^m) \cong \pi^i((T^n)^{m-1}) \oplus j^* \pi^i((T^n)^m)$, $(T^n)^{m-1})$ is represented by the elements $g_{j,i}^x \circ p_j \circ \text{proj}_{m,j}^{\beta'}$ except that the long exact sequences are not long enough to the left to allow us to decide whether $g_{j,i}^x \circ p_j \circ \text{proj}_{m,j}^{\beta'}$ is homotopic to 0 or not.

The start of the induction is trivial by the long exact sequences.

Let the above be true for $m-1$. We see that all the elements $g_{j,i}^x \circ p_j \circ \text{proj}_{m-1,j}^{\beta'}$ of $\pi^i((T^n)^{m-1})$ where $\beta' \in \{1, \ldots, (n)\}$ and $j \leq m-1$, have trivial extensions to $(T^n)^m$: $g_{j,i}^x \circ p_j \circ \text{proj}_{m,\beta'}^{\beta'}$, which proves that $i^* = i^m_\xi$ is surjective. Further,

$$g_{j,i}^x \circ p_j \circ \text{proj}_{m-1,j}^{\beta'} \text{ non} \cong 0 \iff g_{j,i}^x \circ p_j \text{ non} \cong 0 \iff g_{j,i}^x \circ p_j \circ \text{proj}_{m,j}^{\beta'} \text{ non} \cong 0.$$

From the isomorphisms $\pi^i(S^i) \cong \pi^i(S^i, s_0) \cong \pi_j(S^i, s_0)$, $i \leq 2j-2$, we see for $p \in \mathbb{Z}$ that $p(g_{j,i}^x \circ f) = (pg_{j,i}^x) \circ f: X \to S^j \to S^i$ when $\pi^i(X)$ and $\pi^i(S^j)$ are cohomotopy groups $(f^* \text{ is a homomorphism } \pi^i(S^j) \to \pi^i(X))$. From this we see that $pg_{j,i}^x \circ p_j \circ \text{proj}_{m-1,j}^{\beta'}$ and its extension to $(T^n)^m$: $pg_{j,i}^x \circ p_j \circ \text{proj}_{m,j}^{\beta'}$ are zero homotopic at the same time. This and the fact that the elements $g_{j,i}^x \circ p_j \circ \text{proj}_{m-1,j}^{\beta'}$ (when not zero homotopic) represent a basis for $\pi^i((T^n)^{m-1})$ gives us a well defined homomorphism $h: \pi^i((T^n)^{m-1}) \to \pi^i((T^n)^m)$ such that $i^* \circ h = 1\pi^i((T^n)^{m-1})$. This and the exactness of the long cohomotopy sequence proves that the sequence above is split exact. Thus

$$\pi^i((T^n)^m) \cong \pi^i((T^n)^{m-1}) \oplus j^* \pi^i((T^n)^m), \quad (T^n)^{m-1})$$

where $j^* \pi^i((T^n)^m), \quad (T^n)^{m-1})$ is generated by
If \( m = 2i-2 \) we do not know whether \( g_{m,i}^\prime \circ p_m \circ \text{proj}_{m,m}^\prime \approx 0 \) or not.

If \( m < 2i-3 \), we get from the above that \( j_i^m \) is injective, because \( i_{i-1}^m \) is surjective and the sequence exact. The following proof is valid also if \( m = 2i-3 \). Because all the generators \( g_{j,i-1}^\prime \circ p_j \circ \text{proj}_{m-1,j}^\prime \): \((T^m)^{m-1} \to S^{i-1}\) of \( \pi^i((T^m)^{m-1}) \) have extensions to \((T^m)^m \): \( g_{j,i-1}^\prime \circ p_j \circ \text{proj}_{m,j}^\prime \): \((T^m)^m \to S^{i-1}\), the exactness gives us:

\[
\delta_{i-1}^m([g_{j,i-1}^\prime \circ p_j \circ \text{proj}_{m-1,j}^\prime]) = 0 \Rightarrow \delta_{i-1}^m = 0 \Rightarrow j_i^m \text{ injective, i.e. for } m < 2i-2 \text{ we have } \pi^i((T^m)^m) \cong \pi^i((T^m)^{m-1}) \oplus \pi^i((T^m)^m, (T^m)^{m-1}), \text{ and a basis for } \pi^i((T^m)^m) \text{ is represented by all the }
\]

\[
g_{j,i}^\prime \circ p_j \circ \text{proj}_{m,j}^\prime \quad (\beta \in \{1, \ldots, \binom{m}{i}\}; j \leq m).
\]

**Theorem 5.** The cohomotopy groups \( \pi^i(T^m)^m \) are finitely generated and for \( 2i \geq m + 3 \) we have

\[
\pi^i(T^m)^m \cong \begin{cases} 0 & \text{if } i > m \\ \pi_i(S^i, s_0)^{(m)}_i \oplus \pi_{i+1}(S^i, s_0)^{(m)}_{i+1} \oplus \ldots \oplus \pi_m(S^i, s_0)^{(m)}_m & \text{if } i \leq m \end{cases}
\]

and the last expression is valid for \( m = 2i-2 \), if the last factor is replaced by a suitable factor group.

For \( m = n \) we get:

**Theorem 6.** For \( 2i \geq n + 3 \) we have

\[
\pi^i(T^n) \cong \begin{cases} 0 & \text{if } i > n \\ \pi_i(S^n, s_0)^{(n)}_i \oplus \pi_{i+1}(S^n, s_0)^{(n)}_{i+1} \oplus \ldots \oplus \pi_n(S^n, s_0) & \text{if } i \leq n \end{cases}
\]

and the last expression is valid for \( n = 2i-2 \) if the last factor is replaced by a suitable factor group.

It is known that \( \pi_m(S^n, s_0) \) is zero for \( m < n \), \( Z \) for \( m = n \) and a finite abelian group for \( m > n \), except \( \pi_{i-1}(S^{2i}, s_0) \) which is the direct sum of \( Z \) and a finite group. This group is not a cohomotopy group and so it has no influence on the cohomotopy groups of \((T^n)^m\). Hence:

**Corollary.** \( \pi^i(T^n) \) is the direct sum of \( Z^{\binom{n}{i}} \) and a finite abelian group, when \( i \geq \frac{1}{2}n + 1 \) (i.e. rank \( \pi^i(T^n) = \binom{n}{i} \)).
Chapter 3

Elementary Properties of Tori

Before I discuss the problems of reducibility mentioned in the introduction, I deduce a few elementary results:

Lemma 1. The homotopy classes of homeomorphisms \((T^n,t_0) \rightarrow (T^n,t_0)\) are in one to one correspondence with the unimodular \(n \times n\) matrices.

Proof. [A unimodular \(n \times n\) matrix \(A\) has elements in the integers \(\mathbb{Z}\) and determinant \(\pm 1\).] The lemma follows easily from the isomorphism \(F_n: [T^n,T^n] \rightarrow \text{Hom}(\pi_1(T^n,t_0), \pi_1(T^n,t_0)) \cong \text{Hom}(\mathbb{Z}^n,\mathbb{Z}^n) \cong \mathbb{Z}^n\) defined by \(F_n[f] = f_*\).

Lemma 2. The homotopy classes in \(T^n\) of \(T^m\)'s imbedded in \(T^n\) for which \(T^n = T^m \times X\) are in one to one correspondence with the direct summands of \(\pi_1(T^n,t_0)\) of rank \(m\).

This follows easily from Lemma 1.

Theorem 7. \(f: T^n \rightarrow X\) is \(m\)-reducible iff \(f\) is homotopic to the projection of \(T^n\) onto a \(T^m\) imbedded in \(T^n = T^m \times Y\) (possibly after a shift of coordinates in \(T^n\)), followed by a map \(T^m \rightarrow X\).

Proof. Obviously, “if” is trivial. We shall prove “only if”. It is enough to show that every map \(f: (T^n,t_0) \rightarrow (T^m,t_0)\) can be projected through a \(T^m_1\) imbedded in \(T^n\). Because \(\pi_1(T^m,t_0)\) is free, \(f_*(\pi_1(T^n,t_0))\) is also free and so \(f_*(\pi_1(T^n,t_0))\) is a direct summand in \(\pi_1(T^n,t_0)\) of rank \(m' \leq m\). Let \(G \subseteq \pi_1(T^n,t_0)\) be a direct summand in \(\pi_1(T^n,t_0)\) of rank \(m\) including the isomorphic image of \(f_*(\pi_1(T^n,t_0))\). From Lemma 2 \(G\) corresponds to a \(T^m_1\) imbedded in \(T^n = T^m_1 \times Y\) (possibly after a shift of coordinates of \(T^n\)). Thus \(f\) is homotopic to the projection in new coordinates of \(T^n\) onto \(T^m_1\) followed by the map \(f|T^m_1: T^m_1 \rightarrow T^m\).

Chapter 4

Remarks about Finitely Generated Abelian Groups

We know that there is an isomorphically unique decomposition of a finitely generated abelian group \(G\) as a direct sum of cyclic groups \(\mathbb{Z}^n \oplus \mathbb{Z}_{n_1} \oplus \ldots \oplus \mathbb{Z}_{n_p}\), where \(n_i\) divides \(n_{i-1}\).

The number \(n\) is the rank of \(G\) and the \(n_i\)'s are called the torsion coefficients of \(G\).
We know that every subgroup of a free abelian group $F \cong \mathbb{Z}^m$ of rank $m$ is a free abelian group of rank $m' \leq m$. A finite abelian group $G'$ has in general many decompositions as a direct sum of cyclic groups, but we can define a dimension of a finitely generated abelian group $G$ as

$$\text{dim } G = \text{the smallest number of generators of } G.$$ 

The usual proof of the theorem above starts with $m$ generators $g_1, \ldots, g_m$ of $G$ and then shows that there exists an $r \leq m$ and $\varepsilon_1, \ldots, \varepsilon_r$ with $\varepsilon_i \varepsilon_{i-1}$ so that $G$ is isomorphic to $\mathbb{Z}^{m-r} \oplus \mathbb{Z}^{\varepsilon_1} \oplus \ldots \oplus \mathbb{Z}^{\varepsilon_r}$, where $1 = 0$. This gives us that

$$\text{dim } G = \text{rank } G + \text{the number of torsion coefficients of } G.$$ 

The dimension of $G$ has the following properties:

1) $\text{dim } G = m \Rightarrow \forall q \leq m \exists G' \subseteq G: \text{dim } G' = q.$

2) If $f \in \text{Hom}(\mathbb{Z}^n, \hat{G})$, $\hat{G}$ a group, then $f(\mathbb{Z}^n)$ is a finitely generated abelian group with $\text{dim } f(\mathbb{Z}^n) \leq n.$

3) $G' \subseteq G \Rightarrow \text{dim } G' \leq \text{dim } G.$

Chapter 5

Torus Reducibility Problems

Let $X \neq \emptyset$ be a topological space, $x_0 \in X$ a fixed point of $X$. Looking at torus maps $T_n \to X$, we always assume them as already mentioned, continuous, and if $X$ is path connected, we assume that $t_0$ is mapped into $x_0$. We are interested only in the homotopy classes of maps $T_n \to X$. We know that $\pi_0(X,x_0)$ is the set of path components of $X$ with the path component including $x_0$ as 0-element, that, for all $i \geq 1$ and all $x_0, x_1 \in X$ in the same path component, $\pi_i(X,x_0)$ is isomorphic to $\pi_i(X,x_1)$ and that $T^n$ itself is path connected. Hence a theorem about torus mappings into path connected spaces under some conditions on the $\pi_i(X,x_0)$'s for fixed $x_0$ can be translated to a theorem about torus mappings into spaces not necessarily path connected, under the same conditions on the $\pi_i(X,x_1)$'s for all $x_1 \in X$. In the following I therefore assume the spaces $X$ path connected even if this is not written explicitly.

We start with some trivial remarks concerning the definitions 1-3 in the introduction:

For $f: (T^n, t_0) \to (X, x_0)$ we have: If $f$ is $m$-reducible, then $\dim f_*(\pi_1(T^n, t_0)) \leq m$ and $\dim f_#(H_p(T^n)) \leq \binom{m}{p}$ for every $p \in \mathbf{N}$, where
\[ f_\ast \in \text{Hom}(\pi_1(T^n, t_0), \pi_1(X,x_0)), f_{\#} \in \text{Hom}(H_p(T^n), H_p(X)) \]

and
\[ \pi_1(T^n, t_0) \cong \mathbb{Z}^n, H_p(T^n) \cong \mathbb{Z}^{(n)}_p. \]

The map \( F_{m+1} : [T^{m+1}, X] \to \text{Hom}(\pi_1(T^{m+1}, t_0), \pi_1(X,x_0)) \) defined by \( F_{m+1}[f] = \{f_\ast\} \) is surjective iff every homomorphism \( h : \pi_1(T^{m+1}, t_0) \to \pi_1(X,x_0) \) is induced by some map \( f : (T^{m+1}, t_0) \to (X,x_0) \). From Theorems 1 and 3 it follows that this happens when \( X \) is an \( m \)-space of type \((\pi, 1)\) and when \( X \) is an \( H \)-space.

**Theorem 8.** If \( F_{m+1} \) is surjective, and \( X \) is \([n \text{ dimensionally } (n > m)]\) \( m \)-reducible, then every finitely generated abelian subgroup of \( \pi_1(X,x_0) \) has dimension \( \leq m \).

**Remark.** If \( \pi_1(X,x_0) \) is itself a finitely generated abelian group, then the above condition is equivalent to dim \( \pi_1(X,x_0) \leq m \).

Proof of Theorem 8. Suppose \( \pi_1(X,x_0) \) has a finitely generated abelian subgroup of dimension greater than \( m \), then \( \pi_1(X,x_0) \) also has a subgroup \( G \) of dimension \( m+1 \). Because \( \pi_1(T^{m+1}, t_0) \cong \mathbb{Z}^{m+1} \), we have a surjective \( h \in \text{Hom}(\pi_1(T^{m+1}, t_0), G) \), which because of the assumption corresponds to an \( f : T^{m+1} \to X \) with \( f_\ast = h \). If we define \( f' \) as \( T^n \xrightarrow{\text{proj}} T^{m+1} \xrightarrow{f} X \), then \( f_\ast = h(\pi_1(T^{m+1}, t_0)) = m+1 \). Thus \( f' \) is not \( m \)-reducible.

**Theorem 9.** Suppose \( F_n : [T^n, X] \to \text{Hom}(\pi_1(T^n, t_0), \pi_1(X,x_0)) \) defined by \( F_n[f] = \{f_\ast\} \) is bijective. (From Theorem 1 we know that this is the case when \( X \) is an \( n \)-space of type \((\pi_1, 1)\)). Then
\[ f : (T^n, t_0) \to (X,x_0) \text{ is } m \text{-reducible iff } f_\ast(\pi_1(T^n, t_0)) \]

(an abelian group of dimension \( \leq n \)) is of dimension \( \leq m \).

Proof. We have already proved "only if". If \( \text{dim } f_\ast(\pi_1(T^n, t_0)) \leq m \), then \( f_\ast \) can be factorized through \( \pi_1(T^n, t_0) \cong \mathbb{Z}^n \) so that we can choose \( \tilde{h} \in \text{Hom}(\pi_1(T^n, t_0), \pi_1(T^n, t_0)) \) and \( \tilde{g} \in \text{Hom}(\pi_1(T^n, t_0), \pi_1(X,x_0)) \) such that \( f_\ast = \tilde{g} \circ \tilde{h} \). Because \( F_n \) surjective implies that \( F_m \) is surjective for \( m \leq n \) there exists a \( g : T^m \to X \) with \( g_\ast = \tilde{g} \), We also have an \( h : T^n \to T^m \) with \( h_\ast = \tilde{h} \). Thus \( g_\ast \circ h_\ast = f_\ast \) or \( g \circ h \cong f \), i.e. \( f \) is \( m \)-reducible.

This gives us the following theorems:

**Theorem 10.** Let \( F_n \) be bijective. (This is the case when \( X \) is an \( n \)-space of type \((\pi_1, 1)\)). Then \( X \) is \( n \text{-dimensionally } m \text{-reducible } (1 \leq m < n) \) iff the dimension of any finitely generated abelian subgroup of \( \pi_1(X,x_0) \) is \( \leq m \).
Remark. Instead of "$F_n$ bijective" we could write "$F_n$ injective and $F_{m+1}$ surjective".

Theorem 11. Let $F_n$ be bijective for all $n > m$. (This is the case when $X$ is a space of type $(\pi_1,1)$). Then $X$ is $m$-reducible iff the dimension of any finitely generated abelian subgroup of $\pi_1(X,x_0)$ is $\leq m$.

We now show

Theorem 12. $X$ is $n$-dimensionally 1-reducible $(n > 1)$ iff every finitely generated abelian group of $\pi_1(X,x_0)$ is cyclic and $F_n$ is injective.

Corollary. Let $\pi_1(X,x_0) = 0$. Then $X$ is $n$-dimensionally 1-reducible $(n > 1)$ iff $[T^n,X] = 0$.

Proof of Theorem 12. Because $F_2$ is always surjective, it follows from the remark above that it only remains to show that $F_n$ injective is a necessary condition.

Let $f,g : T^n \to X$ be given such that $f \sim g$ (i.e. $f|(T^n)^1 \approx g|(T^n)^1$). We suppose that $X$ is $n$-dimensionally 1-reducible and then want to show that $f \sim g$. Because $f$ and $g$ are 1-reducible and

$$[T^n,S^1] \approx \text{Hom}(\pi_1(T^n,t_0), \pi_1(S^1,s_0)),$$

$$[S^1,X] \approx \text{Hom}(\pi_1(S^1,s_0), \pi_1(X,x_0))/\sim,$$

we can choose the factorizations $T^n \xrightarrow{h_1} S^1 \xrightarrow{g_1} X$ and $T^n \xrightarrow{h_2} S^1 \xrightarrow{g_2} X$ of $f$ and $g$ such that $g_1 \pi_1(S^1,s_0) = f_\# \pi_1(T^n,t_0)$ and $g_2 \pi_1(S^1,s_0) = g_\# \pi_1(T^n,t_0)$ and $h_{1*} = h_{2*}$. Then $h_1 \approx h_2$ and $g_1 \approx g_2$. Thus

$$f \sim g_1 \circ h_1 \approx g_2 \circ h_2 \approx g.$$

This gives us

Theorem 13. $X$ is 1-reducible iff every finitely generated abelian subgroup of $\pi_1(X,x_0)$ is cyclic and every $F_n$ is injective.

Theorem 14. Let $X$ be a $p$-space of type $(\pi_p,p)$ with $\pi_p$ abelian. Then we have:

1) If $m \leq n < p$, we have $[T^n,X] \approx [T^m,X] \approx 0$ so $X$ is trivially $n$-dimensionally $m$-reducible.

2) If $m < p \leq n$, we have $[T^m,X] = 0$, but $[T_n,X]$ is non trivial if $\pi_p$ is non trivial, so a necessary condition for $X$ to be $n$-dimensionally $m$-reducible is that $\pi_p = 0$. 

2*
3) If \( p \leq m < n \) and \( G_{m+1} : [T^{n+1}, X] \rightarrow \text{Hom}(H_p(T^{m+1}), \pi_p) \) defined by \( G_{m+1}[f] = \xi^{-1}f_{x_0} \) is surjective (which is the case when \( X \) is an \( m \)-space of type \((\pi_p, p)\) or when \( X \) is an \( H \)-space of \( p \)-type \((\pi_p, p)\)), then a necessary condition for \( X \) to be \( n \)-dimensionally \( m \)-reducible is that the dimension of any finitely generated subgroup of \( \pi_p \) is \( \leq \binom{m}{p} = \dim H_p(T^m) \).

The proof of 3) is analogous to the proof of Theorem 8 because of the remark p. 17.

**Corollary.** Let \( X \) be a \( p \)-space of type \((\pi_p, p)\) with \( \pi_p \) abelian. Then we have:

1) For \( m < p \) a necessary condition for \( X \) being \( m \)-reducible is \( \pi_p = 0 \).

2) For \( m \geq p \) and \( G_{m+1} \) surjective a necessary condition for \( X \) being \( m \)-reducible is that the dimension of any finitely generated subgroup of \( \pi_p \) is \( \leq \binom{m}{p} \).

**Examples.**

1) Because the fundamental groups of the complex and quaternionic projective spaces are zero but not all the homotopy groups are zero, these spaces are not \( 1 \)-reducible.

2) \( S^n(n > 1) \) is not \( m \)-reducible for \( m < n \).

3) \( P^n(n > 1) \) is not \( m \)-reducible for \( m < n \):

We have the covering projection \( p : S^n \rightarrow P^n \) and a map \( p_n : T^n \rightarrow S^n \) which is not \( 0 \)-homotopic. \( p_n \) is not \( m \)-reducible, nor is \( p_n \circ (\times N)_n : T^n \rightarrow T^n \rightarrow S^n \) \( m \)-reducible. We now look back at \( p \circ p_n : T^n \rightarrow S^n \rightarrow P^n \). The map \( p \circ p_n \circ (\times N)_n \) is not \( 0 \)-homotopic since \( p \) is a covering projection. If \( p \circ p_n \) were \( m \)-reducible for some \( m < n \), i.e., \( p \circ p_n \cong g \circ h : T^n \xrightarrow{h} T^m \xrightarrow{g} P^n \), then, since \([T^m,P^n] \cong \text{Hom}(\mathbb{Z}^m,\mathbb{Z}_2)\), we would obtain

\[
p \circ p_n \circ (\times 2)_n \cong g \circ h \circ (\times 2)_n = [g \circ (\times 2)_m] \circ h \cong 0 \circ h = 0
\]

(we assume \( h \) linear because \([T^n,T^m] \cong \text{Hom}(\pi_1(T^n,t_0), \pi_1(T^m,t_0))\), contradicting \( p \circ p_n \circ (\times 2)_n \) non \( \cong 0 \). Thus \( p \circ p_n \) is not \( m \)-reducible.

4) From the computation of the cohomotopy groups of \( T^n \) and the results in this chapter one can get further results about the reducibility of spheres; for instance:

If \( 2i \geq n+3, i < n \) and \( \pi_n(S^i,s_0) \neq 0 \) then we get from Theorem 6 that the map \( g_{n,i}^x \circ p_n : T^n \rightarrow S^n \rightarrow S^i \) with \( g_{n,i}^x \) non \( \cong 0 \) is not homotopic to zero. This map \( f \) is algebraically trivial, i.e., its induced homomorphisms between the homology and cohomology groups are trivial. If \( f \) were \( i \)-reducible, then
Theorem 7 would give that $f$ could be projected through a $T_i(T^n = T^i \times T^{n-i}$, perhaps after a shift of coordinates). Thus $f$ would be homotopic to a composed map

$$T^n \xrightarrow{h} T^n \xrightarrow{proj} T^i \xrightarrow{g} S^i,$$

where $g$ non $\simeq 0$, and $h$ is a change of basis. Because $[T^i, S^i] \cong \text{Hom}(H_i(T^i), H_i(S^i)) \cong \text{Hom}(Z, Z)$, $g_{#i} \neq 0$ and so $g \circ \text{proj} \circ h$ would not be algebraically trivial, contradicting $f \cong g \circ \text{proj} \circ h$. Thus if $2n > 2i \geq n+3$ and $\pi_n(S^i, s_0) \neq 0$, then $S^i$ is not $i$-reducible.

5) In an analogous way we get:

$S^2$ is not 2-reducible,

because the map $T^2 \xrightarrow{p} S^2 \xrightarrow{S^2}$, where $p$ is the Hopf fibration, is known to be non $\simeq 0$ when $p\neq 0$.

**Chapter 6**

**Almost Periodic Movements**

**Definition.** A topological space $X$ is called continuously locally arcwise connected when to every compact subspace $K$ of $X$ there exists a neighbourhood $O \subseteq K \times K$ of the diagonal $\Delta_K$ in $K \times K$ and a continuous map $\Phi: O \times I \to X$ so that $\Phi(x,x,t) = x$, $\Phi(x,y,0) = x$, $\Phi(x,y,1) = y$.

For metric spaces this is equivalent to the definition (for metric spaces only) used in [4].

**Remarks.** 1) When $X$ is a metric space for which any two points $x$ and $y$ whose distance remains below a certain number can be connected by a geodetic arc which depends continuously on $x$ and $y$, then $X$ is continuously locally arcwise connected. 2) Any CW-complex is continuously locally arcwise connected. Indeed it satisfies the following:

There exists a covering $(U_j| j \in J)$ of $X$ with open sets and a continuous function $\Phi: O \times I \to X$, where $O = \bigcup U_j \times U_j$, such that $\Phi(x,x,t) = x$, $\Phi(x,y,0) = x$, $\Phi(x,y,1) = y$. This can be shown by induction: Assuming $(U_j)$ chosen and $\Phi$ constructed on the $n-1$ skeleton $X^{n-1}$ of $X$, it can be shown that the definitions can be extended to $X^n$, which is obtained from $X^{n-1}$ by adjoining $n$-cells and which has the topology coherent with $X^{n-1}$ and the $n$-cells.

**Definition.** A continuous movement in a metric space $X$ is a continuous function $x = f(t)$, $t \in \mathbb{R}$, $x \in X$. A number $\tau = \tau_f(\varepsilon)$ is called a translation
number of \( f(t) \) corresponding to \( \varepsilon > 0 \) if the condition \( \operatorname{dist}(f(t), f(t+\tau)) \leq \varepsilon \) is satisfied for all \( t \in \mathbb{R} \). The movement \( x = f(t) \) is called almost periodic if the range of \( f(t) \) lies in a compact subset of \( X \) and the set \( \{ \tau_f(\varepsilon) \} \) is relatively dense for every \( \varepsilon > 0 \) (i.e. there exists an \( l > 0 \) so that every interval of length \( l \) contains at least one of the \( \tau_f(\varepsilon) \)'s).

**Definition.** A function \( x = f(t, v) ; \ t \in \mathbb{R}, \ v \in [0,1] \), \( x \in X \) is called a uniformly continuous family of almost periodic movements when 1) the range of \( f(t, v) \) lies in a compact subset of \( X \), 2) for all \( v_0 \in [0,1] \) \( f(t, v_0) \) is almost periodic, 3) to all \( \varepsilon > 0 \) and all \( v_0 \in [0,1] \) corresponds a neighbourhood \( U_\varepsilon(v_0) \) of \( v_0 \) such that for all \( t \in \mathbb{R}, \ \forall v \in U_\varepsilon(v_0) : \ \operatorname{dist}(f(t,v_0), f(t,v)) \leq \varepsilon \).

**Remark.** Tornehave's definitions are the same except that he does not demand that the ranges lie in compact subsets. Instead he is mainly interested in complete metric spaces and he shows that the closure of the range of an almost periodic movement \( f(t) \) in a complete metric space \( X \) is a compact subspace. In the same way it can be shown that the closure of the range of a uniformly continuous family of a.p. movements in a complete metric space is a compact subspace. This gives us that the definitions coincide for complete metric spaces, and this is all I need. With the new definitions Tornehave's results about complete metric spaces can easily be extended to arbitrary metric spaces. The new definition can be extended further to arbitrary topological spaces because it is possible to introduce one and only one uniformity structure on compact spaces.

Let \( \mathcal{C} \) denote the class of metric spaces which are continuously locally arcwise connected. Then \( \mathcal{C} \) includes the class \( \mathcal{C}' \) of metrizable \( CW \) complexes, which again includes the class \( \mathcal{C}'' \) of locally compact polyhedrons.

**Definition.** Two almost periodic movements \( x_1 = f_1(t) \) and \( x_2 = f_2(t) \) are called almost periodic homotopic iff there exists a uniformly continuous family \( x = f(t, v) \) with \( f(t,0) = f_1(t) \) and \( f(t,1) = f_2(t) \).

This relation is obviously an equivalence relation in the set of a.p. movements in \( X \) and it leads to a subdivision of this set into a.p. homotopy classes.

**Remark.** A continuous function \( \tilde{f} : \mathbb{R}^n \to X \) which is periodic in all the variables with the same period \( r \in \mathbb{R} \) (called a torus map by Tornehave) corresponds to torus maps \( T^n \to X \) (we can look at \( T^n \) as \( \mathbb{R}^n/(qr\mathbb{Z})^n \), \( q \in \mathbb{Z} \)). The function \( g : \mathbb{R}^n \to X : g(t) = \tilde{f}(pt) \), \( p \in \mathbb{R} \) has period \( \frac{r}{p} \) and corresponds to the same torus maps as \( \tilde{f} \) (we can look at \( T^n \) as \( \mathbb{R}^n/(qr\mathbb{Z})^n \), \( q \in \mathbb{Z} \)). Usually
I think of $T^n$ as $\mathbb{R}^n/\mathbb{Z}^n$ and I call the variables $(u_1, \ldots, u_n)$ with $u_t \in [0,1]$. If $(\beta_1, \ldots, \beta_n)$ are real numbers and $t$ is also real, and $f: T^n \to X$ is a map, I write $f(\beta_1t, \ldots, \beta_nt)$ for $f(\{\beta_1t\}, \ldots, \{\beta_nt\})$, where $\{\beta t\}$ is the (one of the) $u \in [0,1]$ for which $\beta t = u \pmod{1}$. It is well known that $x = f(\beta_1t, \ldots, \beta_nt)$ is an almost periodic movement in $X$. In the opposite direction we get from [4], p. 28:

**Theorem.** Every almost periodic movement $x_1 = f(t)$ in $X \in \mathcal{C}$ is a.p. homotopic to a certain $x_3 = g(\beta_1t, \ldots, \beta_nt)$ with $g: T^n \to X$ and $(\beta_1, \ldots, \beta_n)$ rationally independent real numbers (i.e. independent as vectors in a $\mathbb{Q}$-vector space).

The following theorem is [4] lemma 22 with a correction\(^1\).

**Theorem 15.** Let $\tilde{f}(t)$ and $\tilde{g}(t)$ be two almost periodic movements in $X \in \mathcal{C}$. In order to investigate whether $\tilde{f}$ and $\tilde{g}$ are a.p. homotopic we choose a common set of $n$ rationally independent real numbers $(\beta'_1, \ldots, \beta'_p)$ and torus maps $f', g': T^p \to X$ such that $\tilde{f}(t)$ is a.p. homotopic to $f'(\beta'_1t, \ldots, \beta'_pt)$ and $\tilde{g}(t)$ is a.p. homotopic to $g'(\beta'_1t, \ldots, \beta'_pt)$. Then $\tilde{f} \sim_{a.p.} \tilde{g}$ iff there exists a natural number $N$, such that $f' \circ (\times N) \approx g' \circ (\times N)$, where $\times : T^p \to T^p$ is defined by $(\times N)(u_1, \ldots, u_p) = (Nu_1, \ldots, Nu_p)$.

I shall now discuss the notions introduced in definitions 4–7 in the introduction.

**Theorem 16.** Let $X = \tilde{f}(t)$ be an a.p. movement in $X \in \mathcal{C}$. Let $f: T^n \to X$ be one of its corresponding torus maps (i.e. $f(\beta_1t, \ldots, \beta_nt) \sim_{a.p.} \tilde{f}(t)$ and $(\beta_1, \ldots, \beta_n)$ are rationally independent). Then $\tilde{f}$ is a.p. $m$-reducible iff $f \circ (\times N)$ is $m$-reducible for some natural number $N$.

**Proof.** We may assume $m < n$. We shall first prove “only if”. Let $\tilde{f}$ be a.p. $m$-reducible. We choose $g: T^m \to X$ and rationally independent numbers $(\gamma_1, \ldots, \gamma_m)$ such that

$$g(\gamma_1t, \ldots, \gamma_mt) \sim_{a.p.} \tilde{f}(t) \sim_{a.p.} f(\beta_1t, \ldots, \beta_nt).$$

We now look at the vector space $V$ over $\mathbb{Q}$ spanned by $(\beta_1, \ldots, \beta_n, \gamma_1, \ldots, \gamma_m)$. Then $p = \dim V \geq \max\{m, n\}$. We supplement $(\beta_1, \ldots, \beta_n)$ to a basis $(\beta'_1, \ldots, \beta'_n, \beta'_{n+1}, \ldots, \beta'_p)$ with $\beta'_v = \beta_n$, $v = 1, \ldots, n$. Then the $\gamma_i$'s are rational linear combinations of $(\beta'_j)$. Thus there exists a natural number $M$

\(^1\) Tornhave told me about this mistake. He overlooked the possibility of the factor $(\times N)$ such that his condition is too strong. The examples proving the existence of non-trivial a.p. movements are never less correct, because the relevant obstructions belong to infinite cyclic groups.
so that the $M_\gamma$'s are integral linear combinations $A_i$, $i = 1, \ldots, m$ of $(\beta'_j)$. We now define a map $f_1: T^p \to T^n \to X$ by $f_1(u_1, \ldots, u_n, u_{n+1}, \ldots, u_p) = f(Mu_1, \ldots, Mu_n)$ and a map $g_1: T^p \to T^m \to X$ by $g_1(u_1, \ldots, u_p) = g(A_1(u), \ldots, A_m(u))$.

Because $f(\beta_1 t, \ldots, \beta_n t) \sim g(\gamma_1 t, \ldots, \gamma_m t)$ we have

$$f(M\beta_1 t, \ldots, M\beta_n t) = f(\beta'_1 t, \ldots, \beta'_n t) \sim g(M\gamma_1 t, \ldots, M\gamma_m t) = g(\beta'_1 t, \ldots, \beta'_n t)$$

(and $(\beta'_1, \ldots, \beta'_p)$ are rationally independent). Theorem 15 tells us that there exists a natural number $N_1$ such that $f_1 \circ (\times N_1) \cong g_1 \circ (\times N_1)$. Then the restrictions to the $T^n \subseteq T^p$ corresponding to $u_{n+1} = \ldots = u_p = 0$ are homotopic, i.e.

$$f_1(N_1u_1, \ldots, N_1u_n, 0, \ldots, 0) = f(MN_1u_1, \ldots, MN_1u_n) \cong g_1(N_1u_1, \ldots, N_1u_n, 0, \ldots, 0)$$

$$= g(N_1A_1(u_1, \ldots, u_n, 0, \ldots, 0), \ldots, N_mA_m(u_1, \ldots, u_n, 0, \ldots, 0)).$$

Denoting by $h: T^n \to T^m$ the map defined by

$$(u_1, \ldots, u_n) \mapsto (N_1A_1(u_1, \ldots, u_n, 0, \ldots, 0), \ldots, N_mA_m(u_1, \ldots, u_n, 0, \ldots, 0))$$

we have $f \circ (\times N_1 M) \cong g \circ h$, where $h: T^n \to T^m$ and $g: T^m \to X$, so that $f \circ (\times N_1 M)$ is $m$-reducible.

Next, we shall prove "if". Suppose $f' = f \circ (\times N)$ is $m$-reducible. We have $f'(\beta'_1 t, \ldots, \beta'_n t) \sim f'(t)$, where $(\beta'_1 = \beta_1/N, \ldots, \beta'_n = \beta_n/N)$ are rationally independent. Because $[T^n, T^m] \cong \text{Hom}(\pi_1(T^n, t_0), \pi_1(T^m, t_0))$ we can choose $h: T^n \to T^m$ linear (corresponding to $A_{m,n}$) and $g: T^m \to X$ so that $f' \cong g \circ h$. If we put

$$\begin{pmatrix} \beta'_1 \\ \vdots \\ \beta'_m \end{pmatrix} = A_{m,n} \begin{pmatrix} \beta'_1 \\ \vdots \\ \beta'_n \end{pmatrix},$$

then

$$f'(t) \sim f'(\beta'_1 t, \ldots, \beta'_n t) \sim g \circ h(\beta'_1 t, \ldots, \beta'_n t) = g(\beta_1 t, \ldots, \beta_m t),$$

so that $f'$ is a.p. $m$-reducible.

Remarks. 1) $(f \circ (\times N))_* = Nf_*$, so rank $f_*\pi_1(T^n, t_0) = r$ iff there exists a natural number $N$ such that $(f \circ (\times N))_*\pi_1(T^n, t_0) \cong \mathbb{Z}^r$.

2) $(f \circ (\times N))_\# = N^p f_\#p$, so rank $f_\#p H_p(T^n) = r$ iff there exists a natural number $N$ such that $(f \circ (\times N))_\# \cong \mathbb{Z}^r$. 
Statement 2) follows immediately from the expression of elements of $H_p(T^n)$ as cross products of elements from $H_1(S^1)$. The remarks and Theorem 16 gives us that the theorems from chapter V can be changed to theorems about almost periodic movements in $X \in \mathcal{C}$, by translating “dim” to “rank”. Let $\mathcal{C}_0$ denote the class of path connected spaces in $\mathcal{C}$.

**Remark.** Let $x = \tilde{f}(t)$ be an a.p.-movement in $X \in \mathcal{C}_0$ corresponding (for some $(\beta_1, \ldots, \beta_n)$) to $f: T^n \to X$. If $\tilde{f}(t)$ is a.p. $m$-reducible, then rank $f_*\pi_1(T^n,t_0) \leq m$ and rank $f_{#p} H_p(T^n) \leq \binom{m}{p}$ for every $p$.

**Theorem 17.** Suppose that $F_{m+1}: [T^{m+1}, X] \to \text{Hom}(\pi_1(T^{m+1}, t_0), \pi_1(X, x_0))$ is surjective and that $X \in \mathcal{C}_0$ (e.g. $X$ is an $m$-space of type $(\pi_1, 1)$ or an $H$-space). If, further, $X$ is a.p. $m$-reducible, then the rank of any abelian subgroup of $\pi_1(X, x_0)$ is $\leq m$.

**Theorem 18.** Suppose that $F_n: [T^n, X] \to \text{Hom}(\pi_1(T^n, t_0), \pi_1(X, x_0))$ is bijective and that $X \in \mathcal{C}_0$ (e.g. $X$ is an $n$-space of type $(\pi_1, 1)$). Then an almost periodic movement corresponding to $f: T^n \to X$ is a.p. $m$-reducible if and only if the rank of any abelian subgroup of $\pi_1(X, x_0)$ is $\leq m$.

**Theorem 19.** Let $F_n: [T^n, X] \to \text{Hom}(\pi_1(T^n, t_0), \pi_1(X, x_0))$ be bijective for every $n > m$ (e.g. $X$ is a space of type $(\pi_1, 1)$), and let $X \in \mathcal{C}_0$. Then $X$ is a.p. $m$-reducible if and only if the rank of every abelian subgroup of $\pi_1(X, x_0)$ is $\leq m$.

**Theorem 20.** A space $X \in \mathcal{C}_0$ is a.p. $1$-reducible (i.e. every a.p. movement in $X$ is a.p. homotopic to a periodic movement) if and only if the rank of every abelian subgroup of $\pi_1(X, x_0)$ is $\leq 1$ and for every natural number $n$ every pair of maps $f, g: T^n \to X$ satisfying $F_n[f] = F_n[g]$ also satisfies the condition $f \circ (\times N) \cong g \circ (\times N)$ for some natural number $N$.

**Theorem 21, 22.** Let $X \in \mathcal{C}_0$ be a $p$-space of type $(\pi_p, p)$ with $\pi_p$ abelian. Then

1) if $X$ is a.p. $m$-reducible for some $m < p$, then rank $\pi_p = 0$.

2) if $p \leq m$ and $G_{m+1}: [T^{m+1}, X] \to \text{Hom}(H_p(T^{m+1}), \pi_p)$ defined by $G_{m+1}[f] = \pi^{-1}f_{#p}$ is surjective (e.g. $X$ is an $m$-space of type $(\pi_p, 1)$) and $X$ is an $H$-space of $p$-type $(\pi_p, p)$) and $X$ is a.p. $m$-reducible, then rank $\pi_p \leq \binom{m}{p}$.

**Examples.** 1) Because the fundamental groups of the complex and quaternionic projective spaces are zero and the first nontrivial homotopy groups are isomorphic to $\mathbb{Z}$, these spaces are not a.p. 1-reducible.
2) $S^n (n > 1)$ is not a.p. $m$-reducible for $m < n \ (\pi_m (S^n, s_0) \cong \mathbb{Z}).$

3) $P^n (n > 1)$ is not a.p. $m$-reducible for $m < n$ (proof analogous to the proof p. 20).

4) $S^2$ is not a.p. 2-reducible ($[T^3, S^3] \cong \mathbb{Z}$).

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**Bibliography**


