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GENERAL RELATIVISTIC FLUID SPHERE AT MECHANICAL AND THERMAL EQUILIBRIUM

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Synopsis

The field equations of general relativity for a static fluid sphere composed of ideal gas and radiation are solved numerically under the assumption that the fluid is at thermal equilibrium throughout a configuration. The law of thermal equilibrium is used in the general relativistic form.

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1. Introduction

Only few solutions describing the static gravitational field inside a spherical mass distribution are known. This is caused by the complicated non-linear character of the field equations. In order to ensure solutions in terms of known functions, one has been obliged to pay more attention to a mathematical simplification than to a physical situation. Therefore the resulting solutions, derived by R. C. Tolman[1], M. Wyman[2] and B. Kuchowicz[3], are not of much physical interest.

A more satisfactory procedure is to use an equation of state as an auxiliary equation. The system of equations can then be solved only numerically. The simplest cases are the relativistic generalizations of the classical polytropic, standard and isothermal fluid spheres. A polytropic fluid sphere where the pressure and energy density are connected by a power law, has been examined by R. F. Tooper[4, 5]. He has also examined a standard model where the ratio of the gas pressure to the total pressure is constant throughout a configuration[6]. A fluid sphere obeying an isothermal equation of state has been studied by M. L. Mehra[7]. However, it appears that there are some errors in Mehra's paper. Mehra has not taken into account the microscopic kinetic energy of the gas. The law of thermal equilibrium has been used in the classical form. The boundary conditions are not satisfactory, either.

In the present paper a relativistic treatment for a fluid sphere where the matter is at thermal equilibrium, is given again. The proper temperature of the fluid as measured by local observers is not constant throughout a sphere, but varies with gravitational potential according to the general relativistic law of thermal equilibrium[8]. The conditions under which thermal equilibrium might arise are not examined in this paper. The dynamical stability of a relativistic 'isothermal' sphere is not investigated, either.

In section 2 the general relativistic equations of mechanical equilibrium are given for a spherically symmetric, static system. Using the relativistic, statistical expressions for the pressure and energy density[9], the equations of state are derived for a mixture of ideal gas and radiation in section 3. The equations of mechanical equilibrium are transformed in a suitable dimensionless form in section 4. In section 5 the radius, mass, and pressure distribution are expressed in terms of dimensionless variables. The equilibrium equations have been integrated numerically, and results are given and discussed in section 6.

2. Equations of Mechanical Equilibrium

Any time-independent, spherically symmetric general relativistic metric can be transformed to the standard form

$$ds^2 = -e^{\lambda}dr^2 - r^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2) + e^{\nu}c^2dt^2, \tag{1}$$

where the functions e^{λ} and e^{ν} depend on the radial coordinate r only. Assuming the expression for the energy-momentum tensor

$$T^i_j = \left(\varrho + \frac{p}{c^2}\right)u_j u^i - \frac{p}{c^2} g^i_j, \tag{2}$$

where quantities u^i, p and ϱ are the fluid 4-velocity, pressure and total mass density, respectively, the gravitational field equations reduce in the metric (1) to

$$e^{-\lambda}\left(\frac{1}{r}\frac{d\nu}{dr} + \frac{1}{r^2}\right) - \frac{1}{r^2} = \frac{8\pi G}{c^4}p \tag{3}$$

$$\frac{1}{2}e^{-\lambda}\left\{\frac{d^2\nu}{dr^2} + \frac{1}{2}\left(\frac{d\nu}{dr} - \frac{d\lambda}{dr}\right)\left(\frac{d\nu}{dr} + \frac{2}{r}\right)\right\} = \frac{8\pi G}{c^4}p \tag{4}$$

$$e^{-\lambda}\left(\frac{1}{r}\frac{d\lambda}{dr} - \frac{1}{r^2}\right) + \frac{1}{r^2} = \frac{8\pi G}{c^2}\varrho. \tag{5}$$

Here G is the Newtonian gravitational constant. Setting eqs. (3) and (4) equal to each other and making use of eq. (5), lead to the relation

$$\frac{dp}{dr} + \frac{1}{2}(\varrho c^2 + p)\frac{d\nu}{dr} = 0. \tag{6}$$

Relation (6) follows also from the conservation of the energy-momentum. Eqs. (3), (5) and (6) are the relativistic conditions for mechanical equilibrium which, together with an equation of state, form the full set for determination of e^{λ}, e^{ν}, p and ϱ as functions of the coordinate r.

3. Equations of State

The total pressure and the total mass density of a fluid sphere consisting of ideal gas and radiation are

$$p = p_g + p_r \tag{7}$$

$$\varrho = \varrho_g + \varrho_r, \tag{8}$$

where quantities belonging to ideal gas and radiation are denoted with indexes g and r.

A relativistic ideal gas consisting of identical particles has the following equations of state[9]

$$p_g + \varrho_g c^2 = \frac{\gamma}{\tau} K_3(\tau) \tag{9}$$

$$p_g = \frac{\Re}{\mu} \varrho_0 T = \frac{\gamma}{\tau^2} K_2(\tau). \tag{10}$$

Here we denote

$$\tau = \frac{\mu c^2}{\Re T}; \tag{11}$$

$K_n(\tau)$ are modified Bessel functions of the second kind; γ is a coefficient depending on nature constants and on the chemical potential; $\Re$ is the gas constant; μ is the mean molecular weight; ϱ_0 is the rest-mass density, and T is the proper temperature measured in a fluid comoving system of coordinates. Using the recurrence relation

$$K_3(\tau) = K_1(\tau) + \frac{4}{\tau} K_2(\tau), \tag{12}$$

we obtain from eqs. (9) and (10) the expression for $\varrho_g c^2$:

$$\left.\begin{aligned} \varrho_g c^2 &= \frac{\gamma}{\tau} K_1(\tau) + \frac{3\gamma}{\tau^2} K_2(\tau) \\ &= \left(\frac{K_1(\tau)}{K_2(\tau)} + \frac{3}{\tau}\right) \varrho_0 c^2. \end{aligned}\right\} \quad (13)$$

The pressure and energy density of radiation are expressed by

$$p_r = \frac{1}{3} \varrho_r c^2 = \frac{1}{3} a T^4, \quad (14)$$

where a is the radiation constant.

For a mixture of ideal gas and radiation we get from eqs. (7), (8), (10), (13) and (14) the equations of state

$$p + \varrho c^2 = \frac{4}{3} a T^4 + \left(\frac{K_1(\tau)}{K_2(\tau)} + \frac{4}{\tau}\right) \varrho_0 c^2 \quad (15)$$

$$p = \frac{1}{3} a T^4 + \frac{\mathfrak{R}}{\mu} \varrho_0 T. \quad (16)$$

4. Equations of Mechanical Equilibrium in Dimensionless Form

From the general relativistic law of thermal equilibrium(8)

$$T (g_{44})^{1/2} = T e^{\nu/2} = \text{constant} \quad (17)$$

we obtain a relation

$$\frac{d\nu}{dr} = - \frac{2}{T} \frac{dT}{dr} \quad (18)$$

which, substituted in eq. (6), gives

$$\frac{dp}{dr} - \frac{(\varrho c^2 + p)}{T} \frac{dT}{dr} = 0. \quad (19)$$

Inserting expressions (15) and (16) in eq. (19), we obtain after a simplification

$$\frac{1}{\varrho_0} \frac{d\varrho_0}{dr} - \frac{1}{T} \left(3 + \tau \frac{K_1(\tau)}{K_2(\tau)}\right) \frac{dT}{dr} = 0. \quad (20)$$

Eq. (20) is difficult to integrate due to the presence of functions $K_1(\tau)$ and $K_2(\tau)$. However, in the case of large argument τ, we have a convenient approximation for $K_n(\tau)$[10]:

$$K_n(\tau) = \left(\frac{\pi}{2\tau}\right)^{1/2} e^{-\tau}\left\{1 + \frac{4n^2-1}{8\tau} + \frac{(4n^2-1)(4n^2-3)}{2!(8\tau)^2} + \ldots\right\}, \tag{21}$$

from which

$$\frac{K_1(\tau)}{K_2(\tau)} \approx 1 - \frac{3}{2\tau}. \tag{22}$$

This approximation can be used for all constituents of the fluid below the limit $T = 10^9$ degrees where $\tau = 5.9$ for electrons and positrons. For all constituents except electrons and positrons, extension (21) is valid for $T < 10^{12}$ degrees. But in any case we must restrict the treatment to temperatures below $2 \cdot 10^9$ degrees, because above this limit the neutrino emission carries so much energy away that equilibrium is disturbed[11].

Using expression (22), eq. (20) becomes

$$\frac{1}{\varrho_0}\frac{d\varrho_0}{dr} - \frac{1}{T}\left(\frac{3}{2} + \tau\right)\frac{dT}{dr} = 0, \tag{23}$$

which can be integrated to give

$$\varrho_0 = A T^{3/2} e^{-\mu c^2/\Re T}. \tag{24}$$

The constant A of integration is determined by initial conditions. Denoting the rest mass density and the proper temperature at the centre by ϱ_{0c} and T_c, we obtain

$$\frac{\varrho_0}{\varrho_{0c}} = \left(\frac{T}{T_c}\right)^{3/2} e^{-\frac{\mu c^2}{\Re}\left(\frac{1}{T} - \frac{1}{T_c}\right)}. \tag{25}$$

Eq. (5), which can be written also in the form

$$\frac{d(re^{-\lambda})}{dr} = 1 - \frac{8\pi G}{c^2}\varrho r^2, \tag{26}$$

can be integrated only formally because the distribution of the energy density is unknown. We define a new function $M(r)$ by

$$M(r) = \int_0^r 4\pi\varrho r^2 dr. \tag{27}$$

Function $M(r)$ represents the total mass arising from the density ϱ and from the gravitational field, inclosed by the sphere of coordinate radius r. Integrating eq. (26), we get

$$e^{-\lambda} = 1 - \frac{2GM(r)}{c^2 r}. \tag{28}$$

In terms of function M, eq. (26) becomes

$$\frac{dM}{dr} = 4\pi\varrho r^2. \tag{29}$$

Substituting the expression of density ϱ from eqs. (13), (14) and (22) in eq. (29), we obtain

$$\frac{dM}{dr} = \frac{4\pi r^2}{c^2}\left\{aT^4 + \left(1 + \frac{3}{2\tau}\right)\varrho_0 c^2\right\}. \tag{30}$$

Insertion of eqs. (18) and (28) in eq. (3) gives

$$-\frac{2}{r}\left(1 - \frac{2GM}{c^2 r}\right)\frac{1}{T}\frac{dT}{dr} + \frac{1}{r^2}\left(1 - \frac{2GM}{c^2 r}\right) - \frac{1}{r^2} = \frac{8\pi G}{c^4}p. \tag{31}$$

Using the expression

$$p = \frac{c^2}{12\pi r^2}\frac{dM}{dr} - \left(\frac{1}{3} - \frac{1}{2\tau}\right)\varrho_0 c^2 \tag{32}$$

obtained from eqs. (15), (16), (22) and (29), eq. (31) becomes

$$\left(1 - \frac{2GM}{c^2 r}\right)\frac{r}{T}\frac{dT}{dr} + \frac{GM}{c^2 r} + \frac{G}{3c^2}\frac{dM}{dr} - \frac{4\pi G}{c^2}\left(\frac{1}{3} - \frac{1}{2\tau}\right)\varrho_0 r^2 = 0. \tag{33}$$

A change of variables

$$r = \frac{x}{B} \tag{34}$$

$$v(x) = \frac{B^3 M(r)}{4\pi\varrho_{0c}}, \tag{35}$$

where

$$B = \left(\frac{\pi G\varrho_{0c}^2}{p_c}\right)^{1/2} \tag{36}$$

puts eqs. (30) and (33) in dimensionless form. Substituting new variables x and v in eqs. (30) and (33), we obtain the following first order differential equations

$$\frac{dv}{dx} = x^2 \left\{ \frac{aT^4}{\varrho_{0c} c^2} + \left(1 + \frac{3\Re T}{2\mu c^2}\right) \frac{\varrho_0}{\varrho_{0c}} \right\} \tag{37}$$

$$\left(\frac{\varrho_{0c} c^2}{4p_c} - \frac{2v}{x}\right) \frac{x^2}{T} \frac{dT}{dx} + v + \frac{x}{3} \frac{dv}{dx} - \left(\frac{1}{3} - \frac{\Re T}{2\mu c^2}\right) \frac{\varrho_0}{\varrho_{0c}} x = 0. \tag{38}$$

Using eq. (37), eq. (38) can be written also in the form

$$\left(\frac{1}{C} - \frac{2v}{x}\right) \frac{x^2}{T} \frac{dT}{dx} + v + \frac{aT^4}{3\varrho_{0c} c^2} x^3 + \frac{\Re T}{\mu c^2} \frac{\varrho_0}{\varrho_{0c}} x^3 = 0, \tag{39}$$

where we denote

$$C = \frac{4p_c}{\varrho_{0c} c^2} = 4\left(\frac{aT_c^4}{3\varrho_{0c} c^2} + \frac{\Re T_c}{\mu c^2}\right). \tag{40}$$

In terms of variables x and v, the following expression for the metric component $|g_{11}| = e^{\lambda}$ is found from eq. (28)

$$e^{\lambda} = \left(1 - 2C\frac{v}{x}\right)^{-1}. \tag{41}$$

5. Physical quantities

The element of spatial distance measured with a standard measuring-rod is given by[12]

$$dl^2 = \left(g_{ij} - \frac{g_{4i} g_{4j}}{g_{44}}\right) dx^i dx^j. \tag{42}$$

In the metric (1), the metric components g_{4i} are equal to zero, and the physical distance from the centre to a point with radial coordinate r is from eqs. (34), (41) and (42),

$$R = \int_0^r e^{\lambda/2} dr = \frac{1}{B} \int_0^x \left(1 - 2C\frac{v(x)}{x}\right)^{-1/2} dx \tag{43}$$

or measured in units of the radius of the sun $R_\odot = 6.96 \cdot 10^{10}$ cm

$$\frac{R}{R_\odot} = 4.71 \cdot 10^2 \left(\frac{C}{\varrho_{0c}}\right)^{1/2} \int_0^x \left(1 - 2C\frac{v(x)}{x}\right)^{-1/2} dx, \tag{44}$$

where we have also used eqs. (36) and (40). For the mass inside a sphere of the radius r we obtain from eqs. (35) and (36) the expression

$$M = \frac{4p_c^{3/2}}{\pi^{1/2} G^{3/2} \varrho_{0c}^{\;2}} v(x) = 2.22 \cdot 10^8 M_\odot \frac{C^{3/2}}{\varrho_{0c}^{\;1/2}} v(x), \tag{45}$$

where $M_\odot = 1.985 \cdot 10^{33} g$ is the mass of the sun. The expression for the pressure distribution

$$\left.\begin{aligned} \frac{p}{p_c} &= \frac{4}{C}\left(\frac{aT^4}{3\varrho_{0c}^{\;2}} + \frac{\Re T}{\mu c^2}\frac{\varrho_0}{\varrho_{0c}}\right) \\ &= \frac{1}{C}\left(1.13 \cdot 10^{-3}\frac{T^4}{\varrho_{0c}} + 3.70 \cdot 10^{-5}\frac{T}{\mu}\frac{\varrho_0}{\varrho_{0c}}\right) \end{aligned}\right\} \tag{46}$$

is found from eqs. (16) and (40).

6. Numerical Results and Discussion

Eqs. (37) and (39) were integrated numerically under the initial conditions $v(0) = 0$ and $T(0) = T_c$ using the Runge-Kutta method. The molecular weight was taken to be 0.5, corresponding to purely hydrogen configurations. Simultaneously with the determinations of functions $v(x)$ and $T(x)$ integral (43) for the physical radius R was calculated. The values of the radial metric component $|g_{11}| = e^\lambda$, mass and normalized pressure were obtained from eqs. (41), (45) and (46). For the calculation of the metric component $g_{44} = e^\nu$ from eq. (17), it was assumed that, far from the centre, the relation $|g_{11}| = 1/g_{44}$ is valid.

Numerical results in an appropriate case are shown graphically in Fig. 1. The physical radius R is used as the independent variable, so that the geometrical effects arising from the presence of gravitating matter have been taken into account in preparing the graphs.

The behaviour of the results differs in several respects from the corresponding classical case. In spite of thermal equilibrium, the proper temperature measured in a fluid comoving system of coordinates, decreases monotonically as a function of radius. The matter is concentrated more mark-

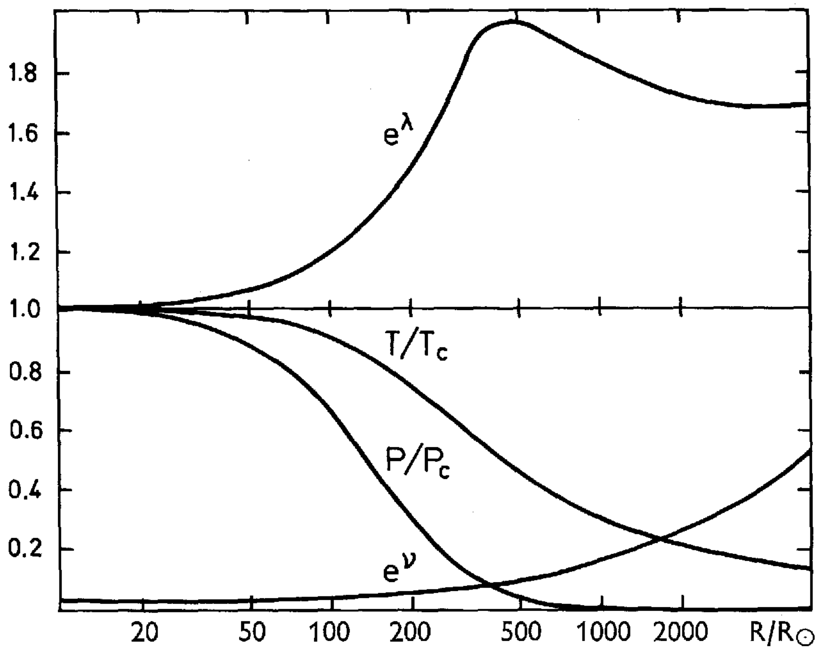

Fig. 1. Metric components, temperature and pressure as functions of the physical radius, when central temperature $T_c = 10^9$ degrees.

edly toward the centre than in a classical isothermal sphere. The normalized pressure and total density fall off rapidly in an outward direction. The metric components e^{λ} and e^{ν} differ strongly from their Euclidian values. The radial component $|g_{11}| = e^{\lambda}$ increases from its minimum value unity at the centre to the first and highest maximum, goes down to the lowest minimum, and then oscillates moderately. The time component e^{ν} is always less than unity, and is a monotonically increasing function of radius.

In Tables 1 and 2 the values of the radius, mass and normalized temperature and pressure are given, corresponding to the first maximum and second minimum of e^{λ} for three central temperatures. Eqs. (37) and (39) include besides the central temperature T_c, the central rest mass density ϱ_{0c} as another parameter. The rest mass density originates from the gas pressure term in the expression of the total pressure. The ratio of gas pressure to radiation pressure decreases very rapidly as a function of radius. Even in cases where gas pressure dominates over radiation pressure at the centre, it is negligible quite soon outside the centre. This is caused by an exponential decrease of the rest mass density with respect to the proper temperature, which is found in eq. (25). For this reason, the metric tensor

TABLE 1. Radius, mass, normalized temperature and pressure corresponding to the first maximum of e^{λ}.

T_c	$R/R_{\odot}$	$M/M_{\odot}$	T/T_c	P/P_c	e^{λ}
10^8	$4.8\cdot10^4$	$4.4\cdot10^9$	0.45	0.041	1.965
$5\cdot10^8$	$2.0\cdot10^3$	$1.8\cdot10^8$	0.45	0.041	1.965
10^9	$4.8\cdot10^2$	$4.4\cdot10^7$	0.45	0.041	1.965

TABLE 2. Radius, mass, normalized temperature and pressure corresponding to the second minimum of e^{λ}.

T_c	$R/R_{\odot}$	$M/M_{\odot}$	T/T_c	P/P_c	e^{λ}
10^8	$3.0\cdot10^5$	$2.2\cdot10^{10}$	0.18	$1.0\cdot10^{-4}$	1.690
$5\cdot10^8$	$1.2\cdot10^4$	$9.1\cdot10^8$	0.18	$1.0\cdot10^{-4}$	1.690
10^9	$3.0\cdot10^3$	$2.2\cdot10^8$	0.18	$1.1\cdot10^{-4}$	1.690

components and physical parameters are nearly independent of the central rest mass density.

It is seen from tables 1 and 2 that the maximum and minimum values of the metric component e^{λ} and the corresponding values of the normalized temperature and pressure are independent of the central temperature. The physical radius and mass are approximately inverse to the square of the central temperature. This is valid, not only for the maximum and minimum values of e^{λ}, but everywhere. Because of the similar feature of the graphs for different central temperatures, we have shown the results graphically only in one special case.

The condition of thermal equilibrium leads to very massive configurations. Such fluid spheres extend to infinity, like classical isothermal models. However, it might be possible to develop theoretical models of a moderately long time-scale where the inner parts are at mechanical and thermal equilibrium, but the outer layers are away from equilibrium. The radial pulsations at the surface should correspond to a "stellar wind".

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