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# ANGULAR MOMENTUM DEPENDENT POTENTIALS

BY

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## Synopsis

The properties of the most general local two-body potential for elastic scattering of scalar particles are investigated. This potential is angular momentum dependent. (Arbitrarily energy dependent potentials are ruled out by general symmetry and invariance arguments.) In particular, we investigate the momentum space representation of the angular momentum dependent potential, and show that it is characterized by a particular off shell behaviour. By considering the partial wave Lippmann-Schwinger equation, we establish in a rather simple manner the existence of a $p$-fold class of phase equivalent potentials (containing $p$ local, angular momentum dependent potentials), where $p$ equals the number of bound states in the partial wave under consideration. In this connection, we show that a potential, which is defined by using a perturbative expansion of the $S$-matrix from field theory, can be chosen to be local and angular momentum dependent, provided the expressions representing the diagrams included in the potential satisfy simple regularity conditions. There does not, however, seem to exist any simple relation (which does not involve the inverse of the Greens function) between a given non-local potential and the corresponding phase equivalent local and angular momentum dependent potential. As an illustration, we make some numerical calculations with a non-local single-particle exchange potential. The adiabatic approximation is investigated in this case, and is shown to be quite inaccurate for a strong attractive potential.



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## 1. Introduction

The object of this paper is to investigate the concept of a local, angular momentum dependent potential and its relevance to the general problem of obtaining a local potential which is equivalent to a given non-local potential.

We consider only the case of scalar particles of equal mass in this paper, leaving the general case of spin-dependent potentials to a forthcoming paper[1].

From the results of the investigations of the inverse problem of scattering, one can deduce, as has been pointed out by S. OKUBO and R. E. MARSHAK[2], that if a potential is reconstructed from a given scattering matrix, it can be chosen to be a function of $r^2$ and $\boldsymbol{L}^2$ only, where $\boldsymbol{L}$ is the angular momentum operator.

Thus, it is in principle possible to construct a (not necessarily unique) local, angular momentum dependent potential, which is equivalent to a given non-local potential, in the sense that both potentials give identical phase shifts.

In practice, however, this problem has apparently no simple and explicit solution, since it seems to be rather difficult to obtain a simple and explicit relation between the local, angular momentum dependent potential and the class of non-local potentials, which have identical phase shifts. On the other hand, in perturbation theory, one can easily establish the explicit relation between the equivalent local potential and the corresponding non-local potentials.

Before proceeding further, we may remark that it is by no means necessary to deal with local potentials instead of non-local potentials in the scattering problem. The scattering problem is certainly more complicated with a non-local potential, since one has to solve the Schrödinger equation as an integro-differential equation, or equivalently the Lippmann-Schwinger integral equation, instead of the Schrödinger (differential) equation; but the calculations are, after all, not overwhelmingly complicated. Despite this fact, almost all the papers (known to the author at least) published on e.g. nucleon-nucleon potentials, present potentials which are local, or at

most quadratically momentum dependent. These potentials are obtained by using approximations involving expansions with respect to the inverse of the nucleon mass. The validity of approximations of this kind is quite doubtful in general.

The formalism developed in this paper may be used in practical calculations to obtain local, angular momentum dependent potentials which approximate given non-local potentials. In such an approximation we completely avoid the use of expansions with respect to the inverse of the mass of the scattered particles. As an example, we have considered a single particle exchange potential and calculated the $^1S_0$-phase shifts for the exact potential, the static approximation to it, and for our local approximation to the potential in question. The details concerning the numerical calculations are found in sections 5–6.

From our calculations we conclude that the static approximation is rather poor for attractive potentials, in particular when the mass of the exchanged particle becomes comparable to the mass of the scattered particles, whereas the phase shifts calculated with our local, angular momentum dependent potential, which approximates the non-local single particle exchange potential, agree quite well with the exact phase shifts.

There have also appeared a few papers on the problem of defining "local" potentials, in which no expansion with respect to the inverse mass is used, but where the resulting potential is energy dependent. Among these papers, we may mention one by L. A. P. BALÁZS[3]. In section 2 we show that the use of an energy dependent potential in an ordinary Schrödinger equation is inconsistent with fundamental symmetry and invariance requirements. This fact casts some doubt on the validity of Balázs' results in particular, and on the use of energy dependent potentials in an ordinary (time independent) Schrödinger equation in general.

In section 3, which is divided into 3 subsections, we investigate under what conditions a potential, given in the momentum representation, may be represented by a local, angular momentum dependent potential in coordinate space, and discuss the properties of such a potential both for physical (integral) values of the angular momentum and for unphysical (complex) values. Section 4 contains a discussion of the equivalence problem, i.e., the problem of obtaining a local potential, which is equivalent to a given non-local one. In section 5 we discuss the single-particle exchange potential, which is used in the numerical calculations.

Finally, in section 6, we present the results of the numerical calculations and a discussion of these results.

In an Appendix, we present a method by which the partial-wave Lippmann-Schwinger equation, which is used in the calculation with the non-local potential, can be reduced to a non-singular equation.

## 2. The general form of the potential

In this section we review some of the properties of a general potential, which describes the interaction between two chargeless scalar particles of equal mass, and discuss the transformation formulae from the momentum representation to coordinate space representation.

From the analysis given in an article by J. GOTO and S. MACHIDA[4], we can deduce that the most general form of a potential between two scalar particles, which fulfils natural invariance requirements, i.e. invariance with respect to coordinate space translation, Galilei transformation, the exchange of the two particles, rotation of space coordinates, space reflections, time reversal, and Hermiticity of the potential, is, in momentum representation,

$$V(\boldsymbol{q},\boldsymbol{p}) = V_0(\boldsymbol{q}^2,\boldsymbol{p}^2,(\boldsymbol{q}\times\boldsymbol{p})^2). \tag{2.1}$$

The function $V_0$ is a real function of its arguments, which are the three independent scalars that can be obtained from the vectors $\boldsymbol{q}$ and $\boldsymbol{p}$, which in turn are defined in terms of the centre of mass (c.m) momenta as follows, (Fig. 1)

$$\boldsymbol{q} = \boldsymbol{k} - \boldsymbol{k}', \boldsymbol{p} = \frac{1}{2}(\boldsymbol{k} + \boldsymbol{k}'). \tag{2.2}$$

Defining

$$\boldsymbol{u} = \frac{1}{2}(\boldsymbol{r} + \boldsymbol{r}'), \boldsymbol{v} = \boldsymbol{r} - \boldsymbol{r}' \tag{2.3}$$

we have the relation between the coordinate space potential

$$V(\boldsymbol{u},\boldsymbol{v}) \text{ and } V(\boldsymbol{q},\boldsymbol{p})$$

$$V(\boldsymbol{u},\boldsymbol{v}) = \frac{1}{(2\pi)^6}\int d^3\boldsymbol{q}\, d^3\boldsymbol{p}\, e^{i\boldsymbol{q}\cdot\boldsymbol{u}} e^{i\boldsymbol{p}\cdot\boldsymbol{v}} V(\boldsymbol{q},\boldsymbol{p}). \tag{2.4}$$

(We shall occasionally use the same symbol to denote mathematically different functions, such as $V(\boldsymbol{q},\boldsymbol{p})$ and $V(\boldsymbol{u},\boldsymbol{v})$, which should not give rise to confusion). The potential $V(\boldsymbol{u},\boldsymbol{v})$, which in general is non-local, is to be inserted in the Schrödinger equation, in the c.m system*

$$(E_{\text{cm}} + \nabla^2)\psi(\boldsymbol{r}) = \int d^3\boldsymbol{r}'\, V(\boldsymbol{u},\boldsymbol{v})\psi(\boldsymbol{r}'). \tag{2.5}$$

* We use natural units with $\hbar = c = 2M$, where $M$ is the reduced mass.

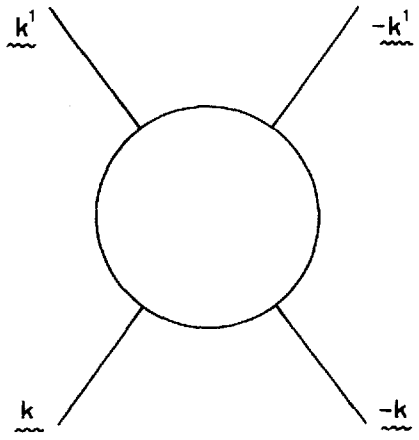


Fig. 1.

The transformation (2.4) is discussed in detail in Ref. (5), where it is shown that $V(\boldsymbol{q},\boldsymbol{p})$ may also be transformed to coordinate space by the formula

$$V(\boldsymbol{r},\hat{p}) = \frac{1}{(2\pi)^3}\int d^3\boldsymbol{q}e^{i\boldsymbol{q}\cdot\boldsymbol{r}}V(\boldsymbol{q},\boldsymbol{p}), \tag{2.6}$$

provided $\hat{p}$ in $V(\boldsymbol{r},\hat{p})$ is understood as $-i$ times the symmetrical gradient operator $\overleftrightarrow{\partial}$;

$$\psi^*(\boldsymbol{r})\hat{p}\psi(\boldsymbol{r}) = -\frac{i}{2}(\psi^*(\boldsymbol{r})\nabla\psi(\boldsymbol{r}) - (\nabla\psi^*(\boldsymbol{r}))\psi(\boldsymbol{r})). \tag{2.7}$$

The function $V(\boldsymbol{r},\hat{p})$ may be considered as a symbolical representation of $V(\boldsymbol{u},\boldsymbol{v})$, which in general is non-local.

Conversely, if $V(\boldsymbol{u},\boldsymbol{v})$ is given, $V(\boldsymbol{q},\boldsymbol{p})$ can be obtained by performing the inverse of the double Fourier transform (2.4). This can also be expressed in terms of $V(\boldsymbol{r},\hat{p})$ in the familiar form

$$V(\boldsymbol{q},\boldsymbol{p}) = \int d^3\boldsymbol{r}e^{-i\boldsymbol{k}'\cdot\boldsymbol{r}}V(\boldsymbol{r},\hat{p})e^{i\boldsymbol{k}\cdot\boldsymbol{r}} \tag{2.8}$$

which is equivalent to the inverse of (2.4).

From the previous discussion we deduce that one can obtain a strictly local potential $V(r)$ in coordinate space from a given $V(\boldsymbol{q},\boldsymbol{p})$ only if $V(\boldsymbol{q},\boldsymbol{p})$ is independent of $\boldsymbol{p}$. If $V(\boldsymbol{q},\boldsymbol{p})$ is an arbitrary function of its arguments $\boldsymbol{q}^2,\boldsymbol{p}^2$ and $(\boldsymbol{q}\times\boldsymbol{p})^2$, the resulting coordinate space potential is completely non-local. Only in the special case when $V(\boldsymbol{q},\boldsymbol{p})$ depends quadratically $\boldsymbol{p}$,

is it possible to obtain an "effective" energy dependent potential in coordinate space from the given $V(\boldsymbol{q},\boldsymbol{p})$, in which case the resulting "effective" energy dependent potential depends linearly on the energy of the two-particle system.

We have considered the most general potential to be used in the ordinary Schrödinger equation, restricted only by the invariance and symmetry requirements stated at the beginning of this section, and shown that the potential in the coordinate space representation cannot be an arbitrary function of energy in addition to the $r$-dependence. We can therefore conclude that the use of an arbitrarily energy-dependent potential in an ordinary Schrödinger equation is inconsistent with the given symmetry and invariance requirements.

There is still one special case in which the potential in coordinate space is neither strictly local nor non-local. When $V(\boldsymbol{q},\boldsymbol{p})$ depends on $\boldsymbol{q}$ and $\boldsymbol{p}$ in a rather special fashion, one obtains in coordinate space a potential $V(r,\boldsymbol{L}^2)$, where $\boldsymbol{L}$ is the angular momentum operator. The next section is devoted to an investigation of this special case.

### 3.I. Basic properties of the angular momentum dependent potential

We now assume that the potential in momentum representation $V(\boldsymbol{q},\boldsymbol{p})$ is given, and investigate under what conditions $V(\boldsymbol{q},\boldsymbol{p})$ can be represented by a local, angular momentum dependent potential $V(r,\boldsymbol{L}^2)$ in coordinate space. If $V(\boldsymbol{q},\boldsymbol{p})$ is represented by a $V(r,\boldsymbol{L}^2)$ in coordinate space, we have the following relation (eq. (2.8)) between $V(\boldsymbol{q},\boldsymbol{p})$ and $V(r,\boldsymbol{L}^2)$

$$V(\boldsymbol{q},\boldsymbol{p}) = \int d^3\boldsymbol{r} e^{-i\boldsymbol{k}'\cdot\boldsymbol{r}} V(r,\boldsymbol{L}^2) e^{i\boldsymbol{k}\cdot\boldsymbol{r}}. \tag{3.1}$$

We now investigate the restrictive conditions implied by (3.1) for the functional dependence on $\boldsymbol{p}$ and $\boldsymbol{q}$ in $V(\boldsymbol{q},\boldsymbol{p})$ and, assuming these conditions to be fulfilled, derive the inverse of (3.1), which gives $V(r,\boldsymbol{L}^2)$ as an integral transform of $V(\boldsymbol{q},\boldsymbol{p})$.

Let us denote an eigen-state of $\boldsymbol{L}^2$ by $|L\rangle$. Then we have, formally,

$$V(r,\boldsymbol{L}^2)|L\rangle = V(r,L(L+1))|L\rangle. \tag{3.2}$$

A function of an operator can in general be defined through a series expansion in powers of the operator in question. To ensure that eq. (3.2) is valid for all physical values of $L$, we shall have to require that $V(r,L(L+1))$ can be expanded into a series of powers of $L(L+1)$, convergent for all

(real and complex) values of $L(L+1)$. In other words, we must require that $V(r,L(L+1))$ is an entire function of $L(L+1)$ (for fixed $r$). When this is the case, the action of $V(r,\boldsymbol{L}^2)$ on an eigenstate $|L\rangle$ is certainly well defined. However, despite the fact that the function $V(r,L(L+1))$ must be an entire function of $L(L+1)$, this function has a well defined meaning only for physical (integral) values of $L$. When we consider the Schrödinger equation with an angular momentum dependent potential for a general complex value of $L$, we must use an extrapolation or continuation of the potential to complex values of $L$, which is such that the Watson-transform[6] can be applied to the resulting scattering amplitude. It is not *a priori* certain that the entire function $V(r,L(L+1))$ offers the required extrapolation. Therefore, when we consider the angular momentum dependent potential for complex values of $L$, we shall mean a function which coincides with $V(r,L(L+1))$ for physical values of $L$, but which is extrapolated to complex values of $L$ in such a manner that the Watson-transform can be applied to the resulting scattering amplitude. (We shall later return to this point in detail).

After these preliminaries, we consider eq. (3.1). We may now expand the plane waves in (3.1) into spherical waves and obtain a series involving $V(r,L(L+1))$ on the right hand side of (3.1). The integrations in (3.1) can now be performed term by term, provided $V(r,L(L+1))$ satisfies certain conditions, which we give below. We do not present the details of the necessary convergence proofs, which are readily obtained, using known properties of Neumann series, given e.g. in Watson's "Theory of Bessel functions"[7], (W. 526, W. 35).* The criterions which we have obtained are as follows. First, we require the existence of a fixed number $\alpha < 3$ such that

$$r^{\alpha}V(r,L(L+1)) \tag{3.3}$$

is bounded for $r \geqq 0$ and for fixed $L$. Then we require that $V(r,L(L+1))$ be bounded by a finite power of $L$, for integral values of $L$, or more precisely, we require the existence of a fixed non-negative integer $p$ such that

$$\sup_{r \geq 0} \frac{|r^{\alpha}V(r,L(L+1))|}{(2L+1)^{2p}} \leq A \tag{3.4}$$

where $A$ is an absolute constant. Let $N$ now be an arbitrary fixed positive integer. We then have to require the existence of the integrals

$$\int_0^{\infty} dr r^2 |V(r,L(L+1))| \tag{3.5}$$

* References to this work will be cited as W. followed by the appropriate page reference.

for non-negative integral values of $L$ such that $L \leq N$, and the existence of the integrals,

$$\int_0^\infty dr r^{2+m}(2L+1)^{-2p}\,|V(r,L(L+1))| \tag{3.6}$$

uniformly with respect to $L$, for integral values of $L > N$. The number $m$ in (3.6) takes the values $m = 0, 1, \ldots, 2p$. We understand by $p$ the smallest non-negative integer for which the conditions (3.4) and (3.6) are fulfilled. The conditions given above are sufficient to guarantee the validity of the term by term integrations in (3.1), when the plane waves are expanded into spherical waves, and also that the resulting series is convergent, uniformly with respect to the angle between $\boldsymbol{k}$ and $\boldsymbol{k}'$, and convergent for all positive fixed values of $k$ and $k'$. We may summarize the discussion as follows. If a given potential $V(r,L(L+1))$ satisfies the conditions (3.3)–(3.6), an integral transform of the form (3.1) is well defined.

### 3.II. The partial wave equations

We now proceed to investigate the consequences for $V(\boldsymbol{q},\boldsymbol{p})$ of the assumption that $V(\boldsymbol{q},\boldsymbol{p})$ is represented by a function $V(r,\boldsymbol{L}^2)$ in coordinate space. Let $F(\boldsymbol{k},\boldsymbol{k}')$ be a given function of $\boldsymbol{k}$ and $\boldsymbol{k}'$. We then define the partial wave projection of $F$ as follows,

$$F_{LL'}(k,k') = \int d\Omega_k d\Omega_{k'} Y_{L'}^{m_{L'}*}(\Omega_{k'}) F(\boldsymbol{k},\boldsymbol{k}') Y_L^{m_L}(\Omega_k) \tag{3.7}$$

where the $Y_L^{m_L}$ are spherical harmonics. Taking the partial wave projection of both sides of eq. (3.1), we obtain

$$V_L(k,k') = 4\pi^2(kk')^{-\frac{1}{2}} \int_0^\infty dr r V(r,L(L+1)) J_{L+\frac{1}{2}}(kr) J_{L+\frac{1}{2}}(k'r). \tag{3.8}$$

The function $V_L(k,k')$ is given by

$$V_L(k,k') = \int_{-1}^{+1} dx P_L(x) V(\boldsymbol{q},\boldsymbol{p}), \tag{3.9}$$

where $x$ is the cosine of the angle between $\boldsymbol{k}$ and $\boldsymbol{k}'$, and $P_L(x)$ the Legendre polynomial. The functions $J_\nu(z)$ in (3.8) are Bessel functions of the first kind.

It is clear that eq. (3.8), considered as an integral equation for $V(r,L(L+1))$, can have a solution only if the functional dependence on $k$ and $k'$ in $V_L(k,k')$ is rather restricted. Let us now suppose that we have obtained a solution $V(r,L(L+1))$ to eq. (3.8) ,which fulfills the conditions given earlier in section 3.I, eqns. (3.3)–(3.6). These conditions imply in particular that there should exist a fixed number $n < 2$, such that $r^n V(r,L(L+1))$ is absolutely summable in $(0, \infty)$. The eq. (3.8) is required to hold for all values of $k$ and $k'$ and must in particular be in force for $k = k'$. This means that $V(r,L(L+1))$ also satisfies the equation

$$kV_L(k,k) = 4\pi^2 \int_0^\infty dr r V(r,L(L+1)) J^2_{L+\frac{1}{2}}(kr). \tag{3.10}$$

Suppose now that we solve (3.10) for $V(r,L(L+1))$. The solution obtained from (3.10) can differ from the solution obtained from (3.8) only by a function which we denote by $\Delta(r,L(L+1))$, and which satisfies the equation

$$0 = \int_0^\infty dr r \Delta(r,L(L+1)) J^2_{L+\frac{1}{2}}(kr). \tag{3.11}$$

The function $\Delta(r,L(L+1))$ must also satisfy the summability condition given above and at the same time satisfy (3.11) for all values of $k$; $0 < k < \infty$. It is therefore clear that $\Delta(r,L(L+1))$ is a null-function, i.e., it equals zero almost everywhere in $(0, \infty)$; therefore, whenever eq. (3.8) has a solution for $V(r,L(L+1))$, this solution can be obtained from eq. (3.10) instead of eq. (3.8). Before solving eq. (3.8), we return to the problem of extrapolating $V(r,L(L+1))$ to complex values of $L$.

From the discussion given by L. Brown et al.[8] on the partial wave Lippmann-Schwinger (L.-S.) equation in momentum space, we may deduce that we get the "correctly" extrapolated scattering amplitude from the L.-S. equation, provided we extrapolate the potential in the L.-S. equation, which is essentially the function $V_L(k,k')$ (eq. (3.9)), in a manner which is consistent with the well known theorem of Carlson[9]. Let $\lambda_0$ be a fixed real number, and let $\lambda$ denote $L+1/2$. Let $\lambda = \lambda_0 + \varrho e^{i\theta}$. The function $A(\lambda;k,k')$, which extrapolates $V_L(k,k')$, is now a function which (for $k,k'$ fixed) (i) coincides with $V_L(k,k')$ at the non-negative integers larger than $\lambda_0$; (ii) is regular at all points inside the angle $-\alpha \leqq \theta \leqq \alpha$, where $\alpha \geqq \frac{\pi}{2}$; (iii) is bounded by $Ae^{B\varrho}$, where $A$ and $B$ are absolute constants and $B < \pi$, throughout the angle $-\alpha \leqslant \theta \leqslant \alpha$. If necessary, one may also allow a finite num-

ber of poles in the extrapolating function $A(\lambda;k,k')$. The coordinate space potential, corresponding to $A(\lambda;k,k')$, $W(r,\lambda)$, is now related to $A(\lambda;k,k')$ by the generalization of eq. (3.8) or equivalently eq. (3.10)

$$kA(\lambda;k,k) = 4\pi^2 \int_0^\infty dr r W(r,\lambda) J_\lambda^2(kr). \tag{3.12}$$

The function $W(r,\lambda)$ defined by (3.12) coincides with $V(r,L(L+1))$ defined by (3.10) at the non-negative integers (exceeding $\lambda_0$), but is in general different for general complex values of $\lambda$.

### 3.III. The inversion formula

In order to solve the eqns. (3.10) and (3.12), we need the following theorem (a proof of which is given in Appendix I);

Theorem: If $xf(x,\lambda)$ is differentiable in $(0, \infty)$, and if $(xf(x,\lambda))' \equiv \frac{d}{dx}(xf(x,\lambda))$ belongs to $L^2(0,\infty)$ uniformly with respect to $\lambda$ within a closed domain to the right of the line $Re(\lambda) = -\frac{1}{4}$, the equation

$$f(x,\lambda) = \int_0^\infty dy g(y,\lambda) J_\lambda^2(xy) \tag{3.13}$$

implies almost everywhere

$$g(x,\lambda) = -2\pi \frac{d}{dx} \int_0^\infty \frac{dy}{y} (yf(y,\lambda))' \int_0^{xy} du u J_\lambda(u) Y_\lambda(u), \tag{3.14}$$

and $g(x,\lambda)$ defined for $Re(\lambda) > -\frac{1}{4}$ by (3.14) also belongs to $L^2(0,\infty)$. The function $Y_\lambda(z)$ in (3.14) is a Bessel function of the second kind.

We now assume that $(k^2A(\lambda;k,k))'$ exists and belongs to $L^2(0,\infty)$ for $Re(\lambda) \geq \lambda' \geq \lambda_0$. Then we obtain

$$rW(r,\lambda) = -\frac{1}{2\pi} \frac{d}{dr} \int_0^\infty \frac{dk}{k} (k^2A(\lambda;k,k))' \int_0^{kr} du u J_\lambda(u) Y_\lambda(u), \tag{3.15}$$

for $Re(\lambda) \geq \max\left(\lambda', -\frac{1}{4} + \varepsilon\right)$, where $\varepsilon$ is a fixed arbitrary positive number, however small.

We know from our inversion theorem that $rW(r,\lambda)$ defined by (3.15) belongs to $L^2(0, \infty)$. Therefore, the potential we obtain from (3.15) cannot be more singular than $o(r^{-\frac{3}{2}})$ for small values of $r$, in contrast to the allowed behaviour $o(r^{-3})$. However, one can probably generalize the inversion theorem to cover a class of functions which behave like $o(r^{-3})$ for small values of $r$.

The expression (3.15) for $W(r,\lambda)$ has been obtained by using only the on-shell part of $A(\lambda;k,k')$. As mentioned above, the function $W(r,\lambda)$, which we obtain by using only the on-shell part of $A(\lambda;k,k')$, coincides with the $W(r,\lambda)$, obtained from the original off-shell equation, whenever the latter has a solution. We can therefore conclude the following. The assumption that $V(\boldsymbol{q},\boldsymbol{p})$ is represented by a $V(r,\boldsymbol{L}^2)$ in coordinate space, implies no restrictions on the on-shell part of $V(\boldsymbol{q},\boldsymbol{p})$, (apart from the differentiability and summability condition for the class of functions considered here), but implies that $V(\boldsymbol{q},\boldsymbol{p})$ should be continued off the energy shell in a particular way.

We can formulate this condition more easily in terms of $V_L(k,k')$ as follows. In order that a $V(\boldsymbol{q},\boldsymbol{p})$ be represented by a $V(r,\boldsymbol{L}^2)$ in coordinate space, it is necessary that the partial wave projection $V_L(k,k')$ of $V(\boldsymbol{q},\boldsymbol{p})$ has a repeated integral representation of the form

$$\left.\begin{aligned} V_L(k,k') = &-2\pi(kk')^{-\frac{1}{2}}\left\{\int_0^\infty dr J_{L+\frac{1}{2}}(kr)J_{L+\frac{1}{2}}(k'r)\right. \\ &\left.\left(\frac{d}{dr}\int_0^\infty \frac{dz}{z}\left[\frac{d}{dz}(z^2V_L(z,z))\right]\int_0^{zr} du u J_{L+\frac{1}{2}}(u)Y_{L+\frac{1}{2}}(u)\right)\right\} \end{aligned}\right\} \tag{3.16}$$

The condition (3.16) looks rather complicated, but the content of it is clear enough, namely, that the off-shell part $V_L(k,k')$ is uniquely given by the on-shell part $V_L(k,k)$ for a potential which is represented in coordinate space by a local, angular momentum dependent potential. This is of course also true for a strictly local potential $V(r)$, as this is a special case of a local angular momentum dependent potential.

We shall now finally have to make sure that $W(r,\lambda)$, given by (3.15), actually reduces to a function $V(r,L(L+1))$ when $L$ in $\lambda$ becomes a non-negative integer, and that this $V(r,L(L+1))$ can be considered as an entire function of $L(L+1)$. When $L$ is a non-negative integer, then we have

$$J_{L+\frac{1}{2}}(z)Y_{L+\frac{1}{2}}(z) = (-1)^{L+1}J_{L+\frac{1}{2}}(z)J_{-(L+\frac{1}{2})}(z), \tag{3.17}$$

and the right hand side of (3.17) is, apart from the factor $(-1)^{L+1}$, an entire function of $L(L+1)$ for fixed $z$. From (3.15) we then obtain

$$rV(r,L(L+1)) = \frac{1}{2\pi}\frac{d}{dr}\int_0^\infty \frac{dk}{k}((-1)^L k^2 V_L(k,k))' \left\{ \int_0^{kr} du u J_{L+\frac{1}{2}}(u) J_{-(L+\frac{1}{2})}(u) \right\}. \tag{3.18}$$

From the definition of $V_L(k,k')$, eq. (3.9), we obtain

$$(-1)^L V_L(k,k) = \int_{-1}^{+1} dx P_L(-x) V(\boldsymbol{q},\boldsymbol{p}), \tag{3.19}$$

where we have used the well known symmetry property of the Legendre polynominal. (It is of course understood that we use the constraint $k = k'$ in $V(\boldsymbol{q},\boldsymbol{p})$ in (3.19)). It is known that the function $P_L(-x)$ considered as a function of $L(L+1)$ is an entire function of $L(L+1)$, when $x$ has any assigned value, such that $-1 \leq x < 1$. The function $(-1)^L V_L(k,k)$ defined by (3.19) for general values of $L$ will therefore be an entire function of $L(L+1)$, provided well known conditions concerning continuity and uniformity of convergence of the integral (3.19) are satisfied. The integrand in (3.18) becomes then an entire function of $L(L+1)$, which means that also $V(r,L(L+1))$, defined by (3.18), is an entire function of $L(L+1)$, provided certain standard conditions are satisfied.

It is a simple matter to show that $V(r,L(L+1))$ is bounded by a finite power of $L(L+1)$ for integral values of $L$, and to derive conditions for the existence of the appropriate number of absolute moments of $V(r,L(L+1))$, in accordance with the discussion in section 3.I. The proofs are neither very difficult nor very interesting, and are therefore omitted.

## 4. The equivalence problem

In the previous sections we have investigated in detail the properties of a local, angular momentum dependent potential, and derived the conditions under which a potential given in momentum space is represented by a local, angular momentum dependent potential in coordinate space.

Let us first discuss the definition of a potential in perturbation theory. We follow the discussion given in a paper by A. A. Logunov et al.[10]. The

authors of this paper define the potential by the requirement that, when inserted into an equation of the Lippmann-Schwinger type considered by them, it will reproduce, to each order in perturbation theory, a $T$-matrix on the energy shell, which is considered given through an expansion in a coupling constant. The equation in question can in fact be exactly reduced to the ordinary non-relativistic Lippmann-Schwinger equation, as we are going to show in the following subsection. The definition formulated above can symbolically be stated as follows

$$[V_2] = [T_2], [V_{2n}] = [T_{2n}] - \sum_{m=1}^{n-1} [V_{2m} \times \bar{T}_{2n-2m}]. \tag{4.1}$$

Here $T_2$ is the second order $T$-matrix, and $V_2$ the second order potential (in momentum space) etc., and the square brackets mean a transition to the energy shell in the corresponding expressions.

Let us now discuss the meaning of eq. (4.1) more in detail. We see that the second order potential $V_2$ becomes fixed only on the energy shell. We can therefore continue the function $V_2$ off the energy shell in any (reasonable) prescribed manner. It seems therefore natural to continue $V_2$ off the energy shell in such a manner that $V_2$ becomes as simple as possible, without imposing restrictions at the same time on the resulting $T$-matrix by the chosen off shell continuation. This principle leads to an off shell continuation of $V_2$, which permits $V_2$ to be represented by a local, angular momentum dependent potential in coordinate space. Whatever off shell continuation we choose for $V_2$, we have as the result that the (new) $T$-matrix part corresponding to the chosen $V_2$ is $\bar{T}_2 = V_2$. (This fact is not indicated in eq. (24) of LOGUNOV et al., which corresponds to our eq. (4.1)). The fourth-order potential $V_4$ becomes again fixed on the energy shell only, be the next equation in (4.1). However, the value of $V_4$ on the energy shell depends now also on the off shell continuation chosen for $V_2$. We can then continue $V_4$ off the energy shell in the same way as $V_2$. The new fourth order $T$-matrix is now $\bar{T}_4 = V_4 + V_2 \times \bar{T}_2$. It is obvious that we can continue this reasoning to any order in perturbation theory. We have thus demonstrated that there is a considerable amount of freedom in choosing the off shell continuation of a potential which is constructed to reproduce a given $T$-matrix on the energy shell only. This does not mean that we can add arbitrary terms vanishing on the energy shell to a given potential without affecting the resulting $T$-matrix on the energy shell. In particular we have shown that a potential, defined by the principle symbolically stated in eq.

(4.1), can be chosen to be local and angular momentum dependent in coordinate space.

From their eq. (24) LOGUNOV et al. conclude that the potential can be chosen to be a local, energy dependent function in coordinate space. This is inconsistent with basic symmetry and invariance requirements, as we demonstrated earlier. (Some of the equations of LOGUNOV et al. manifestly violate the necessary symmetry between in- and out-going momenta.) We note incidentally that we have also in the foregoing discussion explicitly demonstrated that it is possible to construct a local, angular momentum dependent potential, which is equivalent to a given non-local one.

It appears to be rather difficult, however, to obtain a solution to the equivalence problem without resorting to perturbation theory arguments, or without solving directly the whole scattering problem with the non-local potential. To see this clearly, let us consider the partial wave Lippmann-Schwinger equation. We define

$$U_L(k',k) = -\frac{k'}{8\pi} V_L(k',k), \tag{4.2}$$

where $V_L(k',k)$ is the partial wave projection of the potential, defined by eq. (3.9). We then have the partial wave L.-S.-equation

$$T_L(k',k) = U_L(k',k) + \frac{2}{\pi}\int_0^\infty \frac{dk'' k'' U_L(k',k'') T_L(k'',k)}{k''^2 - k^2 - i\varepsilon}, \tag{4.3}$$

where $T_L$ is an off shell amplitude, which on the energy shell becomes

$$T_L(k,k) = e^{i\delta_L(k)} \sin(\delta_L(k)). \tag{4.4}$$

The main ambiguity in the potential is due to the fact that there exists a whole class of functions $U_L(k',k)$ which give rise to the same $T_L$ on the energy shell, but for which $T_L$ off the energy shell is different. This ambiguity is the one we have already discussed in the foregoing perturbation theory discussion. The other ambiguity comes from the possible existence of bound states. To see this clearly, we consider the problem of deducing $U_L(k',k)$ from a given phase shift $\delta_L(k)$, with the aid of eq. (4.3). Let us then suppose that we make an arbitrary (but sufficiently smooth) off shell continuation of the $T_L(k,k)$, which is determined by $\delta_L(k)$. The eq. (4.3) can then be considered as a singular integral equation for $U_L(k',k)$, with $k'$ as a parameter. Consulting the literature on singular integral equations[11], we observe that

eq. (4.3) can be reduced to a Fredholm equation for $U_L(k',k)$. However, we do not in general obtain a unique solution for $U_L(k',k)$ from eq. (4.3). This ambiguity is related to the existence of linearly independent solutions to the dominant part of eq. (4.3). The theorems given in Ref. (11) state that the number of linearly independent solutions equals the index $\varkappa$ of the Hilbert problem connected to the solving of the dominant equation. The index $\varkappa$ can easily be calculated and is, in this case, given by

$$\varkappa = \frac{1}{\pi}(\delta_L(0+) - \delta_L(\infty)). \tag{4.5}$$

The index $\varkappa$ given by (4.5) is equal to the number of bound states in the $L$:th partial wave. We have thus obtained the result: For a given off shell continuation of $T_L$, we obtain $\varkappa$ independent potentials $U_L^{(1)}$, $U_L^{(2)}$, . . ., $U_L^{(\varkappa)}$, which produce the given $T_L$ on the energy shell.

Thus, using an other off shell continuation of the $T_L$ given on the energy shell, we can obtain $\varkappa$ different potentials $\overline{U}_L^{(1)}$, $\overline{U}_L^{(2)}$, . . ., $\overline{U}_L^{(\varkappa)}$. We can thus assert that there exists a $p$-fold class of potentials which produce a given phase shift, where $p$ is the number of bound states in the partial wave under consideration.

However, we have not been able to obtain a simple and explicit relation between two members $U_L^{(i)}$ and $\overline{U}_L^{(i)}$ of this class.

We have now analysed the ambiguities inherent in a potential, which is required to produce a given $T$-matrix on the energy shell only. We may conclude that, although it is in principle possible to use these ambiguities in constructing a local, angular momentum dependent potential, which is equivalent to a given non-local potential, this problem has apparently no simple and practical solution.

The formalism developed in this paper may, however, be used to approximate non-local potentials by local, angular momentum dependent potentials. We conclude this section with a brief discussion of this possibility, and of the conventional methods which have been earlier used to approximate non-local potentials by local ones. Let us consider again the potential

$$V(\boldsymbol{q},\boldsymbol{p}) \tag{4.6}$$

where, as before, $\boldsymbol{q}$ is the momentum transfer and $2\boldsymbol{p}$ is the sum of the in- and outgoing momenta in the c.m. system. Let us then suppose that we can expand (4.6) in powers of $\boldsymbol{p}^2$

$$V(\boldsymbol{q},\boldsymbol{p}) = \sum_{n=0}^{\infty} v_n(\boldsymbol{q})\boldsymbol{p}^{2n} \tag{4.7}$$

The so-called static approximation means that we neglect all higher powers of $\boldsymbol{p}^2$, and use $v_0(\boldsymbol{q})$ as an approximation to $V(\boldsymbol{q},\boldsymbol{p})$. It is clear that $v_0(\boldsymbol{q})$ becomes local in coordinate space. In the next approximation we obtain a quadratically momentum dependent potential: $v_0(\boldsymbol{q}) + v(\boldsymbol{q})\boldsymbol{p}^2$. In coordinate space, this becomes of the form $\bar{v}_0(r) + \bar{v}_1(r)\hat{p}^2$, where $\hat{p}$ is the differential operator defined by eq. (2.7). It is obvious that one cannot continue this approximation method to higher order terms, since potentials of the form $\bar{v}_n(r)(\hat{p}^2)^n$ $(n > 1)$ cannot be used in the Schrödinger equation. We may remark that the potentials derived from meson theory are not entire functions of $\boldsymbol{p}^2$, so that expansions of the type (4.7) do not exist except possibly in a small region around $\boldsymbol{p}^2 = 0$. From the formal point of view, the approximations based on equations of the type (4.7) are therefore meaningless. However, if the function $V(\boldsymbol{q},\boldsymbol{p})$ is a slowly varying bounded function of $\boldsymbol{p}^2$, the static approximation need not be entirely unreliable. If this is the case, then the next approximation $v_0(\boldsymbol{q}) + v_1(\boldsymbol{q})\boldsymbol{p}^2$ is certainly very doubtful, although it might to some extent be remediable by a properly chosen cut off.

Despite the large uncertainties which inevitably are connected with the use of approximations involving expansions with respect to $\boldsymbol{p}^2$, such approximations have been used extensively e.g. in derivations of nucleon-nucleon potentials. The approximation method we suggest is rather obvious. Consider the partial wave projection $V_L(k,k')$ of the potential $V(\boldsymbol{q},\boldsymbol{p})$. We now approximate the function $V_L(k,k')$ by a function $\bar{V}_L(k,k')$, which coincides with $V_L(k,k')$ on the energy shell, but which is continued off the energy shell in the manner prescribed by eq. (3.16). The function $\bar{V}_L(k,k')$ coincides with $V_L(k,k')$ along the lines $k = 0$, $k' = 0$ and $k = k'$. Unless $V_L(k,k')$ varies violently in the sectors between these lines, we may expect that $\bar{V}_L(k,k')$ approximates $V_L(k,k')$ in an acceptable manner in the whole first quadrant of the $kk'$-plane. The coordinate space potential corresponding to $\bar{V}_L(k,k')$ is obtained directly by inserting $V_L(k,k)$ in the formula (3.18), or, for complex $L$, eq. (3.15).

## 5. The single-particle exchange potential

We consider an equation, recently discussed by R. BLANKENBECLER and R. SUGAR[12],

$$M(\boldsymbol{k}',\boldsymbol{k}) = W(\boldsymbol{k}',\boldsymbol{k}) + \frac{1}{4}\int \frac{d^3\boldsymbol{k}'' W(\boldsymbol{k}',\boldsymbol{k}'') M(\boldsymbol{k}'',\boldsymbol{k})}{(2\pi)^3 \sqrt{\boldsymbol{k}''^2 + 1}\,(\boldsymbol{k}''^2 - \boldsymbol{k}^2)} \qquad (5.1)$$

Here $W(\boldsymbol{k}', \boldsymbol{k})$ is the propagator for the exchange of a particle of mass $m$ between two scalar particles of unit mass

$$W(\boldsymbol{k}', \boldsymbol{k}) = g^2(m^2 + (\boldsymbol{k} - \boldsymbol{k}')^2)^{-1}, \tag{5.2}$$

and $\boldsymbol{k}'$ and $\boldsymbol{k}$ the relative initial and final momenta, respectively, in the c.m. system. The eq. (5.1) with $W$ given by (5.2) may be considered as an approximation to the Bethe-Salpeter equation in the ladder approximation. An equation of the type (5.1) has also been discussed earlier, e.g. in the previously mentioned article by A. A. LOGUNOV et al. (Ref. (10)). We recall that there is a difference in normalization between the relativistic amplitude $M$ and the amplitude $T$, which occurs in the ordinary non-relativistic Lippmann-Schwinger equation.

Defining

$$T(\boldsymbol{k}', \boldsymbol{k}) = -\frac{M(\boldsymbol{k}', \boldsymbol{k})}{4\sqrt[4]{\boldsymbol{k}'^2 + 1}\sqrt[4]{\boldsymbol{k}^2 + 1}}, \tag{5.3}$$

$$V(\boldsymbol{k}', \boldsymbol{k}) = -\frac{W(\boldsymbol{k}', \boldsymbol{k})}{4\sqrt[4]{\boldsymbol{k}'^2 + 1}\sqrt[4]{\boldsymbol{k}^2 + 1}}, \tag{5.4}$$

we obtain the ordinary L.-S.-equation from (5.1)

$$T(\boldsymbol{k}', \boldsymbol{k}) = V(\boldsymbol{k}', \boldsymbol{k}) + \frac{1}{(2\pi)^3}\int d^3\boldsymbol{k}'' \frac{V(\boldsymbol{k}', \boldsymbol{k}'')\, T(\boldsymbol{k}'', \boldsymbol{k})}{\boldsymbol{k}^2 - \boldsymbol{k}''^2 - i\varepsilon} \tag{5.5}$$

The function $V(\boldsymbol{k}', \boldsymbol{k})$ in (5.5) is precisely the quantity we have called a potential in momentum space, expressed as a function of the in- and outgoing momenta $\boldsymbol{k}'$ and $\boldsymbol{k}$. (Note the symmetry between $\boldsymbol{k}'$ and $\boldsymbol{k}$ in (5.4)). For later convenience, we introduce explicitly a mass $M$ of the scattered particles in the expression (5.4), and introduce also a strength parameter $\Lambda = -g^2/16\pi$. The expression for the single-particle exchange potential is then

$$V(\boldsymbol{k} - \boldsymbol{k}', \tfrac{1}{2}(\boldsymbol{k} + \boldsymbol{k}')) = \frac{4\pi\Lambda M}{\sqrt[4]{\boldsymbol{k}^2 + M^2}\sqrt[4]{\boldsymbol{k}'^2 + M^2}((\boldsymbol{k} - \boldsymbol{k}')^2 + m^2)}. \tag{5.6}$$

We evaluate the partial wave projection of (5.6) according to eq. (3.9) and obtain

$$V_L(k,k') = \frac{4\pi\Lambda M}{kk'\sqrt[4]{k^2+M^2}\sqrt[4]{k'^2+M^2}}\, Q_L\left(\frac{k^2+k'^2+m^2}{2kk'}\right), \tag{5.7}$$

where $Q_L$ is a Legendre function of the second kind. We can also immediately infer from the properties of the $Q_L$-function that the expression

$$A(\lambda;k,k') = \frac{4\pi\Lambda M}{kk'\sqrt[4]{k^2+M^2}\sqrt[4]{k'^2+M^2}}\, Q_{\lambda-\frac{1}{2}}\left(\frac{k^2+k'^2+m^2}{2kk'}\right), \tag{5.8}$$

is the correct continuation of $V_L(k,k')$.

The static approximation of (5.6) consists of replacing the factor

$$M\left(\sqrt[4]{k^2+M^2}\sqrt[4]{k'^2+M^2}\right)^{-1}$$

by unity. We then obtain, upon transision to coordinate space, the well known Yukawa potential

$$V(r) = \Lambda\frac{e^{-mr}}{r} \tag{5.9}$$

It is not difficult to see that the off shell behaviour of $V_L(k,k')$ in (5.7) does not allow $V_L(k,k')$ to be represented by a local, angular momentum dependent potential in coordinate space. In constructing the potential $W(r,\lambda)$ from $A(\lambda;k,k)$ given by (5.8), we therefore make an approximation of the kind previously discussed. We obtain, according to eq. (3.15), $\left(Re(\lambda) > -\frac{1}{4}\right)$,

$$rW(r,\lambda) = -\Lambda\int_0^\infty dk\left[\frac{d}{dk}\left(\frac{2M}{\sqrt{M^2+k^2}}\, Q_{\lambda-\frac{1}{2}}\left(1+\frac{m^2}{2k^2}\right)\right)\right]\left\{krJ_\lambda(kr)Y_\lambda(kr)\right\}. \tag{5.10}$$

We may check the integral (5.10) by putting $M(M^2+k^2)^{-\frac{1}{2}} = 1$ in the integrand. The integral can then be evaluated and becomes precisely equal to (5.9), as it should.

When $L$ is a non-negative integer, we may use the simple analytic properties of the integrand in (5.10) to write the integral in a more convenient form. We denote by $K_{L+\frac{1}{2}}(z)$ the exponentially damped Bessel function (W. 80)

$$K_{L+\frac{1}{2}}(z) = \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} e^{-z}\sum_{n=0}^{L}\frac{(L+n)!}{n!\,(L-n)!}(2z)^{-n}. \tag{5.11}$$

$$(L = 0,1,2,\ldots.)$$

Let

$$C_L(z) = \frac{d}{dz}(zK^2_{L+\frac{1}{2}}(z)). \tag{5.12}$$

After some exercises in contour integration, we obtain from (5.10)

$$\left.\begin{aligned} V(r, L(L+1)) = &-A\frac{2}{\pi}\int_{\frac{m}{2}}^{M}\frac{dyM}{\sqrt{M^2-y^2}}P_L\left(\frac{m^2}{2y^2}-1\right)C_L(yr) \\ &+A\frac{4}{\pi^2}(-1)^L\int_{M}^{\infty}\frac{dyM}{\sqrt{y^2-M^2}}Q_L\left(1-\frac{m^2}{2y^2}\right)C_L(yr) \\ &(L = 0, 1, 2, \ldots) \end{aligned}\right\} \tag{5.13}$$

It is easily seen that the local, angular momentum dependent potential (5.10) and (5.13) posesses the properties we required in section 3.I.

## 6. Numerical results and discussion

In this section we describe the results of the numerical calculations performed with the non-local potential (5.6) and the two approximations (5.9) and (5.13), respectively, to this non-local potential.

A calculation of phase shifts with a non-local potential in momentum space has been performed e.g. by J. GOTO[13] and by P. SIGNELL and P. S. CONNORS[14]. These authors restricted themselves to calculate phase shifts for a one-pion exchange potential. They also used a rather small cut off, of the order $M - 2M$, where $M$ is the nucleon mass. This makes it difficult to compare their results directly with those obtained in the static approximation with the local potential in coordinate space.

In fact, to the best of the author's knowledge, there exists no systematic investigation of the validity of the static approximation, even for the simple case of a single-particle exchange potential. We have attempted to make such an investigation by calculating the $L = 0$ phase shifts with the potentials mentioned above, for a number of values of the strength parameter $A$ and of the mass $m$ of the exchanged particle. The mass $M$ is given the value of the nucleon mass ($M = 938.5$ MeV). In order to have a physical measure of the strength of the potentials, we present below a table (Table 1), which shows the values of $A$, for which one and two bound $S$-states occur with the Yukawa potential (5.9)[15].

TABLE 1.

| $m$(MeV) | $\Lambda(1)$ | $\Lambda(2)$ |
|---|---|---|
| 140 | −0.25 | −0.95 |
| 280 | −0.50 | −1.9 |
| 400 | −0.72 | −2.7 |
| 700 | −1.25 | −4.8 |

(The strength of the static one-pion exchange potential in the $^1S_0$-state corresponds to $\Lambda = -0.081$.)

For positive values of $\Lambda$ in the range $0 < \Lambda < 3$, there is no appreciable difference ($< 5\%$) between the phase shifts calculated with the potentials (5.6), (5.9) and (5.13), respectively, in the energy region $0 < E_{\mathrm{LAB}} < 280$ (MeV) and for the values of $m$, given in Table 1. For attractive potentials the situation is different. Here the static approximation yields systematically too large phase shifts. In the static approximation we replace the factor $M(M^2+k^2)^{-\frac{1}{4}}(M^2+k'^2)^{-\frac{1}{4}}$ by unity in the expression (5.6) for the potential in momentum space. The resulting coordinate space potential becomes therefore too singular near $r = 0$. When we take this factor into account in the approximation (5.13), we obtain a potential, which behaves like $O(\log^2(r))$ near $r = 0$, in contrast to the $O(r^{-1})$ behaviour of the Yukawa potential. The difference between the phase shifts in the static approximation and the phase shifts resulting from the potential (5.6) increases rapidly with increasing $m$ and $|\Lambda|$. We give examples of this in Fig. 2–Fig. 10. The curves labeled "Exact" in Fig. 2–Fig. 10, are the $L = 0$ phase shifts, which are obtained by solving the partial wave Lippmann-Schwinger equation (see appendix II) with the potential given by (5.7). The curves labeled "Yukawa" are the phase shifts in the static approximation, obtained from the Schrödinger equation with the potential (5.9), whereas the curves labeled "Appr." are the phase shifts obtained from the Schrödinger equation with the local, angular momentum dependent potential (5.13). The phase shifts obtained from the L.S.-equation are given only in the energy region $50 < E_{\mathrm{LAB}} < 280$ (MeV). We have checked the accuracy of the phase shifts resulting from the L.S.-equation by solving also the L.S.-equation with the static approximation to (5.7). The difference between the phase shifts obtained in this way, and those obtained with the Yukawa potential (5.9), gives a measure of the accuracy achieved in solving the L.S.-equation. Table 2 shows these maximal absolute differences for the values of $m$ and $\Lambda$ given in Fig. 2–Fig. 10.

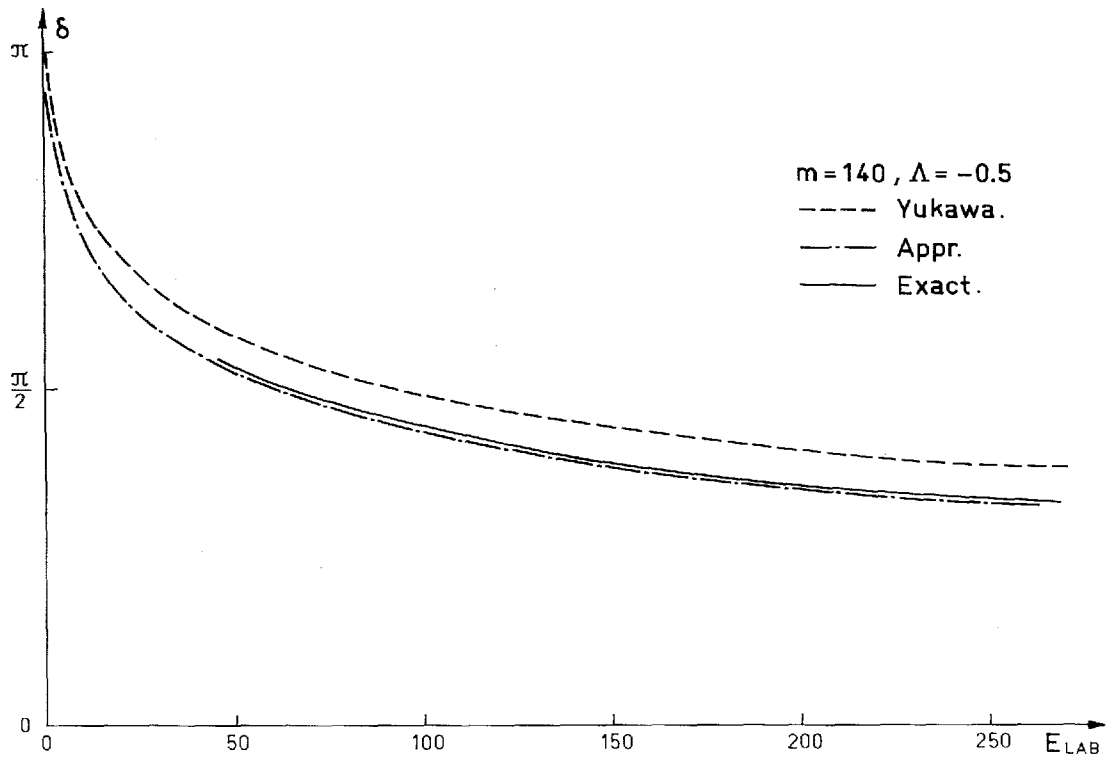


Fig. 2.

TABLE 2.

| $m$(MeV) | Max. num. error |
|---|---|
| 140 | 0.030 |
| 280 | 0.028 |
| 400 | 0.033 |
| 700 | 0.038 |

The numerical errors inherent in the curves labeled "Exact" in Fig. 2–Fig. 10 are therefore rather small, and can hardly produce any detectable effect in the given curves. From Fig. 2–Fig. 10, it is seen that the difference between the exact and the Yukawa phase shifts is approximately constant over the range $50 < E_{\mathrm{LAB}} < 280$ (MeV). A measure of the average error in calculating the phase shifts in the static approximation (5.9) instead of using the potential (5.7) is therefore the difference between the Yukawa phase

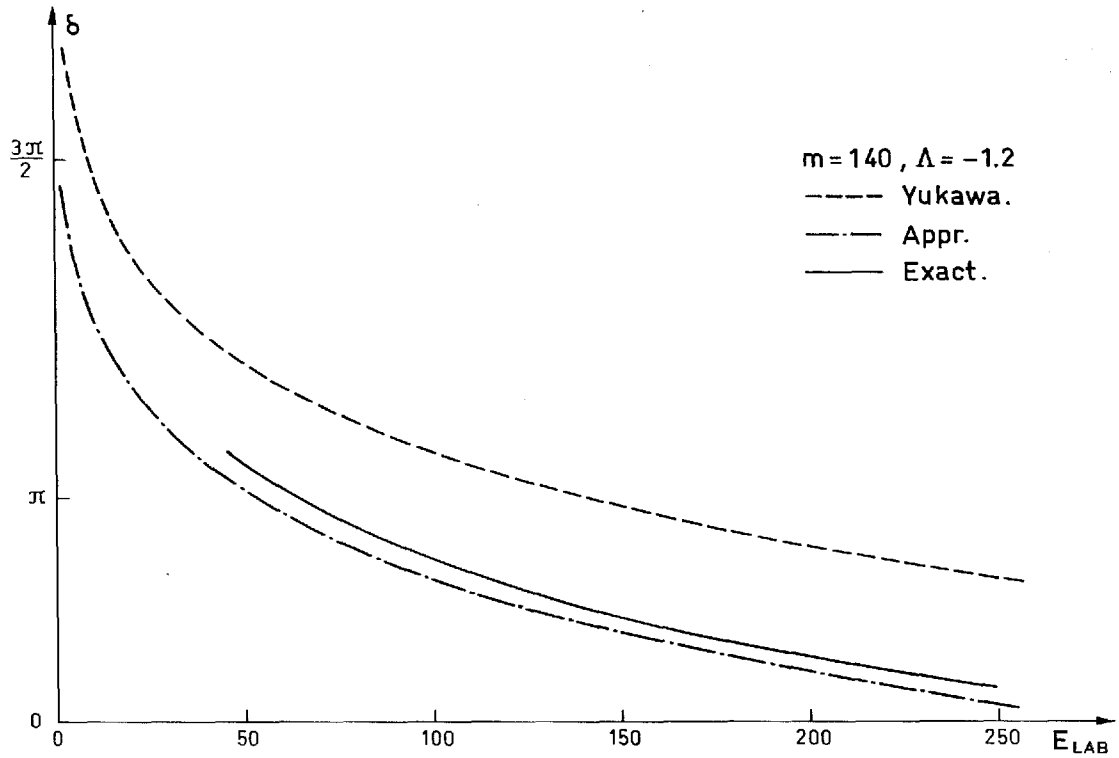


Fig. 3.

TABLE 3.

| $A$ | $m$(MeV) | $D_{\text{LOCAL}}$ % | $D_{\text{STATIC}}$ % |
|---|---|---|---|
| −0.5 | 140 | — | 14 |
| −1.2 | 140 | −2.7 | 20 |
| −0.8 | 280 | −2.7 | 19 |
| −1.5 | 280 | −2.5 | 24 |
| −0.7 | 400 | −2.1 | 19 |
| −1.1 | 400 | −3.7 | 24 |
| −2.0 | 400 | −3.9 | 26 |
| −1.1 | 700 | −4.9 | 32 |
| −2.0 | 700 | −6.5 | 36 |

shift and the exact one at 150 MeV. We give above a table (Table 3) of this difference, D STATIC, expressed in percent of the exact phase shift, and also the corresponding difference, D LOCAL, between the exact phase shift

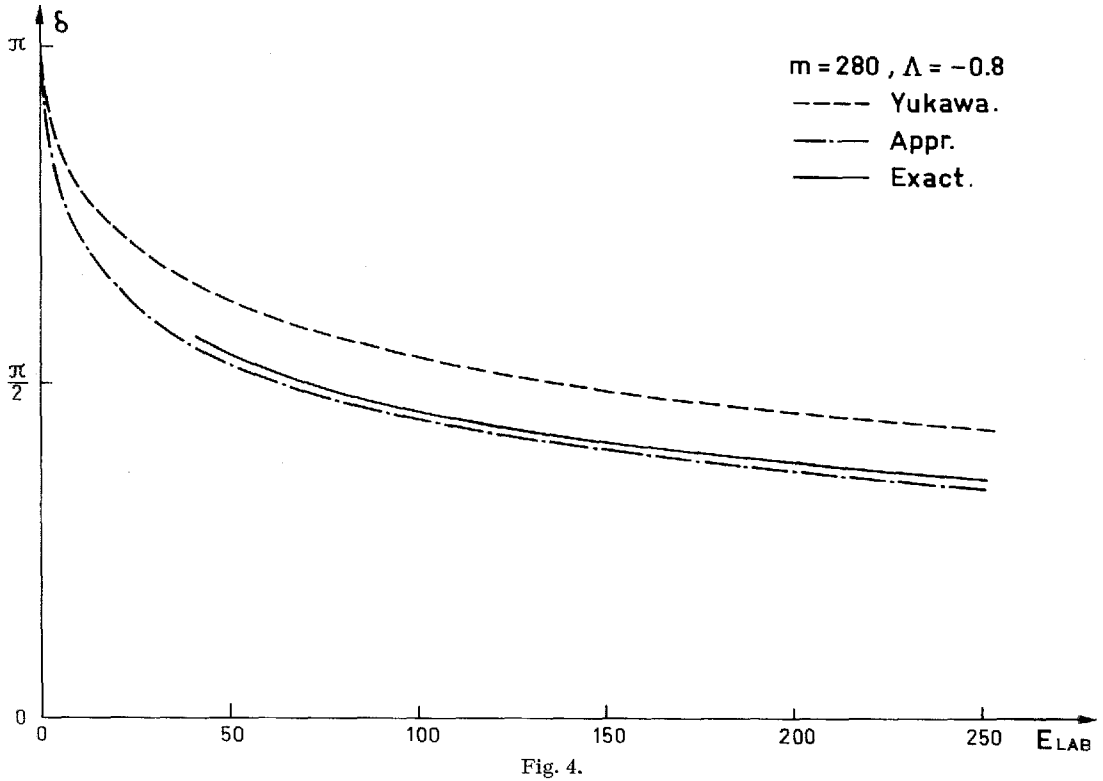


Fig. 4.

and the one obtained with the local, angular momentum dependent potential (5.13).

We have thus shown that the static approximation is quite unreliable for the attractive single-particle exchange potential, in particular when the mass of the exchanged particle becomes large, whereas the phase shifts calculated with the local, angular momentum dependent potential agree quite well with the exact phase shifts. The potential (5.6) is, of course, not exact in an absolute sense, but represents a potential which, when used in the Schrödinger equation, yields an approximation to the sum of all ladder diagrams, as previously pointed out.

In section 4 we showed explicitly that, in perturbation theory, one can choose the potential to be local and angular momentum dependent in coordinate space, since this merely corresponds to a rearrangement of the (infinite number of) equations connecting the quantities $T_{2n}$ and $V_{2n}$. However, the problem of constructing a potential to all orders in perturbation

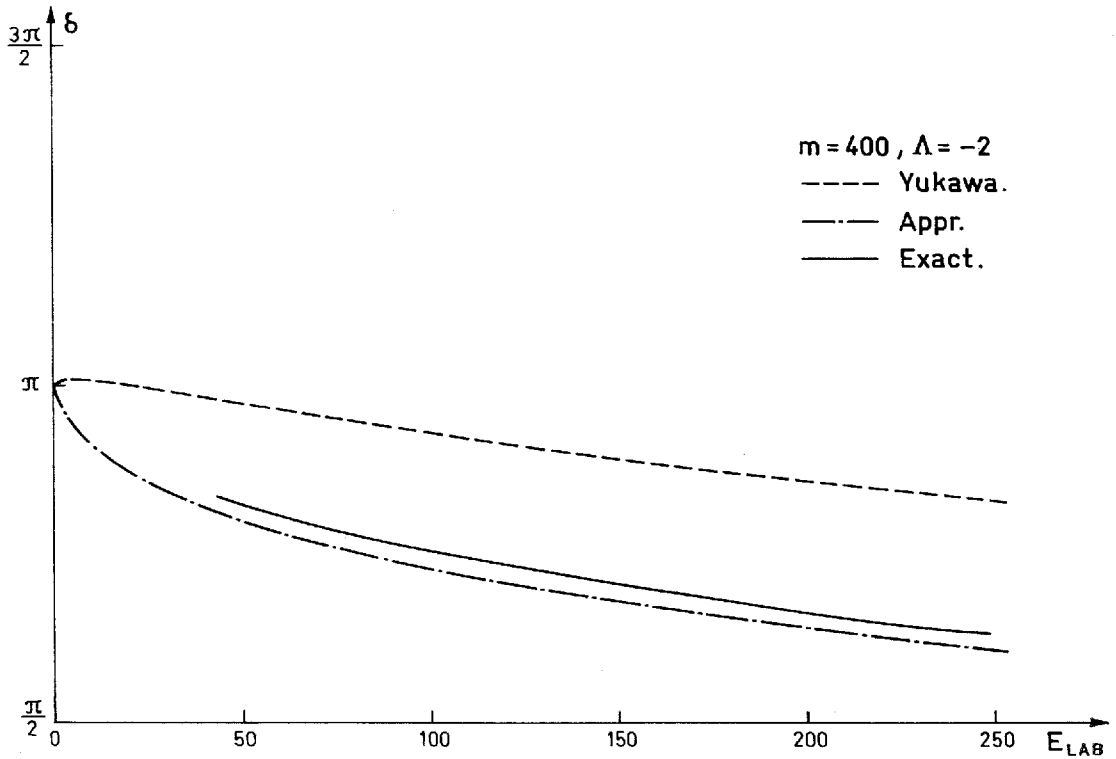


Fig. 5.

theory is, of course, highly academic since, if every $T_{2n}$ is known, and if it is possible to sum these quantities, it is indeed unnecessary to obtain a potential $V$, which on insertion in a Schrödinger equation yields the known $T$-matrix. In practice, the potential $V$ is constructed up to some finite (and small) order, which means that one obtains approximations to the sums of those classes of diagrams, which are the iterations of the diagrams included in $V$, when using this $V$ in the Schrödinger equation. When the potential $V$ is constructed only up to a finite order, the resulting phase shifts do indeed depend on the off shell continuation chosen for $V$. From the pure S-matrix point of view, the question of the off shell continuation is undecidable. However, for reasonable potentials and reasonable off shell continuations, we may expect that the resulting phase shifts do not differ much for two different off shell continuations, even when the potential is constructed up to some small order, as indicated by the numerical example considered in this section.

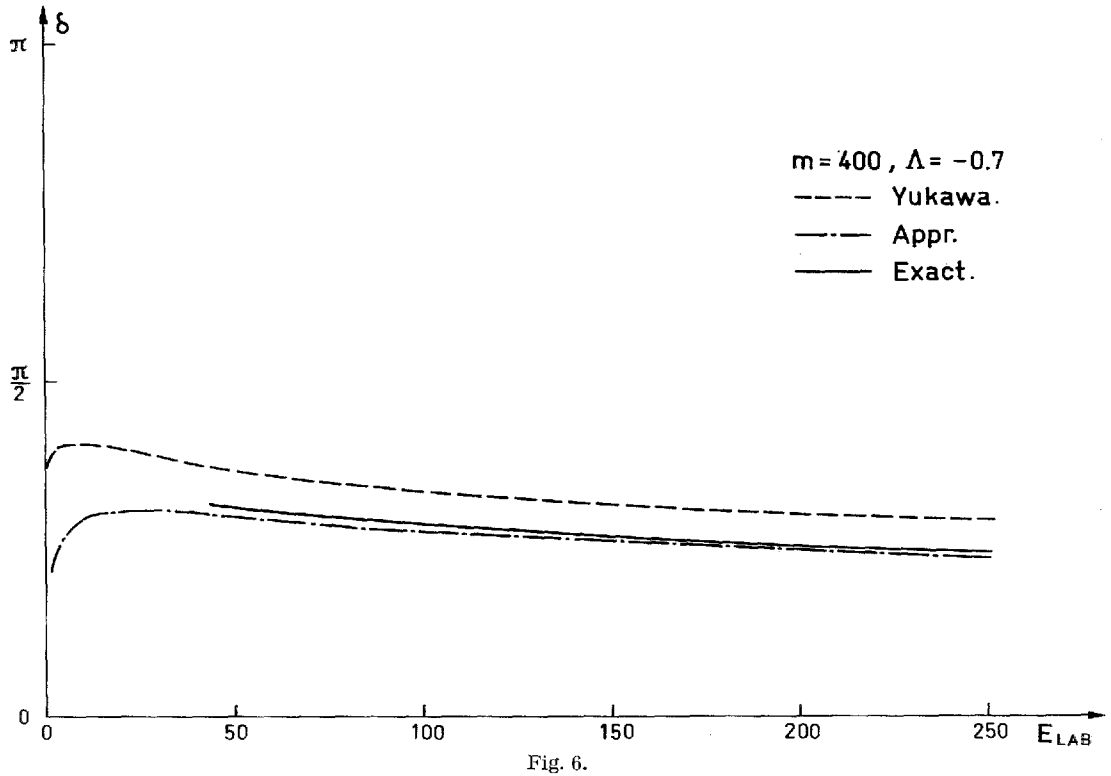


Fig. 6.

We may then assert that within a reasonable and practical perturbation-theoretic definition of a potential (not necessarily based on Feynman-Dyson expansions), it is always possible to obtain a local angular momentum dependent potential in coordinate space, provided the expressions representing the diagrams we include in the potential satisfy simple regularity (differentiability and summability) requirements of the kind given in this paper.

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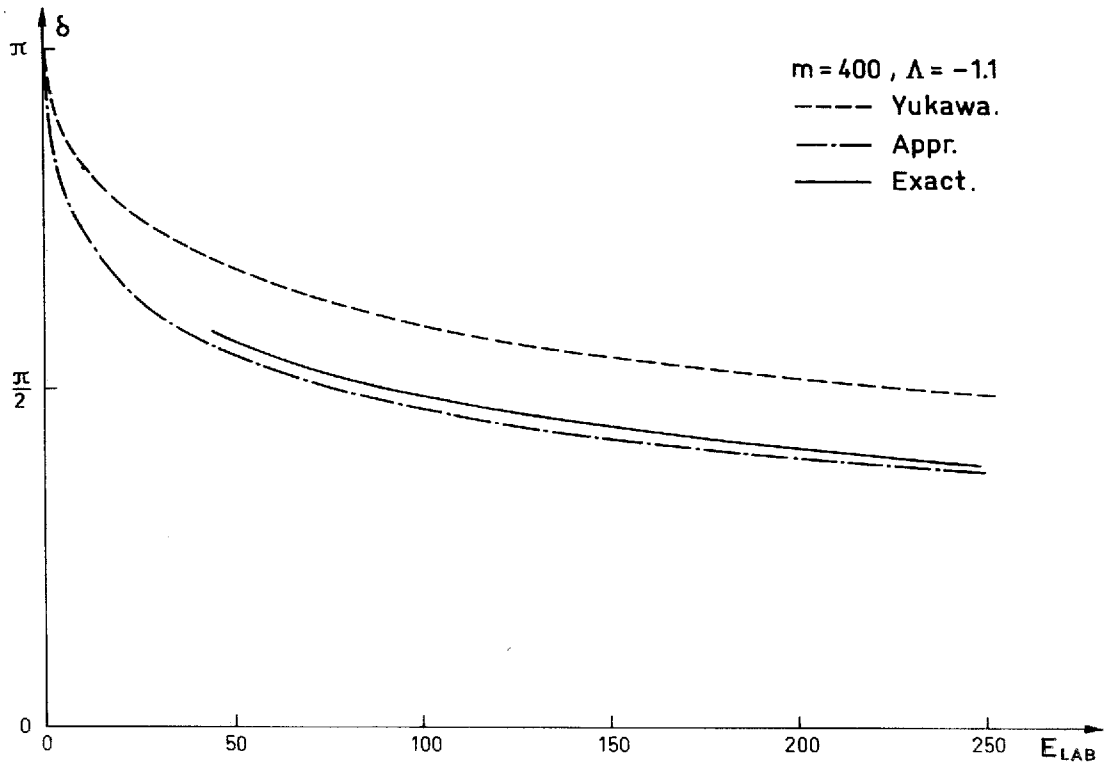


Fig 7.

## Appendix I
## The inversion theorem

In this Appendix we prove the inversion theorem given in section 3.III. The equation to be solved is

$$f(x,v) = \int_0^\infty dy g(y,v) J_v^2(xy). \qquad \text{(A. 1)}$$

Here $f(x,v)$ is given as a function of the real parameter $x$ and of the parameter $v \varepsilon D$, where $D$ is a closed domain which is subsequently determined.

In solving (A.1), we use the theory of "general transforms", an exposition of which can be found in Titchmarsh's "Theory of Fourier Integrals"[16]. We need the following theorem, which is a straightforward generalization of theorem 129 in Ref. (16).

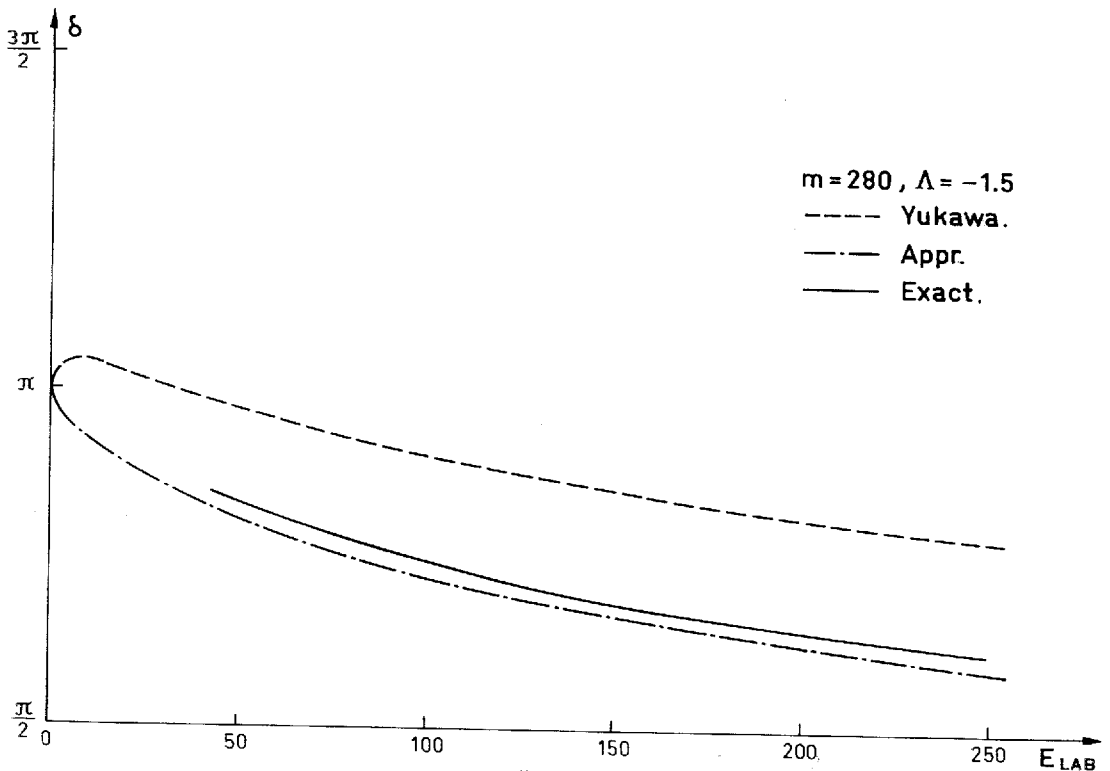


Fig. 8.

Theorem: Let $K\left(\frac{1}{2}+it\right)$ and $H\left(\frac{1}{2}+it\right)$ be any bounded functions of the real parameter $t$, satisfying the condition

$$K\left(\frac{1}{2}+it\right)H\left(\frac{1}{2}-it\right) = 1. \tag{A. 2}$$

Define $k(x)$ and $h(x)$ by the formulae

$$k(x) = \frac{x}{2\pi}\,\mathrm{l.i.m}_{T\to\infty}\int_{-T}^{T} dt\,\frac{K\left(\frac{1}{2}+it\right)}{\frac{1}{2}-it}\,x^{-\frac{1}{2}-it}, \tag{A. 3}$$

$$h(x) = \frac{x}{2\pi}\,\mathrm{l.i.m}_{T\to\infty}\int_{-T}^{T} dt\,\frac{H\left(\frac{1}{2}+it\right)}{\frac{1}{2}-it}\,x^{-\frac{1}{2}-it}. \tag{A. 4}$$

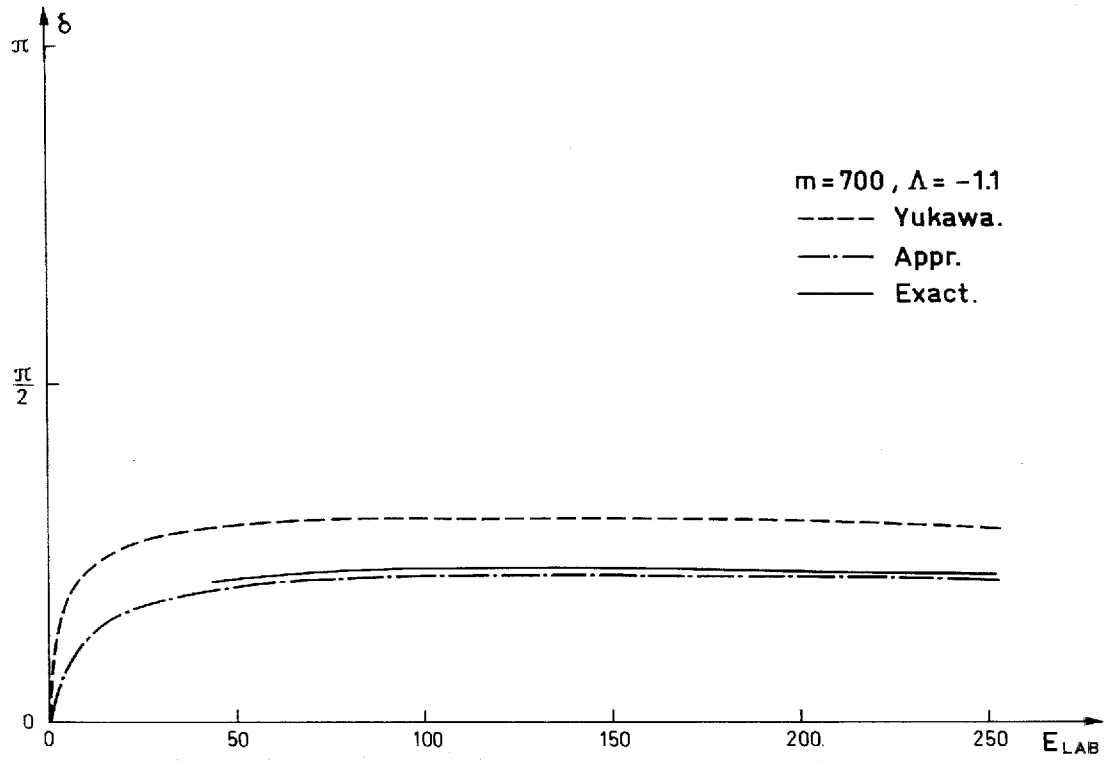


Fig. 9.

Let $f(x)\varepsilon L^2(0,\infty)$. Then the formula

$$g(x) = \frac{d}{dx}\int_0^\infty \frac{du}{u} k(xu)f(u) \qquad \text{(A. 5)}$$

defines almost everywhere a function $g(x)\varepsilon L^2(0, \infty)$, and the reciprocal formula

$$f(x) = \frac{d}{dx}\int_0^\infty \frac{du}{u} h(xu)g(u) \qquad \text{(A. 6)}$$

also holds almost everywhere.

In using this theorem for eq. (A. 1), we proceed as follows. Let $n(x)$ and $m(x)$ be two arbitrary functions of $x$. We multiply and divide (A. 1) by $n$ and $m$

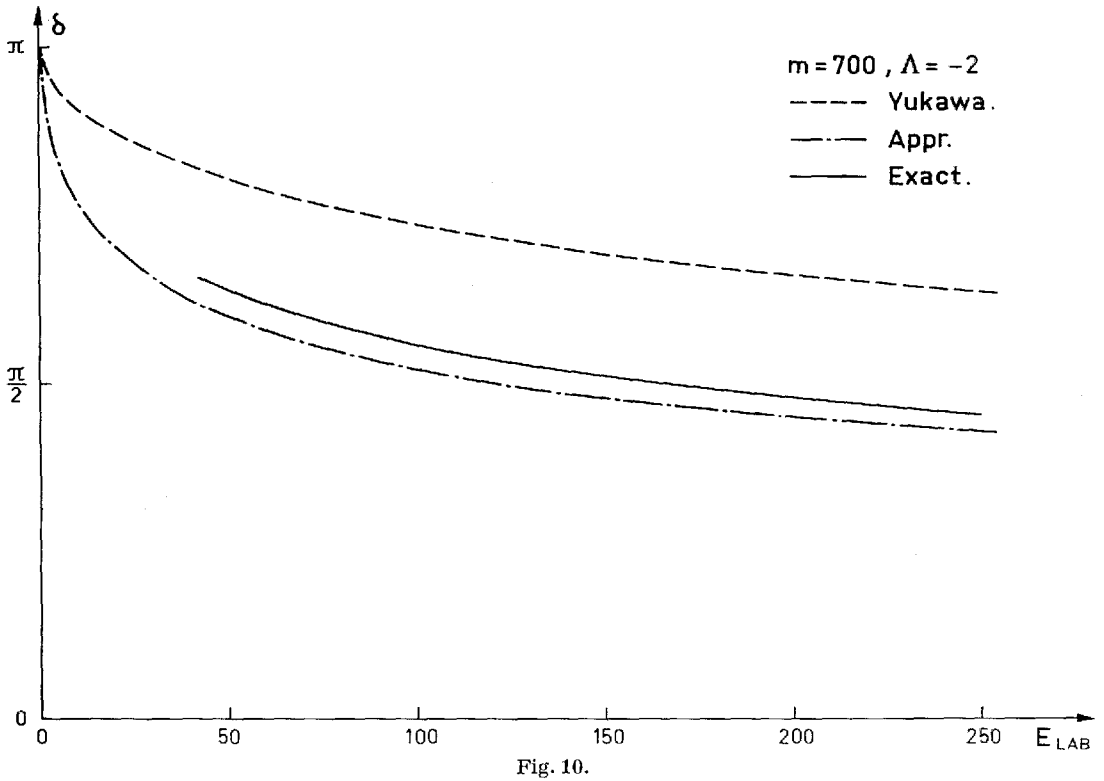


Fig. 10.

$$n(x)f(x,v) = \int_0^\infty dy m(y)g(y,v)\left[\frac{n(x)}{m(y)}J_v^2(xy)\right]. \qquad \text{(A. 7)}$$

When we demand that the kernel (the term within brackets) in (A. 7) be a function of $(xy)$, it follows

$$\frac{n(x)}{m(y)} = (xy)^a, \qquad \text{(A. 8)}$$

where $a$ is an arbitrary (real) number, which we have at our disposal. Hence $n(x) = x^a$ and $m(x) = x^{-a}$. Assume that $x^a f(x,v)$ is differentiable in $(0, \infty)$, when $a$ has a fixed value, which we subsequently determine.

Let $f_1(x,v) = \frac{d}{dx}x^a f(x,v)$ and let $g_1(x,v) = x^{1-a}g(x,v)$. Then we get

$$f_1(x,v) = \frac{d}{dx}\int_0^\infty \frac{dy}{y} g_1(y,v)(xy)^a J_v^2(xy). \qquad \text{(A. 9)}$$

We shall now have to determine the number $a$, so that the conditions of Theorem I are fulfilled for the kernel

$$k(x) = x^a J_v^2(x). \tag{A. 10}$$

The function $K(s)(1-s)^{-1}$ is the Mellin transform of $k(x)x^{-1}$

$$K(s)(1-s)^{-1} = \int_0^\infty dx x^{-r} J_v^2(x), \tag{A. 11}$$

where

$$r = 2 - s - a. \tag{A. 12}$$

The integral (A. 11) can readily be evaluated (W. 403)

$$K(s)(1-s)^{-1} = \frac{\Gamma(r)\Gamma\left(v + \frac{1}{2} - \frac{1}{2}r\right)}{2^r\left(\Gamma\left(\frac{1}{2} + \frac{1}{2}r\right)\right)^2 \Gamma\left(v + \frac{1}{2} + \frac{1}{2}r\right)}, \tag{A. 13}$$

provided

$$Re(r) > 0,\ Re(2v + 1 - r) > 0. \tag{A. 14}$$

The function $K(1/2 + it)$ defined by (A. 13) with $s = 1/2 + it$ is bounded and non-zero for bounded values of $|t|$. The asymptotic behaviour of $K(s)$ for large values of $t = Im(s)$ is

$$|K(s)| = |t|^{Re(s)+a-\frac{3}{2}}(1 + O(t^{-1})). \tag{A. 15}$$

Hence, in order that $K(s)$ be bounded on $Re(s) = 1/2$ for all $t$, we must choose

$$a = 1. \tag{A. 16}$$

As a consequence of the conditions (A. 14), we must have

$$Re(v) > -\frac{1}{4}. \tag{A. 17}$$

Defining $H(s)$ by eq. (A. 2), $K$(s) being given by (A. 13) with $a = 1$, and performing the integral (A. 4) we obtain

$$h(x) = -2\pi \int_0^x du u J_v(u) Y_v(u), \tag{A. 18}$$

where $Y_v(u)$ is the Bessel function of the second kind.

We have thus obtained the result;

If $xf(x,v)$ is differentiable in $(0, \infty)$, and if $(xf(x,v))' \equiv \frac{d}{dx}(xf(x,v)) \varepsilon L^2(0, \infty)$, uniformly with respect to $v$ within a closed domain to the right of the line $Re(v) = -\frac{1}{4}$, the equation

$$f(x,v) = \int_0^\infty dy g(y,v) J_v^2(xy) \tag{A. 19}$$

implies almost everywhere

$$g(x,v) = -2\pi \frac{d}{dx} \int_0^\infty \frac{dy}{y} (yf(y,v))' \int_0^{xy} du u J_v(u) Y_v(u), \tag{A. 20}$$

and $g(x,v)$ defined for $Re(v) > -\frac{1}{4}$ by (A. 20) also $\varepsilon L^2(0, \infty)$.

## Appendix II
## Reduction of the Lippmann-Schwinger equation

We write down the partial wave L.S.-equation, eq. (4.4)

$$T(k',k) = U(k',k) + \frac{2}{\pi} \int_0^\infty \frac{dk'' k'' U(k',k'') T(k'',k)}{k''^2 - k^2 - i\varepsilon} \tag{B. 1}$$

where for simplicity we have omitted the subscript $L$. We define the function

$$F(k',k) = \frac{T(k',k)}{1 + iT(k,k)} \tag{B. 2}$$

It is readily proved that for real non-negative values of $k$ and $k'$

$$arg T(k',k) = arg T(k,k). \tag{B. 3}$$

The function $F(k',k)$ is therefore a real function, which on the energy shell becomes

$$F(k,k) = tan(\delta(k)). \tag{B. 4}$$

Let us then write $F(k',k)$ in the form

$$F(k',k) = \frac{N(k',k)}{D(k)}, \tag{B. 5}$$

where

$$D(k) = 1 - \frac{2}{\pi} P \int_0^\infty \frac{dk'' k'' H(k,k'') N(k'',k)}{k''^2 - k^2}. \tag{B. 6}$$

The function $H(k,k'')$ in (B. 6) should be chosen so that $N$ and $D$ in (B. 5) have convenient properties. From (B. 1), (B. 2), (B. 5) and (B. 6) we obtain the equation for $N(k',k)$

$$N(k',k) = U(k',k) + \frac{2}{\pi} P \int_0^\infty \frac{dk'' k'' [U(k',k'') - U(k',k) H(k,k'')] N(k'',k)}{k''^2 - k^2}. \tag{B. 7}$$

If $H(k',k)$ is chosen to be unity for $k' = k$, the kernel in (B. 7) becomes finite at $k'' = k$, provided $U(k',k)$ satisfies the appropriate Lipschitz condition. If we choose

$$H(k,k'') = \frac{U(k,k'')}{U(k,k)}, \tag{B. 8}$$

we obtain from eq. (B. 7), apart from nugatory changes in the notation, the equation recently discussed by H. P. Noyes[17]. With this choice we force the function $N(k',k)$ to have poles, as function of $k$, at those points where the phase shift goes through an integral multiple of $\pi$. Instead of (B. 8) we therefore take[18]

$$H(k,k'') = 1. \tag{B. 9}$$

We then have

$$N(k',k) = U(k',k) + \frac{2}{\pi} \int_0^\infty \frac{dk'' k'' [U(k',k'') - U(k',k)] N(k'',k)}{k''^2 - k^2} \tag{B. 10}$$

and, from (B. 4) and (B. 5),

$$tan(\delta(k)) = \frac{N(k,k)}{1 - \frac{2}{\pi} P \int_0^\infty \frac{dk'' k'' N(k'',k)}{k''^2 - k^2}}. \tag{B. 11}$$

The phase shifts given in section 6 are obtained by solving a slightly modified version of eq. (B. 10), with $U(k',k)$ given by

$$U(k',k) = -\frac{\Lambda Q_0\left(\dfrac{k^2+k'^2+\left(\dfrac{m}{M}\right)^2}{2kk'}\right)}{2k\sqrt[4]{k^2+1}\,\sqrt[4]{k'^2+1}} \qquad \text{(B. 12)}$$

The method used for solving the equation is the well known matrix inversion method.

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