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A COMMENT ON LANDAU'S METHOD OF INTEGRATION IN QUANTUM ELECTRODYNAMICS

BY

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Synopsis.

It is shown that the integration method of LANDAU et al. is inconsistent even in the energy region below the cut-off. This inconsistency is present also when no ghost states appear. This result implies that Landau's method is not self-consistent even for a finite cut-off.

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§ 1. Introduction and Summary.

Notwithstanding the brilliant success of the renormalization theory of quantum electrodynamics in explaining the experimental results with great accuracy, doubts have been raised from various sides whether or not the theory contains some logical inconsistencies. Thus, it was shown by KÄLLÉN[1] that at least one of the renormalization constants is infinite in magnitude. Even if one connives at this point, there still remain some questions concerning the finite part of the theory, e. g., the problem of the so-called ghost state. In connection with the Lee model[2], KÄLLÉN and PAULI were the first to point at this question, and along these lines many other authors have discussed the mathematical consistency of renormalized quantum electrodynamics[3].

Independently of this approach, LANDAU and others[4] have concluded that difficulties similar to those met with in Lee's model also appear in quantum electrodynamics as long as the concept of point interaction is used. The same conclusion has been reached by other workers in this field who started from somewhat different viewpoints[5]. One very important point in Landau's approach to the consistency problem of quantum electrodynamics lies in the expansion of a certain quantity, the so-called "vertex part", in a power series in e^2. It turns out that every coefficient in this expansion has a very simple limiting form for high energies, and it is then argued that the limiting form of the vertex part itself is given as the sum of the limiting forms of the coefficients. Mathematically, such a conclusion is certainly not very well justified, and counterexamples can easily be given. It is the aim of the present paper to investigate whether or not Landau's method is selfconsistent in spite of these mathematical objections.

To this end, it is very helpful to follow the interesting argument as to the mathematical consistency of this theory, developed by KÄLLÉN in his general consideration[6] developed in a recent lecture in Geneva. By introducing some assumptions on the asymptotic form of the current operator, he has shown that the theory might contain internal inconsistencies of a kind different from the difficulty connected with the ghost state. In the case of Landau's approximation, however, no further assumptions of this kind need to be introduced, since we can take full advantage of his fundamental assumptions from which sufficient information can be derived about the asymptotic behaviour of the current operator as well as a very simple relation between the bare and the renormalized charges. It will appear that the functional equation for the current operator, which is required from the invariance property of the theory under the renormalization transformation, can easily be solved by virtue of Landau's condition, and it turns out that the result is essentially what was conjectured by Källén.

Thus, in accordance with Källén's argumentation, we are finally led to some results contradicting the premises on which our arguments are based. Consequently, we have to conclude that one of the following alternative statements, or both of them simultaneously, are valid.

(i) Landau's approximation is incorrect;

(ii) The theory in itself contains an inconsistency independent of that found in Landau's argumentation and, thus, the present quantum field theory has no mathematically consistent solution at all.

In order to remove from Landau's theory the difficulty connected with the appearance of a ghost state, we have first to introduce a cut-off factor into our formalism and then to consider the consistency of the resulting theory. However, the following question arises immediately. Since a cut-off has been introduced, we are no longer left with the canonical formalism as usual, because we have modified the canonical commutation relations. Does it therefore make any sense to discuss further points of the theory? From the viewpoint of correspondence principle, however, we should expect that those elements in the present theory, where intimate correspondence with reality is established and where the results are strongly supported by experimental evidence, might still survive in a future theory. Our investigation has to be considered from this point of view.

In § 2, the main results of Landau's approximation are summarized and, in § 3, the asymptotic forms of the current operator are discussed. In § 4, we complete our arguments, using the results obtained in the preceding sections, and give some further discussion related to our conclusion.

§ 2. Survey of Landau's Method.

For large momenta $|p^2| \gg m^2$, m being the mass of the electron, the approximation method of Landau et al.[4)] consists in expanding any quantity in a series of the form

$$e_0^n \sum_{m=0}^{\infty} (e_0^2)^m f_m \left[e_0^2 \log \frac{|p^2|}{\Lambda^2} \right] \tag{1}$$

and in retaining only the first term (corresponding to m = 0) in this series. Here, n is the order of the first non-vanishing term in perturbation theory and Λ is the cut-off momentum. A characteristic feature of this approximation method lies in the point that the square of the charge e_0^2 is always accompanied by the logarithm of the cut-off parameter.

The most important result with which we shall be concerned in the following is the relation between the renormalized charge e and the bare one e_0 which is given by

$$e^2 = Z_3(e^2, \Lambda^2)\, e_0^2 = \left(1 - \frac{e^2}{3\pi} \log \frac{\Lambda^2}{m^2}\right) e_0^2 \equiv f(\Lambda^2)\, e_0^2. \qquad (2)^1$$

The relation (2) has also been derived by Taylor who solved a functional equation, required from the renormalization transformation, on the assumption that the cut-off parameter always appears as a product

$$e^2 \log \Lambda^2/m^2 .^{5)}$$

As is immediately seen from (2), the cut-off parameter Λ^2 must be smaller than $\Lambda_c^2 = m^2 \exp\,(3\pi \times 137)$ so that we can avoid the difficulty connected with the ghost state. Consequently, we have to work within the energy region $|p^2| < \Lambda^2$.

Now, it is well known that Dyson's integral equation for the photon propagator[7)] can be written in the form

$$D_F'(p)_{\mu\nu} = D_F(p)\,\delta_{\mu\nu} + D_F(p)\, \Pi_F(p)_{\mu\nu}\, D_F(p). \qquad (3)^2$$

Here, the causal kernel $\Pi_F(p)$ is given by

$$\Pi_F(p) = i \int d^4x e^{ip(x-x')} \langle P(j_\mu(x), j_\nu(x')) \rangle_0, \qquad (4)^2$$

[1] Unless otherwise stated, notations and definitions are the same as in our previous papers, references 5) and 11).

[2] We are referring to the unrenormalized form.

where $j_\mu(x)$ means the current operator in the Heisenberg representation[8]. It is also convenient to make use of Källén's kernel defined by

$$\Pi_K(p) = \frac{V}{-3p^2} \sum_{p(z)=p} \langle 0 | j_\nu | z \rangle \langle z | j_\nu | 0 \rangle . \tag{5}$$

These functions are connected with each other through the relation

$$\left. \begin{aligned} \Pi_F(p)_{\mu\nu} &= \int_0^{\Lambda^2} da\, \Delta_F(p^2, a)\, \Pi_K(-a)(-p^2\delta_{\mu\nu} + p_\mu p_\nu) \\ &= \int_0^{\Lambda^2} da \left[\frac{P}{p^2+a} + i\pi\delta(p^2+a) \right] \Pi_K(-a)(-p^2\delta_{\mu\nu} + p_\mu p_\nu), \end{aligned} \right\} \tag{6}$$

where $\Delta_F(p^2, a)$ means the free propagator for a particle of mass a. As is easily seen, the above relation meets the requirement of causality.[1]

On the other hand, as a general consequence of the renormalization cut-off[11], we obtain from (2) the asymptotic form of $\Pi_F(p)$, i. e. for large $-p^2 \gg m^2$ it behaves like

$$\Pi_F(p)_{\mu\nu} = \frac{\frac{e^2}{3\pi}\log\frac{\Lambda^2}{-p^2}}{1-\frac{e^2}{3\pi}\log\frac{-p^2}{m^2}}(-p^2\delta_{\mu\nu} + p_\mu p_\nu). \tag{7}^2$$

The same result was obtained by LANDAU et al. by solving directly an integral equation.

We can proceed to discuss the properties of Π_K and Π_F by means of (5) and (6), only if we know something about the matrix elements of the current operator $\langle 0 | j_\mu | z \rangle$. Therefore, we shall next investigate the asymptotic behaviour of this matrix element.

§ 3. Asymptomic Behaviour of the Current Operator.

Let us start by recapitulating the main points concerning the discussion of the asymptotic forms of the S-matrix presented in the paper by KONUMA

[1] Rigorously speaking, the relation (6) holds only when $\Lambda^2 = \infty$. STUECKELBERG and WANDERS[10] have shown that, if we expect the cut-off factor to be a result of smearing effects of non-local interactions, the causality relation (6) does no longer hold. Nevertheless, we have to adopt this relation for the reason mentioned at the end of § 1.

Here, we use, for simplicity, the straight cut-off. The discussion in § 3, however, is essentially based on the renormalization cut-off, which we use as a mathematical tool only to obtain the asymptotic behaviour. Both methods give the same result in the asymptotic region[5, 11].

[2] While, in the references 4) and 5), the relations (2) and (7) were derived for the case of $p^2 > 0$ (space-like), we get the same result for $p^2 < 0$ (time-like) by applying the analytic continuation as suggested in reference 4) or by using the method of renormalization cut-off developed in ref. 5).

and UMEZAWA[12]. These authors showed that, from the invariance under the renormalization transformation (or renormalization group[13]), an element of the S-matrix is required to satisfy the following functional equation. For the transition amplitude such that its non-vanishing term first appears in the n^{th} order in perturbation expansion, the functional equation reads

$$\left.\begin{aligned} Z(\Lambda^2)\, g_0^n\, F\left(-\frac{k_i k_j}{m^2}, \ldots; -\frac{k_i k_j}{\Lambda^2}, \ldots; g_0^2\right) \\ = g^n\, F\left(-\frac{k_i k_j}{m^2}, \ldots; -\frac{k_i k_j}{m^2}, \ldots; g^2\right), \end{aligned}\right\} \tag{8}$$

where g_0 and g are the bare and the renormalized charges, respectively, and Z is a product of $Z_2^{1/2}$ and $Z_3^{1/2}$, factors which come from respective external lines of momenta k_i's.

KONUMA and UMEZAWA have shown quite generally how to solve this equation in the asymptotic region if we accept relation (2). Since we now are concerned with matrix elements of the current operator $\langle 0 | j | z \rangle$, their arguments should be somewhat generalized in two points.

In the first place, it should be noted that, in the derivation of equation (8), it was most essential to realize that the charge renormalization could be carried out in the usual way, by extracting factors like Z. In the case of the general matrix elements of Heisenberg operators, the renormalization procedure is somewhat complicated in view of the fact that the "doubled Feynman diagram"[14], has to be used. On the other hand, for the special elements which are considered here, i. e. $\langle 0 | j | z \rangle$, the situation is not more complicated than for the S–matrix. In this case, the matrix element is equal to that obtained in the so-called mixed representation[7], viz. $\langle 0 | j_H | z \rangle = \langle 0 | S^{-1} j_F | z \rangle = \langle S0 | j_F | z \rangle = \langle 0 | j_F | z \rangle$. The last equality is due to the fact that $S | 0 \rangle = | 0 \rangle$ except for a constant phase factor which is to be removed by renormalization.

Secondly, it has to be kept in mind that the quantity which appears in (8) is not the matrix element $\langle 0 | j | z \rangle$ itself, but the dimensionless scalar quantity $-\frac{V}{3p^2} Z_3^{-1} \sum_\mu |\langle 0 | j_\mu | z \rangle|^2$, since this quantity can be written as $[Z_3^{-1/2} D_F'(p^2)\, p^2\, e_0^{\nu+1} Z_3^{\nu/2}]^2\, h(S_F', D_F', \Gamma_\mu', e_0^2) = [D_{Fc}(p^2)\, p^2\, e^{\nu+1}]^2\, h(S_{Fc}, D_{Fc}, \Gamma_{\mu c}, e^2)$[14], where the first factor is the contribution from the external vertex.

Taking into account the above two points, we can set up the functional equation for the matrix element between the vacuum and the state $| z \rangle$ of ν-photons present with momenta k_i's ($i = 1, 2, \ldots \nu$; $\nu \geqslant 3$) in the following form:

$$\left.\begin{aligned}&[Z_3^{-1/2}\, e_0^{\nu+1}\, Z_3^{\nu/2}]^2 F\left(-\frac{k_i k_j}{m^2}, \ldots; -\frac{k_i k_j}{\Lambda^2}, \ldots; e_0^2\right)\\ &= (e^{\nu+1})^2 F\left(-\frac{k_i k_j}{m^2}, \ldots; -\frac{k_i k_j}{m^2}, \ldots; e^2\right), \; (i \neq j;\, p_\mu = \sum_i k_{i\mu}),\end{aligned}\right\} \quad (9)^1$$

where we have put $\frac{V}{-3p^2} Z_3^{-1} \sum_\mu |\langle 0 | j_\mu | z \rangle|^2 = [Z_3^{-1/2} e_0^{\nu+1} Z_3^{\nu/2}]^2 F.$
By virtue of relation (2), this formula may be rewritten in the form

$$Z_3^{-2} F\left(-\frac{k_i k_j}{m^2}, \ldots; -\frac{k_i k_j}{\Lambda^2}, \ldots; e_0^2\right) = F\left(-\frac{k_i k_j}{m^2}, \ldots; -\frac{k_i k_j}{m^2}, \ldots; e^2\right). \quad (10)$$

Now, it is rather complicated, though not impossible, to find the general solution to this functional equation, since many arguments $k_i\, k_j$ are contained in F. For our present purpose, however, it is necessary only to look for the expression for the sum of F over all the possible states of ν-photons which satisfy the condition $p_\mu = \sum_i k_{i\mu}$. If we put $\mathfrak{J} = \sum_{p=\Sigma k} F$, then $\mathfrak{J}$, now being a function of m^2, e_0^2, and $-p^2$ only, should satisfy the following simplified equation:

$$Z_3^{-2}\, \mathfrak{J}\left(-\frac{p^2}{m^2}; -\frac{p^2}{\Lambda^2};\; e_0^2\right) = \mathfrak{J}\left(-\frac{p^2}{m^2};\; -\frac{p^2}{m^2};\; e^2\right). \quad (11)^2$$

When solving this equation, it should be remembered that the Landau approximation makes the cut-off parameter appear together with the bare charge e_0 in the combination $e_0^2 \log \frac{|p^2|}{\Lambda^2}$ in the asymptotic region. Therefore, bearing in mind relation (2), we can rewrite (11) in the form

$$f(\Lambda^2)^{-2}\, \mathfrak{J}\left(-\frac{p^2}{m^2}; \frac{f(-p^2)}{f(\Lambda^2)}\right) = \mathfrak{J}\left(-\frac{p^2}{m^2}; f(-p^2)\right), \quad (12)$$

[1] In our case, F depends only on a single parameter Λ which comes from the cut-off of the photon propagation function. It is also to be remarked that exactly speaking, in the last argument in F on the right hand side of (9), we must substitute the charge e_m defined by $e_m^2/e^2 = k^2 D_{Fc}\left(\frac{k^2}{m^2}, e^2\right)\Big|_{|k^2|=m^2}$. But, under our approximation, $k^2 D_{Fc} = 1$ in the domain $0 \le |k^2| \le m^2$ and so we can put $e_m^2 = e^2$. See in this connection the third paper of reference 4).

[2] The lower limit of each photon energy is conveniently taken as the electron mass, so that the infrared divergence may be neglected. (Otherwise, another dimensional constant has to be introduced into F or $\mathfrak{J}$).

where we have used the relations $\frac{e^2}{f(\Lambda^2)}\log\frac{-p^2}{\Lambda^2} = 3\pi\left(1-\frac{f(-p^2)}{f(\Lambda^2)}\right)$ and $e^2\log\frac{-p^2}{m^2} = 3\pi\ (1-f(-p^2))$ and where we have, moreover, redefined the function $\mathfrak{F}$ in such a way that $\mathfrak{F}(x;\,3\pi(1-y)) \to \mathfrak{F}(x;\,y)$. Equation (12) clearly show sthat the only possible solution $\mathfrak{F}$ is given by

$$\mathfrak{F}\left(-\frac{p^2}{m^2};\,f\right) = \mathfrak{F}_0\left(-\frac{p^2}{m^2}\right)\times f^{-2}. \tag{13}$$

Here, it should be noted that the function $\mathfrak{F}_0$ does no longer contain the charge e^2 and, thus, is nothing but the so-called Born approximation. By this we mean conventionally the first non-vanishing term in the perturbation theory expansion.

In summarizing our results obtained so far, we get the following equation:

$$\frac{V}{-3p^2}\sum_z \langle 0|j_\mu|z\rangle\langle z|j_\mu|0\rangle = \frac{V}{-3p^2}\frac{f(\Lambda^2)}{f(-p^2)^2}\sum_z\langle 0|j_\mu^{Born}|z\rangle\langle z|j_\mu^{Born}|0\rangle \quad (14)^1$$

[1] In some cases, it is still possible to get the concrete asymptotic expression for the current operator itself.

The most general matrix element $\langle 0|j|z\rangle$, for the state in which c pairs of electrons and ν-photons are present, satisfies the following functional equation:

$$\left[Z_3^{-1/2}e_0^{\nu+2c-1}Z_3^{\nu/2}Z_2^c\right]^2 F\left(-\frac{k_ik_j}{m^2},\ldots;-\frac{k_ik_j}{\Lambda^2},\ldots;e_0^2\right)$$
$$= \left[e^{\nu+2c-1}\right]^2 F\left(-\frac{k_ik_j}{m^2},\ldots;-\frac{k_ik_j}{m^2},\ldots;e^2\right).$$

This equation can easily be solved in the case that one of the arguments $-k_ik_j$, say, is extremely large as compared with others, so that $-k_ik_j \approx -p^2$. The final expression then reads

$$\langle 0|j_\mu|z\rangle = \frac{f(\Lambda^2)^{1/2}}{f(-p^2)^c}Z_2(-p^2)^c\langle 0|j_\mu^{Born}|z\rangle.$$

Here, it is interesting to compare the result with the conjecture by KÄLLÉN in his general discussion (without any approximation) [6])

$$\langle 0|j_\mu|z\rangle = \frac{1}{f(-p^2)^{n/2}}Z_2(-p^2)\langle 0|j_\mu^{Born}|z\rangle.$$

In this connection, it should further be remarked that the above functional equation has this solution if, and only if, the function $Z_2(x)$ is equal to some power of the function $f(x)$ or $Z_3(x)$. Landau's approximation meets this requirement in a special gauge, for $Z_2 = f^0 = 1$. See also reference 15).

§ 4. Completion of the Argument. Discussion.

We are now in a position to complete our argument. If we restrict the summation $|z\rangle$ in (5) to the states considered above, and substitute (14) into it, then we obtain an inequality for Π_K.

It has already been shown by KÄLLÉN that the absolute square of $\langle 0 | j_\mu^{Born} | k_1 k_2 \ldots k_\nu \rangle$ gives the probability in lowest order perturbation theory[4], for the emission of ν-photons in a weak external field; thus, Gupta's[16] result can be used and gives the following expression in the asymptotic region

$$\frac{V}{-3p^2} \sum_{k_1 - k_\nu} \langle 0 | j_\mu^{Born} | k_1 \ldots k_\nu \rangle \langle k_\nu \ldots k_1 | j_\mu^{Born} | 0 \rangle = \frac{\alpha^\nu}{\nu !} \left[\log \frac{-p^2}{m^2} \right]^{2\nu} c \tag{15}$$

with a constant c which is of the order of magnitude unity.

By means of (5), (14), and (15) we now get the following inequality for Π_K

$$\Pi_K(p^2) > e^2 \frac{f(\Lambda^2)}{f(-p^2)} c \sum_{\nu = 3, 5, }^{\overline{N}} \frac{\alpha^\nu}{\nu !} \left[\log \frac{-p^2}{m^2} \right]^{2\nu}, \tag{16}$$

where $\overline{N}$, the upper limit of the sum, is not larger than $\sqrt{-p^2/E_0^2}$ with $E_0 \approx m$. If $\overline{N}$ and $\alpha \log^2 \left(\frac{-p^2}{m^2} \right)$ are very large, (16) may be rewritten in the form

$$\Pi_K(p^2) > \frac{c}{2} \frac{e^2 f(\Lambda^2)}{\left(1 - \frac{e^2}{3\pi} \log \frac{-p^2}{m^2}\right)^2} \left(\frac{-p^2}{m^2} \right)^{\alpha \log \left(\frac{-p^2}{m^2} \right)}. \tag{17}$$

From the relation (6), together with (16) and (17), we can immediately deduce the following results. Let us first consider the real part of $\Pi_F(p^2)$. If we take $\Lambda^2 \gg -p^2 \gg m^2$ we obtain the cut-off dependent term which increases much faster than any power of $\log \left(\frac{\Lambda^2}{m^2} \right)$, because in this case we have $\int^{\Lambda^2} \frac{da}{p^2 + a} \left(\frac{a}{m^2} \right)^{\alpha \log \frac{a}{m^2}} \approx \int^{\Lambda^2/m^2} \frac{dx}{x} e^{\alpha \log^2 x} \approx \left(1/\alpha \log \frac{\Lambda^2}{m^2} \right) e^{\alpha \log^2 \frac{\Lambda^2}{m^2}}$.

Secondly, we look at the imaginary part of $\Pi_F(p^2)$. If $-p^2 \gg m^2$ we find that a non-negligible imaginary part must exist, which strongly depends on $-p^2$ and which satisfies the same inequality as do (16) and (17). Therefore, it appears that the absolute value of $\Pi_F(p^2)$ must be very much larger than the value given by (7) (as far as the region $\Lambda_c^2 \gg \Lambda^2 > -p^2 \geqslant e^{100} m^2$ is concerned)[1].

These results are inconsistent with (2) and (7) which have been derived from the same starting point as (17). This puzzling situation can be explained if, and only if, Landau's approximation is incorrect. In spite of this, it is, of course, possible that other difficulties, independent of those found by LANDAU, exist within the framework of present field theory.

If the first alternative is true, it clearly means that there are appreciable contributions from terms neglected in Landau's approximation. This is not so inconceivable since, in the n^{th} order of perturbation expansion, we get, roughly speaking, a contribution $e^{2n}[n!\, c_0 + (n-1)!\, c_1 \log(\Lambda^2/m^2) + \ldots + c_n \log^n(\Lambda^2/m^2)]$ and all the terms, with the exception of the last one, are neglected in his approximation. The ratio of the former to the latter is approximately $n! / \log^n(\Lambda^2/m^2)$ or at least $n! / (137 \times 3\pi)^n$. Consequently, even if the approximation is good up to some orders, it would no longer be justified in higher orders, however large the cut-off may be. In other words, here it is not allowed to interchange the two kinds of limiting processes, viz. $\lim\limits_{\Lambda \to \infty} \lim\limits_{n \to \infty}$ and $\lim\limits_{n \to \infty} \lim\limits_{\Lambda \to \infty}$. The latter limit has recently been studied in great detail by several groups[5]. On the other hand, it was pointed out by KÄLLÉN[6] that our result (16) is consistent with every result obtained in perturbation theory.

Furthermore, if the second alternative is taken to be true, it would necessarily lead to the conclusion that present quantum field theory has no mathematically consistent solution at all.

Finally we like to add a few words about (ps) (ps) meson theory[5], [17]. From the outset, it is evident that here Landau's approximation is not so powerful as in quantum electrodynamics. In this case, the cut-off momentum is of the order of the nucleon mass M because of the large value of the coupling constant. Consequently, we are no longer left with any asymptotic region $-p^2 \gg M^2$ in which this approximation is applicable. However, we might still make a formal argument, artificially supposing the coupling constant to be small and therefore using the same technique. If that is done, we get the same result for meson theories as for quantum electrodynamics.

[1] As far as the renormalized kernel π_{Fc} is concerned, the above statement is true, irrespective of the region of Λ. (Landau's π_{Fc} is given by (7), but with replacement of $\Lambda^2 \to m^2$).

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