I. BREVIK

ELECTROMAGNETIC ENERGY-MOMENTUM TENSOR WITHIN MATERIAL MEDIA

2. DISCUSSION OF VARIOUS TENSOR FORMS

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Synopsis

This paper represents the second part of a study of the electromagnetic energy-momentum tensor within a material medium. Similarly as in the first part, essentially a macroscopical point of view is adopted, and emphasis is laid upon the comparison with experiments, both in the case of static fields and in the case of time-varying fields within bodies at rest and in relativistic motion. For the main part the relative behaviour of Minkowski's and Abraham's tensors is studied, but some attention is also given to the tensors introduced by Einstein and Laub, de Groot and Suttorp, Beck and Marx et al. Deductive procedures are employed, characteristic effects are studied, both within media at rest and in motion, and some attention is given to a critical analysis of earlier treatments. Our main conclusion is that Minkowski's and Abraham's tensors are equivalent in the usual physical cases, while the remaining tensor expressions seem to run into conflict with experimental evidence.

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BIANCO LUNOS BOGTRYKKERI A/S

1. Introduction and Summary

In a previous paper[1]—hereafter referred to as I—we discussed the application of MINKOWSKI's energy-momentum tensor in phenomenological electrodynamics. The medium was assumed to be homogeneous, transparent and usually also nondispersive. Since the essential differences between the various competing tensor forms are present also in the most simple media, the above restrictive assumptions were legitimate in relation to the main purpose of the investigation, namely to examine whether MINKOWSKI's tensor is appropriate to use in the most common and simple situations. And the affirmative answer to this question made it just convenient to restrict the treatment so as to incorporate MINKOWSKI's tensor only.

In the present paper we shall consider also other tensor forms, so let us first write down some expressions. The rest inertial frame of the medium shall be denoted by K^0, while the inertial frame in which K^0 moves with the uniform velocity $\boldsymbol{v}$, shall be denoted by K. MINKOWSKI's tensor reads

$$S_{ik}^{M} = -E_i D_k - H_i B_k + \tfrac{1}{2}\delta_{ik}(\boldsymbol{E}\cdot\boldsymbol{D} + \boldsymbol{H}\cdot\boldsymbol{B}) \tag{1.1a}$$

$$S_{4k}^{M} = i(\boldsymbol{E}\times\boldsymbol{H})_k, \quad S_{k4}^{M} = i(\boldsymbol{D}\times\boldsymbol{B})_k, \quad S_{44}^{M} = -\tfrac{1}{2}(\boldsymbol{E}\cdot\boldsymbol{D} + \boldsymbol{H}\cdot\boldsymbol{B}), \tag{1.1b}$$

or, in covariant form,

$$S_{\mu\nu}^{M} = F_{\mu\alpha}H_{\nu\alpha} - \tfrac{1}{4}\delta_{\mu\nu}F_{\alpha\beta}H_{\alpha\beta} \tag{1.2}$$

(for notation, see I).

Perhaps the main reason why MINKOWSKI's tensor often has been rejected and instead replaced by some other tensor form is the asymmetry of the former, which is present even within isotropic media. The symmetry requirement is met by the following tensor, which we shall call ABRAHAM's tensor,

$$S_{ik}^{A0} = -\tfrac{1}{2}(E_i^0 D_k^0 + E_k^0 D_i^0) - \tfrac{1}{2}(H_i^0 B_k^0 + H_k^0 B_i^0) + \tfrac{1}{2}\delta_{ik}(\boldsymbol{E}^0\cdot\boldsymbol{D}^0 + \boldsymbol{H}^0\cdot\boldsymbol{B}^0) \tag{1.3a}$$

$$S_{k4}^{A0} = S_{4k}^{A0} = i(\boldsymbol{E}^0\times\boldsymbol{H}^0)_k, \quad S_{44}^{A0} = -\tfrac{1}{2}(\boldsymbol{E}^0\cdot\boldsymbol{D}^0 + \boldsymbol{H}^0\cdot\boldsymbol{B}^0 \tag{1.3b}$$

(here given in K^0), although this symmetrized form of the stress tensor S_{ik}^{A0} for anisotropic media seems to have been given first by H. HERTZ[(2)]. When the body is isotropic, the force density in K reads

$$\boldsymbol{f}^{A0} = \boldsymbol{f}^{M0} + \frac{n^2-1}{c^2}\frac{\partial \boldsymbol{S}^{M0}}{\partial t^0}, \quad f_4^{A0} = f_4^{M0}, \tag{1.4}$$

where n is the refractive index. We shall often be concerned with this tensor in the following chapters. Its covariant form can be written as

$$S_{\mu\nu}^A = S_{\mu\nu}^M + \frac{\varkappa}{\mu}\left(F_{\mu\alpha}F_\alpha - \frac{1}{c^2}F_\alpha F_\alpha V_\mu\right)V_\nu, \tag{1.5}$$

where $\varkappa = (\varepsilon\mu - 1)/c^2 = (n^2-1)/c^2$, $F_\alpha = F_{\alpha\nu}V_\nu$ and $V_\mu = \gamma(\boldsymbol{v}, ic)$.

Another proposal was put forward by G. MARX and collaborators[(3)]. They examined a simple radiation field travelling through an isotropic medium, and came to the conclusion that ABRAHAM's tensor, describing the electromagnetic field, must be supplemented with a mechanical tensor to give the symmetrical "radiation" tensor $S_{\mu\nu}^S$ describing the total system: radiation plus connected mechanical field. In K^0 the radiation tensor is given by

$$S_{ik}^{S0} = \frac{1}{n^2}S_{ik}^{A0} = \frac{1}{n^2}S_{ik}^{M0}, \quad S_{4\nu}^{S0} = S_{\nu 4}^{S0} = S_{4\nu}^{A0}, \tag{1.6}$$

for all ν between 1 and 4.* The covariant expression can be written

$$S_{\mu\nu}^S = \frac{1}{n^2}S_{\mu\nu}^M + \frac{\varkappa}{\mu n^2}[V_\mu F_{\nu\alpha}F_\alpha + \tfrac{1}{2}V_\mu V_\nu(\varkappa F_\alpha F_\alpha + \tfrac{1}{2}F_{\alpha\beta}F_{\alpha\beta})]. \tag{1.7}$$

A. EINSTEIN and J. LAUB[(5)] have also examined the problem; by means of simple examples they constructed an expression for the force density in K^0 which corresponds to the following components of the energy-momentum tensor

$$S_{ik}^{E0} = -E_i^0 D_k^0 - H_i^0 B_k^0 + \tfrac{1}{2}\delta_{ik}(\boldsymbol{E}^{02} + \boldsymbol{H}^{02}) \tag{1.8a}$$

$$S_{4k}^{E0} = S_{k4}^{E0} = i(\boldsymbol{E}^0 \times \boldsymbol{H}^0)_k. \tag{1.8b}$$

The energy density component was not given.

The last tensors we shall mention here are due to S. R. DE GROOT and L. G. SUTTORP[(6)]. These authors have examined the problem from a purely

* F. BECK[(4)] has also introduced a tensor which, however, in the case of a radiation field coincides with MARX's radiation tensor. Therefore we shall not pay any special attention to this form in the following sections.

microscopical point of view, and published recently a series of papers on the subject. (See also I, section 7.) They give two tensor expressions, dependent on whether the total interaction between field and matter is taken into account or not. In the case of an isotropic medium their first proposal reads in K^0

$$S_{ik}^{G0} = -E_i^0 D_k^0 - H_i^0 B_k^0 + \delta_{ik}(\tfrac{1}{2}\boldsymbol{E}^{02} + \tfrac{1}{2}\boldsymbol{B}^{02} - \boldsymbol{M}^0 \cdot \boldsymbol{B}^0) \tag{1.9a}$$

$$S_{4k}^{G0} = S_{k4}^{G0} = i(\boldsymbol{E}^0 \times \boldsymbol{H}^0)_k, \quad S_{44}^{G0} = -\tfrac{1}{2}(\boldsymbol{E}^{02} + \boldsymbol{B}^{02}), \tag{1.9b}$$

where $\boldsymbol{M}^0 = \boldsymbol{B}^0 - \boldsymbol{H}^0$. It is apparent that for $\boldsymbol{M}^0 = 0$, the components (1.8) of the EINSTEIN-LAUB tensor agree with the corresponding components of the DE GROOT-SUTTORP tensor (1.9).

The second tensor expression proposed by DE GROOT and SUTTORP was defined as the difference between the total energy-momentum tensors with and without external electromagnetic fields. This tensor thus corresponds to taking the whole interaction between field and matter into account. By omitting the variations of the material constants with density and temperature, as we mainly do throughout our work, we find that their second field tensor agrees with ABRAHAM's tensor within an isotropic body.

There exist also other proposals that have been put forward, and we shall have the opportunity to comment upon some of them in the detailed considerations later on. Mostly we shall be concerned with the relative merits of ABRAHAM's and MINKOWSKI's tensors, since these tensors, combined with their appropriate interpretations, are found to be both adequate and equivalent in most of the simple physical situations considered.

Further introductions to the subject are given in the books by C. MØLLER(7) and W. PAULI(8), and in the review article by G. MARX(9).

The main task of the subsequent exposition can be conveniently divided into three parts. Firstly, we want to apply some deductive methods in order to see how the various tensors adapt themselves to the formalism. As indicated already in I it must be borne in mind that the power of this kind of method is restricted in the sense that the expressions one obtains are not unique. Secondly, we wish to examine the applicability of the various tensor forms to the description of definite phenomena. The description of the experiments is here a crucial point. Thirdly, we shall spend some effort to comment upon parts of the earlier literature. There has been published a large number of papers on the subject, which are often mutually contradictory and moreover scattered over a number of different journals. We find it therefore of importance to point out some crucial points in the various derivations as an attempt to find the deeper reason why the results are seemingly incompatible.

Throughout this work we take a phenomenological point of view and refer only occasionally to the simple microscopical treatment in I. This is done for practical reasons, a thorough scrutiny of the microscopical aspects would require a separate treatment. However, we think there is also a reason of principle why it is sensible first to choose the macroscopical line of approach in order to obtain a satisfactory description of the physical phenomena: In the simple cases considered, the results obtained by means of these macroscopic or semi-macroscopic methods are both consistent and moreover fit the observed data in an excellent way. From a pragmatic point of view the macroscopical kind of method is therefore not only a possible kind of approach but in fact the *appropriate* one as a first step, and microscopical methods with their complicated formalism should properly be considered to represent a later stage of the development.

Let us now review the subsequent sections. Section 2 is devoted to an analysis of electrostatic fields. We considor again the variational method which was employed in section 3 of I, and show how MINKOWSKI's and ABRAHAM's tensors emerge from the formalism in an equivalent way. It is found that, as far as a dielectric body is surrounded by a vacuum or an isotropic liquid, no experiment testing electromagnetic forces or torques on the body can decide between these tensors. The two tensors correspond merely to different distributions of forces and torques throughout the body: According to MINKOWSKI the torque is essentially a *volume* effect, described by the tensor asymmetry, while according to ABRAHAM the torque is described completely in terms of the force density. We consider a typical example, in which ABRAHAM's torque naturally comes out as a *surface* effect.

In the remainder of section 2 we discuss to some extent the EINSTEIN-LAUB (or the DE GROOT-SUTTORP) tensor. It is found that also in this case no force or torque experiments on a body surrounded by a vacuum or an isotropic fluid represent a critical test for the tensors in question. However, there is actually one effect which represents a critical test, namely the pressure increase in a dielectric liquid because of the field. In order to apply the theory to this case it is necessary to extend the variational method mentioned earlier (the HELMHOLTZ method) so as to include also the electrostriction effect, although we are otherwise ignoring this effect in our work. S. S. HAKIM and J. B. HIGHAM have tested the pressure increase experimentally, and they found that the HELMHOLTZ force describes the observed data very well. On the contrary, the pressure increase predicted by the EINSTEIN-LAUB force (which is also called the KELVIN force) was found to be in disagreement with the experiment.

In section 3 we continue the consideration from I, section 6 concerning the propagation of an electromagnetic wave within an isotropic body at rest. By means of the semi-macroscopic method that we are adopting, and by taking the radiation pressure experiment due to R. V. JONES and J. C. S. RICHARDS into account, we find that ABRAHAM's and MINKOWSKI's tensors are equivalent in the following sense: ABRAHAM's force density excites the constituent dipoles of the material and produces a mechanical momentum which travels together with the field. If we count this mechanical momentum together with ABRAHAM's momentum as a field momentum, we obtain MINKOWSKI's tensor. By considering the situation in the frame where the mean motion of the constituent particles vanishes we find that, in the case of an infinite medium, the energy-momentum tensor of the total system can be written as the sum of ABRAHAM's tensor and the mechanical tensor in the absence of fields.

We continue section 3 by discussing an example in which the boundary between two media is involved. Finally we consider alternative tensor forms, and find that the radiation pressure predicted by the radiation tensor is in disagreement with the JONES-RICHARDS experiment.

In section 4 we discuss possibilities for torque experiments, especially when MINKOWSKI's or ABRAHAM's tensors are taken as field tensors. For a stationary optical wave in interaction with a dielectric body we find that the two tensors will always yield the same value for the torque. Thereafter we propose an experiment involving a low-frequency combination of electric and magnetic fields. This experiment should be appropriate for the detection of ABRAHAM's force, which is hidden in the case of optical fields. Finally it is concluded that the case of an optical field travelling through a dielectric body immersed in a dielectric liquid should represent a possible means for a further experimental check of the radiation tensor and the EINSTEIN-LAUB tensor.

Section 5 is devoted to a critical review of some parts of the earlier literature, especially those parts which seem to run into conflict with our own interpretations. We are otherwise commenting upon passages from earlier treatments also in our ordinary exposition of various topics, but there remain interesting arguments which cannot so naturally be dealt with in the ordinary treatment. We think such a critical analysis is desirable in a study of the present problem, since an important part of the task is just to clear up a situation which is confused by mutually contradictory opinions.

For the main part we discuss gedanken experiments which have been put forward to support either MINKOWSKI's or ABRAHAM's tensor, and show how these situations are to be explained with the use of the formerly rejected alternative. In the remaining part of the section we mainly discuss some aspects of the EINSTEIN-LAUB paper.

In the subsequent sections we discuss topics connected with relativity, and, except for the last section, limit the consideration to the case of isotropic media. Section 6 is devoted to a study of the torque acting on a moving body when an electromagnetic wave is travelling within it. We first calculate ABRAHAM's and MINKOWSKI's torque expressions when the body is assumed infinitely extended, and show thereafter that both these expressions are relativistically consistent. In this context we draw into consideration an analogous situation encountered in relativistic mechanics: An elastic body subjected to stresses in its rest system may in other inertial systems require a torque in order to maintain steady motion. A similar situation is found to be present also here in electrodynamics: We require stady motion of matter plus field and find that there must then exist a rate of change of electromagnetic momentum which is just equal to the previously calculated torque, with the opposite sign.

If the body is *finite*, we find that the most natural division of the total angular momentum into a field part and a mechanical part is obtained with the use of ABRAHAM's tensor for the field.

Section 7 contains a discussion of various relativistic phenomena. We begin by considering the velocity $\boldsymbol{u} = \boldsymbol{S}/W$ of the energy in an optical wave. In section 9 of I we found that $\boldsymbol{u}$ transforms like a particle velocity if MINKOWSKI's tensor is used. We now find that ABRAHAM's tensor cannot fulfil the transformation criterion due to the fact that this tensor does not describe the total travelling wave. We analyse the background for the transformation criterion, and give a rather general form of a tensor that fulfils it. The radiation tensor falls within this category.

Next we consider the relativistic centre of mass of a finite, but practically monochromatic, field. In section 12 of I we found that the various centres obtained with the use of MINKOWSKI's tensor in general do not coincide when considered simultaneously in one frame. Actually, by considering in the rest frame K^0 the centres of mass obtained by varying the direction and magnitude of the medium velocity, we found that they are located on a circular disk lying perpendicular to the inner angular momentum vector in K^0 with centre at the centre of mass in K^0. Now the various centres of mass are found to behave in exactly the same way if the ABRAHAM tensor or the radiation tensor is adopted.

The ČERENKOV effect is thereafter briefly analysed in the inertial frame in which the emitting particle is at rest. From a study of the momentum balance in this situation, I. TAMM has given preference to MINKOWSKI's tensor. We show how the momentum balance appears with the use of ABRAHAM's tensor. Section 7 is closed by some further remarks upon the literature.

In the last section we employ a variational method which implies the application of curvilinear coordinates as a formal remedy. For a closed system this method in general leads to a determination of the energy-momentum tensor, but the method is shown to leave a certain ambiguity here due to the fact that the LAGRANGIAN leading to the electromagnetic field equations corresponds to a non-closed physical system. Section 8 is rather detailed, since this subject has caused some confusion.

Finally we consider again the Sagnac-type experiment due to C. V. HEER, J. A. LITTLE and J. R. BUPP, which was discussed in section 9 of I. We find that this experiment, although it gives an excellent verification of the predictions of macroscopic electrodynamics, does *not* represent a critical test for MINKOWSKI's tensor, such as it was originally claimed. In fact, the experiment is found to be explained equivalently also by ABRAHAM's tensor and the radiation tensor.

The Appendix gives in tabular form a summary of the behaviour of the various examined energy-momentum tensors in some physical situations.

2. Static Fields

We begin with an examination of the various tensors applied to the simplest physical case, namely the static fields. Actually, only electrostatic fields shall be considered since, for the simple case with linear inductive magnetization here considered, the corresponding results in the magnetostatic case can be taken over by analogy. In this section we first consider the important point concerning the relative behaviour of MINKOWSKI's and ABRAHAM's tensors, and show how they in general lead to equivalent experimental results. Thereafter we consider various other tensor possibilities. Since all quantities are taken in the rest frame, the superscript zero on them shall simply be omitted.

Minkowski's versus Abraham's tensor

From (1.1a) and (1.3a) it is apparent that MINKOWSKI's and ABRAHAM's tensors are equal in the electrostatic case for isotropic media. We therefore generalize the situation and consider the same physical system as in I, section 3, namely a dielectric, anisotropic medium containing an electric field which is produced by some external devices. The linear relation $E_i = \eta_{ik} D_k$ is assumed to be valid. By varying the free energy

$$\mathscr{F} = \tfrac{1}{2}\int \boldsymbol{E}\cdot\boldsymbol{D}\,dV \tag{2.1}$$

and equating $-d\mathscr{F}/dt$ to the rate of mechanical work $\int \boldsymbol{f}\cdot\boldsymbol{u}\,dV$ exerted by the volume *forces*, we found in I $\boldsymbol{f} = \boldsymbol{f}^A$, where

$$\boldsymbol{f}^A = \varrho\boldsymbol{E} + \tfrac{1}{2} D_i D_k \nabla\eta_{ik} - \tfrac{1}{2}\partial_k(\boldsymbol{E}D_k - E_k\boldsymbol{D}). \tag{2.2}$$

This corresponds to the stress tensor

$$S_{ik}^A = -\tfrac{1}{2}(E_i D_k + E_k D_i) + \tfrac{1}{2}\delta_{ik}\boldsymbol{E}\cdot\boldsymbol{D}. \tag{2.3}$$

By comparison with (1.3a) it is thus evident that we have obtained ABRAHAM's tensor. However, by invoking the "dipole model" and assuming the existence of a *torque* density $\tau = \boldsymbol{D}\times\boldsymbol{E}$ with a corresponding extra contribution $\int \tau\cdot(d\varphi/dt)dV$ to the rate of mechanical work (φ being the rotational angle), we found instead MINKOWSKI's result

$$\boldsymbol{f}^M = \varrho\boldsymbol{E} + \tfrac{1}{2} D_i D_k \nabla\eta_{ik} \tag{2.4}$$

$$S_{ik}^M = -E_i D_k + \tfrac{1}{2}\delta_{ik}\boldsymbol{E}\cdot\boldsymbol{D}. \tag{2.5}$$

According to this description, the result is dependent explicity on the assumption of an extra torque density.

In order to make a more distinct comparison between the two tensor forms, it is convenient to reformulate the balance equation in terms of the rotational angle φ rather than the velocity $\boldsymbol{u} = d\boldsymbol{s}/dt$. Since $\boldsymbol{f}\cdot\boldsymbol{s} = (\boldsymbol{r}\times\boldsymbol{f})\cdot\varphi$ we have from (I,3.10,9)

$$\left.\begin{aligned}\int[\boldsymbol{r}\times(\varrho\boldsymbol{E} + \tfrac{1}{2}D_i D_k\nabla\eta_{ik})]\cdot\varphi dV + \int(\boldsymbol{D}\times\boldsymbol{E})\cdot\varphi dV \\ = \int(\boldsymbol{r}\times\boldsymbol{f})\cdot\varphi dV + \int\tau\cdot\varphi dV,\end{aligned}\right\} \tag{2.6}$$

where $\boldsymbol{f}$ and τ are as yet unspecified. As φ is arbitrary, we obtain

$$\boldsymbol{r}\times\boldsymbol{f} + \tau = \boldsymbol{r}\times(\varrho\boldsymbol{E} + \tfrac{1}{2}D_i D_k\nabla\eta_{ik}) + \boldsymbol{D}\times\boldsymbol{E}. \tag{2.7}$$

This relation is fulfilled directly with MINKOWSKI's tensor, and only then. However, let us add the vanishing quantity

$$-\tfrac{1}{2}\int\limits_{\text{cond}} (E_i \boldsymbol{D}\cdot\boldsymbol{n} - \boldsymbol{E}\cdot\boldsymbol{n}D_i)s_i\,dS, \tag{2.8}$$

taken over the external conductors that produce the field, and let us combine

$$\left.\begin{aligned}&\int(\boldsymbol{D}\times\boldsymbol{E})\cdot\boldsymbol{\varphi}dV - \tfrac{1}{2}\int\limits_{\text{cond}}(E_i\boldsymbol{D}\cdot\boldsymbol{n} - \boldsymbol{E}\cdot\boldsymbol{n}D_i)s_i dS\\ &= -\tfrac{1}{2}\int\partial_k(\boldsymbol{E}D_k - E_k\boldsymbol{D})\cdot\boldsymbol{s}dV = -\tfrac{1}{2}\int[\boldsymbol{r}\times\partial_k(\boldsymbol{E}D_k - E_k\boldsymbol{D})]\cdot\boldsymbol{\varphi}dV.\end{aligned}\right\} \tag{2.9}$$

Then (2.6) is equivalent to

$$\left.\begin{aligned}&\int[\boldsymbol{r}\times(\varrho\boldsymbol{E} + \tfrac{1}{2}D_iD_k\nabla\eta_{ik} - \tfrac{1}{2}\partial_k(\boldsymbol{E}D_k - E_k\boldsymbol{D}))]\cdot\boldsymbol{\varphi}dV\\ &= \int(\boldsymbol{r}\times\boldsymbol{f})\cdot\boldsymbol{\varphi}dV + \int\boldsymbol{\tau}\cdot\boldsymbol{\varphi}\,dV,\end{aligned}\right\} \tag{2.10}$$

and we obtain now $\boldsymbol{f} = \boldsymbol{f}^A$, $\tau = 0$, i.e. ABRAHAM's tensor. In this case the torque is described in terms of the force density, while in the former case it was described by the asymmetry of the stress tensor. We must conclude that, as far as the dielectric body is surrounded by an isotropic medium (here vacuum), no unambigeous answer can be given for electrostatic systems. And this result is connected with the fact that the total body torque is the same for both tensors in this case: We may put the torque formula into the form

$$N_l = \int(x_if_k - x_kf_i + S_{ik} - S_{ki})\,dV = -\int\limits_{\text{surface}}(\boldsymbol{r}\times\boldsymbol{S}_n^{\text{vac}})_l\,dS, \tag{2.11}$$

where $S_{ni} = S_{ik}n_k$. Thus the total torque can be evaluated from the vacuum values of the field, and MINKOWSKI's and ABRAHAM's tensors must yield the same result. Similarly, the total body force can also be put into a form which involves the vacuum field values only; by starting from the balance equation for total momentum we obtain readily for the total body force

$$F_i = -\int\limits_{\text{surface}} S_{ni}^{\text{vac}}\,dS, \tag{2.12}$$

in accordance with (2.11).

It should be emphasized that in order to obtain MINKOWSKI's tensor in the first procedure above, we had to take into account the existence of extra body torques with the density $\boldsymbol{D}\times\boldsymbol{E}$. In the second procedure, however, the equivalence between MINKOWSKI's and ABRAHAM's tensors was demonstrated simply by adding the vanishing term (2.8) in the energy balance. The additional assumption concerning the torque $\boldsymbol{D} \times \boldsymbol{E}$ will thus lead to an equivalent description with respect to observable effects for the whole dielectric body, only the distribution of torques and forces within the body will in general be different.

It is clear that the above reasoning will not be changed if we assume that an isotropic, dielectric liquid fills the space between the body and the conductors, since MINKOWSKI's and ABRAHAM's tensors are equal in such a liquid.

The arguments hitherto have dealt with the dielectric system considered as a whole. If several insulators are present between the conductors, then the torque acting on an individual insulator is still independent of which tensor we use. That follows immediately from the fact that we obtain expressions like the last term in eq. (2.11) for each insulator in question.

An example

For the sake of illustration, let us consider again the same physical situation as in I, section 3: A dielectric sphere is located in a homogeneous electrostatic field such that the principal axes of the sphere coincide with the coordinate axes. The external field is given as $\boldsymbol{E}^0 = (E_1^0, E_2^0, E_3^0)$. With the use of MINKOWSKI's tensor, we obtained in I for the single nonvanishing component of the torque

$$N_3^M = \int_{\text{body}} (S_{12}^M - S_{21}^M)\, dV = \int_{\text{body}} (\boldsymbol{D} \times \boldsymbol{E})_3\, dV = (\boldsymbol{p} \times \boldsymbol{E}^0)_3, \tag{2.13}$$

where $\boldsymbol{p} = 3V[(\varepsilon_1 - 1)E_1^0/(\varepsilon_1 + 2), (\varepsilon_2 - 1)E_2^0/(\varepsilon_2 + 2), 0]$, V being the volume of the sphere. According to (2.13), it is natural to interpret the effect as a volume effect.

Let us now insert ABRAHAM's tensor into the torque formula (2.11) so as to obtain

$$\left.\begin{aligned} N_3^A &= \int_{\text{body}} (\boldsymbol{r} \times \boldsymbol{f}^A)_3\, dV + \int_{\text{surface}} [\boldsymbol{r} \times (\boldsymbol{S}_n^A - \boldsymbol{S}_n^{\text{vac}})]_3\, dS \\ &= -\int_{\text{surface}} (\boldsymbol{r} \times \boldsymbol{S}_n^{\text{vac}})_3\, dS = (\boldsymbol{p} \times \boldsymbol{E}^0)_3. \end{aligned}\right\} \tag{2.14}$$

The expressions (2.14) and (2.13) are equal, as they should be. But for ABRAHAM's tensor the volume effect vanishes, as is apparent also from the fact that $\boldsymbol{f}^A = 0$ in the homogeneous field in the body. In this case it is natural to interpret the effect as arising from the volume forces in the boundary layer.

Other tensor forms

Let us now examine the various other tensor proposals mentioned in section 1. The radiation tensor due to MARX *et al* is defined for radiation fields within isotropic media only, and shall not be considered here. But there remains the EINSTEIN-LAUB tensor (1.8a) and the DE GROOT-SUTTORP tensor (1.9a), which actually are seen to be equal in the electrostatic case. The force density is

$$\boldsymbol{f}^E = \varrho \boldsymbol{E} + (\boldsymbol{P} \cdot \nabla) \boldsymbol{E}, \tag{2.15}$$

which is different from both (2.4) and (2.2). This force is also called the KELVIN force. The difference is expected to be connected with the fact that the force densities (2.2) and (2.4) were obtained from a variational principle based on the free energy in the form (2.1), which includes the interaction energy between field and matter. And this energy is not directly compatible with the energy $_2\int E^2 dV$ following from (1.9b).

As regards the possibility for an experimental check of the force (2.15) we have first to point out that, as far as the dielectric body is surrounded by a *vacuum*, the total body force and torque obtained from S^E_{ik} must both be equal to those obtained from the two tensors considered earlier. That this is so follows immediately from (2.11) and (2.12); the effects can be calculated directly from the vacuum tensor. We therefore next have to consider the situation where the body is surrounded by an isotropic *liquid*. There exist certainly electrostatic effects for which the influence of a dielectric liquid is essential; we may think of the rising of a liquid between two charged condenser plates partly dipped into the liquid(10), or the force acting on a grounded metal sphere immersed in a liquid and surrounded by an inhomogeneous field.

However, none of these experiments represent critical tests for the validity of either MINKOWSKI's or EINSTEIN's force. This can be seen in a simple way by first noting that the force difference is a gradient term:

$$\boldsymbol{f}^E = \tfrac{1}{2}\nabla(\boldsymbol{E}\cdot\boldsymbol{P}) + \varrho \boldsymbol{E} + \tfrac{1}{2} D_i D_k \nabla \eta_{ik} = \tfrac{1}{2}\nabla(\boldsymbol{E}\cdot\boldsymbol{P}) + \boldsymbol{f}^M. \tag{2.16}$$

Compared to MINKOWSKI's tensor, EINSTEIN's tensor thus gives rise to an extra isotropic pressure

$$p^E - p^M = \tfrac{1}{2} \boldsymbol{E}\cdot\boldsymbol{P}. \tag{2.17}$$

In accordance with (2.11) and (2.12) the total force and the total torque on the solid body are determined by the values of $\boldsymbol{S}_n$ in the liquid just outside the body. We have

$$S_n^E = S_n^M - \tfrac{1}{2}\boldsymbol{n}(\boldsymbol{E}\cdot\boldsymbol{P}), \tag{2.18}$$

but the effect from the last term in (2.18) (acting outwards) is just balanced by the extra pressure (2.17) which the liquid exerts on the solid. Hence MINKOWSKI's and EINSTEIN's tensors give the same values for the body force and torque. This compensation effect is the direct reason why a measurement of the total force on a metal sphere in the liquid represents no critical test: With EINSTEIN's tensor there are additional forces in the boundary layer of the sphere which just counterbalance the additional forces in the liquid tending to press the liquid into regions of higher field.*) If we suppose that the system producing the inhomogeneous electric field (for instance a small, charged metal sphere) is maintained at constant charge when it is surrounded by the dielectric liquid, we find that the total force $\boldsymbol{F}^M = \boldsymbol{F}^E$ on the test sphere will drop in the ratio $1/\varepsilon$ in comparison with the total force in the absence of the liquid, $\boldsymbol{F}^M = (1/\varepsilon)\boldsymbol{F}^{\text{vac}}$.

In the remaining example mentioned above, where two parallel condenser plates are partly immersed in a dielectric liquid, the main reason for the equivalence is simply the compensating forces in the liquid itself: The total electromagnetic force in the liquid between the condenser plates which balances the gravity force at equilibrium is found by integrating the force density over a volume which starts in a domain of the liquid where the field vanishes and ends just above the surface where $\varepsilon = 1$. Thus the effect from the gradient term in (2.16) vanishes, and a measurement of the height of the liquid between the condenser plates cannot serve as a means to determine the validity of either $\boldsymbol{f}^E$ or $\boldsymbol{f}^M$. This point has been emphasized also by S. S. HAKIM(11).

[As stated above, MINKOWSKI's and EINSTEIN's tensors must be equivalent also with respect to the torque on the body. Actually, this latter kind of equivalence can be seen already by inspection of the expressions (2.5) and (1.8a). For the difference between the tensors is contained entirely in the terms multiplying δ_{ik}, and the torque effect from such a term is found simply by integrating $-\frac{1}{2}\boldsymbol{E}\cdot\boldsymbol{D}(\boldsymbol{r}\times\boldsymbol{n})$ and $-\frac{1}{2}E^2(\boldsymbol{r}\times\boldsymbol{n})$, respectively, where the field variables refer to the fluid, over the body surface. If the body is a sphere, it follows immediately that this torque effect vanishes. Further, the same result also applies if the body does not have a spherical form: In this case we may lay a fictitious spherical surface in the fluid outside the body so that $\boldsymbol{r}\times\boldsymbol{n} = 0$ on the surface, and from the stability of the fluid it follows that the torque exerted on the fictitious surface from the outside must be

* We are as usual assuming a rapid but continuous variation of ε across the boundary layers.

equal to the torque acting on the real body surface. In all cases the body torque is determined entirely by the first terms in (1.8a) or (2.5).]

While MINKOWSKI's and EINSTEIN's tensors thus lead to the same expressions for forces and torques, we shall now see that there actually exists another effect which is measurable and which represents a critical test of the two tensors, namely the *pressure* increase in a dielectric non-polar liquid because of the field. Let us then first point out which electromagnetic forces may produce this excess fluid pressure. MINKOWSKI's force density is, in accordance with (2.4),

$$\boldsymbol{f}^M = \varrho \boldsymbol{E} - \tfrac{1}{2} E^2 \nabla \varepsilon, \tag{2.19}$$

and so the only pressure-producing term within the fluid, where $\varrho = 0$, is the term $-\frac{1}{2}E^2\nabla\varepsilon$. This term is of importance in the boundary region between two media. We shall, however, in the following confine ourselves to situations where this term is of no importance, as for instance the situation where a charged condenser is completely immersed in the liquid.*) The condenser is moreover imagined placed horizontally, so that the gravity effect can be ignored.

The next kind of force which may yield an increased pressure effect is the *electrostriction* force. We have hitherto ignored the electrostriction in our work, it has usually no influence upon measurable quantities, but at this point it is indispensable. We then start again from the free energy (2.1) and carry through the variational procedure similarly as in sect. 3 of I, but now with the inclusion of terms showing the dependence of ε on the mass density ϱ_m. For definiteness we shall continue to call the expression (2.19) MINKOWSKI's force, while the complete force expression shall be denoted as HELMHOLTZ' force

$$\boldsymbol{f}^H = \varrho \boldsymbol{E} - \tfrac{1}{2} E^2 \nabla \varepsilon + \tfrac{1}{2} \nabla \left(E^2 \varrho_m \frac{d\varepsilon}{d\varrho_m} \right). \tag{2.20}$$

For the simple non-polar liquids here studied we may eliminate the mass density be means of the CLAUSIUS-MOSSOTTI relation $(\varepsilon - 1)/(\varepsilon + 2) = \text{const.}\ \varrho_m$, and so (2.20) yields the following expression for the excess pressure, produced by the field:

$$\Delta p^H \equiv p^H - p^0 = \tfrac{1}{6}(\varepsilon - 1)(\varepsilon + 2) E^2, \tag{2.21}$$

* However, even in such a case $\nabla\varepsilon$ will not be exactly equal to zero; ε will increase somewhat in the domain between the condenser plates if the fluid pressure here increases due to some other kind of force. With the simple non-polar liquids and moderate pressure changes that we shall be considering (Δp of the order of one atmosphere), the influence from $\nabla\varepsilon$ on the force is, however, negligible. See refs. 11, 12 or *International Critical Tables.*

where p^0 is the fluid pressure when the field is turned off, thus corresponding to a slightly diminished mass density.

Finally we turn our attention to the EINSTEIN force (2.16). Since MINKOWSKI's force yields no pressure effect in the physical situations we consider, it follows immediately from (2.17) that

$$\Delta p^E = p^E - p^0 = \tfrac{1}{2}(\varepsilon - 1)E^2. \qquad (2.22)$$

It is clear from eqs. (2.21) and (2.22) that an experimental detection of the excess pressure represents a critical test of HELMHOLTZ' and EINSTEIN's force expressions. Now this kind of experiment has actually been performed by S. S. HAKIM and J. B. HIGHAM[12]. They used an ingenious method based on the fact that the excess pressure which the field produces gives rise to a slight compression of the liquid and so increases its refractive index. This increase was determined experimentally by means of a TOEPLER-SCHLIEREN optical technique, i.e. by a measurement of the angular deflection of light rays passing through the liquid. The experimental results were found to be in agreement with the formula (2.21) within limits of accuracy of $\pm$ 5 %, while they disagreed completely with the formula (2.22).

The HAKIM-HIGHAM experiment thus yields the important result that the fluid pressure p in the presence of the field can be identified with the HELMHOLTZ pressure p^H. Hence we can draw the conclusion that the validity of the HELMHOLTZ variational method used above, based on the free energy (2.1), is confirmed experimentally. It has sometimes been argued that one has the freedom to define the force density $\boldsymbol{f}$ and the pressure p arbitrarily, also in the electrostatic case, apart from the single restrictive condition that the relation $\boldsymbol{f} = \nabla p$ must be satisfied. We think however that the experiment clearly demonstrates that there is no room for this kind of arbitrariness in the electrostatic case within a dielectric liquid: By an integration of the force density over a volume element one must obtain the total electromagnetic force on that element which is compensated by the external pressure force acting on the surface. Since the excess pressure predicted by HELMHOLTZ' force expression has been verified experimentally, one should not introduce different definitions for pressure and force that would destroy this correspondence.

We also refer to another, theoretical, work[11] by HAKIM in which the HELMHOLTZ force is derived under essentially the same assumptions as those inherent in the usual derivation of the CLAUSIUS-MOSSOTTI equation. Further, HAKIM was able to show that the EINSTEIN force runs into conflict with the CLAUSIUS-MOSSOTTI equation.

Since the electrostatic contribution to the force consists in a gradient term it follows immediately, as indicated above, that the electrostriction will yield no observable effect upon the electromagnetic force or torque acting on a test body. The gradient form implies that there is always a balance between two equally large and oppositely directed forces at the body surface. For this reason HELMHOLTZ' force can usually be replaced by MINKOWSKI's force, as we have done in our work.

It is instructive to give the expression for the *total* stress tensor T_{ik} corresponding to both the liquid and the field:

$$T_{ik} = p^H \delta_{ik} - E_i D_k + \tfrac{1}{2} E^2 \delta_{ik} (\varepsilon - \varrho_m d\varepsilon / d\varrho_m) \tag{2.23a}$$

$$= p^0 \delta_{ik} - E_i D_k + \tfrac{1}{2} \delta_{ik} \boldsymbol{E} \cdot \boldsymbol{D}, \tag{2.23b}$$

$$\partial_k T_{ik} = 0. \tag{2.23c}$$

These equations obviously do not apply to the domains in space wherein external bodies have been placed. Note that the validity of eq. (2.23b) is dependent on the fact that we have confined ourselves to systems for which the excess pressure is due entirely to the electrostrictive force. If on the other hand we had considered a situation in which also the term $-\frac{1}{2}E^2 \nabla \varepsilon$ in the force had a pressure-producing effect (as for instance the situation where the vertical condenser plates are partly immersed in the liquid), the fluid pressure p^H appearing in (2.23a) would no longer have been determined by the simple equation (2.21).

Now we have considered the pressure as a function of the zero-field pressure p^0 and the squared electric field E^2. It is however possible to regard the pressure as a function of the mass density ϱ_m only, where the latter quantity includes also the contribution from the compressional potential energy set up by the electromagnetic forces. We can write the total free energy density F^{tot} as the sum of a mechanical part F^{mech} and an electromagnetic part $F = \frac{1}{2} \boldsymbol{E} \cdot \boldsymbol{D}$:

$$F^{\text{tot}} = F^{\text{mech}}(\varrho_m) + \tfrac{1}{2} \boldsymbol{E} \cdot \boldsymbol{D}, \tag{2.24}$$

where $\varrho_m = \varrho_m^0 + \Delta\varrho_m$, ϱ_m^0 denoting the zero-field value and $\Delta\varrho_m$ denoting the increase on account of the field. The pressure is then derived according to the familiar formula

$$p = -\frac{\partial(\varrho_m^{-1} F^{\text{mech}})}{\partial \varrho_m^{-1}}. \tag{2.25}$$

Thus, although the amount of compressional potential energy transferred to the material from the electrostrictive forces is very small, it is nevertheless important to include also the electrostrictive contribution to ϱ_m when deriving the pressure according to (2.25). Otherwise, if the expression (2.25) is calculated simply when the field is switched off, one will obtain the pressure p^0. Obviously it is not the electric field *per se* which is of main importance; we may well assume that the field is absent in calculating (2.25), but then we have to imagine the presence of some other kind of external force which produces the same value of the density at each point.

We now turn to a comparison of the above results with those obtained by DE GROOT *et al.* As mentioned already in section 1, DE GROOT and SUTTORP(6) have introduced also a second form of the electromagnetic energy-momentum tensor, which is assumed to describe the whole interaction between matter and field. This tensor form is in agreement with ABRAHAM's expression when the latter is supplemented with the appropriate electrostrictive and magnetostrictive terms, and when the terms involving the derivatives of the material constants with respect to the temperature are omitted (these temperature-dependent terms being negligible in the case of non-polar media). We then first note the interesting result that the second tensor introduced by DE GROOT and SUTTORP is in accordance with the HELMHOLTZ force in the electrostatic case, and thus is in agreement with our interpretation above. Now, since this tensor is assumed to describe the whole interaction between field and matter, it is constructed as the difference between the total (field plus matter) tensor in the presence of the field, and the total tensor in the absence of the field but at the same value of the density (and the temperature). This last statement is presumably to be understood so that the total mass density ϱ_m (including the contribution from the compressional potential energy) is required to be kept constant, independent of the field, the authors thus implicitly presupposing the existence of some extra kind of force to maintain the compressional energy when the field is switched off. By looking at the theory in this way we find that their mechanical stress tensor can be written as $p^H \delta_{ik}$, the force balance thus reading $\boldsymbol{f}^H = \nabla p^H$, in accordance with our result earlier obtained.

However, in spite of this formal agreement between the results it turns out that the two procedures are essentially different. (Apart from the already cited papers by DE GROOT and SUTTORP, see also similar treatments by MAZUR and DE GROOT(13, 17).) Let us here therefore sketch some important parts of the mathematical formalism. The authors employ the following, rather unusual, balance equation for free energy per unit mass

$$d(\varrho_m^{-1} F^{\text{tot}}) = -p d(\varrho_m^{-1}) + \boldsymbol{E} \cdot d(\varrho_m^{-1} \boldsymbol{P}). \tag{2.26}$$

Here we have omitted a temperature-dependent term. We shall not penetrate into the background of this equation, but mention that it is connected with the adoption of $\frac{1}{2}E^2$ as the electrostatic energy density. Eq. (2.26) is integrated at constant ϱ_m to give

$$F^{\text{tot}} = F^{\text{mech}}(\varrho_m) + \tfrac{1}{2} \boldsymbol{E} \cdot \boldsymbol{P}, \tag{2.27}$$

where F^{mech} is the free energy density in the absence of the field, but at the same mass density. The authors then invoke eqs. (2.26) and (2.27) to calculate the pressure

$$p = -\left[\frac{\partial(\varrho_m^{-1} F^{\text{tot}})}{\partial \varrho_m^{-1}}\right]_{(\varrho_m^{-1}\boldsymbol{P})} = p^H + \tfrac{1}{2} \boldsymbol{E} \cdot \boldsymbol{P} - \tfrac{1}{2} E^2 \varrho_m \frac{d\varepsilon}{d\varrho_m}. \tag{2.28}$$

This pressure p is now identified with the EINSTEIN pressure p^E and the expression (2.28) is inserted into the force balance $\boldsymbol{f}^E = \nabla p^E$. The force $\boldsymbol{f}^E$ can be expressed in terms of the field quantities by means of eqs. (2.16) and (2.19), and by comparing with the expression (2.20) for the HELMHOLTZ force one sees that

$$\boldsymbol{f}^E = \boldsymbol{f}^H + \tfrac{1}{2} \nabla\left(\boldsymbol{E} \cdot \boldsymbol{P} - E^2 \varrho_m \frac{d\varepsilon}{d\varrho_m}\right). \tag{2.29}$$

Thus, by using eqs. (2.29) and (2.28) the authors obtain that the force balance $\boldsymbol{f}^E = \nabla p^E$ can alternatively be written $\boldsymbol{f}^H = \nabla p^H$, as previously mentioned. Correspondingly, the identification of the pressure p in eq. (2.28) with the EINSTEIN pressure p^E is in accordance with eq. (2.29).

At this stage it should be clear what in reality distinguishes the method employed by DE GROOT *et al* from the method we have employed earlier in this section. First, the expression (2.27) for the free energy density differs essentially from the expression (2.24) and hence does not correspond to the free energy density $\frac{1}{2}\boldsymbol{E} \cdot \boldsymbol{D}$ for the field. The latter density was used in the variational principle based on eq. (2.1), and it must be equal to the work exerted per unit volume in building up the field. Secondly, a relation of the form (2.28) is incompatible with our earlier interpretation according to which the pressure is a function of the total mass density alone, the field playing only a secondary role in establishing the compressional force. Instead of calculating the pressure as a partial derivative of the type (2.28) whose physical meaning does not appear quite clear to us, we have instead employed the usual method according to which the pressure gradient and the electromagnetic force emerge from a variational principle wherein respectively the mechanical free energy density F^{mech} and the field free energy density $F = \frac{1}{2}\boldsymbol{E}\cdot\boldsymbol{D}$ are varied. Thus, after variation of the mechanical part, the fluid pressure can be written simply as a partial derivative in the form (2.25), but this quantity is not explicitly dependent on the field. If we instead had inserted the total free energy density F^{tot} into the variational integral we would have obtained the resulting force density equal to zero, in accordance with the fact that the system consisting of matter plus field is a closed system.

The results obtained in this section can be summarized as follows: The variational method based on the energy (2.1) can lead both to MINKOWSKI's and ABRAHAM's tensors, and as far as the dielectric body is surrounded by an isotropic medium (vacuum or liquid), no experiments testing forces or torques can decide between them. These tensors correspond only to different distributions of forces and torques throughout the body. Within an isotropic medium the tensors become equal, and the increased pressure effect predicted in a dielectric liquid (including the electrostriction effect) has been verified experimentally.

The other proposal considered, but forward among others by EINSTEIN and LAUB (as well as DE GROOT and SUTTORP in their first proposal), is different from the above two expressions even in the isotropic case. The extra pressure effect predicted by this tensor does not agree with experiment.

As usual, we have in this section confined ourselves to the macroscopic approach. It seems to be a rather common feature, however, that the *microscopic* treatments that have been given in this field favour the force expression which we have called EINSTEIN's force. Apart form the already cited papers by MAZUR and DE GROOT(13), DE GROOT and SUTTORP(6), we may refer also to a paper by KAUFMAN(14), in which similar conclusions have been drawn. We shall not, however, go into further considerations at this point.

3. Consideration of an Electromagnetic Wave in an Isotropic Body at Rest

We now turn our attention to simple time-varying fields within a dielectric medium at rest. In the first part of the section we rely upon the semi-microscopical arguments from I, section 6 to point out the connection between MINKOWSKI's and ABRAHAM's tensors for a plane wave travelling within an isotropic and homogeneous body; thereafter the considerations are illustrated by an example where also boundaries are involved. Finally, we examine alternative tensor proposals.

We recall the essential parts of the procedure for constructing the energy-momentum tensor in I: The energy density was taken to be the sum of the electrostatic and magnetostatic energy densities; correspondingly, the stress tensor was constructed as the sum of the electrostatic and magnetostatic stress tensors derived by the usual energy variational method. From the energy density in the form $W = \frac{1}{2}(\boldsymbol{E}\cdot\boldsymbol{D} + \boldsymbol{H}\cdot\boldsymbol{B})$ and from the fact that the four-component of force, f_4, vanishes within the dielectric, we deduced the expression $\boldsymbol{S} = c(\boldsymbol{E}\times\boldsymbol{H})$ for the energy flux. Assuming the relation $\boldsymbol{S} = c^2\boldsymbol{g}$, expressing PLANCK's principle of inertia of energy, to be valid also for the electromagnetic field, we further found the momentum density $\boldsymbol{g} = (1/c)(\boldsymbol{E}\times\boldsymbol{H})$.

In accordance with (1.3) it is apparent that these components form ABRAHAM's tensor. If the remaining part of the total system (the mechanical part) is described by an energy-momentum tensor $U_{\mu\nu}$, the present division of the total system into electromagnetic and mechanical parts may be expressed by the equation

$$-\partial_\nu S^A_{\mu\nu} = f^A_\mu = \partial_\nu U_{\mu\nu}. \tag{3.1}$$

The covariant form of $S^A_{\mu\nu}$ is given in (1.5). ABRAHAM's tensor has been advocated by many authors, and we also agree that it represents a fully adequate description of phenomenological electrodynamics. It must be borne in mind that we are neglecting electrostriction and magnetostriction effects; these effects would lead to additional terms in the tensor components. Actually we find, in the time-dependent case as well as in the static case, that if ABRAHAM's tensor is augmented by the electrostrictive and magnetostrictive terms the resulting expression is just equal to the second tensor expression given by DE GROOT and SUTTORP (apart from terms involving the derivatives of the material constants with respect to temperature).

It must be borne in mind however, that the present problem is to some extent a matter of *convenience*, and the question arises whether there are alternative tensors which can equally well be justified on the basis of (3.1). Our next task is thus to examine the effect induced in the mechanical tensor $U_{\mu\nu}$ on account of the force f^A_μ. According to (1.4) this force has only one nonvanishing component, namely a fluctuating component in the direction of propagation of the plane wave. We take this direction as the x-direction; if the velocity of the constituent dipoles in the x-direction is denoted by u_1, we found in I that the contributions to the components U_{ik} and U_{44} because of this velocity component are at most of the order $(u_1/c)^2$, which are negligible quantities. On the other hand, the components $U_{14} = U_{41} = icg_1^{\text{mech}} = ic\varrho_m u_1$ are of the first order in u_1/c and may thus be appreciable. By invoking the JONES-RICHARDS experiment(15) we actually determined the induced mechanical momentum density as

$$\boldsymbol{g}^{\text{mech}} = \frac{n^2 - 1}{c}(\boldsymbol{E} \times \boldsymbol{H}) \tag{3.2}$$

in the case of an optical wave. This mechanical momentum runs always together with the field. Simply by *including* (3.2) in the field momentum density we obtained MINKOWSKI's value $\boldsymbol{g}^M = (1/c)(\boldsymbol{D} \times \boldsymbol{B})$. This is the *total* electromagnetic and mechanical momentum density associated with a propagating optical wave. Further, this interpretation means that the matter is set into a small motion with the velocity u_1 when the field passes through it; the flux of mechanical energy $S_1^{\text{mech}} = -icU_{41} = -icU_{14}$ being present because of this motion must naturally be included in the mechanical tensor. Note that $f_4 = 0$ ($f_1^A u_1$ being negligible), so that $\partial_\nu U_{4\nu} = 0$.

If we suppose that the optical wave travels within an *infinite* medium, so that there are no forces in the boundary layers to cause stresses in the material, the components U_{ik} of the stress tensor are equal to their values at zero field. In more general cases, the components U_{ik} have to describe the elastic stresses which are set up because of the electromagnetic forces at the boundaries.

Further considerations on these topics are contained in I, section 6, but we shall here write down the tensor scheme which pertains to MINKOWSKI's tensor: The field is described by

$$S_{\mu\nu}^M = \begin{pmatrix} S_{ik}^A, & S_{i4}^A + U_{i4} \\ S_{4k}^A, & S_{44}^A \end{pmatrix}, \quad \partial_\nu S_{\mu\nu}^M = 0, \tag{3.3}$$

and if $T_{\mu\nu}$ is the energy-momentum tensor of the total system the mechanical part is described by

$$T_{\mu\nu} - S_{\mu\nu}^M = \begin{pmatrix} U_{ik}, & 0 \\ U_{4k}, & U_{44} \end{pmatrix}, \tag{3.4}$$

where $U_{44} = -\varrho_m c^2$. The symmetrical and divergence-free tensor $T_{\mu\nu}$ is thus divided into two asymmetrical but divergence-free tensors describing the electromagnetic and mechanical parts of the system. We emphasize again that the reason why this kind of division is convenient lies entirely in *experience*. Further, although the division of course does not affect the angular momentum conservation law for the total system, the asymmetry of the partial tensors gives rise to unfamiliar aspects for the angular momenta of the two subsystems.

It is instructive to consider the system not only in the frame K^0—the original rest frame—but also in the frame K' in which the mean velocity of the matter is zero. In this frame all tensor components retain their old values from K^0, except for the components $U'_{4k} = U'_{k4}$ whose average values are zero. Apart from fluctuating terms the above two kinds of splitting then become equivalent: The field is described by the same ABRAHAM tensor as in the frame K^0, and the remaining matter system is described by the tensor $U'_{\mu\nu}$ which, in the case of an *infinite* medium, can be taken to be equal to the energy—momentum tensor at zero field. If the medium is *finite*, the components U'_{ik} must describe also any mechanical stresses that may arise. With the omission of electrostrictive and magnetostrictive terms we thus obtain *in the frame* K' a division of the total energy-momentum tensor into an electromagnetic and a mechanical part in a way which is in agreement with the division that has been proposed by several other authors(6, 16, 17) in the rest frame. The new element of our analysis is essentially that this kind of division is interpreted *not* to run into conflict with MINKOWSKI's tensor, due to the fact that the experiments lead us to distinguish between the original rest frame K^0 and the frame K' in which the mean velocity vanishes.

Further, there is still another aspect which should be emphasized in connection with the comparison between MINKOWSKI's and ABRAHAM's tensors: ABRAHAM's force density is the *real* force acting on a unit volume, i.e. the force on the matter itself as well as on any charges and currents present within the volume. This force is compensated by the mechanical stresses plus the inertial force, in accordance with the relation

$$f_i^A = \partial_k U_{ik} + (\partial/\partial t) g_i^{\text{mech}}. \tag{3.5}$$

MINKOWSKI's force, on the other hand, amounts to counting the inertial force together with the proper force:

$$f_i^M = f_i^A - (\partial/\partial t) g_i^{\text{mech}} = \partial_k U_{ik}, \tag{3.6}$$

and it has thus a less direct physical meaning than ABRAHAM's force. MINKOWSKI's force does not contain any term which corresponds to the magnetic force on the polarization currents, this term is hidden in the field momentum.

The non-appearance of such a magnetic force term has represented an obstacle for the acceptance of MINKOWSKI's tensor, as reflected for instance in EINSTEIN and LAUB's article.[5]

Example involving the boundary between two media

By the above analysis we have come to the important conclusion that the propagation of an electromagnetic wave through matter is conveniently described by MINKOWSKI's tensor in such a way that the rest of the system (the mechanical part) may usually be ignored. In this subsection, however, we shall examine the total momentum and motion of centre of mass for a total system when boundaries are involved; in this case all kinds of momentum and energy flows have to be taken into account.

Imagine a plane wave with $\boldsymbol{E} = E_0 \boldsymbol{e}_y \sin(k_0 x - \omega t)$ that falls in from vacuum towards an isotropic and homogeneous insulator at zero angle of incidence. We take the boundary as the plane $x = 0$, and put for simplicity $\varepsilon = \mu = n$ so that the reflected wave vanishes. We may consider a certain part of the plane wave, say of length l_0 and cross section unity, and examine the consequences of the application of different forms of the momentum expressions. (The length l_0 is then required to be much smaller than the width L of the body over which the field travels.) But it is more convenient simply to consider the field as a wave parcel with length l_0 and cross section unity, where $l_0 \ll L$, so let us look at the system in this way.

The total field energies in vacuum and in the body are equal, $\mathscr{H}_0 = l_0 E_0^2/2 = n l E_0^2/2 = \mathscr{H}$, where l and $\mathscr{H}$ refer to the body. By taking the divergence of ABRAHAM's tensor we obtain

$$\boldsymbol{f}^A = -\tfrac{1}{2}E^2 \nabla \varepsilon - \tfrac{1}{2}H^2 \nabla \mu + \frac{n^2-1}{c}\frac{\partial}{\partial t}(\boldsymbol{E} \times \boldsymbol{H}) \tag{3.7}$$

(cf.(1.4)), valid also over the boundary if one assumes a continuous variation of ε and μ. We shall first use this force in a computation of the various momenta. As $E = H$ everywhere, the surface force during the penetration period is $(1-n)E^2$, and so the total momentum component in the x-direction transferred to the body on account of this force is

$$G^{\text{surf}} = -(n^2-1)E_0^2 \int_0^{l_0/c} \sin^2 \omega t \, dt = \frac{1-n}{c}\mathscr{H}, \tag{3.8}$$

where we have integrated over the penetration period.

According to our earlier results, the effect of the last term in (3.7) is to excite a mechanical momentum in the body:

$$G^{\text{mech}} = \frac{n^2-1}{c}\int_0^l EHdx = \frac{n^2-1}{nc}\mathscr{H}. \tag{3.9}$$

Finally, the electromagnetic part is

$$G^{\text{el.m.}} = \frac{1}{c}\int_0^l EHdx = \frac{1}{nc}\mathscr{H}. \tag{3.10}$$

Collecting these terms the balance of total momentum can be checked:

$$G^{\text{surf}} + G^{\text{mech}} + G^{\text{el.m.}} = \frac{\mathscr{H}}{c} = G^{\text{vac}}, \tag{3.11}$$

where G^{vac} is the magnitude of the momentum of the incoming field. This simple analysis exhibits the behaviour of the various momentum parts.

If we instead had started from Minkowski's tensor, the last term in (3.7) would have been absent. In this case the momentum component G^{surf} supplied by the forces in the boundary layer, plus the *field* momentum $G^M = G^{\text{el.m.}} + G^{\text{mech}} = n\mathscr{H}/c$, would have added up to give the total momentum $\mathscr{H}/c$.

Let us also examine the centre of mass velocity for the total system. Denoting the coordinates of the centre of mass by $\boldsymbol{X} = (X, 0, 0)$, we have

$$\frac{d}{dt}X^{\text{tot}} = \frac{d}{dt}\left[\frac{1}{\mathscr{H}^{\text{tot}}}\int xW^{\text{tot}}dV\right] = -\frac{ic}{\mathscr{H}^{\text{tot}}}\int\frac{\partial}{\partial x_\nu}(xS_{4\nu}^{\text{tot}})\,dV, \tag{3.12}$$

since the contribution to (3.12) from $\nu = 1,2,3$ vanishes when the boundary surface of the integration volume is chosen sufficiently far away. Hence

$$\frac{d}{dt}X^{\text{tot}} = \frac{1}{\mathscr{H}^{\text{tot}}}\int S^{\text{tot}}dV = \frac{c^2G^{\text{vac}}}{\mathscr{H}^{\text{tot}}} = \frac{\mathscr{H}}{\mathscr{H}^{\text{tot}}}c, \tag{3.13}$$

corresponding to the fact that the parcel travels with the velocity c before it strikes the body. It should be noted that in (3.13) S^{tot} includes also the mechanical energy flux S^{mech} due to the small motion of matter induced by the field.

Since the body has a finite extension L in the x-direction then, during the period when the wave parcel leaves the body, the effect on the body is equal and opposite to that during the entrance period. Further, the motion of matter described by S^{mech} is considered to be absent when the wave has left, so the body will stay at rest. Since the length of the parcel is small, it can be considered to have remained a time $\tau = Ln/c$ in the body. Let M and ξ denote the total mass and displacement of the body in the x-direction; we then find from (3.8), (3.9) and the relation $M\xi/\tau = G^{\text{surf}} + G^{\text{mech}}$ that $\xi = (\mathscr{H}/Mc^2)(n-1)L$.

The gedanken experiment above is one of those considered by N. L. Balazs[18]. We cannot agree to his conclusion, however, when he claims the correctness of $S^A_{\mu\nu}$ in contrast with $S^M_{\mu\nu}$ by an analysis of the total momentum and centre of mass. Let us apply his procedure to the above case: The equation of momentum balance is given in the form

$$G^{\text{vac}} = G' + M\xi/\tau, \tag{3.14}$$

where G' is the magnitude of the field momentum in the body which is to be determined. Further, the law of conservation of the centre of mass velocity is written as

$$\mathscr{H} c\tau = \mathscr{H} c\tau/n + Mc^2\xi. \tag{3.15}$$

From these equations he obtains $G' = \mathscr{H}/(nc)$, which agrees with Abraham's expression only.

But by comparison with our previous treatment it is apparent that the balance equation (3.14) is incomplete. Eq. (3.15) is valid for both tensors, and leads to the expression for ξ found above. But (3.14) implies that the magnitude of the mechanical momentum be given by $M\xi/\tau$, which ,in accordance with (3.15), (3.8) and (3.9), is equal to $G^{\text{surf}} + G^{\text{mech}}$. This is an assumption which is compatible with Abraham's expression only; we see from (3.14) that $G' = G^{\text{vac}} - G^{\text{surf}} - G^{\text{mech}} = G^{\text{el.m.}}$, when the balance equation (3.11) is taken into account.

Finally we should mention that in an examination of an example similar to the one above, E. G. Cullwick[19] has claimed that Abraham's momentum density is satisfactory while Minkowski's momentum density leads to inconsistencies with respect to the momentum balance. His argument is essentially tantamount to saying that, in the situation above, the relations $g^{\text{vac}} = g^A$, $g^{\text{vac}} \neq g^M$, determine the validity of the Abraham expression. It is however evident that in order to check the momentum balance over the boundary one should integrate the equation $\partial_k S_{ik} + \partial g_i/\partial t = -f_i$ in question over a volume which includes a part of the boundary, and thus one must consider instead the momentum *flow* described by the components S_{ik}. Moreover, the surface force must also be taken into account. The paper has been criticised also by P. PENFIELD[20, 21].

Other tensor forms

It is convenient to collect the remarks on the alternative tensor forms in this final subsection. First we recall that the DE GROOT-SUTTORP tensor (1.9) must describe essentially another part of the total system than the part which we have made to correspond to the electromagnetic energy—momentum tensor. This follows from a comparison between the energy density (1.9b) and the energy density $W = \frac{1}{2}(\boldsymbol{E}\cdot\boldsymbol{D} + \boldsymbol{H}\cdot\boldsymbol{B})$ on which we have based our derivations (cf. also the HAKIM-HIGHAM experiment mentioned in section 2). Next, the EINSTEIN-LAUB tensor (1.8) is in conformity with the expressions (1.9) when $\mu = 1$. The most interesting alternative in relation to the topics considered in the present section is the radiation tensor (1.6) introduced by MARX and his collaborators; we recall that this tensor was defined for radiation fields only. The essential point in the construction of the radiation tensor can be visualized by an inspection of the equation (3.1): One assumes that the effect of the force $\boldsymbol{f}^A$ is not to create a mechanical *momentum*, described by the components U_{i4}, but rather to form *stresses*, described by the components U_{ik}. Eq. (3.1) can then be written as $\partial_k(S_{ik}^A + U_{ik}) + \partial g_i^A/\partial t = 0$, leading to $U_{ik} = (n^{-2} - 1)S_{ik}^A$, in accordance with (1.6). However, the main reason why we have not constructed the theory in this way is simply the result of the JONES-RICHARDS experiment, to which we have already referred repeatedly. As we pointed out in the rather detailed consideration in I, section 6, it was essential for the validity of the derived formulas that the electromagnetic energy—momentum tensor in question be a *divergence—free* quantity in the interior of the body. Since the radiation tensor just has this property, and since the relation between the momentum flow components is $S_{ik}^S = = (1/n^2)S_{ik}^M$, it follows that the radiation pressure predicted by the radiation tensor is equal to $1/n^2$ times the MINKOWSKI radiation pressure. By a comparison with the observed data we are thus in a position to draw the decisive conclusion that the characteristic assumption inherent in the derivation of the radiation tensor should be rejected. Note that the electrostriction effect will have no influence on this result.

Although it should therefore not be of importance to go into a detailed examination of the use of the radiation tensor in the example considered in the above subsection, let us yet note the following points. The force density can no longer be written as (3.7), since this expression will violate the law of conservation of momentum. This is so because the last term in (3.7) is no longer associated with a mechanical momentum, and hence the total momentum after the wave has entered the body is $G^{\text{surf}} + G^{\text{el.m.}} \neq G^{\text{vac}}$. In order to fulfil the momentum conservation law the force density must be defined as $f_i^S = -\partial_\nu S_{i\nu}^S$, where the stress components are *not* the sum of the electrostratic and magnetostatic stress components. If we define $S_{i\nu}^S = (1/n^2(x))S_{i\nu}^M$ also in the spatially dispersive region in the boundary layer, we find that the momentum induced by the surface forces is $(G^S)^{\text{surf}} = (1-1/n)\mathscr{H}/c$. The interesting aspect here is that the quantity $(G^S)^{\text{surf}}$ has the opposite *sign* of the quantity G^{surf} calculated earlier in eq. (3.8); while the surface force following from the radiation tensor acts *inwards* to the body the surface force following from Abraham's and Minkowski's tensors acts *outwards* from the body surface. We are not, however, aware of a direct experimental test of this effect (cf. the last part of the next section).

4. Discussion of some Possibilities for Experiments

In this section we examine experimental situations in which time—dependent fields exert torques on dielectric bodies at rest. As usual we first focus our attention on the relative behaviour of Minkowski's and Abraham's tensors. In the first class of experiments considered—the interaction between a stationary radiation field and a dielectric body—the result is that the two tensors lead to the same answers. Thereafter, an example is given of a second type of experiments in which the difference can be measured. Finally, we propose a critical experiment testing the radiation tensor and the Einstein tensor.

Proof of equivalence

As an example of an experiment which traces the angular momentum interaction between an electromagnetic wave and a dielectric body, the old G. Barlow experiment(22) should first be mentioned. He made a careful measurement of the torque produced by a beam of light in oblique refraction through a glass plate, and obtained good agreement with the theory. We refer also to the famous R. A. Beth experiment(23), in which the existence of angular momentum in a light wave was detected by letting the wave pass through an anisotropic crystal. The latter experiment has more recently been repeated by N. Carrara(24) with the use of centimetre waves. These experiments consisted in letting a stationary wave interact with the body and then measuring the deflection when equilibrium was established between the electromagnetic torque and the mechanical torque exerted by a torsional suspension. However, we need not go into detailed considerations of these situations in order to test the relative behaviour of MINKOWSKI's and ABRAHAM's tensors, since we will find the torque $\boldsymbol{N}^A = \boldsymbol{N}^M$, just as we did in the static case. Instead we present a simple argument which shows in general that in a wave-dielectric body situation the two energy-momentum tensors yield the same value for the torque.

Consider then a stationary high-frequency wave interacting with a dielectric body (in general anisotropic). The body is assumed so heavy that no macroscopic motion needs to be taken into account. If the angular momentum of the internal field in the body is denoted by $\boldsymbol{M}^i$, the torque $\boldsymbol{N}$ can be written as

$$\boldsymbol{N} = -d\boldsymbol{M}^{\text{vac}}/dt - d\boldsymbol{M}^i/dt. \tag{4.1}$$

It can readily be seen that each of the two terms on the right hand side of this equation is the same for ABRAHAM's or MINKOWSKI's tensor. In both cases the energy flux is given as $c(\boldsymbol{E} \times \boldsymbol{H})$, therefore the direction and velocity of the travelling field energy is the same, and it follows that the first term on the right of (4.1) is also the same. Further, since we assume that the field is stationary, we can simply put $d\boldsymbol{M}^i/dt = 0$. Hence $\boldsymbol{N}^A = \boldsymbol{N}^M = -d\boldsymbol{M}^{\text{vac}}/dt$: The two energy—momentum tensors are equivalent with respect to torque effects since these effects are determined in terms of the vacuum field.

(Alternatively, we may consider a wave packet in interaction with the body during the time period $t = 0$ to $t = T$, during which the field is assumed to be stationary. Then we can require on physical grounds that $\boldsymbol{N}$ be independent of T at any time t, also in the small transient period when the field leaves the body. We now assume only that $d\boldsymbol{M}^i/dt$ must be equal to some constant during the stationary interaction period, since each component is proportional to the averaged energy density of the incoming wave. When $t > T$, one has $\boldsymbol{M}^i = 0$, but then $d\boldsymbol{M}^i/dt$ can be made arbitrarily large in the transient period when the wave leaves the body, by choosing T large. These features are incompatible with the condition (4.1), hence $d\boldsymbol{M}^i/dt = 0$ in the stationary interaction period.)

Proposal of an experiment

In the preceding we considered an electromagnetic *wave* in interaction with a dielectric system. Now there exists the possibility of combining electric and magnetic fields in a way which, in principle, makes it possible to bring out explicitly the effect arising from ABRAHAM's force. We shall give a proposal similar to one put forward by MARX and GYÖRGYI(3). A cylindric dielectric shell of isotropic matter with large ε is suspended between the surfaces of a cylindric capacitor so that, in the absence of fields, the shell can oscillate about its axis (z) with a frequency ω_0. The internal surface of the capasitor is then charged to the amount q per unit length, and a homogeneous magnetic field $H_0 e^{-i\omega t}$ is impressed parallel to the z-axis. We suppose that the wavelength which corresponds to the frequency ω is large compared with the dimensions of the system, so that within the internal, massive cylindric conductor, we may write $\nabla \times \boldsymbol{H} = \sigma \boldsymbol{E}/c$, where σ is the conductivity. Taking into account that the penetration depth into the conductor is approximately equal to $\sqrt{c^2/\omega\sigma}$, which is a large quantity when ω is small, and putting $\mu = 1$, we obtain within the internal region of the conductor

$$\boldsymbol{H} = \boldsymbol{H}_0 e^{-i\omega t}, \quad E_\varphi = \frac{i\omega}{2c} r H_0 e^{-i\omega t}. \tag{4.2}$$

Within the dielectric shell $E_r = q/(2\pi\varepsilon r)$, while eqs. (4.2) remain valid also in this domain. Thus

$$f_\varphi^A = -\frac{\varepsilon - 1}{\varepsilon}\frac{\partial}{\partial t}(E_r H_z) = \frac{\varepsilon - 1}{\varepsilon}\frac{q\omega H_0}{2\pi c r}\sin\omega t, \tag{4.3}$$

when we take the real part. Hence the torque component is

$$N_3^A = \int r f_\varphi \, dV = \frac{\varepsilon - 1}{\varepsilon}\frac{q H_0 V}{2\pi c}\omega \sin\omega t \equiv K\omega \sin\omega t, \tag{4.4}$$

where V is the volume of the body. We have ignored the surface forces since these act in the same directions as $-\nabla\varepsilon$ and hence have no influence on the oscillations. The equation of motion can be written as

$$\ddot{\varphi} + \gamma\dot{\varphi} + \omega_0^2\varphi = \frac{K}{I}\omega\sin\omega t, \tag{4.5}$$

where γ is the damping constant and I the moment of inertia about the z-axis. The largest oscillations occur when $\omega = \omega_0$ and are given by

$$\varphi = -\frac{K}{I\gamma}\cos\omega_0 t. \tag{4.6}$$

This effect can in principle be measured. With a direct use of MINKOWSKI's tensor one obtains no force that can account for these oscillations, and MINKOWSKI's tensor is thus inappropriate in the present case. (It must be emphasized that the previous derivation of MINKOWSKI's tensor for time-dependent fields in isotropic media applies properly only to the case of *radiation* fields.)

As far as we know, the experiment has not been performed.

We emphasize the essential difference between this situation and those considered in the above subsection: At a given instant, the force component f_φ^A causing the torque does not vanish when integrated over the volume. Further, it is now the total *time oscillations* themselves which are detected and not, as in the previous situation, their effect after integration over a time which is large in comparison with the oscillation period.

Other tensors

Let us consider again the system of a stationary wave field and a dielectric body studied in the first of the subsections above, and first employ the radiation tensor $S_{\mu\nu}^S$. This tensor has been derived for the case of isotropic bodies only, so we shall accordingly assume the body to be isotropic. It is immediately apparent that if the wave comes in from vacuum, interacts with the body and then enters into vacuum again, we can apply just the same argument as before to conclude that the radiation tensor yields the same value for the torque as MINKOWSKI's and ABRAHAM's tensors. But a simple calculation shows that the *direction* and *magnitude* of the surface force will in general be different from what we obtained in the previous cases; it is only the total torque itself that remains unchanged. (For instance, if an appropriately polarised optical wave falls obliquely inwards to the body at BREWSTER's angle of incidence such that the reflected wave vanishes, it can be verified that the surface force acts in a direction parallel to the surface, instead of in a direction outwards along the normal vector, as obtained from MINKOWSKI's or ABRAHAM's tensor.) It has sometimes been claimed that the BARLOW experiment[22] mentioned above, involving a measurement of the torque exerted by a light wave on a glass plate, should actually provide an experimental test of the direction and magnitude of the surface force. But we think that this is not so, although BARLOW himself interprets the effect in a way corresponding to MINKOWSKI's or ABRAHAM's tensor. The only thing measured is the total torque, which is explained equivalently by all tensors in question.

However, an obvious generalization lies at hand in order to change for instance the BARLOW experiment into a critical experiment with respect to the radiation tensor, namely, to immerse the body into an isotropic dielectric *liquid*. The radiation tensor has a value different from the two other tensors mentioned in the liquid, and so a torque measurement can be crucial. In order to derive the appropriate torque expression it is convenient to write the general formula (cf. (I, 1.7))

$$N_l = \int_{\text{body}} (x_i f_k - x_k f_i + S_{ik} - S_{ki})\, dV \tag{4.7}$$

in the following compact form:

$$\boldsymbol{N} = -\int_{\text{surface}} (\boldsymbol{r} \times \boldsymbol{S}_n^{\text{liq}})\, dS - \frac{d}{dt} \int_{\text{body}} (\boldsymbol{r} \times \boldsymbol{g})\, dV \tag{4.8a}$$

$$= -\int_{\text{surface}} (\boldsymbol{r} \times \boldsymbol{S}_n^{\text{liq}})\, dS. \tag{4.8b}$$

For an optical wave the last integral in (4.8a) vanishes because the field is assumed to be stationary and the body remains practically at rest, and the surface integrals are taken in the liquid just outside the body. By means of (1.6) and (4.8b) we find the result $\boldsymbol{N}^S = (1/n^2)\boldsymbol{N}^M = (1/n^2)\boldsymbol{N}^A$, where n is the refractive index of the liquid. The surface integral in (4.8b) can be evaluated in the actual experimental situation with one of the tensors inserted, and one can thus check the tensors by a comparison with the observed torque.

As the next point we consider the EINSTEIN-LAUB tensor $S_{\mu\nu}^E$ applied to the same situation. (For optical fields we can put $\mu = 1$, and it is then apparent from (1.8) and (1.9) that the EINSTEIN-LAUB tensor and the DE GROOT-SUTTORP tensor are in agreement.) This tensor is defined also for anisotropic media. We evaluate this case most simply by noting the following relation in the liquid which surrounds the body:

$$\boldsymbol{S}_n^E = \boldsymbol{S}_n^A - \tfrac{1}{2}\boldsymbol{n}(\boldsymbol{E} \cdot \boldsymbol{P}), \tag{4.9}$$

so that (4.8b) yields

$$\boldsymbol{N}^E = \boldsymbol{N}^A + \tfrac{1}{2} \int_{\text{surface}} (\boldsymbol{r} \times \boldsymbol{n})\boldsymbol{E} \cdot \boldsymbol{P} dS, \tag{4.10}$$

where the surface integral is taken in the liquid. EINSTEIN's tensor thus leads to still another value for the torque, which might be tested experimentally.

The dielectric shell—experiment considered in the second subsection above is not of direct importance for the radiation tensor since this tensor is defined for *radiation* fields only. However, it can readily be seen that both the radiation tensor and EINSTEIN's tensor lead to ABRAHAM's value (4.4) for the torque. To this end we need only examine eq. (4.8a), where now the last term is non-vanishing and where $\boldsymbol{S}_n^{\text{liq}}$ is replaced by $\boldsymbol{S}_n^{\text{vac}}$ in vacuum outside the shell. Since $\boldsymbol{g}^S = \boldsymbol{g}^E = \boldsymbol{g}^A$ it follows that $\boldsymbol{N}^S = \boldsymbol{N}^E = \boldsymbol{N}^A$. (Moreover, the value (4.4) can be checked by inserting the field values (4.2) and the expression for E_r into (4.8a).) This experiment is therefore *not* a critical test of the relative behaviour of the three tensors mentioned. In this case it does not seem either to be an appropriate generalization to immerse the system into a dielectric liquid.

5. Some Remarks on the Literature

Together with the exposition of the various topics we have met up till now—both in I and in the present paper—we have found it desirable to include also some remarks pertinent to essential passages in earlier works on the subject. The literature is however large, and there remain important parts of it that could not naturally be considered or even touched upon in the preceding exposition. We have therefore reserved the present section for a critical review of some earlier (phenomenological) treatments, especially those which seem to be incompatible with the interpretations given above. We think that this avenue is natural to follow, since the present problem is not only a *deductive* task but also a matter of *clarification* of a confused situation. Evidently we cannot give a detailed scrutiny of all the relevant papers of phenomenological nature, but shall rather be concerned with illustrative examples. For a large part we shall be concerned with the analysis of criteria. The present section represents the end of our nonrelativistic treatment; from the next section on we concentrate upon topics connected with relativity.

In the two first subsections we consider two gedanken experiments which have been put forward. The idea behind these gedanken experiments has been that by comparing with results obtained from physical conservation laws, one should be able to decide which energy—momentum tensor is correct. In both the cases we shall consider, MINKOWSKI's tensor has been claimed to be preferable, as a result of a study of the conservation equations for momentum. We shall show how these experiments can be equivalently described with the use of ABRAHAM's tensor. In the subsequent subsection some aspects of the ČERENKOV effect are considered, and finally we mainly dwell on arguments favouring other tensors than MINKOWSKI's and ABRAHAM's expressions.

Propagation of discontinuities

In two papers A. RUBINOWICZ[25] investigated the situation where electromagnetic discontinuities are propagated through an isotropic body at rest. The conservation equations for energy and momentum are integrated over a domain Σ in four-space bounded by the hyperplanes σ_0, σ_1 and σ_3; σ_0 corresponds to the three-dimensional volume V_0 which at the time t_0 is enclosed within the two-dimensional surface Φ; σ_1 corresponds to the volume V_1 at $t = t_1 > t_0$, and σ_3 is the connecting time-like hypersurface. The surface Φ is considered moving inwards with the velocity $u = c/n$ in the direction of its normal.

Then imagine a two-dimensional surface $\Phi^*(t)$ across which the field is discontinuous:

$$\boldsymbol{E}_1 = \boldsymbol{E}, \boldsymbol{H}_1 = \boldsymbol{H};\ \boldsymbol{E}_2 = \boldsymbol{E} + \Delta\boldsymbol{E}, \boldsymbol{H}_2 = \boldsymbol{H} + \Delta\boldsymbol{H}. \tag{5.1}$$

Here 1(2) denote the inner (outer) side of Φ^*. For simplicity, we suppose Φ^* also to move together with the field, with the velocity $u = c/n$ in the direction of its normal.

RUBINOWICZ integrates the energy conservation equation over Σ and finds that Φ^* is associated with no source of energy when either of the two energy-momentum tensors is inserted. We therefore turn our attention to the momentum conservation equation written in the following form (our notation), where the time derivative is taken along the moving volume element:

$$\partial_k(S_{ik} - g_i u_k) + \frac{1}{dV}\frac{d}{dt}(g_i dV) = -f_i \tag{5.2}$$

and integrate over Σ:

$$\Big(\int_{V_1} - \int_{V_0}\Big) g_i\, dV + \int_{t_0}^{t_1} dt \int_{\Phi + \Phi^*} (S_{ik} - g_i u_k) n_k\, dS = -\int_{\Sigma} f_i\, dV dt. \tag{5.3}$$

The contribution from Φ^* to the left hand side of (5.3) can be written in vector form, according to RUBINOWICZ, as

$$\int_{t_0}^{t_1} dt \int_{\Phi^*} [(\boldsymbol{S}_n + \boldsymbol{g}u)_1 + (\boldsymbol{S}_n - \boldsymbol{g}u)_2] dS = \int_{t_0}^{t_1} dt \int_{\Phi^*} \frac{n^2 - p}{n} [\boldsymbol{E} \times \Delta \boldsymbol{H} + \Delta \boldsymbol{E} \times \boldsymbol{H}] dS. \quad (5.4)$$

Here, $\boldsymbol{S}_n$ is a vector with components $S_{ni} = S_{ik} n_k$, and $\boldsymbol{n}$ is taken to point outwards from the integration domains; p is a number, such that $p^M = n^2$, $p^A = 1$. Hence RUBINOWICZ concludes that Φ^* is associated with no source of energy or momentum as far as MINKOWSKI's tensor is employed, in contrast to what is the case with ABRAHAM's tensor, since (5.4) then is non-vanishing. This feature is claimed to favour the former expression.

Let us, however, examine the case $p = 1$. We see that the contribution (5.4) is not yet complete since the effect arising from $\boldsymbol{g}^{\text{mech}}$ has not been incorporated. This effect is connected with the term $(n^2 - 1)/c^2 \partial \boldsymbol{S}/\partial t$ in $\boldsymbol{f}$. Hence, the amount on the left of (5.3) is to be augmented by

$$\left. \begin{aligned} &\frac{n^2 - 1}{c^2} \int_{\Sigma} \left[\frac{d}{dt}(S_i \, dV) - \nabla \cdot (S_i \boldsymbol{u}) \, dV \right] dt \\ &= \frac{n^2 - 1}{c^2} \left[\left(\int_{V_1} - \int_{V_0} \right) S_i dV - \int_{t_0}^{t_1} dt \int_{\Phi + \Phi^*} S_i \boldsymbol{u} \cdot \boldsymbol{n} \, dS \right]. \end{aligned} \right\} \quad (5.5)$$

From (5.5) we see that the contribution from Φ^* equals, in vector form,

$$\frac{n^2 - 1}{c^2} \int_{t_0}^{t_1} dt \int_{\Phi^*} (\boldsymbol{S}_1 - \boldsymbol{S}_2) u dS, \quad (5.6)$$

which, together with (5.1) and (5.4), yields MINKOWSKI's result. We see again that the choice between MINKOWSKI's and ABRAHAM's tensors is mainly a matter of interpretation.

Induced motion of a ferromagnetic test body

Let us next examine the gedanken experiment recently considered by COSTA DE BEAUREGARD(26). The arrangement is rather similar to the one we considered earlier in the second part of section 4: A ferromagnetic shell with mean radius r_0, thickness b and length a is subjected to forces arising from a short current pulse in a rectilinear wire placed along the symmetry axis (z) of the shell. Besides, the wire is charged to a constant charge q per unit length and hence gives rise to the radial electric field $E_r = q/(2\pi r)$, when ε is put equal to 1. When the current is flowing, a tangential magnetic polarization $\boldsymbol{M} = \boldsymbol{B} - \boldsymbol{H}$ is present, and when the current has decreased to zero, there remains an amount $\Delta\boldsymbol{M} = \Delta\boldsymbol{B}$ in the shell which, together with $\boldsymbol{E}$, gives rise to a linear momentum in the z-direction. COSTA DE BEAUREGARD integrates the force component $f_3 = -\varrho\partial A_3/c\partial t$ over time and over the volume of the wire, and obtains

$$\int_{\text{wire}} f_3 dV dt = -\frac{1}{c} qab\Delta M. \tag{5.7}$$

If we use MINKOWSKI's tensor to calculate the remaining momentum component in the z-direction when the current has left, we find

$$\Delta G_3^M = \frac{1}{c}\Delta \int_{\text{body}} DBdV = \frac{1}{c}\int_{\text{body}} E\Delta MdV = \frac{1}{c}qab\Delta M. \tag{5.8}$$

A corresponding calculation with ABRAHAM's tensor yields

$$\Delta G_3^A = \frac{1}{c}\Delta \int_{\text{body}} EHdV = 0. \tag{5.9}$$

Since (5.7) and (5.8) are obviously in accordance with the balance of total momentum, COSTA DE BEAUREGARD concludes that MINKOWSKI's expression for the momentum density should be preferred.

Let us, however, continue to consider ABRAHAM's tensor and write the force density in the form

$$\boldsymbol{f}^A = \boldsymbol{f}^M + \frac{\partial}{\partial t}(\boldsymbol{g}^M - \boldsymbol{g}^A). \tag{5.10}$$

Hence, by integration over the total system

$$\left.\begin{aligned} \left(\int_{\text{wire}} + \int_{\text{body}}\right) f_3^A dVdt &= \int_{\text{wire}} f_3^M dVdt + \Delta \int_{\text{body}} (g_3^M - g_3^A)dV \\ &= \int_{\text{wire}} f_3^M dVdt + \frac{1}{c}\Delta \int_{\text{body}} DBdV = 0, \end{aligned}\right\} \tag{5.11}$$

in view of (5.7) and (5.8). Eqs. (5.11) and (5.9) show how the momentum balance must be interpreted in terms of ABRAHAM's tensor: Although the electromagnetic field represents a non-closed system, eq. (5.9) shows that the electromagnetic momentum is *conserved*. (In the case of MINKOWSKI's tensor this was not so, cf. eq. (5.8).) This conservation is carried into effect by the fact that the action from the force on the wire is equal and opposite to the action on the body, in accordance with (5.11). We note in passing that only ABRAHAM's tensor leads to a mechanical *force* on the test body in the z-direction, due to the fact that the surface forces on the body, which are common for the two tensors, are directed in the radial direction. There are also surface forces at the two end surfaces of the body, but these forces compensate each other. With MINKOWSKI's tensor, the presence of electromagnetic momentum is due to a momentum *flow* into the body.

Following COSTA DE BEAUREGARD we mention that the recent C. GOILLOT[27] experiment might be considered as a possible test of the theory. In this experiment a translational motion of a nature similar to the one described above was detected. However, although the qualitative features are similar, COSTA DE BEAUREGARD reports that the GOILLOT effect is far too large to correspond to the effect deduced from the electromagnetic energy-momentum tensors. The effect of the experiment is presumably a spin effect[28]. The inapplicability of the above theory should be expected in this case, since systems exhibiting remanent magnetization are very different from those described by the simple phenomenological theory we are considering.

On the Čerenkov effect

The ČERENKOV effect is a convenient means for a study of the various energy-momentum tensors. We have touched upon this effect before, in connection with relativistic considerations in I, section 10, and we shall take it up again in the relativistic considerations later on in this paper, but here we examine some of its implications when the medium is at rest. In this kind of problem it is most convenient to use MINKOWSKI's tensor, and let us also employ the phenomenological quantum theory (see, for instance, ref. 29 or ref. 30) according to which the four-momentum of the emitted photon is $\hbar k_\mu = \hbar(\boldsymbol{k}, i\omega/c)$. With MINKOWSKI's tensor the balance equations for energy and momentum for the photon plus its radiating electron with momentum $\boldsymbol{p} \to \boldsymbol{p}'$, are

$$c\sqrt{p^2 + m^2c^2} = \hbar\omega + c\sqrt{p'^2 + m^2c^2} \tag{5.12a}$$

$$\boldsymbol{p} = \hbar\boldsymbol{k} + \boldsymbol{p}', \tag{5.12b}$$

from which we obtain the well-known expression for the angle θ^M between $\boldsymbol{p}$ and $\boldsymbol{k}$:

$$\cos\theta^M = \frac{c}{mu} + \frac{k\hbar}{2p}\left(1 - \frac{1}{n^2}\right). \tag{5.13}$$

Here u is the modulus of the velocity of the incoming electron, $\boldsymbol{u} = \boldsymbol{p}/m(u)$.

From the point of view of ABRAHAM's tensor the above argument is only slightly modified: The momentum of the emitted photon in this case is $\hbar\boldsymbol{k}/n^2$, while the force $\boldsymbol{f}^A$ gives rise to a mechanical momentum $(n^2-1)\cdot\hbar\boldsymbol{k}/n^2$ which runs together with the field. These two contributions together yield the result $\hbar\boldsymbol{k}$ which was used in (5.12b).

Concerning the literature on this subject we should first of all refer to the clear discussion by G. GYÖRGYI(31). He shows the equivalence between MINKOWSKI's and ABRAHAM's tensors along similar lines as above. On the other hand, there has recently appeared a paper by J. AGUDÍN(32) on the ČERENKOV effect in which ABRAHAM's tensor, but not MINKOWSKI's tensor, is claimed to be in accordance with EINSTEIN's mass-energy relation. Let us therefore trace out the reason for this result, when we transform the formalism to our notation and simplify the argument, which consists in a study of the conservation equations for total energy, momentum and centre of mass-velocity. Imagine that the initial electron moves along the x-axis with the velocity u and that it emits a photon with mass m' in the direction θ at the time $t = t_1$. After the emission the electron moves with the velocity $u' = p/m(u')$ in the direction φ. The energy balance is written as

$$m(u) = \hbar\omega/c^2 + m(u'). \tag{5.14a}$$

With ABRAHAM's tensor the magnitude of the momentum of the emitted photon is $\hbar k/n^2 = \hbar\omega/(nc)$, and the balance equation for the x-component of momentum is written as

$$m(u)u = \frac{\hbar\omega}{nc}\cos\theta + G_1^{\mathrm{mech}} + m(u')u'\cos\varphi, \tag{5.14b}$$

where $\boldsymbol{G}^{\mathrm{mech}}$ is the momentum transferred to the medium.

Finally, AGUDÍN introduces an equation expressing the centre of mass-theorem. During the time period $t = 0$ to $t = t_2$, where $0 < t_1 < t_2$, the centre of mass of the total system is displaced by a distance $m(u)c^2ut_2/\mathscr{H}^{\mathrm{tot}}$, and the relation given by AGUDÍN is equivalent to writing

$$\left.\begin{aligned} m(u)ut_2 &= m'\left[ut_1 + \left(\frac{c}{n}\cos\theta\right)(t_2-t_1)\right] \\ &+ G_1^{\mathrm{mech}}(t_2-t_1) + m(u')[ut_1 + (u'\cos\varphi)(t_2-t_1)]. \end{aligned}\right\} \tag{5.15}$$

By inserting eqs. (5.14) into (5.15), one finds that the latter relation is fulfilled if $m' = \hbar\omega/c^2$, which is Einstein's mass-energy relation.

Considering Minkowski's tensor, Agudín uses the same set of equations as above with the single difference that the first term on the right hand side of (5.14b) is multiplied by a factor n^2. The new value for m' one now obtains shows an involved geometrical dependence which must be regarded as unphysical. From this he concludes that Abraham's tensor is the one of the two tensors that should be preferred.

Let us now examine the above argument from the point of view of our earlier interpretation. Since G_1^{mech} in (5.15) refers to the small motion of the medium induced by the photon, we must have $G_1^{\text{mech}} = ((n^2-1)/nc)\hbar\omega\cos\theta$. This value is in accordance with the value for G_1^{mech} appearing in equation (5.14b), which is constructed on the basis of Abraham's tensor. However, with Agudín's construction of the momentum balance in the case of Minkowski's tensor the right hand side of (5.14b) is changed into $(n\hbar\omega/c)\cos\theta + G_1^{\text{mech}} + m(u')u'\cos\varphi$. Thus the two values of G_1^{mech} become different; in (5.15) G_1^{mech} remains unchanged while in the momentum balance $G_1^{\text{mech}} = 0$. This is the reason for the diverging result. It is instructive to recall that the centre of mass-velocity for an arbitrary (limited) total system is given by $c^2\boldsymbol{G}^{\text{tot}}/\mathscr{H}^{\text{tot}}$ (cf. eq. (3.13)), which is a constant in view of the conservation equations for *energy* and *momentum*. Applied to the present case this means that the centre of mass-theorem can yield no more information than what is contained in eqs. (5.14) We are evidently free to assign a mass $m' = \hbar\omega/c^2$ to the photon also in the case of Minkowski's tensor.

Finally we note that the Čerenkov effect provides a convenient opportunity to examine also the radiation tensor (1.6). If we in this case construct the energy and momentum balance similar to (5.12) the only difference is that the term $\hbar\boldsymbol{k}$ in (5.12b) has to be replaced by $\hbar\boldsymbol{k}/n^2$; the radiation tensor is divergence-free and there is no force present to give rise to a mechanical momentum. Thus we find the following expression determining the angle θ^S between $\boldsymbol{p}$ and $\boldsymbol{k}$ in this case:

$$\cos\theta^S = \frac{nc}{u} - \frac{k\hbar}{2p}\left(1 - \frac{1}{n^2}\right). \tag{5.16}$$

Since $k\hbar \ll p$ this equation leads to unphysical values for θ^S. It seems therefore that there are even *formal* difficulties for the application of the radiation tensor to situations where both particles and fields are present.

Final remarks

So far we have limited ourselves to a study of previous treatments advocating the validity of either MINKOWSKI's or ABRAHAM's tensors. In this subsection we discuss briefly, without going into detail, some papers in which diverging tensor expressions have been given preference.

The tensor (1.8) introduced long ago by EINSTEIN and LAUB was encountered already in section 2, in connection with electrostatic phenomena. We recall the important result that the excess pressure effect in a dielectric liquid predicted by this tensor does not fit the HAKIM-HIGHAM experiment. Let us yet write down the complete force expression in the time-varying case:

$$\boldsymbol{f}^E = \varrho \boldsymbol{E} + (\boldsymbol{P}\cdot\nabla)\boldsymbol{E} + (\boldsymbol{M}\cdot\nabla)\boldsymbol{H} + \frac{1}{c}(\boldsymbol{j}\times\boldsymbol{H}) + \frac{1}{c}(\dot{\boldsymbol{P}}\times\boldsymbol{H}) + \frac{1}{c}(\boldsymbol{E}\times\dot{\boldsymbol{M}}). \tag{5.17}$$

It should be noted that according to (5.17) the magnetic force density acting on a stationary current distribution, for instance in the interior of a wire, is equal to $(1/c)(\boldsymbol{j}\times\boldsymbol{H})$, instead of the usual $(1/c)(\boldsymbol{j}\times\boldsymbol{B})$ following from ABRAHAM's or MINKOWSKI's tensors. Now, in order to support their force expression, EINSTEIN and LAUB analyse in their paper(5) two examples involving the presence of stationary currents. The second example considered is the following: An infinitely long, rectilinear wire carrying a stationary current $\boldsymbol{J}$ is assumed to prossess a magnetization $\boldsymbol{M}$ in a direction perpendicular to the wire. When no external field is present, it is clear that the electromagnetic force on the wire vanishes. EINSTEIN and LAUB verify by a direct calculation that their tensor leads actually to a vanishing force F_i per unit length in a direction i perpendicular to the wire. We must point out however, that this result is *not* peculiar for the EINSTEIN-LAUB tensor and thereby does not represent any particular support for this tensor. In fact, *any* of the actual tensor expressions will lead to this result, as an immediate consequence of the relations

$$F_i = -\int\limits_{\substack{\text{unit}\\\text{length}}} \partial_k S_{ik}\, dV = -\int S_{ik}^{\text{vac}}\, n_k\, dS, \tag{5.18}$$

where the value of the last integral goes to zero when the integration surface is taken sufficiently far away from the body.

Concerning the remaining terms in (5.17) we mention that the argument for introducing the term $(1/c)\,(\dot{\boldsymbol{P}}\times\boldsymbol{H})$ was that there must be no distinction in principle between external currents $\boldsymbol{j}$ and polarization currents $\dot{\boldsymbol{P}}$ (cf.

also our remarks in connection with eqs. (3.5) and (3.6)). The magnetic terms in (5.17) were introduced by analogy considerations.

EINSTEIN and LAUB's paper was criticized by R. GANS[33]. He employed the force expression corresponding to MINKOWSKI's tensor, at least for para- and diamagnetic media, and made an explicit calculation of the transversal force on a conductor which carries stationary current and is surrounded by an external magnetic field. Ferromagnetic media were considered separately. In all cases the force was found to vanish when the external field is zero, in accordance with our statement above.

One remark is called for, regarding GANS' claim that the EINSTEIN-LAUB expression comes into conflict with the energy balance. In his argument he uses assumptions that are valid for MINKOWSKI's tensor only, viz. that the energy flux vector is given as $\boldsymbol{S} = c(\boldsymbol{E} \times \boldsymbol{H})$ also when the velocity of the medium is different from zero. The other tensor expressions will lead to an explicit appearance of the velocity in the energy flux expression.

The use of thermodynamic methods in the present problem represents a special kind of approach. We have already employed such a method in this paper, although in a very simple form, in section 2. In this context we should refer to the work by DE SA[34] and to two papers by KLUITENBERG and DE GROOT[35]. KLUITENBERG and DE GROOT postulate a certain relativistic GIBBS relation and assume the material energy-momentum tensor to be symmetric; they obtain from these assumptions a symmetrical electromagnetic tensor which in the rest system is in accordance with eqs. (1.9), apart from a difference in the energy density component. Further, they claim that the formalism yields ABRAHAM's tensor as an equivalent result, if appropriate new definitions for the hydrostatic pressure and the internal energy are imposed. Concerning this latter statement, however, we must point out that the formalism must always be chosen so as to conform to the observed effects, and the HAKIM-HIGHAM experiment does not seem to leave the room for ambiguities in the definition of pressure in the electrostatic case (cf. section 2).

The papers by G. MARX, G. GYÖRGYI and K. NAGY[3, 36, 37, 38] (with further references) contain a series of arguments of different kinds, and represent together one of the most extensive macroscopic treatments of the problem that has been given. We are considering elements of their papers at various places in our work, for instance in the examination of the radiation tensor. Their main conclusion is that ABRAHAM's tensor is the basic electromagnetic tensor, while the radiation tensor (instead of MINKOWSKI's tensor) is claimed to be the result of a combination with the excited matter induced by a propagating field. Since in this section we consider fields within matter at rest, we should mention that the difficulty they claim to exist for MINKOWSKI's tensor in explaining the propagation of the centre of mass for a limited radiation field within an isotropic dielectric, is cleared up of one observes that the time derivative of the quantity $(\boldsymbol{S}^M/c^2 - \boldsymbol{g}^M)$, integrated over the total volume, is equal to zero.

As we have noted, the absence of terms containing polarization and magnetization entities in MINKOWSKI's force has represented an obstacle for the acceptance of this expression (cf. also the book by FANO, CHU, ADLER(39)). In a series of papers published recently(40), P. POINCELOT took the full consequence of the opposite point of view and proposed the introduction of all kinds of polarization and magnetization terms in the force on an equal footing with the free charge and current terms, viz.

$$\boldsymbol{f} = (\varrho - \nabla \cdot \boldsymbol{P})\boldsymbol{E} + \frac{1}{c}(\boldsymbol{j} + \dot{\boldsymbol{P}} + c\nabla \times \boldsymbol{M}) \times \boldsymbol{B} \tag{5.19a}$$

$$f_4 = \frac{i}{c}\boldsymbol{E} \cdot (\boldsymbol{j} + \dot{\boldsymbol{P}} + c\nabla \times \boldsymbol{M}). \tag{5.19b}$$

The tensor corresponding to the force (5.19) can be expressed in terms of $\boldsymbol{E}$ and $\boldsymbol{B}$ in the same form as the electromagnetic tensor in the vacuum-field. However, although (5.19) cannot be rejected on purely formal grounds, we cannot find any argument of convenience or experimental evidence that supports this expression.

6. Angular Momentum in Arbitrary Inertial Systems

In the remaining part of our work we shall be concerned with topics connected with relativity. To some extent we shall have the opportunity to return to a study of situations which were considered already in I, chapter IV, in connection with MINKOWSKI's tensor. From the preceding it should be clear that in a relativistic theory the latter tensor is convenient to use, in order to obtain information about the direction and velocity of the propagating field energy. But it is instructive to consider also the behaviour of the alternative tensors (especially ABRAHAM's tensor) in arbitrary inertial systems, since such an anlysis will exhibit characteristic differences between the tensors. In this section we assume that the medium is homogeneous and isotropic, and let as usual K denote the inertial system in which the rest system K^0 moves with the velocity v along the x-axis.

Evaluation of torques within an infinite medium

Let us image a finite radiation field within a large (infinite) dielectric medium. The angular momentum quantities $M_{\mu\nu}$ are in general defined by the integral

$$M_{\mu\nu} = \int (x_\mu g_\nu - x_\nu g_\mu) dV, \tag{6.1}$$

taken over the whole field, in any frame K. Let us further imagine that for each of the electromagnetic tensors in question we insert the appropriate expression for g_μ into the integral in (6.1) and calculate $M_{\mu\nu}$. In this context it should be emphasized that in each case g_μ is considered as a *field* quantity, $M_{\mu\nu}$ thus being considered as a field angular momentum. This definition is the natural one and we have used it throughout, in I as well as in the present paper, although we have repeatedly pointed out that in the MINKOWSKI case the momentum density g_μ^M in reality includes also a mechanical part which is responsible for the asymmetry of MINKOWSKI's tensor. In other words, MINKOWSKI's angular momentum $M_{\mu\nu}^M$ contains in a strict sense also a contribution from the mechanical part of the total system. To call $M_{\mu\nu}^M$ a field angular momentum is obviously just tantamount to calling G_μ^M a field linear momentum. If on the other hand we take the distinction between the two parts of G_μ^M explicitly into account and exclude the mechanical part of g_μ^M from the expression for field angular momentum, we obtain instead ABRAHAM's expression $M_{\mu\nu}^A$, since that part of g_μ^M which pertains to the electromagnetic field is just g_μ^A. The different ways of dividing the total angular momentum into a field part and a mechanical part obviously have no influence upon the conservation of total angular momentum, which is a consequence of the symmetry and the zero divergence of the total energy—momentum tensor. Thus, in each case we obtain the mechanical angular momentum by inserting that part of the total momentum density which is not counted as a field entity.

As regards MINKOWSKI's tensor it seems appropriate to recall from I, section 11 that the quantities $M_{\mu\nu}^M$ are equivalent to the angular momentum quantities one can most simply construct on the basis of NOETHER's theorem. This is in accordance with the general property of MINKOWSKI's tensor that it readily adjusts itself to the LAGRANGIAN procedures. We recall also that $M_{\mu\nu}^M$ is in general not a tensor.

For a comparison between the various tensors it is however not the angular momentum itself which is of primary interest in each case, but rather its time derivative, i.e. essentially the body *torque*. The torque is defined as $\boldsymbol{N} = -d\boldsymbol{M}/dt$, and we shall in the present subsection start to perform a direct calculation of the torques corresponding to ABRAHAM's and MINKOWSKI's tensors. It will turn out that the two values so obtained in general are different from each other. This difference is what we should expect, since the momentum densities $\boldsymbol{g}^A$ and $\boldsymbol{g}^M$ are themselves essentially different in direction and magnitude.

The last point requires some further explanation. In all electrostatic (or magnetostatic) cases and also in all *high-frequency* electromagnetic cases considered up till now we have found that ABRAHAM's and MINKOWSKI's tensors yield just identical expressions for the torque on a test body immersed either in a vacuum or in a dielectric fluid. The reason for this equality can be understood in a simple way by observing that in those cases the torque could be evaluated as a function of the field stress tensor taken in the domain just *outside* the surface of the body, wherein the equality $S_{ik}^A = S_{ik}^M$ is valid. (Cf. eqs. (2.11) and (4.8b) for the electrostatic and electromagnetic cases, respectively.) In the situations considered in the present section there is however no similar reason why the torque expressions should be the same; we have to lean directly upon the formula (6.1) and evaluate it over the field region within the body. In the MINKOWSKI case the torque can be looked upon as a consequence of the *asymmetry* of the mechanical energy-momentum tensor (this fact having represented as an objection to the acceptance of MINKOWSKI's tensor), while in the ABRAHAM case the torque arises because of the *force* density.

In spite of this difference between the two torque expressions obtained within an infinite medium we shall nevertheless in the next subsection see that the torques are relativistically equivalent from a physical point of view, since both of them are compatible with uniform motion of the physical system in K. In this context we shall draw into consideration the analogous situation encountered in relativistic mechanics of elastic media: An elastic body subjected to stresses in the rest frame will in general require a torque to maintain steady motion in another inertial frame.

Let us now start with ABRAHAM's tensor and perform the calculation. From (6.1) it appears that the torque $\boldsymbol{N}^A = -d\boldsymbol{M}^A/dt$ in K is given by

$$\boldsymbol{N}^A = \int (\boldsymbol{r} \times \boldsymbol{f}^A)\, dV. \tag{6.2}$$

At first sight it seems that one will meet a difficulty in the evaluation of this integral. This difficulty is connected with the non-invariance in four-space of the world lines corresponding to ABRAHAM's energy flux $\boldsymbol{S}^A$ (cf. the next section). On the other hand we pointed out in I, section 9 that the ray velocity $\boldsymbol{u}$, which is the velocity of propagation of the wave energy and which may be written as $\boldsymbol{u} = \boldsymbol{S}^M/W^M$, transforms like a particle velocity. From this it follows that the world lines corresponding to MINKOWSKI's energy flux $\boldsymbol{S}^M$ really have the property that they remain invariant in four-space upon a LORENTZ transformation. Now it is clear that in order to obtain a picture of the wave propagation in K one has to transform the *total* wave, i.e. one must include the effect also from the produced mechanical momentum $\boldsymbol{g}^{\text{mech}\,0}$ in K^0. This feature resolves the apparent dilemma in connection with the evaluation of the integral in (6.2): Even though $\boldsymbol{S}^A$ is different from $\boldsymbol{S}^M$ both in direction and magnitude we have to integrate over that part of space where the field is actually present, i.e. across the world lines corresponding to $\boldsymbol{S}^M$.

It is now convenient to assume that the field travels parallel to the xy-plane in such a way that any wave vector $\boldsymbol{k}$ which is contained in the wave, makes an angle ϑ with the x-axis in K. It can readily be verified that the only non-vanishing component in (6.2) is the z-component, the other components fluctuate away. We evaluate the integral in (6.2) over the domain AB, i.e. over the hypersurface $t = 0$ (cf. I, Fig. 2). We obtain

$$N_3^A = \gamma \int_{AB} (x_1^0 f_2^{A^0} - x_2^0 f_1^{A^0} + \beta c t^0 f_2^{A^0})\, dV. \tag{6.3}$$

This integral is to be transformed into an integral taken at constant time in K^0, and similarly as in I, section 12, we choose the domain CD for which $t^0 = 0$. The world lines determined by $\boldsymbol{S}^M$ will each intersect AB and CD in two points with coordinates $(x_i^0(AB), t^0(AB))$ and $(x_i^0(CD), 0)$ in K^0, such that

$$\left.\begin{aligned} x_1^0(CD) &= x_1^0(AB) - \frac{c}{n} t^0(AB)\cos\vartheta^0 = x_1^0(AB)\left(1 + \frac{\beta}{n}\cos\vartheta^0\right) \\ x_2^0(CD) &= x_2^0(AB) - \frac{c}{n} t^0(AB)\sin\vartheta^0 = x_2^0(AB) + \frac{\beta x_1^0(CD)\sin\vartheta^0}{n + \beta\cos\vartheta^0} \\ x_3^0(CD) &= x_3^0(AB), \;\; t^0(AB) = -\frac{\beta}{c} x_1^0(AB). \end{aligned}\right\} \tag{6.4}$$

The calculation is carried out in a similar way as in section 12 of I, so we abstain from a detailed exposition. The relation between the volume element dV and the element dV^0, taken at constant time in K^0, is given by (I, 12.9). We find

$$N_3^A = \int\limits_{CD} \frac{x_1^0(f_2^{A^0}/\gamma^2 + (\beta/n)\sin\vartheta^0 f_1^{A^0}) - (1+(\beta/n)\cos\vartheta^0)x_2^0 f_1^{A^0}}{(1+(\beta/n)\cos\vartheta^0)^2} dV^0. \qquad (6.5)$$

Since $f^{A^0} = [(n^2-1)/nc]\partial W^0/\partial t^0$, a representative term in (6.5) can be transformed as follows:

$$\int x_1^0 f_2^{A^0} dV^0 = \frac{n^2-1}{nc}\mathscr{H}^0 \sin\vartheta^0 \frac{d}{dt^0}\left[\frac{1}{\mathscr{H}^0}\int x_1^0 W^0 dV^0\right] = \frac{n^2-1}{n^2}\mathscr{H}^0 \sin\vartheta^0 \cos\vartheta^0. \quad (6.6)$$

In the second term we have here used the fact that $d/dt^0[\;] = (c/n)\cos\vartheta^0$, the centre of mass-velocity in the x^0-direction. By a similar treatment of the other terms in (6.5) we find

$$N_3^A = -\beta^2 \frac{n^2-1}{n^2} \frac{\sin\vartheta^0 \cos\vartheta^0}{(1+(\beta/n)\cos\vartheta^0)^2}\mathscr{H}^0. \qquad (6.7)$$

Thus there results a non-vanishing torque also with the symmetrical Abraham tensor. So far we have considered only the case where the domains AB and CD are placed at $t = 0$ and $t^0 = 0$ respectively; however, the same result applies also when AB and CD are placed at arbitrary constant times in K and K^0 due to the fact that the force density fluctuates away when integrated over space. So the expression (6.7) is constant in time.

Let us now consider Minkowski's tensor. From (6.1) we find

$$N_3^M = \int (S_{12}^M - S_{21}^M)\, dV. \qquad (6.8)$$

Now $S_{21}^M - S_{12}^M = [(n^2-1)/n]\beta\gamma W^0 \sin\vartheta^0$, and the integration in (6.8) can be carried out in the same way as above. We get

$$N_3^M = -\beta \frac{n^2-1}{n} \frac{\sin\vartheta^0}{1+(\beta/n)\cos\vartheta^0}\mathscr{H}^0. \qquad (6.9)$$

We see that the expressions (6.7) and (6.9) in general are different from each other, although they both vanish in the rest frame as they should. It is therefore natural to ask whether it is possible to single out one of these these expressions as preferable. As we shall now see this is not so in the case of an infinite medium, since the torque expressions (6.7) and (6.9) may be looked upon as representing relativistic effects of the same nature as the non-observable effect encountered in ordinary relativistic mechanics of an elastic body possessing stresses in its rest frame.

A relativistic effect

Let us first recall the following situation from mechanics: If an elastic body is subjected to stresses in its rest frame it may in other frames exhibit a momentum component at right angle to the direction of motion. Consequently, the body will require a torque in order to maintain its uniform motion.

We find it desirable to go into some details. Let τ^0_{ik} be the mechanical stress tensor of the elastic body in K^0. The mechanical torque in K is

$$\boldsymbol{N} = \int (\boldsymbol{r} \times \boldsymbol{f})\, dV. \tag{6.10}$$

Then make the explicit requirement that the body remain in steady motion in K. This means that we can put $d\boldsymbol{g}/dt = 0$, where $g_i = -i\tau_{i4}/c$ and the time derivative is taken along the volume elements dV which follow the body. Thus, the body experiences a change of angular momentum equal to

$$\frac{d\boldsymbol{M}}{dt} = \int \left(\frac{d\boldsymbol{r}}{dt} \times \boldsymbol{g}\right) dV, \tag{6.11}$$

since also $(d/dt)dV = 0$. Inserting $d\boldsymbol{r}/dt = \boldsymbol{v}$ we obtain

$$\frac{d\boldsymbol{M}}{dt} = \int (\boldsymbol{v} \times \boldsymbol{g})\, dV = \boldsymbol{v} \times \boldsymbol{G}. \tag{6.12}$$

If the torque (6.10) is equal to the amount (6.12) which the body actually requires in order to preserve stationary motion, then the scheme is consistent, and we have an example of a situation where the existence of a torque is not followed by a rotation. We have to stress the difference between the calculations that led to (6.10) and (6.12): In the first case, the velocity of the body was required to be equal to $\boldsymbol{v}$, and we can imagine that this requirement is fulfilled at a certain time in K just after the LORENTZ transformation from K^0 has been performed. But in the latter case, the body velocity was required to be the same at an *arbitrary* instant afterwards, corresponding to the fact that the directions of the world lines of the body were required to be unaltered.

It appears from the text-books (M. VON LAUE(41), R. BECKER(42)) that the equivalence between $\boldsymbol{N}$ and $d\boldsymbol{M}/dt$ has been verified in certain special cases. But the equivalence can also be shown quite generally for an elastic body, by the following simple consideration.

Let us calculate a typical component of the torque in K, say the z-component. We readily find by an insertion into (6.10)

$$N_3 = \int_{AB} [\gamma(x_1^0 + vt^0) f_2^0 - x_2^0 \gamma f_1^0] \gamma^{-1} dV^0 = \int_{CD} (\gamma^{-2} x_1^0 f_2^0 - x_2^0 f_1^0) dV^0. \quad (6.13)$$

Using now the fact that $f_i^0 = \partial_k^0 \tau_{ik}^0$, we can write

$$N_3 = \int_{CD} (\gamma^{-2} x_1^0 \tau_{2k}^0 - x_2^0 \tau_{1k}^0) n_k^0 dS^0 + \beta^2 \int_{CD} \tau_{12}^0 dV^0 = \beta^2 \int_{CD} \tau_{12}^0 dV^0, \quad (6.14)$$

since the surface integral is performed over a surface outside the body where τ_{ik}^0 vanishes.

Further, by means of the relation $\tau_{24} = i\beta\gamma\tau_{12}^0$ we readily obtain by an insertion into (6.12)

$$\frac{dM_3}{dt} = \beta^2 \int_{CD} \tau_{12}^0 dV^0. \quad (6.15)$$

Eqs. (6.14) and (6.15) show the consistency in the case of an elastic body: The body is acted upon by a torque which is equal to the change of momentum required in order to maintain steady motion.

After this digression let us return to the radiation field. The torque on the body is defined as

$$N_l = \int (x_i f_k - x_k f_i + S_{ik} - S_{ki}) dV, \quad (6.16)$$

where i,k,l are cyclic. (Actually, the expression (6.16) has been derived indirectly as $N_l = - dM_{ik}/dt$; however, the coordinate dependent terms in (6.16) appear similarly as in (6.10), and the two last terms in (6.16) must yield the appropriate torque contribution from the tensor asymmetry, cf. for instance the considerations in section 4 of I.) The expressions for N_l that we need here have been derived in (6.7) and (6.9).

Next, require explicitly steady motion in K. The necessary and sufficient conditions are: (1) The body velocity $\boldsymbol{v} = (v,0,0) =$ constant; (2) $d\boldsymbol{r}/dt = \boldsymbol{u} = \boldsymbol{S}^M/W^M$ along the moving wave elements dV. From these conditions it follows that U_μ is a four-vector and that the world lines remain invariant in four-space. Moreover, it follows that $d\boldsymbol{g}/dt = 0$ along the wave trajectories, since g_i (for any field tensor) is proportional to the energy density W^0, which is a function of the invariant wave phase φ, φ being constant along the trajectories. Thus, taking the time derivative of the field angular momentum we obtain in the two cases

$$dM_3^A/dt = (\boldsymbol{u} \times \boldsymbol{G}^A)_3, \quad dM_3^M/dt = (\boldsymbol{u} \times \boldsymbol{G}^M)_3. \tag{6.17}$$

If we here insert the appropriate values for $\boldsymbol{u}$, $\boldsymbol{G}^A$ and $\boldsymbol{G}^M$ we will find the expressions (6.7) and (6.9) respectively, with the opposite sign. If now ABRAHAM's or MINKOWSKI's tensor is taken to describe the field, it follows from the conservation of total angular momentum that the rate of change of the *mechanical* angular momentum is given by the expression (6.17), with the opposite sign. In both cases we therefore find that the scheme is consistent in the same way as it was found to be in the situation considered previously (cf. (6.14) and (6.15)): The body is acted on by a torque which is just equal to the rate of change of mechanical angular momentum being necessary in order to prevent rotation.

At this place we should make a comment on an assertion put forward by VON LAUE in § 19 of his book[41], concerning a verification of the principle of conservation of total angular momentum if MINKOWSKI's tensor is used for the field. This is actually one of the arguments VON LAUE presents in favour of MINKOWSKI's tensor. He first writes the rate of change of field angular momentum similarly as the last of eqs. (6.17), by taking the time derivative along the moving wave elements. Thereafter, and this is the crucial point, the z-component of the torque on the body is claimed to be given by

$$\int (x_1 \partial_k S_{2k}^M - x_2 \partial_k S_{1k}^M)\, dV = \int (S_{12}^M - S_{21}^M)\, dV. \tag{6.18}$$

Since it can be shown that $(\boldsymbol{u} \times \boldsymbol{g}^M)_3 = S_{21}^M - S_{12}^M$, VON LAUE concludes that the conservation of total angular momentum is verified in the present case.

We cannot find, however, any reason why this torque component should be given by the left hand side of eq. (6.18). Moreover, one cannot find expressions for the rate of change of the field angular momentum and the body torque *independently* of each other, and thereafter check the angular momentum balance. Instead, the torque is found by just *requiring* the angular momentum balance to hold, such that $\boldsymbol{N}$ be given by the relation $\boldsymbol{N} = -d\boldsymbol{M}/dt$.

Finite bodies

Hitherto we have restricted ourselves to a consideration of very large (or infinite) dielectric bodies. The case of finite bodies is important, however, since it reveals characteristic features of the angular momentum balance.

Let us therefore consider this case, and for definiteness assume that an optical wave passes through an isotropic and homogeneous glass plate, for instance at BREWSTER's angle of incidence in K^0. The electromagnetic forces are present only in the boundary layers, and we shall assume that an external mechanical surface force $\boldsymbol{F}^{\text{ext}\,0}$ just counterbalances the surface force $\boldsymbol{F}^0$ caused by the field, in such a way that the field is not disturbed. The consequence of the last assumption is that the mechanical angular momentum of the body is conserved in K^0, $\boldsymbol{N}^{A^0} = \boldsymbol{N}^{M^0} = -\boldsymbol{N}^{\text{ext}\,0}$, and that the presence of extra mechanical stresses due to the external forces is avoided.

We now consider the system in the frame K, and adopt ABRAHAM's tensor as the field tensor. From (6.16) it is apparent that the torque is given as $\boldsymbol{r} \times \boldsymbol{f}^A$, integrated over the internal volume, plus $\boldsymbol{r} \times \boldsymbol{F}^A$, integrated over the surfaces. We readily find that the contribution from the first term is zero, and as the electromagnetic surface force $\boldsymbol{F}^A$ transforms similarly as the external force $\boldsymbol{F}^{\text{ext}}$, we can write

$$\boldsymbol{N}^A = -d\boldsymbol{M}^A/dt = -\boldsymbol{N}^{\text{ext}}. \tag{6.19}$$

Thus we obtain the satisfactory explanation that the net torque acting on the body is still zero. If, however, MINKOWSKI's tensor is adopted for the field, the situation is changed. We see that $\boldsymbol{f}^M = 0$ in the interior domain and that $\boldsymbol{F}^M = \boldsymbol{F}^A$ so that the contribution from the forces is the same, but there appears an extra volume effect in the torque because of the asymmetry of the stress tensor S_{ik}^M. According to the theory the body is thus acted upon by a net torque in K, although the motion is uniform and although no account has to be taken of the influence from elastic stresses in K^0. We find this property to be rather inconvenient. It does not mean, however, that MINKOWSKI's torque expression should simply be rejected. For we may carry through an analysis of the same kind as in the previous subsection, where now the time derivatives are to be taken along the moving *body* elements, and will find that also now the MINKOWSKI torque is compatible with the requirement of steady motion. The peculiar property of MINKOWSKI's torque is obviously a consequence of the fact that the momentum density $\boldsymbol{g}^{M^0}$ contains both a pure field quantity $\boldsymbol{g}^{A^0}$ and a mechanical quantity $\boldsymbol{g}^{\text{mech}^0}$, cf. also the remarks in the beginning of this section. In conclusion, the study of the case of finite bodies reveals the characteristic effect that the most natural division of the total angular momentum into a field part and a mechanical part is made when one adopts ABRAHAM's expression for the field. On the other hand, in the case of *infinite* bodies we saw in the previous subsection that no preference could be assigned to either of the two torque expressions.

At this place a remark is in order, in connection with a comparison with the situation where an electromagnetic wave passes through a finite, anisotropic body at rest. Such a situation was considered in section 4, and we recall that the equation $\boldsymbol{N}^{A^0} = \boldsymbol{N}^{M^0}$ was found to hold in general. Now our present situation resembles the wave-crystal situation from section 4, since an isotropic body in K^0 becomes anisotropic in K. We may note that the total angular momentum in the vacuum field when the wave has left the body is independent of which energy-momentum tensor is used for the field, since the direction of the wave propagation in either case is determined from $\boldsymbol{S}^M$. Yet we have found that $\boldsymbol{N}^A$ in general is different from $\boldsymbol{N}^M$ when $\beta \neq 0$.

To point out the difference between these two cases let us once again examine the torque balance (4.1):

$$\boldsymbol{N} = -d/dt\boldsymbol{M}^{\mathrm{vac}} - d/dt\boldsymbol{M}^i, \tag{6.20}$$

where now the time derivatives are taken along the moving body. In addition to the assumption of the independence of $\boldsymbol{M}^{\mathrm{vac}}$ we could, in the case considered in section 4, require on physical grounds that $\boldsymbol{N}$ be independent of the interaction period T (assumed a stationary field during this period), expecially in the small period when the field leaves the medium. The crucial point here is that this latter requirement can no longer be upheld when the body moves. Consequently, $d\boldsymbol{M}^i/dt$ is in general different from zero, i.e. the torque depends in this case also on the internal field. We note that $d\boldsymbol{M}^i/dt \neq 0$ also with ABRAHAM's tensor.

As mentioned above the purpose of assuming $\boldsymbol{F}^0 = -\boldsymbol{F}^{\mathrm{ext}\,0}$ was to obtain a situation in which no complication will arise because of extra mechanical stresses set up in K^0. Let us now briefly consider how the situation is changed if we let the same value of $\boldsymbol{N}^{\mathrm{ext}\,0}$ be obtained by external surface forces which do not compensate the electromagnetic forces at each surface element. In this case there will appear mechanical stresses in K^0, described by the mechanical stress tensor τ^0_{ik}. These stresses may lead to non-vanishing momentum components at right angle to the velocity $\boldsymbol{v}$ in K, and thus be connected with the torque $N_i^{\mathrm{stress}} = -(i/c)\,\delta_{ijk}\int v_j \tau_{k4}\,dV$ which follows from the requirement of steady motion. This amount is equal to the resulting torque exerted by the forces, so that we obtain instead of eq. (6.19) the equation

$$\boldsymbol{N}^A + \boldsymbol{N}^{\mathrm{ext}} = \boldsymbol{N}^{\mathrm{stress}}. \tag{6.21}$$

So far we have considered only ABRAHAM's and MINKOWSKI's tensors. Let us finally for a moment consider the radiation tensor $S^S_{\mu\nu}$, which is symmetric and divergence-free within an isotropic medium. In the situation considered in the first subsection above it follows immediately that $N^S_3 = 0$, so that according to the radiation tensor the angular momenta of the field and the body are conserved separately. If the body is finite, the radiation tensor behaves similarly as ABRAHAM's tensor in the sense that the torque in K is determined by the surface forces only. It should however be borne in mind that the radiation tensor yields already in the rest frame a surface force with another direction and magnitude than ABRAHAM's surface force, although the torques are the same (cf. section 4).

7. Further Considerations on Relativity

In this section we continue the investigation of relativistic phenomena. Only effects involving special relativity will be considered. For the main part we shall be concerned with topics that were studied in chapter IV of I in connection with MINKOWSKI's tensor, and shall relate the phenomena to the other tensors. In the following two subsections we study two subjects that are closely related to each other, namely the velocity of the energy in an electromagnetic wave and the behaviour of the relativistic centre of mass.

Transformation of the energy velocity in a light wave

Consider a plane light wave within an isotropic and homogeneous insulator moving with the uniform four-velocity V_μ in the frame K. Similarly as in I, section 9, the ray velocity $\boldsymbol{u}$ is defined as the velocity of propagation of the light energy. The ray velocity is in general different both in magnitude and direction from the phase velocity. We recall that it is shown in MØLLER's book[7] that the ray velocity transforms like the velocity of a material particle, and further that this transformation property is verified experimentally in the FIZEAU experiment, at least to the first order in v/c.

If now an energy-momentum tensor $S_{\mu\nu}$ shall describe the whole travelling wave, it must be possible to relate the ray velocity $\boldsymbol{u}$ to the components of this tensor by the equation $\boldsymbol{u} = \boldsymbol{S}/W$. For such a tensor the quantity $\boldsymbol{S}/W$ must therefore transform like a particle velocity. To investigate whether $S_{\mu\nu}$ behaves in this way is tantamount to examining whether the quantities

$$U_\mu = \left(\frac{\boldsymbol{S}/W}{\sqrt{1 - S^2/(c^2 W^2)}}, \quad \frac{ic}{\sqrt{1 - S^2/(c^2 W^2)}} \right) \tag{7.1}$$

constitute a four-vector. As stated already in I, MØLLER has shown that the sufficient and necessary condition for U_μ being a four-vector is that

$$R_{\mu\nu} \equiv S_{\mu\nu} + \frac{1}{c^2} S_{\mu\alpha} U_\alpha U_\nu = 0 \tag{7.2}$$

in some inertial system.

We recall that by inserting MINKOWSKI's tensor one really finds $R_{\mu\nu}^{M^0} = 0$ in the case of a most general plane wave. This circumstance thus provides a further support for our general assertion that MINKOWSKI's tensor describes the whole travelling wave. In particular, if a ray travels parallel to the direction of the medium velocity, one obtains immediately by means of MINKOWSKI's tensor the well known formula, to the first order in v/c,

$$u = \frac{S^M}{W^M} = \frac{c}{n} + v\left(1 - \frac{1}{n^2}\right). \tag{7.3}$$

This formula was verified in the FIZEAU experiment.

After this summary of the results obtained in section 9 of I, we investigate how the situation looks from the point of view of ABRAHAM's tensor. In this case one readily finds that $R_{\mu\nu}^{A^0} \neq 0$ in general, so that the equation (7.2) is not fulfilled and $\boldsymbol{S}^A/W^A$ does not transform like a particle velocity. Correspondingly, the last of eqs. (7.3) is replaced by

$$\frac{S^A}{W^A} = \frac{c}{n} + 2v\left(1 - \frac{1}{n^2}\right), \tag{7.4}$$

which is essentially different from (7.3). This kind of behaviour is what we should expect: ABRAHAM's tensor leaves out of consideration the influence from the produced mechanical momentum $\boldsymbol{g}^{\text{mech}^0}$ in K^0, and thus $\boldsymbol{S}^A/W^A$ cannot be expected to be equal to the ray velocity. The non-compatibility between the transformation criterion and the ABRAHAM tensor evidently does not represent a real difficulty for this tensor.

Let us now follow a more general line of approach and try to find a set of reasonable conditions under which the quantity $\boldsymbol{S}/W$, obtained from some energy-momentum tensor $S_{\mu\nu}$, actually obeys the transformation criterion. To this end it is advantageous first to recall the essential assumptions inherent in MØLLER's proof (in § 24 of his book(7)) about the transformation character of the ray velocity $\boldsymbol{u}$: In the first place, the equation for the wave front of an elementary spherical wave in K^0 being emitted from the origin at the time $t^0 = 0$ is written as

$$r^{02} - \frac{c^2}{n^2} t^{02} = 0. \tag{7.5}$$

In the second place, the corresponding equation for the wave front in K is obtained by means of point transformations of each term in (7.5), so that the world lines are assumed to remain invariant in four-space upon a LORENTZ transformation. By means of these conditions MØLLER derives that $\boldsymbol{u}$ transforms like a particle velocity.

Our task is now to transform the above conditions into equivalent conditions imposed on the tensor $S_{\mu\nu}$. In accordance with (7.5) we shall first require that the magnitude of the velocities of propagation of energy and momentum is equal to c/n, as expressed by the equations

$$S^0_{4k} = -\frac{i}{n} S^0_{44} e^0_k, \quad S^0_{ik} = -\frac{i}{n} S^0_{i4} e^0_k, \tag{7.6}$$

where $\boldsymbol{e}^0$ is the wave normal for the plane wave. Note that these conditions actually mean also that the field is closed, i.e. $\partial^0_\nu S^0_{\mu\nu} = 0$, since each frequency component of the plane wave depends on the wave phase $(\boldsymbol{k}^0 \cdot \boldsymbol{r}^0 - \omega^0 t^0)$ so that $e^0_k \partial^0_k$ may be replaced by $-n\partial/(c\partial t^0)$. If we now insert the conditions (7.6) into the expression (7.2) for $R_{\mu\nu}$, we really find that $R^0_{\mu\nu} = 0$.

So far we have only shown that the conditions (7.6) are *sufficient* to satisfy the transformation criterion; we have not verified that they are *necessary*. In fact, if we merely maintain the single restriction that S^0_{ik} be proportional to $S^0_{i4} e^0_k$, we find that the relation

$$S^0_{ik} = -\frac{i}{c} \frac{|\boldsymbol{S}^0|}{W^0} S^0_{i4} e^0_k \tag{7.7}$$

is necessary to yield $R^0_{\mu\nu} = 0$. Evidently eq. (7.7) becomes equal to the last of eqs. (7.6) when $|\boldsymbol{S}^0|/W^0 = c/n$. Note that the weak condition (7.7) does not even imply that $S^0_{\mu\nu}$ be divergence-free. We think that this condition is of minor physical interest, however, since it is preferable to construct the theory so as to conform to the equation (7.5) (or the wave equation) in a simple way, i.e. one should always take $|\boldsymbol{S}^0|/W^0 = c/n$.

It has been pointed out by G. MARX *et al*[3] that the radiation tensor $S^S_{\mu\nu}$ also obeys the transformation criterion. This feature can be explained on the basis of eqs. (7.6), since the radiation tensor satisfies these equations. On the contrary, both ABRAHAM's tensor and the DE GROOT-SUTTORP tensor (1.9) are incompatible with the condition (7.7) as well as the transformation criterion $R_{\mu\nu} = 0$.

Centre of mass

Let us now assume that the interior domain of the radiation field can be taken as a part of a *monochromatic* plane wave with wave vector $\boldsymbol{k}$. Similarly as in section 12 of I we further assume that the small boundary layer—in which the usual plane wave relations between the fields do not hold—contains negligible field energy and momentum.

The spatial coordinates $X_i(K)$ of the centre of mass of the field in K are defined by

$$X_i(K) = \frac{1}{\mathscr{H}} \int x_i W dV, \tag{7.8}$$

whatever energy-momentum tensor is employed. Similarly as in the previous section it must however be borne in mind that in any case the localization of the field in K is determined by MINKOWSKI's tensor, i.e. one integrates over the volume of the field by integrating across the world lines corresponding to $\boldsymbol{S}^M$.

Let us first study the velocity of propagation of the centre of mass in K. From (7.8) we readily find the relation

$$\frac{d}{dt} X_i(K) = \frac{1}{\mathscr{H}} \int S_i dV - \frac{icX_i(K)}{\mathscr{H}} \int f_4 dV + \frac{ic}{\mathscr{H}} \int x_i f_4 dV, \tag{7.9}$$

which in general shows a complicated behaviour for a non-closed system. Inserting ABRAHAM's tensor into the right hand side of eq. (7.9)—and assuming that corresponding world points in K and K^0 are connected by the invariant (MINKOWSKI) world lines—we find however that the two last terms in (7.9) fluctuate away. Moreover, since the field is homogeneous, we find from (7.9) the simple relation

$$\frac{d}{dt} \boldsymbol{X}^A(K) = \frac{\boldsymbol{S}^A}{W^A}. \tag{7.10}$$

By taking into account the result obtained in the previous subsection, we thus find that the velocity $d\boldsymbol{X}^A(K)/dt$ is different from the velocity of propagation of the total field, i.e. the ray velocity $\boldsymbol{u}$. This feature severely limits the validity of the centre of mass as a representative point if ABRAHAM's tensor is used.

With the radiation tensor we get immediately

$$\frac{d}{dt} \boldsymbol{X}^S(K) = \frac{\boldsymbol{S}^S}{W^S} = \boldsymbol{u}, \tag{7.11}$$

in accordance with the general equivalence between the radiation tensor and the MINKOWSKI tensor with regard to wave propagation properties.

So far having studied the *velocity* of propagation of the centre of mass we now turn our attention to its *localization*. From the sudy of MINKOWSKI's tensor in I, we recall that the different centres of mass we obtain by varying the reference frames K, do not in general coincide when considered simultaneously in one frame. In fact, we calculated the difference $\boldsymbol{X}^M(K) - \boldsymbol{X}^M$, where $\boldsymbol{X}^M$ denoted the simultaneous position in K of the *proper* centre of mass. The proper centre was defined as the centre of mass in the rest frame K^0, i.e. $\boldsymbol{X}^M(K^0) \equiv \boldsymbol{X}^{M^0}$. Let us write down again the formula (I, 12.21)

$$\boldsymbol{a}^M(K) \equiv \boldsymbol{X}^M(K) - \boldsymbol{X}^M = \frac{1}{k^0}\frac{\boldsymbol{\beta} \times \boldsymbol{k}^0}{nk^0 + \boldsymbol{\beta} \cdot \boldsymbol{k}^0}, \tag{7.12}$$

where we now have added a superscript M.

Just the same procedure can now be applied to calculate the position $\boldsymbol{X}^A(K)$ when ABRAHAM's tensor is used for the field. In this context we stress that corresponding field points in K and K^0 are required to be connected by the MINKOWSKI world lines, i.e. we simply ignore for a moment the above result $d\boldsymbol{X}^A(K)/dt \neq \boldsymbol{u}$. Since the proper centres coincide in K^0, $\boldsymbol{X}^{A^0} = \boldsymbol{X}^{M^0}$, we evidently have also $\boldsymbol{X}^A = \boldsymbol{X}^M$ in K. We do not give the details of the calculation since it is just similar to the calculation carried through in I, section 12. The result is

$$\boldsymbol{a}^A(K) \equiv \boldsymbol{X}^A(K) - \boldsymbol{X}^A = \boldsymbol{a}^M(K), \tag{7.13}$$

showing that ABRAHAM's tensor yields the same position for the centre of mass as MINKOWSKI's tensor, $\boldsymbol{X}^A(K) = \boldsymbol{X}^M(K)$, if we integrate across the world lines determined by $\boldsymbol{S}^M$.

The radiation tensor exhibits very simple features with respect to the centre of mass. Since $\partial_\sigma(x_\mu S^S_{\nu\sigma} - x_\nu S^S_{\mu\sigma}) = 0$ it follows that the angular momentum quantities $M^S_{\mu\nu}$ constitute a tensor, and by calculating M^S_{i4} in K at $t = 0$ we readily find that

$$X^S_i(K) = -\frac{icM^S_{i4}}{\mathscr{H}^S} = X^M_i(K) \tag{7.14a}$$

$$\boldsymbol{a}^S(K) = \boldsymbol{a}^M(K). \tag{7.14b}$$

The equivalence we now have established between the three energy-momentum tensors with respect to the centre of mass is not accidental. It is connected with the fact that in (I, 12.12) we introduced the radiation tensor as a *formal* remedy in order to extend certain volume integrals, taken over the internal, plane part of the field, into integrals taken over the *whole* field. In the case of the radiation tensor we could just take advantage of the tensor property of $M^S_{\mu\nu}$. It does not seem, however, that the equivalence could easily be foreseen.

The last point we shall dwell on in connection with the study of the centre of mass is a comment concerning a result obtained in a basic paper by C. MØLLER(43). On the basis of some definite assumptions, MØLLER showed that the concept of mass centre for a non-closed system in general is incompatible with the equations of motion. This result seems to run into conflict with the result obtained in the present section, where we have defined the centre of mass even for the ABRAHAM field. However, there is no real discrepancy between the results, since one of the assumptions inherent in MØLLER's proof does not apply to the present situation.

Let us point out in detail the mathematical reason for this circumstance. At an arbitrary point of the world line of the proper centre (with proper time τ) MØLLER assumes that the following relation can be written:

$$\frac{1}{c}\int S_{\mu\nu}d\sigma_\nu = M_0\frac{d}{d\tau}X_\mu, \tag{7.15}$$

where the integration is taken over a hyperplane σ which is normal to the world line. The surface pseudo four-vector $d\sigma_\nu$ is given by $d\sigma_\nu = -i\delta_{\nu\mu\sigma\varrho}dx_\mu \delta x_\sigma \Delta x_\varrho$, $\delta_{1234} = 1$, where dx_μ, δx_σ and Δx_ϱ are four-vectors lying on σ. If σ is orthogonal to the x_4-axis, we choose the latter three vectors so that the non-vanishing component of $d\sigma_\nu$ is $d\sigma_4 = -idV$, when the outward normal lies in the direction of the positive x_4-axis. In (7.15) $M_0 = M_0(\tau)$ is a proportionality constant.

If we now insert ABRAHAM's tensor into (7.15) in the frame K^* where the wave is at rest, we find for $\mu = 4$ the relation $\mathscr{H}^{A*} = M_0 c^2$, while for $\mu = i$ we find that M_0 becomes infinite. This discrepancy shows that an equation of the form (7.15) does not apply here. Hence MØLLER's proof does not come into conflict with the above results in this section. Nor does MINKOWSKI's tensor satisfy the relation (7.15) ,while the radiation tensor does satisfy it.

The Čerenkov effect

As we already have noted, a study of the ČERENKOV effect is very instructive for a comparison between the various energy-momentum tensors. In section 5 of the present paper we studied the ČERENKOV effect in the case that the emitting particle moves within a medium at rest, and in section 10 of I we considered the emitting particle in its own rest system from the point of view of MINKOWSKI's tensor. The reason why we shall here consider the ČERENKOV effect once more, it that we wish to point out how the relativistic theory looks if ABRAHAM's tensor is used for the field. This kind of analysis is desirable, since I. TAMM in his famous paper(44) on the ČERENKOV effect studied the balance of momentum in the rest frame of the particle and came to the conclusion that MINKOWSKI's tensor, but not ABRAHAM's tensor, is able to give a satisfactory description. We shall thus discuss the momentum balance in the ABRAHAM case, since according to our general interpretation MINKOWSKI's and ABRAHAM's tensors ought to be equivalent in such a case.

Consider then the same situation as in I: An electron is moving along the x-axis with a uniform velocity which in K^0 is larger than c/n. The rest frame of the particle is denoted by K; as shown by TAMM, $\boldsymbol{H} = 0$ in K, so that there is no MINKOWSKI energy current in this frame. We integrate the differential conservation law for momentum over a volume which contains the electron and which is enclosed by a cylindric surface S of small radius and infinite length such that the axis of the cylinder coincides with the x-axis. Since the field is stationary in K, one can thus write, in the case of MINKOWSKI's tensor,

$$\int S_{ik}^M n_k dS = -\int f_i^M dV, \tag{7.16}$$

which is the same as (I, 10.3).

As TAMM points out, MINKOWSKI's force must in any case represent the force acting on the electric charge, because the terms which are added to MINKOWSKI's tensor in order to form ABRAHAM's tensor will correspond to additional forces acting on the *medium* itself, and not on the electric charge. The total force on the electron as given by the right hand side of (7.16) can thus be found by transforming the total force from K^0 using the usual transformation formulas. Now TAMM evaluates the integral on the left hand side of (7.16) and verifies that the two sides of the equation are equal. Further, since $S_{ik}^M \neq S_{ki}^M$ for $i = 1$ and $k = 2, 3$, he concludes that a symmetrical "Ansatz" for $S_{\mu\nu}$ would give a different result in disagreement with the force expression on the right hand side of (7.16).

Let us now apply ABRAHAM's tensor to the present case. It is instructive to write the momentum balance in the form

$$\int S_{ik}^A n_k dS + \int (f_i^A - f_i^M) dV = -\int f_i^M dV, \tag{7.17}$$

and so it appears that the second integral on the left may represent a source (or sink) of electromagnetic momentum which also has to be taken into account. Since the force on the *matter* cannot make up an appreciable magnitude in a small volume element just enclosing the electron, we can exclude this element from the second integration in (7.17) and thus obtain

$$\int S_{ik}^{A} n_k dS + \int' f_i^A dV = -\int f_i^M dV, \tag{7.18}$$

where $\int'$ means integration over the remaining part of the volume. However, also the second term on the left in (7.18) vanishes due to the rapid oscillation of the integrand, so that eqs. (7.18) and (7.16) become identical, i.e. $\boldsymbol{S}_n^M = \boldsymbol{S}_n^A$. In fact, the relation $S_{ik}^M = S_{ik}^A$, valid for all combinations of i, k that occur in (7,18), can be checked directly by expressing S_{ik}^M and S_{ik}^A in terms of the tensor components in K^0. Note that it is just the latter relation that represents the main reason why the (macroscopic) descriptions corresponding to MINKOWSKI's and ABRAHAM's tensors are identical in this case; properties of symmetry or asymmetry of the energy-momentum tensors are of no direct importance.

In the remainder of the present section we shall be concerned with a study of the so-called "principle of virtual power". Before embarking upon this subject, let us however pause to make the following brief remarks in connection with the topics considered in I: In sections 4 and 5 in I we gave two sets of conditions from which we showed that MINKOWSKI's tensor is uniquely determined. It should be clear that both these sets of conditions automatically exclude from consideration the alternative tensor forms that we have been studying: The first set because eqs. (I, 4.1) and (I, 4.2) require the tensor to be asymmetric and divergence-free; the second set essentially because eq. (I, 5.1) requires the tensor not to contain the four-velocity V_μ explicitly (cf. (1.5), (1.7) and the fact that also $S_{\mu\nu}^G$ will contain V_μ in a complicated way).

In section 10 of I we discussed the negative field energy which appears with the use of MINKOWSKI's tensor in a certain class of inertial systems due to the space-like character of the four-momentum G_μ^M. This property is peculiar to MINKOWSKI's tensor and is not shared by the other tensor forms. We may check by direct calculation that $W^A > 0$ and $W^G > 0$ in any K, while the result $W^S > 0$ follows immediately from the fact that the four-momentum G_μ^S is time-like. If a plane wave moves parallel to the x-axis we may conveniently write the total energy density of matter and field as

$$W^{\text{tot}} = \gamma^2(1 + 2n\beta + \beta^2)W^{A^0} + \gamma^2 W^{\text{mech}\,0}, \tag{7.19}$$

where the contributions arising from $S^{A^0}_{\mu\nu}$, $\boldsymbol{g}^{\text{mech}\,0}$ and $\boldsymbol{S}^{\text{mech}\,0}$ are collected in the first term.

Principle of virtual power

Quite recently, P. PENFIELD and H. A. HAUS published a book[45] on the electrodynamics of moving media which is a synthesis of work they performed with various collaborators; especially the earlier article[46] by CHU, HAUS and PENFIELD is of particular interest to us. The authors adopt a phenomenological point of view. In addition to employing the usual formulation (I, 1.1) of MAXWELL's equations in a moving medium (the MINKOWSKI formulation), which we also have employed throughout our work, they consider the so-called CHU formulation introduced in the book by FANO, CHU and ADLER[39]. It is outside the scope of our work to go into a study of the CHU formulation. What really is of interest to us, is that the authors, within the frame of the MINKOWSKI formulation, derive an expression for the electromagnetic energy-momentum tensor which is equal to ABRAHAM's expression in an isotropic fluid, while MINKOWSKI's tensor is claimed not to describe the electromagnetic system in a meaningful way. We find it therefore of interest to trace out the reason why the authors have arrived at this result. The keystone of the derivation presented is the "principle of virtual power", invented by the authors, so let us first sketch how the principle looks in the present case. An isotropic fluid is considered, where the fluid velocity $\boldsymbol{u}(\boldsymbol{r},t)$ may be a nonuniform function of the position at a certain time. We simplify the formalism (thereby ignoring the dependence of the field energy on the material density), and transform it to our notation.

Consider an arbitrary space-time point and denote by K^0 the inertial frame in which the velocity of a fluid element around this point momentarily is zero. Thus $\boldsymbol{u}^0 = 0$ for the element, but one assumes that virtual deformations can be applied to the material to produce arbitrary values of $\partial_k u_i^0$ and $\partial u_i^0/\partial t$.

Then let K denote the frame in which K^0 moves with a small velocity $\boldsymbol{u}$. To the first order in u/c we have

$$S_i = S_i^0 + u_k S_{ki}^0 + u_i W^0 \tag{7.20a}$$

$$W = W^0 + \boldsymbol{u}\cdot\boldsymbol{g}^0 + \frac{1}{c^2}\boldsymbol{u}\cdot\boldsymbol{S}^0, \tag{7.20b}$$

and these equations are introduced into the energy balance

$$\nabla \cdot \boldsymbol{S} + \partial W/\partial t = icf_4. \tag{7.21}$$

The authors then let K approach K^0 so that terms containing $\boldsymbol{u}$ (but not the derivatives of $\boldsymbol{u}$) vanish. The resulting equation is

$$\nabla \cdot \boldsymbol{S}^0 + \frac{1}{c^2}\boldsymbol{S}^0 \cdot \frac{\partial \boldsymbol{u}^0}{\partial t} + \frac{\partial W^0}{\partial t} + W^0 \nabla \cdot \boldsymbol{u}^0 - icf_4^0 = -S_{ik}^0 \partial_k u_i^0 - \boldsymbol{g}^0 \cdot \frac{\partial \boldsymbol{u}^0}{\partial t} \tag{7.22}$$

(note that the differential operators ∂_μ are not transformed). The essential point is now that a knowledge of the physical quantities appearing on the left hand side of (7.22), i.e. of $\boldsymbol{S}^0$, W^0 and f_4^0, is claimed to be sufficient to provide a determination of the remaining tensor components S_{ik}^0 and $\boldsymbol{g}^0$ appearing on the right hand side of (7.22). The following expressions are chosen:

$$\boldsymbol{S}^0 = c(\boldsymbol{E}^0 \times \boldsymbol{H}^0) \tag{7.23a}$$

$$\partial W^0/\partial t = \boldsymbol{E}^0 \cdot \partial \boldsymbol{D}^0/\partial t + \boldsymbol{H}^0 \cdot \partial \boldsymbol{B}^0/\partial t \tag{7.23b}$$

$$W^0 = \tfrac{1}{2}(\boldsymbol{E}^0 \cdot \boldsymbol{D}^0 + \boldsymbol{H}^0 \cdot \boldsymbol{B}^0), \quad f_4^0 = 0. \tag{7.23c}$$

The authors now argue that it is convenient to express the fields $\boldsymbol{E}^0$, $\boldsymbol{D}^0$, $\boldsymbol{H}^0$, $\boldsymbol{B}^0$ appearing in (7.23) in terms of the fields pertaining to the inertial frame K before inserting (7.23) into (7.22) (note again that K^0 means the frame where the fluid element momentarily is at rest). By inserting (7.23) into the expression on the left hand side of (7.22) they thus obtain

$$\left.\begin{aligned} &c\nabla \cdot (\boldsymbol{E} \times \boldsymbol{H}) + \boldsymbol{E} \cdot \frac{\partial \boldsymbol{D}}{\partial t} + \boldsymbol{H} \cdot \frac{\partial \boldsymbol{B}}{\partial t} \\ &+ [E_i D_k + H_i B_k - \tfrac{1}{2}\delta_{ik}(\boldsymbol{E} \cdot \boldsymbol{D} + \boldsymbol{H} \cdot \boldsymbol{B})]\,\partial_k u_i - \frac{1}{c}(\boldsymbol{E} \times \boldsymbol{H}) \cdot \frac{\partial \boldsymbol{u}}{\partial t}. \end{aligned}\right\} \tag{7.24}$$

The three first terms add up to zero because of MAXWELL's equations. By letting K approach K^0, identifying (7.24) with the right hand side of (7.22) and taking into account the arbitrariness of the derivatives of $\boldsymbol{u}^0$, the authors finally obtain

$$S_{ik}^0 = -E_i^0 D_k^0 - H_i^0 B_k^0 + \tfrac{1}{2}\delta_{ik}(\boldsymbol{E}^0 \cdot \boldsymbol{D}^0 + \boldsymbol{H}^0 \cdot \boldsymbol{B}^0) \tag{7.25a}$$

$$\boldsymbol{g}^0 = \frac{1}{c}(\boldsymbol{E}^0 \times \boldsymbol{H}^0). \tag{7.25b}$$

This is ABRAHAM's expression. (Actually, the expression given in ref. 46, containing the detailed derivation, was somewhat different from (7.25) but, according to a private communication by the authors, this difference is due to a printing error.)

If we now proceed to examine this principle of virtual power, we ought first to note that one must distinguish between the derivatives of the relative velocity $\boldsymbol{v}$ between the frames K^0 and K, and of the fluid velocity $\boldsymbol{u}$. The formulas (7.20) relate the tensor components in the frame K to the tensor components in the momentary rest frame K^0 moving with the *constant* velocity $\boldsymbol{v}$ with respect to K; although $\boldsymbol{v} = \boldsymbol{u}$ at the space-time point considered the corresponding equality between the derivatives is generally not true. Thus each of the factors $\partial_\mu \boldsymbol{u}^0$ in (7.22) should properly be replaced by $\partial_\mu \boldsymbol{v}^0$, which is zero. In fact, by performing the transformation $K \rightarrow K^0$ the only result one can obtain is the covariant properties of the conservation equations $\partial_\nu S_{\mu\nu} = -f_\mu$. By starting from the relation (7.21), and assuming the velocity v to be small, one will thus end up with the same relation written in K^0. If we really subtract the equation $\nabla \cdot \boldsymbol{S}^0 + \partial W^0/\partial t - icf_4^0 = 0$ from eq. (7.22), we see that obvious inconsistencies will appear in the remaining equation if arbitrarily adjustable terms $\partial_\mu \boldsymbol{u}^0$ are present.

However, the above remark does not elucidate the essential reason why a definite form of the electromagnetic energy-momentum tensor was obtained. To this end let us in the following simply *assume* the validity of eq. (7.22) as it stands. The essence of the principle of virtual power seems in reality to be that one starts from the energy balance (7.21) in K, then transforms the field quantities to K^0 and inserts some physical information in this frame, and finally transforms back to K. Within the frame of the physical information inserted in K^0 the formalism can therefore, if it is carried through consistently, yield only a mathematical *identity*. The reason why the authors instead obtained Abraham's expression in (7.25) is that they implicitly introduced into the formalism a physical assumption which is compatible with Abraham's tensor, but not with Minkowski's tensor. Let us go into some detail at this point. It is then necessary first to focus our attention on the force component f_4 in (7.21). In the conventional theory f_μ transforms like a four-vector, so that, in the limiting case of small u, $f_4 = f_4^0$. This equation was used by the authors in the construction of eq. (7.22). In particular, if $f_4^0 = 0$, as assumed in (7.23c), one should obtain $f_4 = 0$ also in K. However, if we use the covariant expression for $S_{\mu\nu}$ and calculate f_4 in K according to the basic equation $f_4 = -\partial_\nu S_{4\nu}$, we may obtain a different result. For example, both in the Abraham case and the Minkowski case we know that $f_4^0 = 0$, while the covariant expressions (1.5) and (1.1) yield

$$f_4^A = 0, \quad f_4^M = -i(n^2-1)(\boldsymbol{E}\times\boldsymbol{H})\cdot\partial\boldsymbol{u}/\partial t \tag{7.26}$$

to the lowest order. In the Minkowski case there is thus a conflict; it is incorrect to transform f_μ^M as if it were a four-vector.

Due to this peculiar transformation property of f_4 (which evidently is closely connected with the covariance problem of the conservation equations discussed above), it follows that f_4 should properly not have been replaced by f_4^0 in (7.22) but should rather have been retained unchanged. Accordingly, it follows that eq. (7.24) implies the relation $f_4 = 0$. This is a choice which, according to (7.26), implicitly singles out Abraham's tensor. The appearance of Abraham's expression in (7.25) is therefore what we should expect. It is also possible to make Minkowski's tensor emerge from the formalism; to this end we must insert the explicit expression for f_4^M, given by (7.26), into (7.22). Generally speaking, the introduction of a specific expression for f_4 implicitly implies the acceptance of a specific tensor, the remaining formalism thus effectively expressing an identity.

8. Analysis by Means of Curvilinear Coordinates

In connection with the study of the canonical procedure in section 8 of I we mentioned that it is possible, in the case of a *closed* field, to make the canonical energy-momentum tensor complete by means of a symmetrization procedure. Now it is well known that in the presence of a gravitational field one can obtain the complete energy-momentum tensor directly, without having to perform a symmetrization, by means of a variational method involving the variation of the metric tensor. Actually, and it is this case which is of interest to us, the variational method can be applied also in the absence of a gravitational field. Then the transition to curvilinear coordinates occurs formally as an intermediate step in the calculation.

Curvilinear coordinates have been used rather extensively in earlier studies of the electrodynamics of material media, although one here is confronted with a non-closed field. Incorrect use of the variational method caused a great deal of confusion in the literature some years ago. The ambiguity inherent in the calculation seems first to have been pointed out by J. I. HORVÁTH[47] (see also ref. 48). However, we think that it is still of interest to give a careful analysis of the electromagnetic field in terms of these coordinates, to point out the detailed reason why the power of the variational method is restricted, and to supplement with remarks pertaining to alternative variational methods. The main part of the present section is devoted to this task. In the last subsection we shall study again the Sagnac-type experiment from section 9 of I, in connection with an application of the various tensor forms. The cavity frame in this experiment is evidently non-inertial.

A variational method

Let us now leave out the imaginary x_4 coordinate and work with the real coordinates x^1, x^2, x^3, $x^0 = ct$. The square of the line element is $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ (μ, ν running over the numbers 1, 2, 3, 0), whence in GALILEAN coordinates

$$\left.\begin{aligned} g_{11} = g_{22} = g_{33} = 1, \; g_{00} = -1 \\ g = \det g_{\mu\nu} = -1, \; g_{\mu\nu} = 0 \text{ for } \mu \neq \nu. \end{aligned}\right\} \quad (8.1)$$

Further, in GALILEAN coordinates,

$$\left.\begin{aligned} \boldsymbol{E} = (F_{10}, F_{20}, F_{30}), \; \boldsymbol{B} = (F_{23}, F_{31}, F_{12}) \\ \boldsymbol{D} = (H_{10}, H_{20}, H_{30}), \; \boldsymbol{H} = (H_{23}, H_{31}, H_{12}), \end{aligned}\right\} \quad (8.2)$$

and the connection between field and potentials is in general

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad (8.3)$$

as the covariant derivative ∇_μ can be replaced by the ordinary derivative when $F_{\mu\nu}$ is antisymmetric.

For a radiation field MAXWELL's equations take the form

$$\nabla_\lambda F_{\mu\nu} + \nabla_\mu F_{\nu\lambda} + \nabla_\nu F_{\lambda\mu} = \partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0 \quad (8.4a)$$

$$\nabla_\nu H^{\mu\nu} = \frac{1}{\sqrt{-g}}\partial_\nu\left(\sqrt{-g}\,H^{\mu\nu}\right) = 0. \quad (8.4b)$$

We have here assumed arbitrary coordinates where the $g_{\mu\nu}$ are given functions of the coordinates. Then proceed to determine the constitutive relations. We shall keep the formalism so general that it includes the case of an anisotropic dielectric medium, but we shall assume magnetic isotropy with $\mu = 1$. (The procedure runs similarly, however, also if μ_{ik} is a tensor.) Introducing in the small region around each point a local rest system of inertia $\mathring{K}$ with the metric tensor given by (8.1), we may write

$$\mathring{H}_{i0} = \mathring{\varepsilon}_i^k \mathring{F}_{k0}. \tag{8.5}$$

Moreover, in $\mathring{K}$ we introduce the quantities

$$\mathring{\varepsilon}_0^\nu = \mathring{\varepsilon}_\nu^0 = 0 \; (\nu = 1, 2, 3, 0) \tag{8.6}$$

and let in the arbitrary coordinate system the symmetric tensor $\varepsilon^{\mu\nu}$ be defined in such a way that its mixed components in $\mathring{K}$ coincide with $\mathring{\varepsilon}_\nu^\mu$ given by (8.5) and (8.6). The constitutive relations written in covariant form are then

$$H^{\mu\nu} = F^{\mu\nu} + \frac{1}{c^2}(F^\mu - \varepsilon^{\mu\alpha}F_\alpha)V^\nu - \frac{1}{c^2}(F^\nu - \varepsilon^{\nu\alpha}F_\alpha)V^\mu, \tag{8.7}$$

where $F_\alpha = F_{\alpha\beta}V^\beta$, and V^β is the four-velocity of the medium. In isotropic media eq. (8.7) can be written

$$H^{\mu\nu} = F^{\mu\nu} - \varkappa(F^\mu V^\nu - F^\nu V^\mu), \quad \varkappa = (\varepsilon - 1)/c^2. \tag{8.8}$$

This relation between (8.7) and (8.8) can readily be verified, since for an isotropic body in $\mathring{K}$

$$\mathring{\varepsilon}^{\mu\alpha}\mathring{F}_\alpha = \mathring{\varepsilon}_\alpha^\mu \mathring{F}^\alpha = \mathring{\varepsilon}\mathring{g}_\alpha^\mu \mathring{F}^\alpha = \mathring{\varepsilon}\mathring{F}^\mu. \tag{8.9}$$

Here g_α^μ is the metric tensor in GALILEAN form and $\mathring{\varepsilon}$ is the dielectric constant in $\mathring{K}$. Note that $\mathring{\varepsilon}_0^0 = 0$ according to (8.6) while $\mathring{g}_0^0 = 1$; however, this does not matter, since $\mathring{F}^0 = 0$. Writing (8.9) covariantly as $\varepsilon^{\mu\alpha}F_\alpha = \mathring{\varepsilon}F^\mu$, we obtain (8.8) from (8.7). Thus, while $\varepsilon^{\mu\alpha}$ in (8.7) is a tensor, the transformation (8.9) in the case of isotropic media causes the dielectric constant in (8.8) to be treated as a four-dimensional scalar.

It can be verified that an appropriate LAGRANGIAN is

$$\left.\begin{aligned} L &= -\tfrac{1}{4}F_{\mu\nu}H^{\mu\nu} \\ &= -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2c^2}F_\mu F^\mu + \frac{1}{2c^2}\varepsilon^{\mu\nu}F_\mu F_\nu = L_0 + L' + L''. \end{aligned}\right\} \tag{8.10}$$

Multiplying with the pseudo-invariant $\sqrt{-g}\,dx = \sqrt{-g}\,dx^1dx^2dx^3dx^0$ and integrating over a region Σ in four-space lying between two space-like surfaces and extending to infinity in the space directions, we get the action integral

$$J = \int_{\Sigma} L(x)\sqrt{-g}\,dx = \int_{\Sigma} \mathscr{L}(x)dx. \tag{8.11}$$

Since (8.10) corresponds to the field and its interaction with the matter, a variation of (8.11) with respect to the potentials will yield the field equations (8.4b). However, we are primarily interested in the invariance property of J under coordinate transformations.

Let an infinitesimal coordinate transformation be given by $x'^{\mu} = x^{\mu} + \delta x^{\mu} = x^{\mu} + \xi^{\mu}$, where the ξ^{μ} are small, but arbitrary functions of the coordinates, so that terms quadratic in ξ^{μ} may be neglected. The corresponding change of (8.11) is

$$\delta J = \int_{\Sigma'} \mathscr{L}'(x')\,dx' - \int_{\Sigma} \mathscr{L}(x)\,dx. \tag{8.12}$$

By transforming this expression and using the assumption that ξ^{μ} vanish on the boundary, we obtain[8]

$$\delta J = \int_{\Sigma} \delta^{*}\mathscr{L}(x)\,dx = 0, \tag{8.13}$$

where $\delta^{*}\mathscr{L}(x) = \mathscr{L}'(x) - \mathscr{L}(x)$ is the local variation. Eq. (8.13) has the form of a variational principle even though L does not correspond to a closed system; only it must be remembered that all variations are generated by the infinitesimal coordinate transformations.

We proceed then to calculate these variations. By a vector transformation we find

$$V'^{\mu}(x') = \partial_{\nu} x'^{\mu} V^{\nu}(x) = (\delta^{\mu}_{\nu} + \partial_{\nu}\xi^{\mu})V^{\nu}(x), \tag{8.14}$$

whence

$$\delta^{*}V^{\mu}(x) = V^{\nu}(x)\partial_{\nu}\xi^{\mu} - \xi^{\nu}\partial_{\nu}V^{\mu}(x) = V^{\nu}\nabla_{\nu}\xi^{\mu} - \xi^{\nu}\nabla_{\nu}V^{\mu}. \tag{8.15}$$

Here we have for example $\nabla_{\nu}V^{\mu} = \partial_{\nu}V^{\mu} + \Gamma^{\mu}_{\nu\alpha}V^{\alpha}$, where $\Gamma^{\mu}_{\nu\alpha}$ is the CHRISTOFFEL symbol. It appears that $\delta^{*}V^{\mu}$ is a four-vector, as should be the case, since this variation is the difference of the values of two four-vectors at the same point. Correspondingly for the potentials

$$\delta^{*}A_{\mu}(x) = -A_{\nu}\nabla_{\mu}\xi^{\nu} - \xi^{\nu}\nabla_{\nu}A_{\mu}. \tag{8.16}$$

The $g^{\mu\nu}$ will also be affected by the coordinate transformation, and we have

$$g'^{\mu\nu}(x') = g^{\alpha\beta}(x)\partial_\alpha x'^\nu \partial_\beta x'^\nu = g^{\mu\nu}(x) + g^{\mu\alpha}\partial_\alpha \xi^\nu + g^{\nu\alpha}\partial_\alpha \xi^\mu. \quad (8.17)$$

Thus

$$\delta^* g^{\mu\nu}(x) = g'^{\mu\nu}(x') - g^{\mu\nu}(x) - \xi^\alpha \partial_\alpha g^{\mu\nu}(x) = \nabla^\mu \xi^\nu + \nabla^\nu \xi^\mu. \quad (8.18)$$

Similarly

$$\delta^* \varepsilon^{\mu\nu} = \varepsilon^{\mu\alpha}\nabla_\alpha \xi^\nu + \varepsilon^{\nu\alpha}\nabla_\alpha \xi^\mu - \xi^\alpha \nabla_\alpha \varepsilon^{\mu\nu}. \quad (8.19)$$

The part J_0 of the action integral corresponding to L_0 in (8.10) is to be varied with respect to A_μ and $g^{\mu\nu}$. This term is present also in the case of an electromagnetic field in vacuum. One obtains after some calculation (for details, see Fock(49), §§ 47, 48)

$$\left.\begin{aligned} \delta J_0 = & -\tfrac{1}{2}\int (F_{\mu\alpha}F_\nu^{\cdot\alpha} - \tfrac{1}{4} g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta})\delta^* g^{\mu\nu}\sqrt{-g}\,dx \\ & - \int \nabla_\nu F^{\mu\nu}\delta^* A_\mu \sqrt{-g}\,dx. \end{aligned}\right\} \quad (8.20)$$

Here use has been made of the relations $\delta^*\sqrt{-g} = -\frac{1}{2}\sqrt{-g}\,g_{\mu\nu}\delta^* g^{\mu\nu}$, $g^{\mu\nu}\delta^* g_{\mu\nu} = -g_{\mu\nu}\delta^* g^{\mu\nu}$. By virtue of (8.18) the first term in (8.20) can be transformed, so that

$$\delta J_0 = \int \nabla_\nu (F_{\mu\alpha}F^{\nu\alpha} - \tfrac{1}{4} g_\mu^\nu F_{\alpha\beta}F^{\alpha\beta})\xi^\mu \sqrt{-g}\,dx - \int \nabla_\nu F^{\mu\nu}\delta^* A_\mu \sqrt{-g}\,dx. \quad (8.21)$$

We shall now give the detailed calculation for the action term J' corresponding to L' in (8.10). Variations are here to be taken with respect to A^μ, $g^{\mu\nu}$ and V^μ. Let us first calculate the contribution from the potentials and write

$$\delta_A J' = -\frac{1}{c^2}\int F^\mu V^\alpha \delta^* F_{\mu\alpha}\sqrt{-g}\,dx = -\frac{1}{c^2}\int F^\mu V^\alpha (\partial_\mu \delta^* A_\alpha - \partial_\alpha \delta^* A_\mu)\sqrt{-g}\,dx,$$

since ∂_μ and δ^* commute. By partial integrations then

$$\delta_A J' = -\frac{1}{c^2}\int \nabla_\nu (F^\mu V^\nu - F^\nu V^\mu)\delta^* A_\mu \sqrt{-g}\,dx, \quad (8.22)$$

where we have exploited the antisymmetry property of the expression in the parenthesis.

The variation with respect to the metric tensor is handled in the same way, and we get by means of (8.18)

$$\left.\begin{aligned}\delta_g J' &= -\frac{1}{2c^2}\int F_\mu F_\nu(\sqrt{-g}\,\delta^* g^{\mu\nu} - \tfrac{1}{2}\sqrt{-g}\,g^{\mu\nu} g_{\alpha\beta}\delta^* g^{\alpha\beta})\,dx \\ &= -\frac{1}{c^2}\int (F_\mu F \;\; - \tfrac{1}{2} g^\alpha_\mu F_\beta F^\beta)\nabla_\alpha \xi^\mu \sqrt{-g}\,dx \\ &= -\frac{1}{c^2}\int \{\nabla_\alpha[(F_\mu F^\alpha - \tfrac{1}{2} g^\alpha_\mu F_\beta F^\beta)\xi^\mu] \\ &\quad - \nabla_\alpha(F_\mu F^\alpha - \tfrac{1}{2} g^\alpha_\mu F_\beta F^\beta)\xi^\mu\}\sqrt{-g}\,dx.\end{aligned}\right\} \tag{8.23}$$

Since the first term in this expression involves the covariant derivative of the product of a scalar and a four-vector, we can write this term as

$$-\frac{1}{c^2}\int \partial_\alpha[\sqrt{-g}(F_\mu F^\alpha - \tfrac{1}{2} g^\alpha_\mu F_\beta F^\beta)\xi^\mu]\,dx \tag{8.24}$$

and transform into an integral over the boundary. Therefore this term vanishes. It remains

$$\delta_g J' = \frac{1}{c^2}\int \nabla_\nu(F_\mu F^\nu - \tfrac{1}{2} g^\nu_\mu F_\alpha F^\alpha)\xi^\mu \sqrt{-g}\,dx. \tag{8.25}$$

Finally we consider the variations connected with the velocity. By means of (8.15) we have

$$\delta_V J' = -\frac{1}{c^2}\int F_{\alpha\mu} F^\alpha (V^\nu \nabla_\nu \xi^\mu - \xi^\nu \nabla_\nu V^\mu)\sqrt{-g}\,dx. \tag{8.26}$$

Performing a partial integration we obtain, apart from an integral similar to (8.24)

$$\delta_V J' = -\frac{1}{c^2}\int [\nabla_\nu(F_{\mu\alpha} F^\alpha V^\nu) + F_{\nu\alpha} F^\alpha \nabla_\mu V^\nu]\xi^\mu \sqrt{-g}\,dx. \tag{8.27}$$

Similarly we can evaluate the contributions from the term L'' in (8.10). We give the results:

$$\delta_A J'' = \frac{1}{c^2}\int \nabla_\nu[(\varepsilon^{\mu\alpha} V^\nu - \varepsilon^{\nu\alpha} V^\mu) F_\alpha]\,\delta^* A_\mu \sqrt{-g}\,dx \tag{8.28a}$$

$$\delta_g J'' = \frac{1}{2c^2}\int \nabla_\mu(\varepsilon^{\alpha\beta} F_\alpha F_\beta)\xi^\mu \sqrt{-g}\,dx \tag{8.28b}$$

$$\delta_V J'' = -\frac{1}{c^2}\int [\nabla_\nu(\varepsilon^{\alpha\beta} F_{\alpha\mu} F_\beta V^\nu) + \varepsilon^{\alpha\beta} F_{\alpha\nu} F_\beta \nabla_\mu V^\nu]\xi^\mu \sqrt{-g}\,dx \tag{8.28c}$$

$$\delta_\varepsilon J'' = -\frac{1}{2c^2}\int[2\nabla_\nu(\varepsilon^{\nu\alpha}F_\mu F_\alpha) + F_\nu F_\alpha \nabla_\mu \varepsilon^{\nu\alpha}]\xi^\mu \sqrt{-g}\,dx. \qquad (8.28d)$$

In the last equation we have made use of (8.19).

Now we are able to write down the total variation δJ, where $\delta = \delta_A + \delta_g + \delta_V + \delta_\varepsilon$. We obtain

$$\left.\begin{aligned} 0 = \delta J = \delta J_0 + \delta J' + \delta J'' = -\int \nabla_\nu\Big[F^{\mu\nu} + \frac{1}{c^2}(F^\mu - \varepsilon^{\mu\alpha}F_\alpha)V^\nu \\ -\frac{1}{c^2}(F^\nu - \varepsilon^{\nu\alpha}F_\alpha)V^\mu\Big]\delta^* A_\mu \sqrt{-g}\,dx + \int\Big[\nabla_\nu(F_{\mu\alpha}H^{\nu\alpha} - \tfrac{1}{4}g^\nu_\mu F_{\alpha\beta}H^{\alpha\beta}) \\ -\frac{1}{c^2}F_{\nu\alpha}(F^\alpha - \varepsilon^{\alpha\beta}F_\beta)\nabla_\mu V^\nu - \frac{1}{2c^2}F_\nu F_\alpha \nabla_\mu \varepsilon^{\nu\alpha}\Big]\xi^\mu \sqrt{-g}\,dx. \end{aligned}\right\} \qquad (8.29)$$

In this relation $\delta^* A_\mu$ and ξ^μ are not independent, but related through (8.16). However, we do not have to express $\delta^* A_\mu$ by ξ^μ in (8.29) since we know that L is the LAGRANGIAN for the field in interaction with the medium. Therefore the coefficient of $\delta^* A_\mu$ must be equal to zero, as we also see by virtue of (8.7) and (8.4b).

Now the ξ^μ are arbitrary at each point. This means that during the displacement period the dielectric in general will not move as a rigid body, but the bulk density will vary throughout the body. However, even under this deformation process the LAGRANGIAN (8.10) is permitted, since MAXWELL's equations are assumed to be valid within the body also when it becomes inhomogeneous, with the small velocity changes that appear because of the deformations. So MAXWELL's equations do not restrict the variations ξ^μ, and we obtain from (8.29)

$$\nabla_\nu \overset{M}{S}{}^{\cdot\nu}_{\mu} = \frac{1}{c^2}F_{\nu\alpha}(F^\alpha - \varepsilon^{\alpha\beta}F_\beta)\nabla_\mu V^\nu + \frac{1}{2c^2}F_\nu F_\alpha \nabla_\mu \varepsilon^{\nu\alpha}, \qquad (8.30)$$

where

$$\overset{M}{S}{}^{\cdot\nu}_{\mu} = F_{\mu\alpha}H^{\nu\alpha} - \tfrac{1}{4}g^\nu_\mu F_{\alpha\beta}H^{\alpha\beta}$$

is MINKOWSKI's tensor. We now introduce GALILEAN coordinates and use that $\partial_\mu V^\nu = 0$ for the undisturbed body, whence

$$\partial_\nu \overset{M}{S}{}^{\cdot\nu}_{\mu} = \frac{1}{2c^2}F_\nu F_\alpha \partial_\mu \varepsilon^{\nu\alpha}. \qquad (8.31)$$

We should like to mention the possibility of requiring the body to move as a rigid body under the deformation period in some coordinate system. Then the variations of one world line can be chosen arbitrarily, but the variations on the surface t = constant will now be determined by the metric tensor. Because of the relativity of simultaneity however, deformations will in general occur in another coordinate system. Besides, this type of variation does not lead to the strong result (8.30). To see this, let us confine ourselves to GALILEAN coordinates, in which the restriction reads ξ^{μ} = constant on an arbitrary hypersurface t = constant in some inertial system. If we let χ_{μ} mean the difference between the left and the right sides of (8.31), we can write (8.29) as

$$0 = \int \chi_{\mu} \xi^{\mu} dx = \int dx^0 \xi^{\mu} \int d^3 x \chi_{\mu}, \tag{8.32}$$

from which we can only conclude that the volume integral $\int d^3 x \chi_{\mu} = 0$.

Let us now return to the main result (8.31) emerging from the formalism. It should be clear that this result is only a certain combination of MAXWELL's equations. We could equivalently write eq. (8.31) in terms of ABRAHAM's tensor, or any other expression. Apart from the statement of the LAGRANGIAN (8.10), the subsequent calculation is of merely mathematical nature.

The present behaviour arises from the fact that the LAGRANGIAN (8.10) does not describe the total physical system. If the LAGRANGIAN had been complete, then we could further have reduced the expression for the variation of the action integral in view of the mechanical equations of motion, and would have been left with the total energy-momentum tensor as a result of the remaining variations. In some earlier treatments the electromagnetic energy-momentum tensor was claimed to be determined simply by the variation of the action integral with respect to the metric tensor. As mentioned above, HORVÁTH[47, 48] has emphasized the ambiguity of such a procedure. Further, H. G. SCHÖPF[50] has objected against certain calculational inconsistencies in the earlier attempts. The works of HORVÁTH and SCHÖPF contain references to the preceding literature.

In the treatment up till now we have generated all variations from *coordinate transformations,* since this seems to be the simplest kind of approach. However, one will commonly find another method used in order to calculate the variation of the velocity [3, 50, 51]. Namely to preserve the relation

$$V_{\mu} V^{\mu} = -c^2 \tag{8.33}$$

also after the variation, one introduces LAGRANGE variables a^λ ($\lambda = 1, 2, 3, 0$) to describe the medium, where $a^0 = p$ is an arbitrary invariant parameter of the nature of a time. Then, writing

$$V^\mu = \frac{c\partial x^\mu/\partial p}{\sqrt{-g_{\alpha\beta}\partial x^\alpha/\partial p\,\partial x^\beta/\partial p}}, \tag{8.34}$$

the relation (8.33) is identically satisfied. But when evaluating the variation of V^μ given by (8.34), the change in the $g_{\alpha\beta}$ must also be taken into account. In this way the ξ^μ are considered as arbitrary. However, we see that this procedure is necessary only if the LAGRANGIAN obeys an *action principle* with respect to the x^μ. In the case of an electromagnetic field in vacuum interacting with incoherent matter, as treated by FOCK[49] for example, the given LAGRANGIAN corresponds to the total system and must therefore yield the equations of motion of matter when the arbitrary ξ^μ-variations are taken in a *fixed* system of reference. Therefore one must take the restriction given by (8.33) into account, for instance by the parametrical representation (8.34). Another method has been given by L. INFELD[52, 53]; the method consists in introducing a LAGRANGIAN multiplier λ to take care of the degree of freedom being lost by (8.33).

In our case, the LAGRANGIAN L given by (8.10) obeys an action principle only with respect to the potentials; the ξ^μ-variations are consequences of coordinate transformations which preserve the condition (8.33) automatically. Therefore no attention was paid to the restriction (8.33) in the calculation above. But it is not incorrect to use the representation (8.34). We then obtain instead of (8.15) the velocity variation

$$\delta'^* V^\mu = V^\nu \nabla_\nu \xi^\mu - \xi^\nu \nabla_\nu V^\mu + \frac{1}{c^2} V^\mu V_\sigma V^\nu \nabla_\nu \xi^\sigma, \tag{8.35}$$

when the change in the $g_{\alpha\beta}$ is taken into account. However when evaluating the δ_g^*-variations, we vary also the $g_{\alpha\beta}$ in (8.34), so that

$$\delta_g^* V^\mu = -\frac{1}{2c^2} V^\mu V_\sigma V_\nu \delta^* g^{\nu\sigma} = -\frac{1}{c^2} V^\mu V_\sigma V^\nu \nabla_\nu \xi^\sigma, \tag{8.36}$$

where (8.18) has been inserted. We see that in the total velocity variation $(\delta'^* + \delta_g^*)V^\mu$ the expression (8.36) compensates the last term in (8.35), so that we end up with a certain combination of MAXWELL's equations, as before.

Similarly, by using INFELD's method, the multiplier λ drops out of the calculation.

We mention that in the case of isotropic media (fluids) some attempts(54, 50) have been made to complete the LAGRANGIAN so as to make the system closed. In such a case the LAGRANGIAN has to obey a variational principle also with respect to coordinate variations, so that one may use the representation (8.34). In this way the total energy-momentum tensor has been found to be given by ABRAHAM's tensor plus the hydrodynamical tensor.

The consistency of such a procedure may be illustrated by the following consideration. We first tentatively write the LAGRANGIAN density for the total system as

$$L^{\text{tot}} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{\varkappa}{2}F_{\mu}F^{\mu} - \mathring{\varrho}_m c^2, \tag{8.37}$$

where $\varkappa = (n^2 - 1)/c^2$, $F_\mu = F_{\mu\nu}V^\nu$, and $\mathring{\varrho}_m$ is the invariant rest mass density of the fluid. If we now perform coordinate variations (for fixed metric) and evaluate the contribution to the action integral which arises from the second term to the right in (8.37), we find the expression $\int f_\mu^A \xi^\mu \sqrt{-g}\, dx$ due to the velocity variations (8.35). Here f_μ^A means ABRAHAM's force density written in general coordinates. Therefore the coordinate variations, which effect only the two last terms in (8.37), lead to the hydrodynamical equations of motion with ABRAHAM's force as the external force. This result is compatible with the interpretation we found in section 3, and this is the crucial point, since it permits the adoption of (8.37) as the correct LAGRANGIAN density for the total system. If we then perform an infinitesimal coordinate transformation so that the action integral remains invariant, we see that the coefficients in front of $\delta^* A_\mu$ and ξ^μ vanish in view of the field equations and the equations of motion, so that we are left with a divergence-free total energy-momentum tensor in front of $\delta^* g^{\mu\nu}$ which is equal to the sum of ABRAHAM's tensor and the hydrodynamical tensor.

Note that the present direct connection between the variation of the metric tensor and the energy-momentum tensor, and between the remaining variations and the equations of motion, is lost if we employ our first method and generate all variations from coordinate transformations. Thus, if we use the LAGRANGIAN (8.10), a variation of the action integral (8.11) with respect to the metric tensor leads to ABRAHAM's tensor only if both (8.18) and (8.36) are taken into account. However, in order to analyse how the conservation equations emerge from the formalism when (8.10) is used, our first method is simpler.

Final remarks on the Sagnac-type experiment

The last task that we shall take up in our work is to give an extended analysis of the recent Sagnac-type experiment due to HEER, LITTLE and BUPP(55) which we considered in sect. 9 of I in connection with MINKOWSKI's tensor. We shall examine how this experiment is explained by the other tensors.

Let us briefly recall the essential features of the experiment. The apparatus is a triangular ring laser giving rise to two travelling electromagnetic waves in the cavity, one circulating clockwise and the other counterclockwise. A dielectric medium is placed in the light path. When the system is at rest the photon frequencies in the two wave modes are equal. If the cavity is set into rotation with an angular velocity Ω, the photon frequencies of the two beams become different from each other and the beams interfere to produce beats which are counted. With MINKOWSKI's tensor the energy density W^M for one of the modes in the noninertial cavity frame is related to the energy density W^0 for this mode in an instantaneous inertial rest frame by

$$W^M = W^0 + \frac{1}{c}\boldsymbol{\Omega} \cdot [\boldsymbol{r} \times (\boldsymbol{E} \times \boldsymbol{H})], \qquad (8.38)$$

where the fields refer to the mode considered, and are evaluated for $\Omega = 0$ since only effects to the first order in Ω are investigated. Further, within this approximation the total field energy in the cavity frame is a conserved quantity, so that we obtain the formula (I, 9.6) for the relative frequency shift

$$\left(\frac{\Delta\nu}{\nu}\right)^M = \frac{4}{c}\frac{\boldsymbol{\Omega} \cdot \int \boldsymbol{r} \times (\boldsymbol{E} \times \boldsymbol{H})\, dV}{\int (\boldsymbol{E} \cdot \boldsymbol{D} + \boldsymbol{H} \cdot \boldsymbol{B})\, dV}. \qquad (8.39)$$

In the plane wave approximation the agreement between (8.39) and the observed data is excellent, and the authors conclude that their experiment supports the asymmetric MINKOWSKI's tensor.

As we shall see now, the above conclusion should be somewhat modified: The experiment represents a nice verification of the predictions of phenomenological electrodynamics, but it is not a *critical* test of the convenience of MINKOWSKI's tensor as compared to all other tensor forms. In fact, both ABRAHAM's tensor and the radiation tensor give an equivalent description of the experiment. For we have in any case, to the first order in Ω, the following formula for the energy density in the cavity frame:

$$W = W^0 + \frac{g_{4k}}{g_{44}} \mathring{S}_4^{\cdot k}, \tag{8.40}$$

where $g_{\mu\nu}$ is the metric tensor in the cavity frame and the superscript zero refers to the instantaneous rest inertial frame. Since the tensor components $\mathring{S}_4^{\cdot\nu}$ are equal for MINKOWSKI's and ABRAHAM's tensors and also for the radiation tensor, we must obtain the same value for W. Therefore, in any of these cases, we can put the conserved total field energy of each mode proportional to the corresponding photon frequency, and obtain again the fundamental formula (8.39).

Note that the equivalence of the above three tensors with respect to the energy balance in the cavity frame holds for all participating terms. The energy balance reads in general

$$\nabla_\nu S_4^{\cdot\nu} = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\nu}(\sqrt{-g}\, S_4^{\cdot\nu}) - \Gamma_{\varrho,4\nu} S^{\varrho\nu} = -f_4, \tag{8.41}$$

but it can be verified that the term involving the CHRISTOFFEL symbol yields no contribution to the first order in Ω. Moreover, by performing a coordinate transformation between the inertial frame and the cavity frame we find that $f_4 = 0$, even in the ABRAHAM case, and that the components $S_4^{\cdot k}$ take on common values. In all the three cases considered we can thus write the energy balance as $\partial_\nu S_4^{\cdot\nu} = 0$, with common values for the tensor components.

Finally we note that with the DE GROOT-SUTTORP tensor (1.9), complications arise because the expression for W^0 is changed. In this case the force component f_4 is different from zero, yet the total field energy is a conserved quantity in the cavity frame since f_4 fluctuates away when integrated over the volume. However, we do not now obtain the expression (8.39) for the relative frequency shift; in fact, if we put the total energy proportional to the photon frequency for each mode we find the formula $(\Delta v/v)^G = (\mathscr{H}^{M^0}/\mathscr{H}^{G^0})(\Delta v/v)^M$, in disagreement with experiment. This tensor seems in general not to be suitable for the description of propagating waves, since in an inertial rest frame the magnitude of the quantity $\boldsymbol{S}^{G^0}/W^{G^0}$ is different from c/n.

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Appendix

The table below gives a summary of the behaviour of the various energy-momentum tensors in those examined physical situations which are of experimental interest. References are given to those sections of Part I or Part II where the actual subject has been investigated. Cf. also the summaries in the introductory sections of I and II.

Situation considered	a) Minkowski	b) Abraham	c) Radiation tensor (Marx et al; Beck)	d) Einstein – Laub	e) de Groot-Suttorp (first version)
Dielectric isotropic or anisotropic body surrounded by a vacuum or isotropic liquid and acted upon by an electrostatic field: Measurement of force or torque.	Within an anisotropic body the tensor *asymmetry* is of main importance for the torque. I, sect. 3; II, sect. 2.	Torque always described in terms of the *force*. II, sect. 2.	Not defined in this case.	Same experimental result as in the cases a) and b). II, sect. 2.	
	No experimental distinction possible. II, sect. 2.				
Excess pressure produced in a dielectric liquid by an electrostatic field: Hakim-Higham experiment.	In this case the electrostrictive terms must be taken into account. Thereby one obtains a tensor which yields Helmholtz' force, and which is in agreement with the second tensor form put forward by de Groot and Suttorp. Good agreement with experiment. II, sect. 2.		Not defined in this case.	Force density equal to Kelvin's force. Disagreement with experiment. II, sect. 2.	
Radiation pressure exerted by an electromagnetic wave travelling through a dielectric liquid: Jones-Richards experiment.	Good agreement with experiment. Simple interpretation. I, sect. 6; II, sect. 3.	Equivalent to case a), when the appropriate interpretation is imposed. II, sect. 3.	Disagreement with experiment. II, sect. 3.	Inconvenient.	
Dielectric isotropic or anisotropic body surrounded by a *vacuum* and acted upon by a high-frequency field: Measurement of force or torque (Barlow experiment, Beth experiment, etc.).	No experimental distinction possible. II, sect. 4.		Defined for isotropic media only. Same experimental result as in the cases a) and b), although the direction and magnitude of the surface force in general are different. II, sect. 4.	Same experimental result as in the cases a)-c).	
Dielectric isotropic or anisotropic body surrounded by a liquid and acted upon by a high-frequency field: Measurement of force or torque (experiment not performed).	No experimental distinction possible. II, sect. 4.		Experiment of the Barlow type should represent a critical test. II, sect. 4.	Experiment of the Barlow type should also here be critical. The torque formula is different from the formulas corresponding to the cases a)-c). II, sect. 4.	

Situation considered	a) Minkowski	b) Abraham	c) Radiation tensor (Marx et al; Beck)	d) Einstein-Laub	e) de Groot-Suttorp (first version)
Low-frequency variation of electric and magnetic fields: Measurement of oscillations of a suspended dielectric shell (experiment not performed).	Does not predict oscillations. The equivalence between the tensors does not apply to this case. An experimental distinction should be possible. II, sect. 4.	Predicts oscillations.	Same behaviour as in the case b). II, sect. 4.		
Čerenkov effect.	Good agreement with the experiments. Simple interpretation. I, sect. 10; II, sect. 5 and 7.	Equivalent to case a), when the appropriate interpretation is imposed. II, sect. 5 and 7.	Leads to unphysical value for the Čerenkov angle. II, sect. 5.	Inconvenient.	
Velocity of the energy of an optical wave in a uniformly moving body: Fizeau type experiments.	Good agreement with the experiments. The von Laue-Møller transformation criterion is fulfilled. I, sect. 9; II. sect. 7.	Equivalent to case a), when the appropriate interpretation is imposed, II, sect. 7.	Same behaviour as in the case a).	Inconvenient.	
Sagnac-type experiment performed by Heer, Little, Bupp.	Good agreement with experiment. I, sect. 9; II, sect. 8.			Inconvenient. II, sect. 8.	

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