ON THE PROBLEM OF
THREE BODIES IN THE PLANE

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København 1957
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Synopsis.

Three bodies with finite masses are assumed to move in a plane, subject to Newton's law of gravitation. By the introduction of suitable auxiliary variables the equations of motion are transformed into a system of differential equations of the second degree, permitting to expand the unknown quantities in powers of the time $t$, the coefficients of $t^n$ being calculated by means of a set of recurrence formulas. Sufficient conditions for the convergence of the resulting series are given, and the practical working of the method is illustrated by a numerical example.
1. On a former occasion I have shown how a particular case of the Problem of Three Bodies can be dealt with by transforming the equations of motion into a system of differential equations of the second degree in the unknown variables, permitting to expand these in powers of the time $t$, the coefficients of $t^r$ being calculated by a set of recurrence formulas. The same method can, in principle, be employed in other cases of the dynamical astronomy, and I propose in the present paper to extend it to the problem of three finite bodies moving in the same fixed plane and subject to Newton's law of gravitation. The number of recurrence formulas naturally increases, but without becoming unwieldy, as will be shown by a numerical example.

Let the three masses be $m_1$, $m_2$ and $m_3$, the coordinates of $m_i$ being $(x_i, y_i)$, and let us put

\[
\begin{align*}
  r_1^2 &= (x_2 - x_3)^2 + (y_2 - y_3)^2 \\
  r_2^2 &= (x_3 - x_1)^2 + (y_3 - y_1)^2 \\
  r_3^2 &= (x_1 - x_2)^2 + (y_1 - y_2)^2
\end{align*}
\]

so that $r_1$ is the distance between $m_2$ and $m_3$, etc.

Then the equations of motion are

\[
\begin{align*}
  \frac{d^2 x_1}{dt^2} &= m_2 \frac{x_2 - x_1}{r_1^3} + m_3 \frac{x_3 - x_1}{r_2^3} \\
  \frac{d^2 x_2}{dt^2} &= m_3 \frac{x_3 - x_2}{r_2^3} + m_1 \frac{x_1 - x_2}{r_3^3} \\
  \frac{d^2 x_3}{dt^2} &= m_1 \frac{x_1 - x_3}{r_3^3} + m_2 \frac{x_2 - x_3}{r_1^3}
\end{align*}
\]

and corresponding equations with $y$ instead of $x$.


2 It is assumed throughout that none of the distances $r_1$, $r_2$, $r_3$ vanishes.
We now introduce for \( i = 1, 2, 3 \) the auxiliary variables
\[ q_i = r_i^2, \quad \sigma_i = r_i^3, \]
so that
\[ 2q_i q_i' \sigma_i + 3q_i \sigma_i' q_i = 0, \]

\[
\begin{align*}
q_1 &= (x_2 - x_3)^2 + (y_2 - y_3)^2 \\
q_2 &= (x_3 - x_1)^2 + (y_3 - y_1)^2 \\
q_3 &= (x_1 - x_3)^2 + (y_1 - y_3)^2
\end{align*}
\]

while the equations of motion become
\[
\begin{align*}
\frac{d^2 x_1}{dt^2} &= m_3 (x_2 - x_1) \sigma_3 + m_3 (x_3 - x_1) \sigma_2 \\
\frac{d^2 x_2}{dt^2} &= m_3 (x_3 - x_2) \sigma_1 + m_1 (x_1 - x_2) \sigma_2 \\
\frac{d^2 x_3}{dt^2} &= m_1 (x_1 - x_3) \sigma_2 + m_3 (x_2 - x_3) \sigma_1
\end{align*}
\]

and corresponding equations with \( y \) instead of \( x \).

For the determination of the 12 unknowns \( x_i, y_i, \sigma_i, q_i \) we now have the 12 equations (4), (5), (6) and the corresponding equations in \( y \), which are all of the second degree in the unknowns and can be treated in the way indicated above.

2. If, however, only the distances of the masses from each other at any given time are required, the number of equations can be reduced to 10. In that case the absolute positions in the plane can be determined afterwards, if desired. This is the relativistic point of view, familiar from the treatment of the Restricted Problem of Three Bodies. In the present case we put
\[
\xi_1 = x_2 - x_3, \quad \xi_2 = x_3 - x_1, \\
\eta_1 = y_2 - y_3, \quad \eta_2 = y_3 - y_1
\]

and for abbreviation
\[ M_1 = m_2 + m_3, \quad M_2 = m_1 + m_3. \]

We then obtain from (5)
\[
\begin{align*}
q_1 &= \xi_1^2 + \eta_1^2, \\
q_2 &= \xi_2^2 + \eta_2^2 \\
q_3 &= q_1 + q_2 + 2 \xi_1 \xi_2 + 2 \eta_1 \eta_2
\end{align*}
\]
and from (6)
\[
\frac{d^2 \xi_1}{dt^2} = m_1 (\xi_2 \sigma_3 - \xi_1 \sigma_3 - \xi_2 \sigma_3) - M_1 \xi_1 \sigma_1
\]
\[
\frac{d^3 \xi_2}{dt^2} = m_2 (\xi_1 \sigma_1 - \xi_1 \sigma_3 - \xi_2 \sigma_3) - M_2 \xi_2 \sigma_3
\]

and, replacing \( \xi \) by \( \eta \) in this,
\[
\frac{d^2 \eta_1}{dt^2} = m_1 (\eta_2 \sigma_3 - \eta_1 \sigma_3 - \eta_2 \sigma_3) - M_1 \eta_1 \sigma_1
\]
\[
\frac{d^3 \eta_2}{dt^2} = m_2 (\eta_1 \sigma_1 - \eta_1 \sigma_3 - \eta_2 \sigma_3) - M_2 \eta_2 \sigma_3.
\]

(4) and (9)-(11) are 10 equations for determining the 10 unknowns \( \xi_1, \xi_2, \eta_1, \eta_2, \sigma_1, \sigma_2 \). We propose to satisfy them by power series in \( t \), putting
\[
\xi_1 = \sum \alpha_v t^v, \quad \xi_2 = \sum \beta_v t^v
\]
\[
\eta_1 = \sum \gamma_v t^v, \quad \eta_2 = \sum \delta_v t^v
\]
\[
\sigma_1 = \sum \rho_v t^v, \quad \sigma_2 = \sum \sigma_v t^v
\]
\[
\sigma_3 = \sum \sigma_v t^v
\]
the summation being everywhere from \( v = 0 \) to \( v = \infty \).

Inserting these expansions in the aforesaid equations and demanding that the coefficients of \( t^v \) shall vanish, we obtain recurrence formulas for the determination of the coefficients.

We write for abbreviation
\[
\varepsilon_v = \alpha_v + \beta_v, \quad \zeta_v = \gamma_v + \delta_v
\]
and for the product-sums
\[
(\alpha d)_n = \sum_{v=0}^{n} \alpha_v d_{n-v}, \text{ etc.}
\]

In this notation we obtain from (10) and (11)
\[
(n+2)^{th} \alpha_{n+2} = m_1 [(\beta \epsilon)_n - (\epsilon f)_n] - M_1 (\alpha d)_n
\]
\[
(n+2)^{th} \beta_{n+2} = m_2 [(\alpha d)_n - (\epsilon f)_n] - M_2 (\beta \epsilon)_n
\]
\[
(n+2)^{th} \gamma_{n+2} = m_1 [(\delta \epsilon)_n - (\xi f)_n] - M_1 (\gamma d)_n
\]
\[
(n+2)^{th} \delta_{n+2} = m_2 [(\gamma d)_n - (\xi f)_n] - M_2 (\delta \epsilon)_n
\]
where \((n+2)^{th}\) as usual is short for \((n+2)(n+1)\).
Further, we obtain from (9) 
\[
\begin{align*}
\alpha_n &= (x\alpha)_n + (\gamma\gamma)_n \\
\beta_n &= (\beta\beta)_n + (\delta\delta)_n \\
\gamma_n &= \alpha_n + \beta_n + 2(x\beta)_n + 2(\gamma\delta)_n
\end{align*}
\]
and finally from (4) 
\[
\begin{align*}
-2n\alpha_0 \delta_n &= \sum_{v=0}^{n-1} (3n-v) \alpha_0 \sigma_{n-v} \\
-2n\beta_0 \gamma_n &= \sum_{v=0}^{n-1} (3n-v) \beta_0 \sigma_{n-v} \\
-2n\gamma_0 \delta_n &= \sum_{v=0}^{n-1} (3n-v) \gamma_0 \sigma_{n-v}.
\end{align*}
\]  

3. The number of constants of integration in (10) and (11), where the \( \sigma_i \) are known functions of the \( \xi_i \) and \( \eta_i \), is only 8 instead of 12 in the original statement of the problem. It is natural to choose as initial values the values of \( \xi_i, \eta_i, \frac{d\xi_i}{dt} \text{ and } \frac{d\eta_i}{dt} \) for \( t = 0 \), that is 
\[
\sigma_0, \alpha_1, \beta_0, \beta_1, \gamma_0, \gamma_1, \delta_0, \delta_1.
\]

We then obtain first from (17) 
\[
\begin{align*}
\alpha_0 &= \alpha_0^2 + \gamma_0^2 \\
\beta_0 &= \beta_0^2 + \delta_0^2 \\
\gamma_0 &= -\alpha_0 + \beta_0 - 2(\alpha_0 \beta_0 + 2\gamma_0 \delta_0)
\end{align*}
\]  
while the relation \( \sigma_i^2 \xi_i^3 = 1 \), resulting from (3), yields, \( \sigma_i \) and \( \varphi_i \) being positive, 
\[
d_0 = \frac{1}{a_0^2 \alpha_0^2}, \quad e_0 = \frac{1}{b_0^2 b_0}, \quad f_0 = \frac{1}{c_0^2 c_0}.
\]

After this we find by (17) and (18) 
\[
\begin{align*}
\alpha_1 &= 2(\alpha_0 \alpha_1 + \gamma_0 \gamma_1) \\
\beta_1 &= 2(\beta_0 \beta_1 + \delta_0 \delta_1) \\
\gamma_1 &= \alpha_1 + \beta_1 + 2(x\beta)_1 + 2(\gamma\delta)_1.
\end{align*}
\]  

\[
-2\alpha_0 d_1 = 3d_0 a_1, \quad -2\beta_0 e_1 = 3e_0 b_1, \quad -2\gamma_0 f_1 = 3f_0 c_1.
\]
The following coefficients are calculated in succession by (16)–(18). The first few of them are

\[ 2 \alpha_2 = m_1 (\beta_0 e_0 - e_0 f_0) - M_1 a_0 d_0 \]
\[ 2 \beta_2 = m_2 (a_0 d_0 - e_0 f_0) - M_2 \beta_0 e_0 \]
\[ 2 \gamma_2 = m_1 (\delta_0 e_0 - \zeta_0 f_0) - M_1 \gamma_0 d_0 \]
\[ 2 \delta_2 = m_2 (\gamma_0 d_0 - \zeta_0 f_0) - M_2 \delta_0 e_0 \]
\[ 2 \alpha_3 = \alpha_1^2 + \gamma_1^2 + 2 (x_0 a_2 + y_0 \gamma_2) \]
\[ 2 \beta_3 = \beta_1^2 + \delta_1^2 + 2 (\beta_0 \beta_2 + \delta_0 \delta_2) \]
\[ 2 \gamma_3 = \gamma_1^2 + \delta_1^2 + 2 (\gamma_0 \gamma_2) \]
\[ 2 \delta_3 = \delta_1^2 + \gamma_1^2 + 2 (\gamma_0 \gamma_2) \]
\[ \alpha_3 = 2 (x_0 a_3 + x_1 a_2 + y_0 \gamma_3 + y_1 \gamma_2) \]
\[ \beta_3 = 2 (\beta_0 \beta_3 + \beta_1 \beta_2 + \delta_0 \delta_3 + \delta_1 \delta_2) \]
\[ \gamma_3 = 2 (\gamma_0 \gamma_3 + \gamma_1 \gamma_2) \]
\[ \delta_3 = 2 (\delta_0 \delta_3 + \delta_1 \delta_2) \]

For the purpose of examining the convergence we put

\[ H_\nu = \frac{2^\nu}{\nu + 2} \quad (\nu > 0) \]

and

\[ s_n = \sum_{\nu=1}^{n} \frac{1}{\nu}. \]
We have then identically

\[ \frac{1}{n+1} + \frac{1}{n-v+1} \left( \frac{1}{n+3} \right) - \frac{1}{n+2} = \frac{1}{n-v+2} \left( \frac{1}{n+4} \right) \]  

whence

\[ \sum_{v=0}^{n} (n+4) \left( n+4 \right)^{3} \sum_{\nu=0}^{n-1} (n+4) \left( n+4 \right)^{3} \left( n+4 \right)^{3} \]  

\[ \sum_{\nu=0}^{n-1} \frac{3}{(n+4)} \left( n+4 \right)^{3} \left( n+4 \right)^{3} \left( n+4 \right)^{3} \]  

\[ \sum_{\nu=0}^{n-1} \frac{3}{(n+4)} \left( n+4 \right)^{3} \left( n+4 \right)^{3} \left( n+4 \right)^{3} \]  

Furthermore we have the identity

\[ \lambda^{-n} H_r H_n = \frac{n+1 + \frac{1}{n-v+1} \left( \frac{1}{n+3} \right) - \frac{1}{n-v+2} \left( \frac{1}{n+4} \right)} {n+2} \]  

whence

\[ \sum_{\nu=0}^{n-1} \frac{3}{(n+4)} \left( n+4 \right)^{3} \left( n+4 \right)^{3} \left( n+4 \right)^{3} \]  

\[ \sum_{\nu=0}^{n-1} \frac{3}{(n+4)} \left( n+4 \right)^{3} \left( n+4 \right)^{3} \left( n+4 \right)^{3} \]  

5. After these preliminaries we begin with the first of the equations (17) which, keeping the constants of integration apart and assuming \( n > 3 \), we write in the form

\[ a_n = 2 \left( \sum_{\nu=0}^{n-1} \left( \frac{1}{n-v+1} \left( \frac{1}{n+3} \right) - \frac{1}{n-v+2} \left( \frac{1}{n+4} \right) \right) \right) \]  

where the sum is interpreted as zero, if \( n = 3 \).

We now assume that for a certain \( n > 3 \) and for \( 2 \leq \nu \leq n \) is

\[ |\alpha| \leq \alpha H_r, \quad |\beta| \leq \beta H_r, \quad |\gamma| \leq \gamma H_r, \quad |\delta| \leq \delta H_r. \]  

In that case we get from (38)

\[ |a_n| \leq 2 \left( x |\alpha| + y |\gamma| \right) H_n + 2 \left( x |\alpha| + y |\gamma| \right) H_{n-1} + \left( x^2 + y^2 \right) \sum_{\nu=0}^{n-2} H_r H_{n-r}. \]  

By (35) we obtain from this
\[ |a_n| \leq 2 (\lambda \alpha_0 + \gamma \gamma_0) \frac{\lambda^n}{(n+2)^2} + 2 (\lambda \alpha_1 + \gamma \gamma_1) \frac{\lambda^{n-1}}{(n+1)^2} + \frac{2}{3} (\alpha^2 + \gamma^2) \left[ \frac{6 s_n + n - 10 - \frac{12}{n}}{n} \right] \frac{\lambda^n}{(n+4)^2}. \]

(41)

In this, the last term is left out for \( n < 4 \), but since it vanishes for \( n = 3 \), (41) is valid for \( n > 3 \).

A sufficient condition for \( |a_n| \leq AH_n \) for \( n > 3 \) is therefore that the right-hand side of (41) is \( \leq A \frac{\lambda^n}{(n+2)^2} \) which, after multiplication by \( \frac{1}{2} (n+2)^2 \lambda^{-n} \) may be written
\[ |\lambda | \alpha_0 + \gamma \gamma_0 | + (\lambda | \alpha_1 + \gamma \gamma_1 |) \frac{n+2}{n \lambda} + \frac{2}{3} (\alpha^2 + \gamma^2) \left( 6 s_n + n - 10 - \frac{12}{n} \right) \frac{n+1}{n(n+4)^2} \leq A \frac{\lambda^n}{2}. \]

(42)

From (42) we derive a sufficient condition which is independent of \( n \), replacing the factors depending on \( n \) by absolute numbers which are at least as large. We first have
\[ \frac{n+2}{n} = 1 + \frac{2}{n} \leq 3 \quad (n > 3) \]

(43)

and proceed to prove that
\[ \left( 6 s_n + n - 10 - \frac{12}{n} \right) \frac{n+1}{n(n+4)^2} < 2 \quad (n > 3). \]

(44)

Now it is verified directly that (44) is valid for \( n = 3 \) and \( n = 4 \), so that in the remainder of the proof we may assume \( n > 5 \). But we have obviously
\[ s_n \leq s_k + \frac{n-k}{k+1} \quad (n > k) \]

(45)

whence, in particular,
\[ s_n \leq s_5 + \frac{n-5}{6} = \frac{n-5}{6} + \frac{29}{20} \quad (n > 5) \]

(46)

and inserting this in (44) we get the more rigid inequality
\[ \frac{2n-1 \cdot 3 - \frac{12}{n}}{n+4} \cdot \frac{n+1}{n+3} < 2 \]
which is obvious, the first factor on the left being less than 2, and the second less than 1.

By (43) and (44) we finally obtain from (42) the following sufficient condition, which does not depend on \( n \), for \(| a_n | \leq AH_n\)

\[
x | x_0 | + y | y_0 | + \frac{5}{3} (x | x_1 | + y | y_1 |) + \frac{2}{3} (x^2 + y^2) \leq \frac{A}{2},
\]

always provided that \( n > 3 \).

After this, a comparison of the two first equations (17) shows that we obtain from (47), by a simple exchange of letters, as a sufficient condition for \(| b_n | \leq BH_n\) for \( n > 3 \)

\[
\beta | \beta_0 | + \delta | \delta_0 | + \frac{5}{3} (\beta | \beta_1 | + \delta | \delta_1 |) + \frac{2}{3} (\beta^2 + \delta^2) \leq \frac{B}{2}.
\]

As regards the third equation (17) we begin by writing it in the form, valid for \( n > 3 \),

\[
c_n = a_n + b_n + 2 \left( x_0 \beta_n + x_1 \beta_{n-1} + \beta_0 x_n + \beta_1 x_{n-1} + y_0 \delta_n + y_1 \delta_{n-1} + \delta_0 y_n + \delta_1 y_{n-1} \right) + 2 \sum_{\nu = 2}^{n-2} (x \beta_{n-\nu} + y \delta_{n-\nu}) \nonumber \tag{49}
\]

From this we obtain in the same way as above

\[
| c_n | \leq 2 H_n \left[ \frac{A + B}{2} - x | \beta_0 | + \beta | x_0 | + y | y_0 | + \frac{5}{3} (x | x_1 | + y | y_1 |) + \frac{2}{3} (x^2 + y^2) \right] + 2 \sum_{\nu = 2}^{n-2} H_{n-\nu} H_{n-\nu} \nonumber \tag{50}
\]

A sufficient condition for \(| c_n | \leq CH_n\) is therefore that the right-hand side of (50) is \(\leq \frac{C}{(n + 2)(n + 2)} \), and this may, by (30) and (35) and after multiplication by \(\frac{1}{2} (n + 2)(n + 2)^2\), be written

\[
\frac{A + B}{2} - x | \beta_0 | + \beta | x_0 | + y | y_0 | + \frac{5}{3} (x | x_1 | + y | y_1 |) + \frac{2}{3} (x^2 + y^2) \left( 6 s_n + 10 + \frac{12}{n} \right) \leq \frac{C}{n(n + 4)^2} \nonumber \tag{51}
\]

By (43) and (44) we obtain finally the more severe, but of \( n \) independent, condition, valid for \( n > 3 \), for \(| c_n | \leq CH_n\).
6. We now consider (18), assuming that for a certain \( n > 2 \) we have proved that for \( 1 \leq \nu \leq n \)

\[
|a_\nu| \leq A H_\nu, \quad |b_\nu| \leq B H_\nu, \quad |c_\nu| \leq C H_\nu
\]  

and for \( 0 \leq \nu \leq n - 1 \) that

\[
|d_\nu| \leq D H_\nu, \quad |e_\nu| \leq E H_\nu, \quad |f_\nu| \leq F H_\nu.
\]  

We then obtain from the first of the equations (18)

\[
2 n a_0 |d_n| \leq D A \left( 3 n \sum_{\nu}^{n-1} H_\nu H_{n-\nu} + \frac{n-1}{(n+4)^{\nu}} \right)
\]  

whence by (34) and (37), after reduction

\[
2 \alpha_0 |d_n| \leq D A \left( 14 s_{n+1} + 5 n - 5 - \frac{12}{n+1} \right) \frac{\lambda^n}{(n+4)^{\nu}}.
\]  

A sufficient condition for \( |d_n| \leq D H_n \) is therefore that the right-hand side of (56) is \( \leq 2 \alpha_0 \frac{\lambda^n}{(n+2)^{\nu}} \), which may be written

\[
\left( 14 s_{n+1} + 5 n - 5 - \frac{12}{n+1} \right) \frac{n+1}{(n+4)^{\nu}} \leq \frac{2 \alpha_0}{A}.
\]  

We will now show that this condition may be replaced by the more restricted sufficient condition

\[
3 A \leq \alpha_0
\]  

which is independent of \( n \). This comes to proving that

\[
\left( 14 s_{n+1} + 5 n - 5 - \frac{12}{n+1} \right) \frac{n+1}{(n+4)^{\nu}} \leq 6
\]  

or

\[
s_{n+1} \leq \frac{n-1}{14} + 3 + \frac{24}{7(n+1)}.
\]
Now it is seen by a table of $s_n$ that (60) is satisfied for $n < 12$, while for $n > 12$ we may employ

$$s_{n+1} \leq s_{13} + \frac{n-12}{14},$$

which inserted in (60) gives, after reduction, the more rigid condition

$$s_{13} \leq \frac{53}{14} + \frac{24}{7(n+1)}$$

which is also satisfied. Hence, (60) and therefore (58) are proved.

Since the second and third of the equations (18) are obtained from the first by a simple exchange of letters, we may now write down as a sufficient condition for the validity of (54) for $0 \leq \nu \leq n$

$$3 A \leq a_0, \quad 3 B \leq b_0, \quad 3 C \leq c_0. \quad (61)$$

7. As regards finally (16), we isolate the constants of integration, assume $n > 2$ and write the first of these equations in the form

$$(n + 2)^2 a_{n+2} = m_1 (\beta_0 x_n + \beta_1 x_{n-1} - \epsilon_0 f_n - \epsilon_1 f_{n-1}) - M_1 (\alpha_0 d_n + \alpha_1 d_{n-1})$$

$$+ m_1 \sum_{\nu=2}^n (x_\nu e_{n-\nu} - \epsilon_\nu f_{n-\nu}) - M_1 \sum_{\nu=2}^n \alpha_\nu d_{n-\nu}. \quad (62)$$

We write for abbreviation

$$P_1 = m_1 (F + E), P_2 = m_2 (F + D), Q_1 = m_1 F + M_1 D, Q_2 = m_2 F + M_2 E \quad (63)$$

and assume that (54) is satisfied for $0 \leq \nu \leq n$, (39) for $2 \leq \nu \leq n$. From (32) we obtain

$$\sum_{\nu=2}^n H_n H_{n-\nu} = \frac{\lambda^n}{(n+4)(3)} \left( 4 s_{n-1} + \frac{4 n - 7}{3} + \frac{2}{n+1} \right) \quad (n > 2) \quad (64)$$

and thereafter from (62) for $n > 2$

$$(n + 2)^2 | a_{n+2} | \leq (| x_0 | Q_1 + | \beta_0 | P_1) \frac{\lambda^n}{(n+2)^6} + (| \alpha_1 | Q_1 + | \beta_1 | P_1) \frac{\lambda^{n-1}}{(n+1)^6}$$

$$+ (x Q_1 + \beta P_1) \left( 4 s_{n-1} + \frac{4 n - 7}{3} + \frac{2}{n+1} \right) \frac{\lambda^n}{(n+4)^6}. \quad (65)$$

A sufficient condition for $|x_{n+2}| \leq \alpha H_{n+2}$ is therefore that the right-hand side of (65) is $\leq (n+2)^{\frac{(n+4)(n+3)}{(n+2)^3}} \alpha H_{n+2}$, which after multiplication by $\lambda^{-\frac{n}{(n+2)^3}}$ may be written

$$
(|x_0| Q_1 + |\beta_0| P_1) \frac{(n+4)(n+3)}{(n+2)^3(n+1)^3} + (|x_1| Q_1 + |\beta_1| P_1) \frac{(n+4)(n+3)\lambda^{-1}}{(n+2)(n+1)^3} n
$$

$$
\frac{\alpha Q_1 + \beta P_1}{(n+2)^3(n+1)} \left( 4 s_{n-1} + \frac{4 n - 7}{3} + \frac{2}{n+1} \right) \leq \alpha \lambda^2.
$$

In order to find a sufficient condition that does not depend on $n$ we observe that, since we have assumed $n > 2$,

$$
\frac{(n+4)(n+3)}{(n+2)^3(n+1)^3} = \left(1 + \frac{2}{n+2}\right) \left(1 + \frac{1}{n-2}\right) \frac{1}{(n+1)^3} \leq \frac{5}{24}
$$

and

$$
\frac{(n+4)(n+3)}{(n+2)(n+1)^3 n} = \left(1 + \frac{2}{n+2}\right) \left(1 + \frac{3}{n}\right) \frac{1}{(n+1)^3} \leq \frac{5}{12}.
$$

We will finally show that for $n > 2$

$$
\frac{1}{(n+2)^3(n+1)} \left( 4 s_{n-1} + \frac{4 n - 7}{3} + \frac{2}{n+1} \right) \leq \frac{5}{48}.
$$

For $n = 2$ it is seen directly that this holds. For $n > 3$ we insert the inequality

$$
s_{n-1} \leq \frac{1}{2} + \frac{n}{3}, \quad (n > 3)
$$

resulting from (45) for $k = 2$. The result may be written

$$
128 \leq 5 (n+2)^3 + 48 \frac{3 n + 1}{(n+1)^3} \quad (n > 3)
$$

which is easily verified. Hence (69) is proved for $n > 2$.

If now we insert (67)–(69) in (66), we obtain the sufficient condition, valid for $n > 2$, but otherwise independent of $n$,

$$
2 (|x_0| Q_1 + |\beta_0| P_1) + \frac{4}{\lambda^2} (|x_1| Q_1 + |\beta_1| P_1) + \alpha Q_1 + \beta P_1 \leq \frac{48}{5} \alpha \lambda^2.
$$

Since the three last equations (16) are obtained from the first by a simple exchange of letters, we may now by (71) write down the following sufficient conditions, valid for $n > 2$
8. We may summarize the result of the preceding investigation thus:

If (39) is satisfied for $2 < v < 3$, (53) for $1 < v < 2$, (54) for $0 < v < 2$, and if, besides, all the inequalities (47), (48), (52), (61), (71)–(74) are satisfied, then (12) and (13) are convergent provided that $\Sigma H_\nu p^n$ converges, that is, for $|t| \leq \frac{1}{\lambda}$.

It may be observed that the condition (52) implies that $A + B < C$.

The question arises whether it is always possible, when the initial values (19) are arbitrarily given, to find such values of $\lambda, \alpha, \beta, \gamma, \delta, A, B, C, D, E, F$ that the aforesaid inequalities are all satisfied. This question must be answered in the affirmative. To begin with, $\lambda$ can always be chosen so large that (71)–(74) are satisfied and that (47), (48) and (52) are reduced to

$$\alpha |a_0| + \gamma |\gamma_0| + \frac{2}{3} (\alpha^2 + \gamma^2) < \frac{A}{2}$$

$$\beta |b_0| + \delta |\delta_0| + \frac{2}{3} (\beta^2 + \delta^2) < \frac{B}{2}$$

$$\alpha |a_0| + \beta |b_0| + \gamma |\gamma_0| + \frac{4}{3} (\alpha \beta + \gamma \delta) < \frac{1}{2} (C - A - B)$$

while (61) is unchanged. We now choose $A, B$ and $C$ so small that (61) is satisfied and, besides, $A + B < C$. After this $\alpha, \beta, \gamma, \delta$ may be chosen so small that the three reduced inequalities are satisfied. Small values of $A, B, C, \alpha, \beta, \gamma, \delta$ can always be compensated by an increase of $\lambda$.

9. As a simple numerical example to show the practical working of the recurrence formulas we choose

$$m_1 = 1, m_2 = 2, m_3 = 3$$

so that

$$M_1 = 5, M_2 = 4, M_3 = 3$$
and for the initial values
\[\begin{align*}
    x_0 &= 0.5, \quad \beta_0 = 0.9, \quad \gamma_0 = 1.2, \quad \delta_0 = -1.2 \\
    x_1 &= 1.5, \quad \beta_1 = -1, \quad \gamma_1 = -2, \quad \delta_1 = -3.
\end{align*}\] (77)

From these I derive by (20)-(29) the coefficients in the table below where the exact values of \(d_0, e_0, f_0\) are
\[\begin{align*}
    d_0 &= \frac{1}{2.197}, \quad e_0 = \frac{1}{3.375}, \quad f_0 = \frac{1}{2.744} \quad (78)
\end{align*}\]

and where at the time \(t = 0\)
\[\begin{align*}
    r_1 &= \sqrt{a_0} = 1.3, \quad r_2 = \sqrt{b_0} = 1.5, \quad r_3 = \sqrt{c_0} = 1.4. \quad (79)
\end{align*}\]

As regards the convergence, the sufficient conditions established above are satisfied if we choose, for instance
\[\begin{align*}
    \lambda &= 20, \quad \alpha = 0.021, \quad \beta = 0.025, \quad \gamma = 0.047, \quad \delta = 0.038, \\
    A &= 0.14, \quad B = 0.17, \quad C = 0.60, \quad D = 0.92, \quad E = 0.70, \quad F = 0.73. \quad (80)
\end{align*}\]

The expansions (12) and (13) are therefore at least convergent for \(|t| \leq \frac{1}{20}\).

I find for \(t = \frac{1}{20}\)
\[\begin{align*}
    g_1 &= 1.66270, \quad \sigma_1 = 466385 \\
    g_2 &= 2.26598, \quad \sigma_2 = 293155 \\
    g_3 &= 1.95711, \quad \sigma_3 = 365223 \\
\end{align*}\] (81)

and from \(r_t = \sqrt{g_t}\)
\[\begin{align*}
    r_1 &= 1.28946, \quad r_2 = 1.50532, \quad r_3 = 1.39897. \quad (82)
\end{align*}\]

10. A considerable simplification is obtained in the particular case where
\[\begin{align*}
    x_1 &= \beta_1 = \gamma_1 = \delta_1 = 0. \quad (83)
\end{align*}\]

Under these circumstances there are only the four arbitrary constants \(x_0, \beta_0, \gamma_0, \delta_0\) left. The significance of (83) is that at the outset we have
\[\begin{align*}
    \frac{d\xi_1}{dt} = \frac{d\xi_2}{dt} = 0, \quad \frac{d\eta_1}{dt} = \frac{d\eta_2}{dt} = 0 \quad (t = 0) \quad (84)
\end{align*}\]

1 The number of decimals retained in the table has been cut down to seven.
Table.

<table>
<thead>
<tr>
<th>( v )</th>
<th>( \alpha_v )</th>
<th>( \beta_v )</th>
<th>( \gamma_v )</th>
<th>( \delta_v )</th>
<th>( \epsilon_v )</th>
<th>( \zeta_v )</th>
</tr>
</thead>
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<td>0</td>
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<td>1.4</td>
<td>0</td>
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<td>-0.0135695</td>
<td>-0.0370782</td>
</tr>
</tbody>
</table>

or, expressed by the coordinates in the absolute movement,

\[
\frac{dx_1}{dt} = \frac{dx_2}{dt} = \frac{dx_3}{dt}, \quad \frac{dy_1}{dt} = \frac{dy_2}{dt} - \frac{dy_3}{dt} \quad (t = 0).
\] (85)

From (83) follows at once by (22) and (23) that

\[
a_1 = b_1 = c_1 = d_1 = e_1 = f_1 = 0,
\] (86)

whereafter the general recurrence formulas (16)-(18) show that all the coefficients of the odd order vanish.

11. We shall finally call attention to another particular case where considerable simplifications occur. Let \( h \) be an arbitrary constant, and let us for \( v = 0 \) and \( v = 1 \) choose

\[
\gamma_v = h \alpha_v, \quad \delta_v = h \beta_v, \quad \text{whence} \quad \zeta_v = h \epsilon_v.
\] (87)

In that case comparison between the first and third, and between the second and fourth, of the equations (16) shows that (87) is valid for all \( v \). It follows that \((\omega \delta)_v = (\beta \gamma)_v\), so that

\[
\xi_1 \eta_2 - \eta_1 \xi_2 = \xi_3 \eta_2 - \eta_3 \xi_2 \quad (88)
\]
or

\[
\frac{y_3 - y_2}{x_3 - x_2} = \frac{y_3 - y_1}{x_3 - x_1} \quad (89)
\]
But a simple geometrical consideration shows that this means that the three bodies are always situated on a straight line.

Under these circumstances the calculation of the coefficients is simplified, because (87) shows that the two last equations (16) are identical with the two first and can be left out, while (17) is reduced to

\begin{align*}
a_n &= (1 + h^2) (\alpha \beta)_n \\
b_n &= (1 + h^2) (\beta \beta)_n \\
c_n &= a_n + b_n + 2 (1 + h^2) (\alpha \beta)_n.
\end{align*}

12. It was mentioned at the outset that if the absolute positions in the plane of the three bodies are required they can be determined afterwards. We will briefly indicate how this may be done.

Writing the third of the equations (6) in the form

\[ \frac{d^2 x_3}{dt^2} = m_2 \xi_1 a_1 - m_1 \xi_2 a_2 \]

and putting

\[ u_n = m_2 (\alpha d)_n - m_1 (\beta e)_n \]

we have

\[ \frac{d^2 x_3}{dt^2} = \sum_{n=0}^{\infty} u_n l^n, \]

and integrating this twice, introducing thus two more arbitrary constants, we have the expansion of \( x_3 \), whereafter by (7)

\[ x_2 = x_3 + \xi_1, \quad x_1 = x_3 - \xi_2. \]

The equation for \( y_3 \)

\[ \frac{d^2 y_3}{dt^2} = m_2 \gamma_1 a_1 - m_1 \gamma_2 a_2 \]

may be treated in the same way. Putting

\[ v_n = m_2 (\gamma d)_n - m_1 (\delta e)_n \]

we have

\[ \frac{d^2 y_3}{dt^2} = \sum_{n=0}^{\infty} v_n l^n \]
whence, introducing two more arbitrary constants, we find $y_3$ and finally by (7)

$$y_2 = y_3 + \eta_1, \quad y_1 = y_3 - \eta_2.$$  \hspace{1cm} (98)

If the values at $t = 0$ of $x_i, \frac{dx_i}{dt}, \frac{dy_i}{dt}$ are chosen arbitrarily, the corresponding values of (19), or $\xi_i, \frac{d\xi_i}{dt}, \eta_i, \frac{d\eta_i}{dt}$ at $t = 0$, result immediately from (7).