ESKO SUHONEN

ON THE ENERGY TRANSFER BY MATERIAL PARTICLES AND RADIATION IN A GENERAL RELATIVISTIC GAS

Det Kongelige Danske Videnskabernes Selskab
Matematisk-fysiske Meddelelser **36**, 13

Kommissionær: Munksgaard
København 1968

Synopsis

A general relativistic form of the energy transfer equation is developed for material particles interacting with a spherically symmetric and static medium. Using the relativistic expressions of the energy density and the pressure in the different statistics, the radial energy flow is obtained from the transfer equation for Maxwellian particles, bosons and fermions as a diffusion approximation. The thermal conductivity of ionization electrons is derived for a medium consisting of only one kind of nuclei. The equations of the energy transfer by photons and neutrinos are obtained in the special case of massless energy carriers. The condition of no net energy flow is found to lead to Tolman's law of thermodynamical equilibrium. The mean absorption coefficient of material particles is introduced and an approximative method for the calculation of this coefficient is given analogously to the Rosseland mean opacity.

PRINTED IN DENMARK
BIANCO LUNOS BOGTRYKKERI A/S

1. Introduction

In order to investigate the state of matter in the interior of dense stars, it is necessary to derive the energy transfer by radiation as well as by material particles. In most cases the energy transfer by material particles or the thermal conductivity accounts only for a negligible proportion of the energy transfer. But under certain conditions the conductivity of the electron gas plays an important role, particularly in the white dwarf stars(1).

The discovery of the quasistellar radiosources in the past few years has given support to the theory of relativistic astrophysics. The relativistic methods are necessary because of those special circumstances which are probably present in these radio sources(2). The radiative energy transfer has been studied both in the static(3) and in the timedependent(4) radially symmetric medium in a general covariant way. But the general relativistic problem of the conductive energy transfer seems to be unexplored.

In this paper the energy transfer by material particles is examined by the same method as for the radiative transfer problem. The treatment is restricted to the radially symmetric and static case. We also limit ourselves to the case of energy flow due to identical particles which are interacting with the external medium while their interactions with each other are neglected. Further, the study is restricted to the condition that the gas is close to thermodynamical equilibrium.

After some auxiliary equations the transfer equation is presented in section 2. In section 3 the energy density, the radial energy flow and the pressure of the particles carrying the energy are introduced and the transfer equation is expressed as a relation between these quantities. By using the relativistic expressions for the energy density and the pressure in the different statistics, the radial energy flow is found proportional to the gradients of the thermal and the chemical potentials, in chapter 3A for Maxwellian particles and in chapter 3B for fermions and bosons.

The thermal conductivity of ionization electrons which are interacting with a medium consisting of only one kind of nuclei is derived in chapter 4A.

In chapter 4B the expressions of the radiative energy transfer and the neutrino diffusion are obtained in the special case of massless energy carriers. The mean absorption coefficient of material particles is introduced, and an approximative method for the calculation of this coefficient is given for electrons, photons and neutrinos in section 5.

2. Transport equation

We shall study a stream of identical particles, which at a definite point is characterized by a four-dimensional elementary solid angle $\delta\Omega$, a space-like direction N and an intensity I_Ω $\left(\text{in dimension } \frac{|\text{energy}|}{|\text{area}|\,|\text{time}|}\right)$. The elementary solid angle $\delta\Omega$ is given by

$$\delta\Omega = \sinh^2\xi\delta\xi\delta\omega, \tag{1}$$

where $\delta\omega$ is the three-dimensional solid angle in the local rest-system of the external medium with which the particles are interacting. The number ξ denotes a pseudoangle which measures the angle between the unit tangent of nearly parallel world lines in $\delta\Omega$ and the four-velocity vector of the medium.

The metric of the radially symmetric and static medium has the form

$$ds^2 = -e^{\lambda(r)}dr^2 - r^2(d\nu^2 + \sin^2\nu d\varphi^2) + e^{\mu(r)}c^2dt^2. \tag{2}$$

We denote the space-like unit vector in the radial direction by N_r and the time-like unit vector in the direction of time by U. In the metric (2) they have the following contravariant components:

$$\left.\begin{aligned} N_r^\alpha &= \left\{e^{-\frac{1}{2}\lambda}, 0, 0, 0\right\} \\ U^\alpha &= \left\{0, 0, 0, e^{-\frac{1}{2}\mu}\right\}. \end{aligned}\right\} \tag{3}$$

Now we let θ denote the angle between N and N_r. The vectors N, N_r and U then have the properties:

$$\left.\begin{aligned} N\cdot N &= N_r\cdot N_r = -1 \\ U\cdot U &= 1 \\ N\cdot U &= N_r\cdot U = 0 \\ N\cdot N_r &= -\cos\theta. \end{aligned}\right\} \tag{4}$$

We assume that the particle paths are geodesics except in the regions where scatterings occur. The geodesics-approximation is good, if we idealize the scattering processes as point interactions. This is a rather useful approximation, even for electrons in an ionized medium. The electromagnetic interactions between the electrons and between electrons and nuclei cancel on a large scale the effect of each other. We can then write for the particles the equations of motion between scattering regions in the form

$$U_\Omega \cdot (\nabla U_\Omega) = 0, \tag{5}$$

where

$$U_\Omega = U \cosh \xi + N \sinh \xi \tag{6}$$

is the four-velocity vector of the particles belonging to the solid angle $\delta\Omega$ and ∇U_Ω is the covariant gradient of U_Ω.

Taking the scalar product of eq. (5) and the vector U, we obtain

$$U_\Omega \cdot (\nabla U_\Omega) \cdot U = U_\Omega \cdot \nabla \cosh \xi - U_\Omega \cdot (\nabla U) \cdot U_\Omega = 0. \tag{7}$$

Since we have assumed the situation to be static, the operator $U \cdot \nabla$ applied to any scalar function is equal to zero. When this is taken into account, eq. (7) is reduced to the form

$$\sinh \xi N \cdot \nabla \cosh \xi = U_\Omega \cdot (\nabla U) \cdot U_\Omega . \tag{8}$$

Taking the gradient of U in component form, we get the equation

$$N \cdot \nabla \cosh \xi = -\frac{1}{2} \frac{d\mu}{dr} e^{-\frac{\lambda}{2}} \cosh \xi \cos \theta . \tag{9}$$

Performing similar calculations by multiplying eq. (5) scalarly by N_r, we obtain the result

$$N \cdot \nabla \cos \theta = \frac{1}{2} e^{-\frac{\lambda}{2}} \left(\frac{2}{r} - \frac{d\mu}{dr} \frac{\cosh^2 \xi}{\sinh^2 \xi} \right) \sin^2 \theta , \tag{10}$$

where we have also made use of eq. (9).

The energy-momentum for the stream under consideration is

$$\delta T_\Omega = c^2 \varrho_\Omega U_\Omega U_\Omega \delta\Omega , \tag{11}$$

where ϱ_Ω is the density of the rest mass of the particles in the solid angle $\delta\Omega$. In writing eq. (11) we have ignored the interactions of particles with each other. The quantity $-cU\cdot\delta T_\Omega\cdot N$ describes the energy flux in the direction N per unit proper time in the rest-system defined by U. This energy flux is also equal to $I_\Omega\delta\Omega$. Then we have

$$I_\Omega\delta\Omega = -cU\cdot\delta T_\Omega\cdot N$$
$$= c^3\varrho_\Omega\sinh\xi\cosh\xi\,\delta\Omega,$$

so that

$$\varrho_\Omega = \frac{I_\Omega}{c^3\sinh\xi\cosh\xi}. \tag{12}$$

Substitution of eq. (12) into eq. (11) yields the expression for δT_Ω

$$\delta T_\Omega = \frac{I_\Omega U_\Omega U_\Omega\delta\Omega}{c\sinh\xi\cosh\xi}. \tag{13}$$

The covariant divergence of $\frac{1}{c}\,\delta T_\Omega$ describes the generation of momentum per unit time and unit volume. The U-component of this vector expresses the generation of energy in mass units. We define the coefficients of absorption and emission $\varkappa_\Omega$ and j_Ω, such that the energy flux absorbed from the beam and emitted into it by the medium is respectively $\varkappa_\Omega\varrho_0 I_\Omega\delta\Omega$ and $\frac{1}{4\pi}j_\Omega\varrho_0\delta\Omega$, where ϱ_0 is the proper mass density of the external medium. The gain of energy must balance the loss at every point, so that in energy units we have the equation

$$c(\nabla\cdot\delta T_\Omega)\cdot U = -\varkappa_\Omega\varrho_0 I_\Omega\delta\Omega + \frac{1}{4\pi}j_\Omega\varrho_0\,\delta\Omega. \tag{14}$$

Inserting δT_Ω from eq. (13) into the left side of eq. (14), we obtain

$$c(\nabla\cdot\delta T_\Omega)\cdot U = \nabla\cdot\left(\frac{I_\Omega U_\Omega\delta\Omega}{\sinh\xi}\right) - \frac{I_\Omega\delta\Omega}{\sinh\xi\,\cosh\xi}U_\Omega\cdot(\nabla U)\cdot U_\Omega. \tag{15}$$

The first term on the right side can be developed as follows:

$$\nabla\cdot\left(\frac{I_\Omega U_\Omega\delta\Omega}{\sinh\xi}\right) = U\cdot\nabla\left(\frac{\cosh\xi}{\sinh\xi}I_\Omega\delta\Omega\right) + \frac{\cosh\xi}{\sinh\xi}(\nabla\cdot U)I_\Omega\delta\Omega + \nabla\cdot(I_\Omega N\delta\Omega),$$

where the first term is equal to zero, since the situation is static. The second term vanishes too, which we can see by computing $\nabla \cdot U$ from eq. (3). We have then

$$\nabla \cdot \left(\frac{I_\Omega U_\Omega \delta\Omega}{\sinh \xi}\right) = \nabla \cdot (I_\Omega N \delta\Omega). \tag{16}$$

Substituting eqs. (8) and (16) into eq. (15) and then further into eq. (14), we obtain the equation of the energy transfer

$$\left.\begin{aligned} \nabla \cdot (I_\Omega N \delta\Omega) &= -\left(\varkappa_\Omega \varrho_0 + \frac{1}{2}\frac{d\mu}{dr} e^{-\frac{1}{2}\lambda} \cos\theta\right) I_\Omega \delta\Omega \\ &\quad + \frac{1}{4\pi} j_\Omega \varrho_0 \delta\Omega. \end{aligned}\right\} \tag{17}$$

3. Radial Energy Flow

We restrict our examination to the very nearly isotropic case. Instead of working with the function I_Ω, we want to express eq. (17) by the moments of this function. The energy density, the energy flux in the direction N_r and the gas pressure are given by the formulas

$$\left.\begin{aligned} E_\Omega &= U \cdot \delta T_\Omega \cdot U = \frac{1}{c}\frac{\cosh \xi}{\sinh \xi} I_\Omega \delta\Omega \\ H_\Omega &= -cU \cdot \delta T_\Omega \cdot N_r = I_\Omega \cos\theta\, \delta\Omega \\ P_\Omega &= N_r \cdot \delta T_\Omega \cdot N_r = \frac{1}{c}\frac{\sinh \xi}{\cosh \xi} I_\Omega \cos^2\theta\, \delta\Omega. \end{aligned}\right\} \tag{18}$$

Now we introduce the quantities E_ξ, H_ξ, P_ξ and I_ξ, belonging to particles of the same ξ, by the expressions

$$\left.\begin{aligned} E_\xi &= \sinh^2 \xi \int E_\Omega\, d\omega \\ H_\xi &= \sinh^2 \xi \int H_\Omega\, d\omega \\ P_\xi &= \sinh^2 \xi \int P_\Omega\, d\omega \\ I_\xi &= \sinh^2 \xi\, I_\Omega. \end{aligned}\right\} \tag{19}$$

Substituting eq. (18) into eq. (19), we have the following integrals:

$$\left.\begin{aligned} E_\xi &= \frac{1}{c}\frac{\cosh\xi}{\sinh\xi}\int I_\xi\, d\omega \\ H_\xi &= \int I_\xi \cos\theta\, d\omega \\ P_\xi &= \frac{1}{c}\frac{\sinh\xi}{\cosh\xi}\int I_\xi \cos^2\theta\, d\omega. \end{aligned}\right\} \quad (20)$$

In order to construct the appropriate relation between the integrals (20) we need the auxiliary equation

$$\left.\begin{aligned} \nabla\cdot(I_\Omega N\cos\theta\,\delta\Omega) &= \varDelta\cdot(I_\xi N\cos\theta\,\delta\xi\,\delta\omega) \\ &= \cos\theta\,\nabla\cdot(I_\Omega N\delta\Omega) + I_\xi\delta\xi\,\delta\omega\, N\cdot\nabla\cos\theta \end{aligned}\right\} \quad (21)$$

which, after using eqs. (10) and (17), can be written in the form

$$\left.\begin{aligned} \nabla\cdot(I_\xi N\cos\theta\,\delta\xi\,\delta\omega) &= -\left(x_\xi\varrho_0 + \frac{1}{2}\frac{d\mu}{dr}e^{-\frac{\lambda}{2}}\cos\theta\right)I_\xi\cos\theta\,\delta\xi\,\delta\omega \\ &+ \frac{1}{4\pi}j_\xi\varrho_0\cos\theta\,\delta\xi\,\delta\omega + \left(\frac{1}{r} - \frac{1}{2}\frac{d\mu}{dr}\frac{\cosh^2\xi}{\sinh^2\xi}\right)e^{-\frac{\lambda}{2}}I_\xi\sin^2\theta\,\delta\xi\,\delta\omega. \end{aligned}\right\} \quad (22)$$

We have applied the following definitions in eq. (22)

$$\left.\begin{aligned} x_\xi &= x_\Omega \\ j_\xi &= \sinh^2\xi\, j_\Omega. \end{aligned}\right\} \quad (23)$$

Integrating eq. (22) over all three-dimensional solid angles and taking into account the radial symmetry and the expressions (20), we get the relation

$$\left.\begin{aligned} \nabla\cdot\left(N_r\frac{\cosh\xi}{\sinh\xi}P_\xi\delta\xi\right) &= \left(\frac{1}{r}\frac{\sinh\xi}{\cosh\xi} - \frac{1}{2}\frac{d\mu}{dr}\frac{\cosh\xi}{\sinh\xi}\right)e^{-\frac{\lambda}{2}}E_\xi\delta\xi \\ &- \frac{\varrho_0}{c}x_\xi H_\xi\delta\xi - \left(\frac{1}{r}\frac{\cosh\xi}{\sinh\xi} - \frac{1}{2}\frac{d\mu}{dr}\frac{\cosh\xi}{\sinh^3\xi}\right)e^{-\frac{\lambda}{2}}P_\xi\delta\xi. \end{aligned}\right\} \quad (24)$$

The divergence in eq. (24) can be evaluated using the component representation (3) of the vector N_r and the expression

$$\frac{d\xi}{dr} = -\frac{1}{2}\frac{d\mu}{dr}\frac{\cosh\xi}{\sinh\xi} \tag{25}$$

which is obtained from eq. (9). After a simple calculation we get

$$\left.\begin{aligned}\nabla\cdot\left(N_r\frac{\cosh\xi}{\sinh\xi}P_\xi\delta\xi\right) &= \frac{2}{r}e^{-\frac{\lambda}{2}}\frac{\cosh\xi}{\sinh\xi}P_\xi\delta\xi\\ +\frac{1}{2}\frac{d\mu}{dr}e^{-\frac{\lambda}{2}}\frac{\cosh^3\xi}{\sinh^3\xi}P_\xi\delta\xi &+ e^{-\frac{\lambda}{2}}\frac{\cosh\xi}{\sinh\xi}\frac{d(P_\xi\delta\xi)}{dr}.\end{aligned}\right\} \tag{26}$$

Inserting eq. (26) in eq. (24) ,we find

$$\left.\begin{aligned}\frac{d(P_\xi\delta\xi)}{dr} &= \left(\frac{1}{r}\frac{\sinh^2\xi}{\cosh^2\xi}-\frac{1}{2}\frac{d\mu}{dr}\right)E_\xi\delta\xi-\frac{\varrho_0}{c}e^{\frac{\lambda}{2}}\frac{\sinh\xi}{\cosh\xi}x_\xi H_\xi\delta\xi\\ &-\left(\frac{3}{r}+\frac{1}{2}\frac{d\mu}{dr}\right)P_\xi\delta\xi.\end{aligned}\right\} \tag{27}$$

Now we introduce the integrals

$$\left.\begin{aligned}E &= \int_0^\infty E_\xi\delta\xi\\ H &= \int_0^\infty H_\xi d\xi\\ P &= \int_0^\infty P_\xi d\xi\end{aligned}\right\} \tag{28}$$

which represent the total energy density due to the particles we consider, the total radial energy flow defined as the flux of energy relative to the particle stream, and the total gas pressure due to these particles. We further define the mean absorption coefficient x of material particles analogously to the Rosseland mean opacity(5) by

$$xH = \int_0^\infty \frac{\sinh\xi}{\cosh\xi}x_\xi H_\xi d\xi, \tag{29}$$

which we analyze in section 5.

The integration over the quantity ξ in eq. (27) gives the equation

$$\left.\begin{aligned} \frac{dP}{dr} = \frac{1}{r}\int_0^\infty \frac{\sinh^2\xi}{\cosh^2\xi} E_\xi \delta\xi - \frac{1}{2}\frac{d\mu}{dr} E - \frac{\varrho_0}{c} e^{\frac{\lambda}{2}} xH \\ -\left(\frac{3}{r} + \frac{1}{2}\frac{d\mu}{dr}\right) P. \end{aligned}\right\} \quad (30)$$

We have worked with moments of the intensity function rather than the intensity itself in deriving eq. (30). Because of that we need an independent auxiliary equation. It is then customary to use the diffusion approximation, where the statistical expressions of the energy density and the gas pressure in thermal equilibrium are applied. We shall eliminate the quantities E and P from eq. (30) in the case of relativistic Boltzmann-statistics and quantum statistics in chapters 3A and 3B respectively. The quantities belonging to quantum statistics have an index d.

3A. Radial Energy Flow due to Relativistic Boltzmann-particles

The basic assumption of the relativistic kinetic theory is the equal a priori probabilities of equal cells $\delta v \delta \Omega$, where δv is the volume element which the tangential vectors of geodesics intersect orthogonally in the four-dimensional elementary solid angle $\delta\Omega$. The cell $\delta v \delta \Omega$ is invariant under general space-time coordinate transformations[6]. The statistical, special relativistic expressions of the energy density and the pressure first derived by F. JÜTTNER[7] are then valid also in the general theory of relativity.*

In a relativistic Boltzmann-gas consisting of material particles of proper mass m, the following relations are valid[7]:

$$\left.\begin{aligned} E &= 3P + \frac{\gamma}{\tau} K_1(\tau) \\ P &= \frac{\gamma}{\tau^2} K_2(\tau) \end{aligned}\right\} \quad (31)$$

where we denote

* This is physically obvious, because the general relativistic effects are negligible in small distances of the mean free paths of particles.

$$\left.\begin{aligned} \tau &= \frac{mc^2}{kT} \\ \gamma &= 4\pi mc^2\left(\frac{mc}{h}\right)^3 e^{\frac{\psi+mc^2}{kT}}. \end{aligned}\right\} \quad (32)$$

Further, the functions $K_n(\tau)$ are modified Bessel functions of the second kind; k is the Boltzmann constant, h the Planck constant, T is temperature measured in the rest frame of the medium and ψ is the chemical potential. The expressions (31) are derived for thermal equilibrium, but they are also valid to a high degree of accuracy when the gas is close to equilibrium. We recall the definition of $K_n(\tau)$(8)

$$K_n(\tau) = \frac{\tau^n}{1\cdot 3\cdots(2n-1)}\int_0^\infty e^{-\tau\cosh\xi}\sinh^{2n}\xi\, d\xi \quad (33)$$

and two recurrence relations for them

$$\left.\begin{aligned} nK_n(\tau) - \tau\frac{dK_n(\tau)}{d\tau} &= \tau K_{n+1}(\tau) \\ K_{n+1}(\tau) - K_{n-1}(\tau) &= \frac{2n}{\tau}K_n(\tau). \end{aligned}\right\} \quad (34)$$

For the quantity E_ξ we get from eqs. (28), (31) and (33) the expression

$$E_\xi = \gamma e^{-\tau\cosh\xi}\sinh^2\xi\cosh^2\xi \quad (35)$$

which, substituted in the first term of the right side of eq. (30), gives

$$\frac{1}{r}\int_0^\infty \frac{\sinh^2\xi}{\cosh^2\xi}E_\xi\, d\xi = \frac{\gamma}{r}\int_0^\infty e^{-\tau\cosh\xi}\sinh^4\xi\, d\xi = \frac{3}{r}P. \quad (36)$$

By use of eqs. (31), (33), (34) and (36) we obtain from eq. (30)

$$\left.\begin{aligned} -\frac{\varrho_0}{c}e^{\frac{\lambda}{2}}xH &= \frac{dP}{dr} + 2\frac{d\mu}{dr}P + \frac{1}{2}\frac{d\mu}{dr}\frac{\gamma}{\tau}K_1(\tau) \\ &= \frac{K_2(\tau)}{\tau^2}\frac{d\gamma}{dr} + \frac{\gamma}{\tau^3}\left[\tau\frac{dK_2(\tau)}{d\tau} - 2K_2(\tau)\right]\frac{d\tau}{dr} + \frac{1}{2}\frac{d\mu}{dr}\frac{\gamma}{\tau}\left[\frac{4}{\tau}K_2(\tau) + K_1(\tau)\right] \\ &= \frac{K_2(\tau)}{\tau^2}\frac{d\gamma}{dr} + \frac{\gamma k}{mc^2}K_3(\tau)e^{-\frac{\mu}{2}}\frac{d\left(Te^{\frac{\mu}{2}}\right)}{dr}. \end{aligned}\right\} \quad (37)$$

The expression of $\frac{d\gamma}{dr}$ can be written in the form

$$\frac{d\gamma}{dr} = -\frac{\gamma(\psi + mc^2)}{kT^2} e^{-\frac{\mu}{2}} \frac{d\left(Te^{\frac{\mu}{2}}\right)}{dr} + \frac{\gamma}{kT} e^{-\frac{\mu}{2}} \frac{d\left[(\psi + mc^2)c^{\frac{\mu}{2}}\right]}{dr} \qquad (38)$$

which, substituted into eq. (37), yields the result

$$\left.\begin{aligned} H = -\frac{\gamma k}{mc\varrho_0 x} e^{-\frac{\lambda+\mu}{2}} \Bigg\{ & \left[K_3(\tau) - \left(1 + \frac{\psi}{mc^2}\right) K_2(\tau)\right] \frac{d\left(Te^{\frac{\mu}{2}}\right)}{dr} \\ & + \frac{T}{mc^2} K_2(\tau) \frac{d\left[(\psi + mc^2) e^{\frac{\mu}{2}}\right]}{dr} \Bigg\}. \end{aligned}\right\} \qquad (39)$$

The radial energy flow has been separated into two parts to remove differences of relativistic thermal and chemical potentials in the gas. Because the chemical potential is also a function of the temperature and the density of the gas, the first term in eq. (39) is not the total expression of the heat flow, but the other term will give a contribution to it.

3B. Radial Energy Flow due to Bosons and Fermions

In the relativistic quantum statistics, the following relations are valid[9]:

$$\left.\begin{aligned} E_{\xi d} &= \cosh\xi\, u_d \\ P_{\xi d} &= \frac{1}{3}\frac{\sinh^2\xi}{\cosh\xi} u_d, \end{aligned}\right\} \qquad (40)$$

where the distribution function u_d is

$$\left.\begin{aligned} u_d &= \frac{\gamma_d \cosh\xi \sinh^2\xi}{\exp\left[-\frac{\psi}{kT} + \tau(\cosh\xi - 1)\right] + \eta} \\ \gamma_d &= 4\pi mc^2 \left(\frac{mc}{h}\right)^3 \end{aligned}\right\} \qquad (41)$$

and η is equal to $+1$ for Fermi-Dirac-gas and -1 for Bose-Einstein-gas.

Using eq. (25), we obtain from eq. (40)

$$\left.\begin{aligned}\frac{dP_{\xi d}}{dr} = & -\frac{1}{6}\frac{d\mu}{dr}\left(\frac{1}{\cosh\xi}+\cosh\xi\right)u_d - \frac{1}{6}\frac{d\mu}{dr}\sinh\xi\frac{\partial u_d}{\partial\xi}\\ & +\frac{1}{3}\frac{\sinh^2\xi}{\cosh\xi}\left(\frac{\partial u_d}{\partial T}\frac{dT}{dr}+\frac{\partial u_d}{\partial\psi}\frac{d\psi}{dr}\right).\end{aligned}\right\} \quad (42)$$

Insertion from eqs. (40) and (42) into eq. (30) gives the equation

$$\left.\begin{aligned}-\frac{\varrho_0}{c}\frac{\lambda}{e^2}x_d H_d = & \frac{1}{3}\frac{d\mu}{dr}\int_0^\infty \cosh\xi\, u_d\, d\xi - \frac{1}{6}\frac{d\mu}{dr}\int_0^\infty \sinh\xi\frac{\partial u_d}{\partial\xi}d\xi\\ & +\frac{1}{3}\int_0^\infty\frac{\sinh^2\xi}{\cosh\xi}\left(\frac{\partial u_d}{\partial T}\frac{dT}{dr}+\frac{\partial u_d}{\partial\psi}\frac{d\psi}{dr}\right)d\xi + \frac{1}{6}\frac{d\mu}{dr}\int_0^\infty\frac{\sinh^2\xi}{\cosh\xi}u_d\,d\xi.\end{aligned}\right\} \quad (43)$$

From eq. (41) we have for $\frac{\partial u_d}{\partial\xi}$ and $\frac{\partial u_d}{\partial T}$ the expressions:

$$\left.\begin{aligned}\frac{\partial u_d}{\partial\xi} = & -\frac{\tau\sinh\xi\exp\left[-\frac{\psi}{kT}+\tau(\cosh\xi-1)\right]}{\exp\left[-\frac{\psi}{kT}+\tau(\cosh\xi-1)\right]+\eta}u_d + \left(\frac{\sinh\xi}{\cosh\xi}+2\frac{\cosh\xi}{\sinh\xi}\right)u_d\\ \frac{\partial u_d}{\partial T} = & \frac{\tau}{T}\left(\cosh\xi-1-\frac{\psi}{mc^2}\right)\frac{\exp\left[-\frac{\psi}{kT}+\tau(\cosh\xi-1)\right]}{\exp\left[-\frac{\psi}{kT}+\tau(\cosh\xi-1)\right]+\eta}u_d.\end{aligned}\right\} \quad (44)$$

Substituting the expression

$$\left.\begin{aligned}\sinh\xi\frac{\partial u_d}{\partial\xi}+\frac{\sinh^2\xi}{\cosh\xi}T\frac{\partial u_d}{\partial T} = & \left(\frac{\sinh^2\xi}{\cosh\xi}+2\cosh\xi\right)u_d\\ & -\frac{\left(\tau+\frac{\psi}{kT}\right)\exp\left[-\frac{\psi}{kT}+\tau(\cosh\xi-1)\right]}{\exp\left[-\frac{\psi}{kT}+\tau(\cosh\xi-1)\right]+\eta}\frac{\sinh^2\xi}{\cosh\xi}u_d,\end{aligned}\right\} \quad (45)$$

obtained from eq. (44) into eq. (43), we get after grouping the terms

$$-\frac{\varrho_0}{c}e^{\frac{\lambda}{2}}x_d H_d = \frac{1}{3}\left(\frac{dT}{dr}+\frac{T}{2}\frac{d\mu}{dr}\right)\int_0^\infty \frac{\sinh^2\xi}{\cosh\xi}\frac{\partial u_d}{\partial T}d\xi$$

$$+\frac{1}{3kT}\left[\frac{d\psi}{dr}+\frac{1}{2}(\psi+mc^2)\frac{d\mu}{dr}\right]\int_0^\infty \frac{\exp\left[-\frac{\psi}{kT}+\tau(\cosh\xi-1)\right]}{\exp\left[-\frac{\psi}{kT}+\tau(\cosh\xi-1)\right]+\eta}\frac{\sinh^2\xi}{\cosh\xi}u_d\,d\xi$$

from which

$$\left.\begin{aligned} H_d = -\frac{c\gamma_d}{3}\frac{\tau e^{-\frac{\lambda+\mu}{2}}}{\varrho_0 T x_d}&\left\{\frac{d\left(Te^{\frac{\mu}{2}}\right)}{dr}\int_0^\infty \frac{\left(\cosh\xi-1-\frac{\psi}{mc^2}\right)\exp\left[-\frac{\psi}{kT}+\tau(\cosh\xi-1)\right]}{\left(\exp\left[-\frac{\psi}{kT}+\tau(\cosh\xi-1)\right]+\eta\right)^2}\sinh^4\xi\,d\xi\right.\\ &\left.+\frac{T}{mc^2}\frac{d\left[(\psi+mc^2)e^{\frac{\mu}{2}}\right]}{dr}\int_0^\infty \frac{\exp\left[-\frac{\psi}{kT}+\tau(\cosh\xi-1)\right]}{\left(\exp\left[-\frac{\psi}{kT}+\tau(\cosh\xi-1)\right]+\eta\right)^2}\sinh^4\xi\,d\xi\right\}.\end{aligned}\right\} \quad (46$$

4A. Electronic Heat Conductivity

As an important special case of energy transfer by material particles we examine in this chapter the electronic heat conductivity. We limit ourselves to the case with energy flow due to ionization electrons which are interacting with a medium consisting of only one kind of nuclei of proper mass M and atomic number Z.

In order to obtain an expression of thermal conductivity from eq. (39) or eq. (46) we have to find a relation between the expressions $d[(\psi+mc^2)e^{\mu/2}]/dr$ and $d(Te^{\mu/2})/dr$. We regard the medium as a perfect fluid, whose energy-momentum tensor is given by

$$(T_0)_\alpha{}^\beta = \left(\varrho+\frac{p_0}{c^2}\right)u_\alpha u^\beta - g_\alpha{}^\beta\frac{p_0}{c^2}, \quad (47)$$

where ϱ is the total mass-energy density, p_0 is the pressure in the medium, and $g_\alpha{}^\beta$ are the components of the metric tensor.

From the covariant conservation law

$$\nabla \cdot T^{0} = 0 \tag{48}$$

we have in the metric (2) the equation

$$\frac{1}{c^2}\frac{dp_0}{dr} + \frac{1}{2}\left(\varrho + \frac{p_0}{c^2}\right)\frac{d\mu}{dr} = 0. \tag{49}$$

The medium consisting of nuclei is a Boltzman-gas also when the electron gas is weakly degenerate. The equations of state are

$$\left.\begin{aligned} \varrho + \frac{p_0}{c^2} &= \frac{\gamma_0}{\tau_0} K_3(\tau_0) \\ p_0 &= \frac{\gamma_0 c^2}{\tau_0^{\,2}} K_2(\tau_0) = \frac{c^2}{\tau_0}\varrho_0, \end{aligned}\right\} \tag{50}$$

where we denote

$$\left.\begin{aligned} \gamma_0 &= 4\pi M \left(\frac{Mc}{h}\right)^3 e^{\frac{\psi_0 + Mc^2}{kT}} \\ \tau_0 &= \frac{Mc^2}{kT}. \end{aligned}\right\} \tag{51}$$

The chemical potential of nuclei is given[(10)] by

$$e^{\frac{\psi_0}{kT}} = \left(\frac{h^2}{2\pi MkT}\right)^{3/2} \frac{\varrho_0}{M}. \tag{52}$$

Comparing this with

$$e^{\frac{\psi}{kT}} = \left(\frac{h^2}{2\pi mkT}\right)^{3/2} \frac{Z\varrho_0}{M}, \tag{53}$$

which is the expression of the chemical potential in a non-degenerate electron gas, we find the relation

$$\psi_0 = \psi - kT \log\left[\left(\frac{M}{m}\right)^{3/2} Z\right]. \tag{54}$$

Substituting the expressions (50) into eq. (49) and using the first eq. (34), we obtain

$$\frac{K_2(\tau_0)}{\tau_0{}^2}\frac{d\gamma_0}{dr}+\frac{\gamma_0 K_3(\tau_0)}{\tau_0 T}\frac{dT}{dr}+\frac{1}{2}\frac{\gamma_0 K_3(\tau_0)}{\tau_0}\frac{d\mu}{dr}=0. \tag{55}$$

Using eqs. (51) and (54), we get from eq. (55) after simple calculations

$$\left.\begin{aligned}&\frac{d\left[(\psi+mc^2)e^{\frac{\mu}{2}}\right]}{dr}\\&=\left[\frac{\psi+mc^2}{T}-\frac{Mc^2}{T}\frac{K_3(\tau_0)}{K_2(\tau_0)}\frac{T\left(\frac{\partial\psi}{\partial T}+\frac{\partial\psi}{\partial\varrho_0}\frac{d\varrho_0}{dT}\right)-\psi-mc^2}{T\left(\frac{\partial\psi}{\partial T}+\frac{\partial\psi}{\partial\varrho_0}\frac{d\varrho_0}{dT}\right)-\psi-Mc^2}\right]\frac{d\left(Te^{\frac{\mu}{2}}\right)}{dr}.\end{aligned}\right\} \tag{56}$$

The quantity $\frac{d\varrho_0}{dT}$ can further be derived as a function of T and ϱ_0. We obtain from eqs. (50) and (51) the expression

$$\frac{d\varrho_0}{dT}=\frac{T\frac{\partial\psi}{\partial T}-\psi-Mc^2-kT+Mc^2\frac{K_3(\tau_0)}{K_2(\tau_0)}}{T\left(\frac{Mc^2}{\gamma_0 K_2(\tau_0)}-\frac{\partial\psi}{\partial\varrho_0}\right)}. \tag{57}$$

Insertion of ψ from eq. (53) yields

$$\frac{d\varrho_0}{dT}=\frac{\left(\frac{K_3(\tau_0)}{K_2(\tau_0)}-1\right)\tau_0-\frac{5}{2}}{T\left(\frac{\tau_0}{\gamma_0 K_2(\tau_0)}-\frac{1}{\varrho_0}\right)}. \tag{58}$$

Substituting eqs. (53) and (56) into eq. (39), the equation of the radial heat flow reduces to the form

$$H=-\Lambda\frac{d\left(Te^{\frac{\mu}{2}}\right)}{dr}, \tag{59}$$

where

$$\left.\begin{aligned} A &= \frac{4\pi k m^3 c^4}{h^3 \varrho_0 x} e^{\frac{\psi + mc^2}{kT}} e^{-\frac{\lambda+\mu}{2}} \\ &\left[K_3\left(\frac{mc^2}{kT}\right) - K_2\left(\frac{mc^2}{kT}\right) \frac{M K_3\left(\frac{Mc^2}{kT}\right)}{m K_2\left(\frac{Mc^2}{kT}\right)} \frac{\left(\frac{T}{\varrho_0}\frac{d\varrho_0}{dT} - \frac{mc^2}{kT} - \frac{3}{2}\right)}{\left(\frac{T}{\varrho_0}\frac{d\varrho_0}{dT} - \frac{Mc^2}{kT} - \frac{3}{2}\right)} \right] \end{aligned}\right\} \quad (60)$$

is the expression of thermal conductivity of a non-degenerate electron gas.*

The chemical potential of a degenerate electron gas can be derived[10] from the integral equation

$$F\left(e^{\frac{\psi_d}{kT}}\right) = \frac{4}{\pi^{\frac{1}{2}}} \int_0^\infty \frac{x^2 dx}{\exp\left(-\frac{\psi_d}{kT} + x^2\right) + 1} = \left(\frac{h^2}{2\pi m k T}\right)^{\frac{3}{2}} \frac{Z\varrho_0}{M}. \quad (61)$$

For small values of $e^{\psi_d/kT}$ we have the expansion

$$F\left(e^{\frac{\psi_d}{kT}}\right) = e^{\frac{\psi_d}{kT}} - \frac{e^{\frac{2\psi_d}{kT}}}{2^{3/2}} + \frac{e^{\frac{3\psi_d}{kT}}}{3^{3/2}} - \frac{e^{\frac{4\psi_d}{kT}}}{4^{3/2}} + \cdots \quad (62)$$

From eqs. (61) and (62) we obtain

$$e^{\frac{\psi_d}{kT}} = \left(\frac{h^2}{2\pi m k T}\right)^{3/2} \frac{Z\varrho_0}{M} + \frac{1}{2^{3/2}} \left(\frac{h^2}{2\pi m k T}\right)^3 \frac{Z^2 \varrho_0^2}{M^2} + \cdots. \quad (63)$$

Repeating the calculations from eq. (54) to eq. (60), using eq. (46) instead of eq. (39) and eq. (63) instead of eq. (53), we obtain the expression of heat flow due to a weakly degenerate electron gas

$$H_d = -\left\{ \frac{4\pi m^5 c^8 e^{-\frac{\lambda+\mu}{2}}}{3 k h^3 \varrho_0 T^2 x_d} \int_0^\infty \frac{A_{\xi d} \exp\left[-\frac{\psi_d}{kT} + \frac{mc^2}{kT}(\cosh\xi - 1)\right]}{\exp\left[-\frac{\psi_d}{kT} + \frac{mc^2}{kT}(\cosh\xi - 1)\right] + 1} \right\} \frac{d\left(T e^{\frac{\mu}{2}}\right)}{dr}, \quad (64)$$

* The relativistic law of heat-diffusion (59), which agrees with the expression for the heat flow derived by C. Eckart[11], includes the gradient of the quantity $Te^{\mu/2}$ instead of the temperature gradient of the classical case. The dependence on the metrics is due to inertia of the heat. In the classical limit eq. (59) reduces to Fourier's law of heat conduction.

where we denote

$$A_{\xi d} = \frac{\cosh\xi - \dfrac{MK_3\left(\dfrac{Mc^2}{kT}\right)\left[\left(\dfrac{T}{\varrho_0}\dfrac{d\varrho_0}{dT} - \dfrac{3}{2}\right)\dfrac{1 + \dfrac{1}{2^{1/2}}\left(\dfrac{h_2}{2\pi mkT}\right)^{3/2}\dfrac{Z\varrho_0}{M}}{1 + \dfrac{1}{2^{3/2}}\left(\dfrac{h^2}{2\pi mkT}\right)^{3/2}\dfrac{Z\varrho_0}{M}} - \dfrac{mc^2}{kT}\right]}{mK_2\left(\dfrac{Mc^2}{kT}\right)\left(\dfrac{T}{\varrho_0}\dfrac{d\varrho_0}{dT} - \dfrac{3}{2} - \dfrac{Mc^2}{kT}\right)}}{\exp\left[-\dfrac{\psi_d}{kT} + \dfrac{mc^2}{kT}(\cosh\xi - 1)\right] + 1}\sinh^4\xi, \quad (65)$$

and $\dfrac{d\varrho_0}{dT}$ is the same function (58) of T and ϱ_0 as in the case of a non-degenerate electron gas.

From eqs. (59) and (64) we get as a special case of no net energy flow due to a non- or weakly degenerate electron gas the general law[12]

$$Te^{\frac{\mu}{2}} = T\sqrt{g_{44}} = \text{constant} \qquad (66)$$

for a relativistic fluid in thermal equilibrium.

The expressions of thermal conductivity of a semi- or strongly degenerate electron gas can be derived only with numerical methods.

4B. Radiative Energy Transfer and Neutrino Flux

The radiative energy transfer is obtained in the limiting case of the energy transfer by material particles. The calculations up to eq. (30) are valid for a photon gas, if the expressions $m\cosh\xi$ and $m\sinh\xi$ are changed to $h\nu/c^2$, where ν denotes the frequency in the rest-system defined by the vector U. Eq. (30) then takes the form

$$\frac{dP_f}{dr} = \left(\frac{1}{r} - \frac{1}{2}\frac{d\mu}{dr}\right)E_f - \frac{\varrho_0}{c}e^{\frac{\lambda}{2}}\varkappa_f H_f - \left(\frac{3}{r} + \frac{1}{2}\frac{d\mu}{dr}\right)P_f, \qquad (67)$$

where the mean absorption coefficient of photons or the Rosseland mean opacity $\varkappa_f$ is defined by

$$\varkappa_f H_f = \int_0^\infty \varkappa_\nu H_\nu d\nu. \qquad (68)$$

Using the isotropic approximation

$$E_f = 3P_f = aT^4, \tag{69}$$

where a is the radiation pressure constant, we obtain from eq. (67) the equation of the radiative energy transfer

$$H_f = -\frac{acT^3}{3\varrho_0 \varkappa_f} e^{-\frac{\lambda+\mu}{2}} \frac{d\left(Te^{\frac{\mu}{2}}\right)}{dr} \tag{70}$$

agreeing with the corresponding results in references 3 and 4.

We notice that the condition of no net radiation yields the same law (66) of thermal equilibrium as when the equilibrium is established by the thermal conductivity of electrons.

The energy transfer by neutrinos is obtained similarly as the radiative energy transfer, if one assumes almost isotropic neutrino distribution. The only modification which must be done is the change of the radiation constant to the constant $7/8\,a$, since the energy density of an isotropic neutrino gas is not aT^4, but $7/8 aT^4$ (13). Defining the mean absorption coefficient of neutrinos by

$$\varkappa_n H_n = \int_0^\infty \varkappa_{n\nu} H_{n\nu}\, d\nu, \tag{71}$$

we get the equation of the neutrino flux

$$H_n = -\frac{7acT^3}{6\varrho_0 \varkappa_n} e^{-\frac{\lambda+\mu}{2}} \frac{d\left(Te^{\frac{\mu}{2}}\right)}{dr}. \tag{72}$$

Because the interactions of neutrinos and antineutrinos with matter are generally different, the Rosseland mean must be taken as the average of that for neutrinos $\varkappa_n^+$ and for antineutrinos $\varkappa_n^-$ or (14).

$$\frac{2}{\varkappa_n} = \frac{1}{\varkappa_n^+} + \frac{1}{\varkappa_n^-}. \tag{73}$$

Eq. (67), which has been derived using an isotropic approximation, can not be applied to the case of a purely radial outward-moving neutrino flow. A purely radial neutrino emission has been used by C. W. Misner (15) to describe a flux of neutrinos produced in the deep interior of a supernova.

5. Mean Absorption Coefficient

Eqs. (59), (64), (70) and (72) for the radial energy flow contain as a parameter the mean absorption coefficient derived in eq. (29) for material particles, in eq. (68) for photons and in eq. (71) for neutinos. In this section we give an approximative method for the calculation of this coefficient. The procedure is the same[16] as in computing the Rosseland mean opacity from monocromatic absorption coefficients.

First we want to eliminate $\delta\xi$ from eq. (27). Since we are concerned with the radially symmetric and static case, we have from eqs. (3) and (4)

$$N \cdot \nabla \cosh \xi = e^{-\frac{\lambda}{2}} \cos \theta \frac{d(\cosh \xi)}{dr}. \tag{74}$$

Comparing eqs. (9) and (74), we find the relation

$$\frac{1}{\cosh \xi} \frac{d(\cosh \xi)}{dr} = -\frac{1}{2}\frac{d\mu}{dr}. \tag{75}$$

For another stream of particles having the same space-like direction N but a little different pseudo-angle ξ, eq. (75) can be applied in the form

$$\frac{1}{\cosh \xi + \delta(\cosh \xi)} \frac{d[\cosh \xi + \delta(\cosh \xi)]}{dr} = -\frac{1}{2}\frac{d\mu}{dr}. \tag{76}$$

Subtracting eq. (75) from eq. (76) and using eq. (25), we get

$$\frac{d(\delta\xi)}{dr} = \frac{1}{2}\frac{d\mu}{dr}\frac{\delta\xi}{\sinh^2 \xi}. \tag{77}$$

Elimination of $\delta\xi$ from eq. (27) by means of eq. (77) yields

$$\left.\begin{aligned} \frac{dP_\xi}{dr} = \frac{1}{r}\frac{\sinh^2 \xi}{\cosh^2 \xi} E_\xi - \frac{1}{2}\frac{d\mu}{dr} E_\xi - \frac{\varrho_0 x_\xi}{c} e^{\frac{\lambda}{2}} \frac{\sinh \xi}{\cosh \xi} H_\xi \\ - \frac{1}{2}\frac{d\mu}{dr}\frac{\cosh^2 \xi}{\sinh^2 \xi} P_\xi - \frac{3}{r} P_\xi. \end{aligned}\right\} \tag{78}$$

Dividing this equation by $\frac{\sinh\xi}{\cosh\xi}x_\xi$ and integrating over the quantity ξ, we obtain

$$\left.\begin{aligned} -\frac{\varrho_0}{c}e^{\frac{\lambda}{2}}H = & -\frac{1}{r}\int_0^\infty \frac{\sinh\xi}{\cosh\xi}\frac{E_\xi}{x_\xi}d\xi + \frac{1}{2}\frac{d\mu}{dr}\int_0^\infty \frac{\cosh\xi}{\sinh\xi}\frac{E_\xi}{x_\xi}d\xi \\ & +\int_0^\infty \frac{\cosh\xi}{\sinh\xi}\frac{1}{x_\xi}\frac{dP_\xi}{dr}d\xi + \frac{1}{2}\frac{d\mu}{dr}\int_0^\infty \frac{\cosh^3\xi}{\sinh^3\xi}\frac{P_\xi}{x_\xi}d\xi + \frac{3}{r}\int_0^\infty \frac{\cosh\xi}{\sinh\xi}\frac{P_\xi}{x_\xi}d\xi. \end{aligned}\right\} \quad (79)$$

We shall eliminate the quantities E_ξ and P_ξ from eq. (79) separately for a non- and weakly degenerate electron gas and for a photon and neutrino gas analogously to the calculations in chapter 3B.

5A. Mean Absorption Coefficient of Non- and Weakly Degenerate Electron Gas

From eqs. (28), (31), and (33) we obtain for a non-degenerate electron gas the relations

$$\left.\begin{aligned} E_\xi &= \cosh\xi\, u \\ P_\xi &= \frac{1}{3}\frac{\sinh^2\xi}{\cosh\xi}u, \end{aligned}\right\} \quad (80)$$

where we denote

$$u = \gamma e^{-\tau\cosh\xi}\cosh\xi\sinh^2\xi. \quad (81)$$

Using eq. (80) we can write eq. (79) in the form

$$\left.\begin{aligned} -\frac{\varrho_0}{c}e^{\frac{\lambda}{2}}H = & \frac{1}{3}\int_0^\infty\left(\frac{1}{\cosh\xi}+\cosh\xi\right)\frac{d\xi}{dr}\frac{u}{x_\xi}d\xi \\ & +\frac{1}{3}\int_0^\infty\left(\frac{\partial u}{\partial\xi}\frac{d\xi}{dr}+\frac{\partial u}{\partial T}\frac{dT}{dr}+\frac{\partial u}{\partial\psi}\frac{d\psi}{dr}\right)\frac{\sinh\xi}{x_\xi}d\xi + \frac{2}{3}\frac{d\mu}{dr}\int_0^\infty\frac{\cosh^2\xi}{\sinh\xi}\frac{u}{x_\xi}d\xi. \end{aligned}\right\} \quad (82)$$

Substitution of the expression

$$\sinh\xi\frac{\partial u}{\partial\xi}+\frac{\sinh^2\xi}{\cosh\xi}T\frac{\partial u}{\partial T} = \frac{\sinh^2\xi}{\cosh\xi}\left(1-\tau-\frac{\psi}{kT}\right)u + 2\cosh\xi\, u \quad (83)$$

into eq. (82) yields

$$\left.\begin{aligned} -\frac{\varrho_0}{c} e^{\frac{\lambda}{2}} H = \frac{4}{3}\int_0^\infty \cosh\xi \frac{d\xi}{dr}\frac{u}{x_\xi} d\xi - \frac{1}{3}\left(\tau + \frac{\psi}{kT}\right)\int_0^\infty \frac{\sinh^2\xi}{\cosh\xi}\frac{d\xi}{dr}\frac{u}{x_\xi} d\xi \\ +\frac{1}{3}\int_0^\infty \frac{\sinh\xi}{x_\xi}\frac{\partial u}{\partial\psi}\frac{d\psi}{dr} d\xi - \frac{1}{3}\int_0^\infty \left(\frac{\sinh^2\xi}{\cosh\xi} T \frac{d\xi}{dr} - \sinh\xi \frac{dT}{dr}\right)\frac{\partial u}{\partial T}\ \frac{1}{x_\xi} d\xi \\ +\frac{2}{3}\frac{d\mu}{dr}\int_0^\infty \frac{\cosh^2\xi}{\sinh\xi}\frac{u}{x_\xi} d\xi. \end{aligned}\right\} \tag{84}$$

Taking into account eq. (25), we obtain from eq. (84)

$$\left.\begin{aligned} -\frac{\varrho^0}{c} e^{\frac{\lambda}{2}} H = \\ \frac{1}{3} e^{-\frac{\mu}{2}}\left\{\frac{d\left(Te^{\frac{\mu}{2}}\right)}{dr}\int_0^\infty \frac{\sinh\xi}{x_\xi}\frac{\partial u}{\partial T} d\xi + \frac{1}{kT}\frac{d\left[(\psi + mc^2)e^{\frac{\mu}{2}}\right]}{dr}\int_0^\infty \frac{\sinh\xi}{x_\xi} u d\xi\right\}. \end{aligned}\right\} \tag{85}$$

Making insertion of $u, \frac{\partial u}{\partial T}$ and the relation (56) into eq. (84), and denoting

$$A_\xi = \frac{\cosh\xi - \dfrac{MK_3\left(\frac{Mc^2}{kT}\right)\left(\frac{T}{\varrho_0}\frac{d\varrho_0}{dT} - \frac{3}{2} - \frac{mc^2}{kT}\right)}{mK_2\left(\frac{Mc^2}{kT}\right)\left(\frac{T}{\varrho_0}\frac{d\varrho_0}{dT} - \frac{3}{2} - \frac{Mc^2}{kT}\right)}}{\exp\left[-\frac{\psi}{kT} + \frac{mc^2}{kT}(\cosh\xi - 1)\right]} \sinh^4\xi, \tag{86}$$

we finally get

$$H = -\left\{\frac{4\pi m^5 c^8}{3kh^3\varrho_0 T^2} e^{-\frac{\lambda+\mu}{2}}\int_0^\infty \frac{A_\xi}{x_\xi}\frac{\cosh\xi}{\sinh\xi} d\xi\right\}\frac{d\left(Te^{\frac{\mu}{2}}\right)}{dr}. \tag{87}$$

Comparing eqs. (59) and (60) with eq. (87) we find the following expression for the mean absorption coefficient of a non-degenerate electron gas:

$$\frac{1}{\varkappa} = \frac{\frac{1}{3}\left(\frac{mc^2}{kT}\right)^2 \int_0^\infty \frac{A_\xi}{\varkappa_\xi} \frac{\cosh \xi}{\sinh \xi} d\xi}{K_3\left(\frac{mc^2}{kT}\right) - K_2\left(\frac{mc^2}{kT}\right) \frac{MK_3\left(\frac{Mc^2}{kT}\right)\left(\frac{T}{\varrho_0}\frac{d\varrho_0}{dT} - \frac{3}{2} - \frac{mc^2}{kT}\right)}{mK_2\left(\frac{Mc^2}{kT}\right)\left(\frac{T}{\varrho_0}\frac{d\varrho_0}{dT} - \frac{3}{2} - \frac{Mc^2}{kT}\right)}}. \tag{88}$$

The calculations for the mean absorption coefficient of a weakly degenerate electron gas are similar to the ones made above. We shall give only the final results. The equation of thermal conductivity of a weakly degenerate electron gas is

$$H_d = -\left\{\frac{4\pi m^5 c^8}{3kh^3 \varrho_0 T^2} e^{-\frac{\lambda+\mu}{2}} \int_0^\infty \frac{A_{\xi d} \exp\left[-\frac{\psi_d}{kT} + \frac{mc^2}{kT}(\cosh \xi - 1)\right]}{\varkappa_{\xi d}\left[\exp\left[-\frac{\psi_d}{kT} + \frac{mc^2}{kT}(\cosh \xi - 1)\right] + 1\right]} \frac{\cosh \xi}{\sinh \xi} d\xi\right\} \frac{d\left(Te^{\frac{\mu}{2}}\right)}{dr}. \tag{89}$$

For the absorption coefficients $\varkappa_{\xi d}$ and $\varkappa_\xi$ the following relation is valid[17]

$$\varkappa_{\xi d} = \bar{\eta}\varkappa_\xi = \left\{1 - \frac{1}{\exp\left[-\frac{\psi_d}{kT} + \frac{mc^2}{kT}(\cosh \xi - 1)\right] + 1}\right\}\varkappa_\xi, \tag{90}$$

where $\bar{\eta}$ is the probability that the final state after a scattering of an electron is free. Inserting eq. (90) into eq. (89), it reduces to the form

$$H_d = -\left\{\frac{4\pi m^5 c^8}{3kh^3 \varrho_0 T^2} e^{-\frac{\lambda+\mu}{2}} \int_0^\infty \frac{A_{\xi d}}{\varkappa_\xi} \frac{\cosh \xi}{\sinh \xi} d\xi\right\} \frac{d\left(Te^{\frac{\mu}{2}}\right)}{dr}. \tag{91}$$

The expression for the inverse value of the mean absorption coefficient for a weakly degenerate electron gas is

$$\frac{1}{x_d} = \frac{\int_0^\infty \frac{A_{\xi d}}{x_\xi} \frac{\cosh \xi}{\sinh \xi} d\xi}{\int_0^\infty A_{\xi d} d\xi}. \tag{92}$$

5B. Mean Absorption Coefficient of Photons and Neutrinos

The relativistic Rosseland mean opacity has been studied by K. A. Hämeen-Anttila(16). For the sake of completeness, the calculation is now outlined starting from the theory of material particles. In the special case of $mc^2 \cosh \xi \rightarrow h\nu$ and $mc^2 \sinh \xi \rightarrow h\nu$, we obtain from eqs. (25) and (77) the corresponding relations

$$\frac{d\nu}{dr} = -\frac{1}{2}\frac{d\mu}{dr}\nu \tag{93}$$

$$\frac{d(\delta\nu)}{dr} = -\frac{1}{2}\frac{d\mu}{dr}\delta\nu. \tag{94}$$

With the quantities now measured per unit frequence interval, eq. (27) can be written in the form

$$\frac{d(P_\nu \delta\nu)}{dr} = \left(\frac{1}{r} - \frac{1}{2}\frac{d\mu}{dr}\right) E_\nu \delta\nu - \frac{\varrho_0}{c} e^{\frac{\lambda}{2}} x_\nu H_\nu \delta\nu - \left(\frac{3}{r} + \frac{1}{2}\frac{d\mu}{dr}\right) P_\nu d\nu. \tag{95}$$

Elimination of $\delta\nu$ by means of eq. (94) gives

$$\frac{dP_\nu}{dr} = \left(\frac{1}{r} - \frac{1}{2}\frac{d\mu}{dr}\right) E_\nu - \frac{\varrho_0}{c} e^{\frac{\lambda}{2}} x_\nu H_\nu - \frac{3}{r} P_\nu. \tag{96}$$

Dividing this equation by x_ν and integrating over ν, we get

$$-\frac{\varrho_0}{c} e^{\frac{\lambda}{2}} H_f = -\left(\frac{1}{r} - \frac{1}{2}\frac{d\mu}{dr}\right) \int_0^\infty \frac{E_\nu}{x_\nu} d\nu + \frac{3}{r} \int_0^\infty \frac{P_\nu}{x_\nu} d\nu + \int_0^\infty \frac{1}{x_\nu} \frac{dP_\nu}{dr} d\nu. \tag{97}$$

The classical laws of black body radiation

$$\left.\begin{aligned} E_\nu &= \frac{4\pi}{c} B_\nu \\ P_\nu &= \frac{4\pi}{3c} B_\nu \\ B_\nu &= \frac{2h\nu^3}{c^2} \frac{1}{\exp\left(\frac{h\nu}{kT}\right) - 1} \end{aligned}\right\} \tag{98}$$

can be calculated from eqs. (40) and (41) in the special case of photons, after putting ψ_d equal to zero and multiplying the expressions with 2 because of the polarization of photons. Insertion of eq. (98) into eq. (97) yields

$$-\frac{\varrho_0}{c} e^{\frac{\lambda}{2}} H_f = \frac{2\pi}{c} \frac{d\mu}{dr} \int_0^\infty \frac{B_\nu}{x_\nu} d\nu + \frac{4\pi}{c} \int_0^\infty \frac{1}{x_\nu} \left(\frac{\partial B_\nu}{\partial \nu} \frac{d\nu}{dr} + \frac{\partial B_\nu}{\partial T} \frac{dT}{dr} \right) d\nu. \tag{99}$$

Using eq. (93) and the expression

$$3B_\nu = \nu \frac{\partial B_\nu}{\partial \nu} + T \frac{\partial B_\nu}{\partial T}, \tag{100}$$

we get, after grouping the terms,

$$-\frac{\varrho_0}{c} e^{\frac{\lambda}{2}} H_f = \frac{\pi}{3c} e^{-\frac{\mu}{2}} \frac{d\left(T e^{\frac{\mu}{2}}\right)}{dr} \int_0^\infty \frac{1}{x_\nu} \frac{\partial B_\nu}{\partial T} d\nu. \tag{101}$$

Comparing eqs. (70) and (101), we find the following expression for the mean absorption coefficient of photons:

$$\frac{1}{x_f} = \frac{\pi}{acT^3} \int_0^\infty \frac{1}{x_\nu} \frac{\partial B_\nu}{\partial T} d\nu. \tag{102}$$

Differentiation of the expression for the total energy density of black body radiation

$$\frac{4\pi}{c}\int_0^\infty B_\nu\, d\nu = aT^4 \tag{103}$$

with respect to T and the insertion into eq. (102) gives another representation for $\frac{1}{x_f}$:

$$\frac{1}{x_f} = \frac{\int_0^\infty \frac{1}{x_\nu}\frac{\partial B_\nu}{\partial T}\, d\nu}{\int_0^\infty \frac{\partial B_\nu}{\partial T}\, d\nu}. \tag{104}$$

Eqs. (102) and (104) for the Rosseland mean opacity are just the same as in classical astrophysics[(5)].

The mean absorption coefficient of neutrinos is obtained similarly as the Rosseland mean opacity by using Fermi-Dirac statistics instead of Bose-Einstein statistics. We get the expression

$$\frac{1}{x_n} = \frac{\int_0^\infty \frac{1}{x_{\nu n}}\frac{\partial B_{n\nu}}{\partial T}\, d\nu}{\int_0^\infty \frac{\partial B_{n\nu}}{\partial T}\, d\nu}, \tag{105}$$

where

$$B_{n\nu} = \frac{2h\nu^3}{c^2}\frac{1}{\exp\left(\frac{h\nu}{kT}\right)+1} \tag{106}$$

is the Fermi-Dirac distribution function for massless particles.

Acknowledgment

The author wishes to thank Professor K. A. Hämeen-Anttila for suggesting the problem and for many helpful discussions. He is also much indebted to Professor C. W. Misner and especially to Professor C. Møller for reading the manuscript and for valuable advice.

References

1. T. D. Lee, Astrophys. J. **111**, 625 (1950).
2. I. Robinson, A. Schild and E. L. Schucking: Quasi-Stellar Sources and Gravitational Collapse (University of Chicago Press, Chicago, 1965).
3. K. A. Hämeen-Anttila and E. Anttila, Ann. Acad. Sci. Fennicae, Ser. A. VI, No 191 (1966).
4. R. W. Lindquist, Ann. Phys. **37**, 487 (1966).
5. S. Rosseland: Theoretical Astrophysics (Clarendon Press, Oxford, 1936), p. 110.
6. K. A. Hämeen-Anttila, Ann. Acad. Sci. Fennicae, Ser. A. VI, No 251 (1967).
7. F. Jüttner, Ann. Phys. (Leipzig) **34**, 856 (1911); **35**, 145 (1911).
8. G. N. Wattson: Theory of Bessel Functions, (Cambridge University Press, London, 1958) pp. 76–77.
9. F. Jüttner, Z. Phys. **47**, 542 (1928).
10. K. Huang: Statistische Mechanik II (Bibliographisches Institut, Mannheim, 1966), pp. 72, 100–103.
11. C. Eckart, Phys. Rev. **58**, 919 (1940).
12. R. C. Tolman: Relativity, Thermodynamics and Cosmology Clarendon Press, Oxford, 1962) p. 313.
13. L. D. Landau and E. M. Lifshitz: Statistical Mechanics (Pergamon Press, London, 1958), p. 326.
14. R. A. Schwartz, Ann. Phys. **43**, 42 (1967).
15. C. W. Misner, Phys. Rev. **137**, B 1360 (1965).
16. K. A. Hämeen-Anttila, Ann. Acad. Sci. Fennicae, Ser. A. VI, No 192 (1966).
17. D. H. Sampson, Astrophys. J. **129**, 734 (1959).

Indleveret til Selskabet den 20. oktober 1967.
Færdig fra trykkeriet den 18. april 1968.