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FOUNDATIONS OF THE THEORY OF DYNAMICAL SYSTEMS OF INFINITELY MANY DEGREES OF FREEDOM. I

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CONTENTS

Synopsis

It is shown effectively that, while there are many unitarily inequivalent representations of the canonical Bose-Einstein field variables, the S-matrix can be theoretically specified without using any particular representation. The central idea is that the bounded functions of finite subsets of the canonical variables, together with their limits in a physically meaningful sense, are substantially the same for all representations. On the other hand, convergence questions may depend strongly on the representation; in fact, formal operators fairly typical of divergent interaction Hamiltonians may be rendered hermitian operators in Hilbert space by a suitable choice of representation. Applications are made to the 'clothing' of field kinematics, statistics, and canonical variables, for a theory in which only the transformation properties of single particles under an arbitrary covariance group, and a covariant interaction, need to be specified. The results are mostly of a general character, such as the existence of a physical vacuum, and the possibility of ascribing definite constitutions in terms of primary elementary particles to bound states. It is shown also that the canonical variables and occupation numbers of a field can be described by the so-called 'free-field representation' only if the physical vacuum is invariant under the dynamical development of the system in essentially the interaction representation.

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Introduction

In both classical and quantum mechanics of systems of a finite number of degrees of freedom, one has to do with so-called canonical variables $p_1, p_2, \ldots, p_n$ and $q_1, q_2, \ldots, q_n$, which are now fairly well understood mathematically. The theory of unbounded operators in Hilbert space has provided a foundation for the use of these variables that is made in quantum mechanics. When the number of degrees of freedom is infinite, the situation is different. The canonical variables are relatively most useful in quantum mechanics, and it is here that the greatest mathematical difficulties appear. From the beginning, the success of quantum field theory has been attended by 'infinities' in even the simplest cases. It was assumed that the canonical variables $p_1, p_2, \ldots,$ and $q_1, q_2, \ldots,$ were operators, but the vectors on which they acted had for the most part only hypothetical character, which did not appear to matter substantially for the computational use to which the theory was put, but made it difficult to get at the bottom of the trouble.

This difficulty about 'interacting' systems was nevertheless accompanied by unclouded optimism concerning the mathematical basis for the theory of free systems, which went unexamined for many years. One had the p's and q's satisfying the basic relations $p_j q_k - q_k p_j = -i\delta_{jk}$ $(j, k = 1, 2, \ldots)$, and it was assumed that any two such systems were equivalent, apart from the irrelevant complication of multiplicity, which was generally suppressed by assuming the p's and q's to act irreducibly; upon this informal axiomatic basis the theory rested. Within the past decade these matters have been gone into, and it was found that this central assumption was literally in great error. There were at least continuum many inequivalent such systems; some of the most commonly applied so-called canonical transformations, – even that of multiplying the p's by a positive constant and dividing the q's by the same constant $(\neq 1)$–, led to inequivalent systems; and these facts had a concrete analytical basis, having nothing to do with the 'pathology' of unbounded operators, etc., persisting when the relations were strengthened into the formally equivalent ones due to WEYL, which involved only algebraic operations on bounded operators.

A unique theory of 'free' systems could be salvaged by defining the canonical p's and q's to be the specific ones treated rigorously by COOK, following the basic physical ideas of FOCK, and of BOHR and ROSENFELD. This set of variables proved to be especially well adapted to the program of treating systems in terms of their transformation properties, as well as to the correlation of their wave and particle aspects. Apart from the unpleasantly strong inhibitions on the application of canonical transformations, the mathematical situation in the theory of free systems developed in a satisfactory way, and there seemed little doubt that the Fock-Cook system was at least appropriate in connection with the theoretical idea of a field of completely non-interacting particles, and possibly appropriate for the physically fictitious 'bare' particles. But it was not clear how the latter field and the free physical-particle field (whose 'freedom' is experimental, and not necessarily mathematical, since in all theories the physical particles involve self-interactions) were related, and the precise character of the p's and q's for interacting fields remained a mystery,–which in fact had conceivably to await a new physical idea for its elucidation.

This paper gives a mathematical interpretation of the canonical variables which, from a foundational viewpoint, obviates the uniqueness troubles mentioned, and does so by eliminating from the mathematical formalism features lacking in physically operational significance. While this shows that in spirit the assumption about the essential uniqueness of the canonical variables was in a way partially sound, it also indicates that many tacit assumptions need to be revised,– e. g. it shows that the zero-interaction and the free physical-particle canonical variables cannot be expected to be unitarily equivalent, even in comparatively simple cases. A significant application is to make possible the physically effective quantization of fields of particles whose transformation properties are non-unitary, such as the important type admitting an indefinite invariant inner product. It also narrows slightly, but perhaps illuminatingly, the gap between the mathematical formulations of classical and quantum mechanics, a dynamical transformation in either theory being an automorphism of a certain C^*-algebra, the unitary transformations of quantum mechanics being seen to play an essentially technical role.

In a purely mathematical way, this work relates essentially to an abstract C^*-algebra $\mathfrak{A}$ which is invariantly attached to a real topological linear space $\mathfrak{H}$ admitting a Hilbert space structure, and to a representation of the symplectic group on the direct sum of $\mathfrak{H}$ with its dual by automorphisms of this algebra $\mathfrak{A}$. A distinguished automorphism of $\mathfrak{A}$ which commutes with the action on $\mathfrak{A}$ of a distinguished subgroup of the symplectic group plays a fundamental role in the later developments. The canonical variables are in a sense affiliated with $\mathfrak{A}$, which consists essentially of all bounded functions of a finite number of the canonical variables, together with their uniform limits.

When $\mathfrak{H}$ is finite-dimensional, $\mathfrak{A}$ consists of all bounded linear transformations on $L_2(\mathfrak{H})$ (within algebraic isomorphism). The general linear group on $\mathfrak{H}$ induces automorphisms of this algebra through its action on $L_2(\mathfrak{H})$ in an obvious manner, involving normalization in terms of the determinant of the transformation, and this action extends to the symplectic group over the direct sum of $\mathfrak{H}$ with its dual. In the case of present interest in which $\mathfrak{H}$ is infinite-dimensional, the Hilbert space $L_2(\mathfrak{H})$ can be effectively formulated, and an analogous canonical unitary representation on $L_2(\mathfrak{H})$ obtained for the orthogonal group on $\mathfrak{H}$, relative to any admissible Hilbert space structure on $\mathfrak{H}$; but the full general linear group on $\mathfrak{H}$ cannot act on $L_2(\mathfrak{H})$ in any reasonable fashion, one relevant difficulty being the absence of a determinant for such transformations. Nevertheless, it induces an automorphism of $\mathfrak{A}$, which is all that is essential for, or operationally relevant to, kinematical and dynamical purposes, and which determines the corresponding unitary transformation on $L_2(\mathfrak{H})$, when it happens to exist, uniquely within a scalar factor.

This has been stated in terms of a particular representation for the algebra $\mathfrak{A}$ of 'field observables', but it is of crucial importance that the space on which $\mathfrak{A}$ acts can be eliminated, and that essentially only $\mathfrak{A}$ as an abstract algebra is fundamentally necessary or logically relevant. It is more than conceivable that the interacting field p's and q's cannot be represented as operators in the same representation as that in which the free-field p's find q's and operators (a partially heuristic result to this effect for certain fields is due to HAAG), but it is still possible to obtain the interacting ones from the free-field ones essentially by an automorphism of $\mathfrak{A}$, by virtue of its essentially abstract algebraic character. Such an automorphism is just as good as the unitary operator in terms of which it has been conventionally assumed that a dynamical transformation in quantum mechanics is to be mathematically expressed, for it preserves spectral values, expectation values in states, the purity of states, etc.; it has, in fact, a certain advantage, in that it has greater operational meaning (in particular there is no phase ambiguity). Even when, in a particular representation, $\mathfrak{A}$ acts irreducibly, relatively few of its automorphisms will be implementable by unitary transformations, the quotient of its full group of automorphisms by the subgroup of those so implementable being infinite-dimensional in any reasonable sense.

Therefore it is not surprising that the Hamiltonians of the quantum theory of interacting fields do not appear to correspond to operators, for it is only in the comparatively rare event that the corresponding integrated motion is representable by unitary transformations that this will be the case. For this reason and for greater operationalism, we define physical particles substantially in terms of the scattering operator and the field kinematics, without reference to the interacting field Hamiltonian, and the relatively explicit theoretical construction of these particles from the 'pristine' ones (the so-called 'clothing' of the particles) makes it clear that, in general, their canonical operators will not be merely a similarity transform of those of the pristine particles. This difficulty, which is parallel to that circumvented by renormalization in conventional field theory, is here treated in a theoretically effective and mathematically rigorous manner. It appears that renormalizable divergences may well arise in substantial part from the attempt to enforce analytically the logically unnecessary and rigorously absent unitary equivalence between pristine and free physical-particle representations.

It may clarify our results to discuss briefly their character for the very simple case when $\mathfrak{H}$ is finite-dimensional. The central uniqueness theorem is then a variant of the fundamental result on the uniqueness of the Schrödinger operators developed by STONE and VON NEUMANN, whose results are used in the proof of our theorem. On the other hand, in its present formulation, it may also be regarded as a variant of a well-known theorem of PLESSNER, stating that a quasi-invariant regular measure on Euclidean space is equivalent to Lebesgue measure. The relevant unitary ray representation on $L_2(\mathfrak{H})$ of the symplectic group on the direct sum of a finite-dimensional $\mathfrak{H}$ with its dual is, however, not treated in the existing literature known to us.

It may also be helpful to compare our approach with one sketched by von Neumann[10], in which field dynamics is likewise to be expressed more or less in terms of automorphisms of an algebra. Apart from this similarity, there appears to be nothing in common between the approaches. The elegant and somewhat formal idea of von Neumann is that all the measurable field-theoretic variables should be expressible in terms of a 'type II_1' ring, whose automorphisms are expressible by unitary operators, which however are in general outside the ring; it is based technically on a *weakly* closed ring. The present intuitive idea is roughly that the only measurable field-theoretic variables are those that can be expressed in terms of a *finite* number of canonical operators, or *uniformly* approximated by such; the technical basis is a *uniformly* closed ring (more exactly, an abstract C^*-algebra). The crucial difference between the two varieties of approximation arises from the fact that, in general, weak approximation has only analytical significance, while uniform approximation may be defined operationally, two observables being close if the maximum (spectral) value of their difference is small. More technically, weak approximation depends on the particular representation of the canonical operators, and also will be affected by an enlargement of the physical system under consideration, while uniform approximation is independent of the particular representation of the canonical variables, and is unaffected by enlargement of the system. The weak closure in analytically relevant concrete representations (e. g., the zero-interaction representation) of the present algebra of field observables may well consist of all bounded operators, and so have little connection with a ring of type II_1.

The specific formal operators of relativistic field theory and of possible extensions of the theory will be treated on a later occasion, to which we defer also the consideration of more general varieties of statistics and the representation of non-linear transformations on the single-particle space $\mathfrak{H}$ by automorphisms of the algebra $\mathfrak{A}$ of field observables. We may note that significant progress relevant to the latter has been made by LEONARD GROSS[(2)], while a discussion of general statistics as well as a survey of the physical background of the present work is given in[(3)].

1. Field Observables

A *single-particle* (state-vector) structure Σ is defined as a system $(\mathfrak{H}, \mathfrak{H}', B)$, where $\mathfrak{H}$ and $\mathfrak{H}'$ are real linear spaces, and B is a real non-singular bilinear form on $(\mathfrak{H}, \mathfrak{H}')$. Thus, for fixed x in $\mathfrak{H}$ (resp. x' in $\mathfrak{H}'$), $B(x, x')$ is a linear function of x' in $\mathfrak{H}'$ (resp. x in $\mathfrak{H}$), while for any non-vanishing x (resp. x'), there exists an x' (resp. x) such that $B(x, x') \neq 0$.

Example 1.1. Let $\mathfrak{H}$ be a real linear topological space admitting a Hilbert space structure, let $\mathfrak{H}'$ be its dual, and set $B(x, x') = x'(x)$.

Example 1.2. Let $\mathfrak{M}$ be a complex Hilbert space with a distinguished conjugation J. Let $\mathfrak{H}$ be the set of all elements of $\mathfrak{M}$ left invariant by J, as a linear space, and $\mathfrak{H}'$ the set of elements taken by J into their negative. Set $B(x, x') = i(x, x')$, where $(.\,,.)$ denotes the canonical inner product in $\mathfrak{M}$; it is easily seen that B is real-valued. This structure will be called *standard.*

A *canonical system over* Σ is a defined as an (ordered) pair of linear maps $p(.)$ and $q(.)$ from $\mathfrak{H}$ and $\mathfrak{H}'$, respectively, to respective commutative families of self-adjoint operators on a (complex) Hilbert space (called the representation space) such that

$$e^{ip(x)}\, e^{iq(x')} = e^{iB(x, x')}\, e^{iq(x')}\, e^{ip(x)}$$

for arbitrary x in $\mathfrak{H}$ and x' in $\mathfrak{H}'$. Linearity, it should be noted, is with reference to the strong operations on the unbounded linear operators on the representation space. That is, it is required that $p(ax + by)$ be the closure of $ap(x) + bp(y)$ for arbitrary x and y in $\mathfrak{H}$ and real numbers a and b, and similarly in the case of $q(.)$. Further, commutativity of unbounded self-adjoint operators is in the sense that any two spectral projections of the operators commute (in the usual sense). It follows that a linear map $p(.)$ of the type described is precisely one such that $U(x) = e^{ip(x)}$ defines a unitary representation $U(.)$ of the additive group of $\mathfrak{H}$, whose restriction to any finite-dimensional submanifold is continuous in the strong operator topology; and similarly for $q(.)$.

Example 1.3. Assume the $\mathfrak{H}$ in Example 1.2 is countably-dimensional, and let $e_1, e_2, \ldots$ be an orthonormal basis for $\mathfrak{H}$. Define $\mathfrak{H}_0$ as the set of finite linear combinations of the e_k $(k = 1, 2, \ldots)$, $\mathfrak{H}'$ as $i\mathfrak{H}_0$, and B_0 as the restriction of B to $\mathfrak{H}_0' \times \mathfrak{H}_0'$. This yields a single-particle space $\Sigma_0 = (\mathfrak{H}_0, \mathfrak{H}_0', B_0)$.

Now let $p_1, p_2, \ldots$ and $q_1, q_2, \ldots$ be two sequences of self-adjoint operators on a complex Hilbert space K such that any two p_k commute and also any two q_k commute, while $e^{ip_j s}\, e^{iq_k t} = e^{i\delta_{jk} st}\, e^{iq_k t}\, e^{ip_j s}$. It is not difficult to show that there exists a unique canonical system over Σ_0 such that $p(e_k) = p_k$ and $q(ie_k) = q_k$; and that, conversely, every canonical system over Σ_0 arises in this way.

A bounded linear operator T (on the representation space) is said to *depend on submanifolds* $\mathfrak{M}$ of $\mathfrak{H}$ and $\mathfrak{M}'$ of $\mathfrak{H}'$ in case T is in the weak closure of the algebra generated (algebraically) by the $\exp(ip(x))$ and $\exp(iq(x'))$ as x and x' range over $\mathfrak{M}$ and $\mathfrak{M}'$, respectively. The collection of all bounded linear operators dependent on a pair of fixed manifolds $\mathfrak{M}$ and $\mathfrak{M}'$ forms a weakly closed ring $\mathfrak{A}_{\mathfrak{M},\mathfrak{M}'}$, while the union over all finite-dimensional $\mathfrak{M}$ and $\mathfrak{M}'$ of the $\mathfrak{A}_{\mathfrak{M},\mathfrak{M}'}$ forms an algebra $\mathfrak{A}$ whose uniform closure is called the *representation algebra of field observables* (over Σ).

Now the algebra $\mathfrak{A}$ evidently depends not only on Σ, but also on the particular canonical system over Σ involved in its definition. The next theorem establishes the essential point that, as an algebra,—or physically, as regards operationally significant aspects—, $\mathfrak{A}$ is independent of the representation employed. The corresponding abstract algebra uniquely associated with Σ (as a pair of linear spaces with a distinguished bilinear form) may then be referred to as the algebra $\mathfrak{A}(\Sigma)$ *of all (bounded) field observables over* Σ. This is essentially, in conventional physical language, the system of observables of the Bose-Einstein field of particles with wave functions in the direct sum $\mathfrak{H} \oplus \mathfrak{H}'$.

THEOREM 1. *For any two canonical systems over a single-particle space* $\Sigma = (\mathfrak{H}, \mathfrak{H}', B)$, *there exists a unique algebraic isomorphism between the corresponding representation algebras of field observables that exchanges the bounded Baire functions of the canonical* $p(x)$ *and* $q(x')$ *for all* x *in* $\mathfrak{H}$ *and* x' *in* $\mathfrak{H}'$.

That is, denoting the one canonical system as above and the other by the use of the superscript $^\sim$, there exists a unique algebraic isomorphism of $\mathfrak{A}$ onto $\mathfrak{A}^\sim$ (where among the algebraic operations is included adjunction) that takes $\varphi(p(x))$ and $\varphi(q(x'))$ into $\varphi(p^\sim(x))$ and $\varphi(q^\sim(x'))$, respectively, for every bounded Baire function φ, and arbitrary $x \varepsilon \mathfrak{H}$ and $x' \varepsilon \mathfrak{H}'$. Actually a stronger result will be established, which implies, e. g., that such an isomorphism exchanges also functions of any finite number of canonical variables.

To prove the theorem, let $\mathfrak{M}$ and $\mathfrak{M}'$ be finite-dimensional subspaces of $\mathfrak{H}$ and $\mathfrak{H}'$, respectively, that are mutually separating, in the sense that the restriction of B to $\mathfrak{M} \times \mathfrak{M}'$ is non-singular. Bases may then be chosen in $\mathfrak{M}$ and $\mathfrak{M}'$ in such a manner that $B(x, x')$ has the form $x_1 x_1' + \ldots + x_r x_r'$ for x and x' in $\mathfrak{M}$ and $\mathfrak{M}'$, respectively, the x_j (resp. x_j') being the coordinates of x (resp. x') relative to the basis in $\mathfrak{M}$ (resp. $\mathfrak{M}'$). We can now employ the uniqueness theorem for canonical systems in the case of finitely many degrees of freedom in the form given in[11]. According to this, the most general such system is, within unitary equivalence, a discrete direct sum of copies of the conventional (Schrödinger) representation. In particular, the restrictions $p_{\mathfrak{M}}(.)$ and $q_{\mathfrak{M}}(.)$ of $p(.)$and $q(.)$ to $\mathfrak{M}$ and $\mathfrak{M}'$, respectively, have the forms

$$p(x) = p_0(x) \times I_L \qquad q(x') = q_0(x') \times I_L \; (x \,\varepsilon\, \mathfrak{M},\; x' \,\varepsilon\, \mathfrak{M}'),$$

where $(p_0(.), q_0(.))$ is an irreducible canonical system over $\Sigma_0 = (\mathfrak{M}, \mathfrak{M}', B_0)$, B_0 being the restriction of B to $\mathfrak{M} \times \mathfrak{M}'$, and I_L is the identity operator on a Hilbert space L dependent on $\mathfrak{M}$ and $\mathfrak{M}'$. Furthermore, $(p_0(.), q_0(.))$ is unitarily equivalent to the Schrödinger system; the precise form of this will not be needed, but only that any two irreducible canonical systems over a finite-dimensional single-particle space are necessarily unitarily equivalent.

Now the bounded Baire functions of the $p_0(x)$ and of the $q_0(x')$ generate, as x and x' range over $\mathfrak{M}$ and $\mathfrak{M}'$, respectively, a (weakly closed) ring of operators $\mathfrak{B}_{M,M'}$. The map $T \to T \times I_L$ is an algebraic isomorphism of $\mathfrak{B}_{M,M'}$ onto the ring $\mathfrak{A}_{M,M'}$ generated by the bounded Baire functions of the $p(x)$ and $q(x')$, as x and x' range over $\mathfrak{M}$ and $\mathfrak{M}'$ respectively. (Here 'ring' is, as generally when referring to operators on a Hilbert space, a weakly closed self-adjoint ring of bounded linear operators, that contains the identity operator on the space). This is clear from the behaviour of direct products with an identity operator, from which it is also apparent that the isomorphism exchanges the $\varphi(p(x))$ and $\varphi(q(x'))$ with the $\varphi(p_0(x))$ and $\varphi(q_0(x'))$, respectively, for arbitrary x and x' in $\mathfrak{M}$ and $\mathfrak{M}'$, respectively.

It follows that there exists an algebraic isomorphism $\theta_{M,M'}$ of $\mathfrak{A}_{M,M'}$ onto $\mathfrak{A}^{\sim}_{M,M'}$ that exchanges the $\varphi(p(x))$ and $\varphi(q(x'))$ with the $\varphi(p^{\sim}(x))$ and $\varphi(q^{\sim}(x'))$ for all bounded Baire functions φ, and x and x' in $\mathfrak{M}$ and $\mathfrak{M}'$, respectively, inasmuch as a ring $\mathfrak{B}^{\sim}_{M,M'}$ analogous to $\mathfrak{B}^{\sim}_{M,M'}$ would exist, and these two rings, by the uniqueness in the finite-dimensional case, would be algebraically isomorphic in the appropriate manner. Moreover, $\theta_{M,M'}$ is the unique isomorphism with the stated property, for otherwise there would exist a non-trivial automorphism of $\mathfrak{A}_{M,M'}$ leaving fixed the $\varphi(p(x))$ and $\varphi(q(x'))$ for x in $\mathfrak{M}$ and x' in $\mathfrak{M}'$. By virtue of the isomorphism of $\mathfrak{A}_{M,M'}$ there would consequently exist a non-trivial automorphism of $\mathfrak{B}_{M,M'}$ leaving fixed the $\varphi(p_0(x))$ and $\varphi(q_0(x'))$ for x in $\mathfrak{M}$ and x' in $\mathfrak{M}'$. But the irreducibility of the canonical system $(p_0(.), q_0(.))$ over Σ_0 implies that $\mathfrak{B}_{M,M'}$ consists of all bounded linear transformations on the representation space of the system. As is well known, every automorphism of this algebra is inner, which implies that an automorphism leaving fixed a set of generators for this ring of operators must be the identity.

Now, if $\mathfrak{N}$ and $\mathfrak{N}'$ are mutually separating subspaces of $\mathfrak{M}$ and $\mathfrak{M}'$, respectively, the restriction of $\theta_{M,M'}$ to $\mathfrak{A}_{N,N'}$ has precisely the properties that characterize $\theta_{N,N'}$, so that $\theta_{M,M'}$ must extend $\theta_{N,N'}$. Therefore there exists an automorphism θ_0 of the algebra $\mathfrak{A}_0$ consisting of the set-theoretic union of the $\mathfrak{A}_{M,M'}$ as $(\mathfrak{M}, \mathfrak{M}')$ varies over the collection of all pairs of finite-dimensional mutually separating subspaces of $\mathfrak{H}$ and $\mathfrak{H}'$, respectively, onto the corresponding subalgebra $\mathfrak{A}_0^{\sim}$ of $\mathfrak{A}^{\sim}$ uniquely determined by the property that $\varphi(p(x))$ and $\varphi(q(x'))$ go into $\varphi(p^{\sim}(x))$ and $\varphi(q^{\sim}(x'))$, respectively, for all x in $\mathfrak{H}$, x' in $\mathfrak{H}'$, and bounded Baire functions φ. In this connection it is relevant to observe that by elementary algebra the foregoing collection forms a directed system under the ordering: $(\mathfrak{M}, \mathfrak{M}') \subset (\mathfrak{N}, \mathfrak{N}')$ in case $\mathfrak{M} \subset \mathfrak{N}$ and $\mathfrak{M}' \subset \mathfrak{N}'$. Now an algebraic isomorphism of one C^*-algebra with another preserves the norm (or operator 'bound'), so that all the $\theta_{M,M'}$ are isometries. Hence so also is θ_0, and it follows by continuity that θ_0 can be uniquely extended to an algebraic isomorphism θ from the closure of $\mathfrak{A}_0$ onto the closure of $\mathfrak{A}_0^{\sim}$, that exchanges the canonical operators in the designated fashion.

Remark 1.1. Theorem 1 shows only the algebraic uniqueness of canonical systems. A system first defined by Fock in a formal manner and given a rigorous analytical interpretation and examination by Cook(1) was shown in (5) to satisfy the Weyl relations, thereby establishing the existence of a canonical system in the present sense. This system has been utilized in heuristic and implicit fashion in much of the literature on field theory. It seems possibly appropriate for the mathematical representation of the partially nebulous idea of a field of 'bare' particles. It is also in extensive although often somewhat unconscious use for the representation of a field of 'free physical' particles. To avoid the tacit assumption,–which will later appear to be an unnecessary and generally incorrect oversimplification,– that the same mathematical representation is appropriate for both the zero-interaction and the free physical-particle canonical operators (which involve the self-interactions of the particles), we shall call the mathematically and physically distinguished representation described the zero-interaction, rather than the more common 'free-field', representation.

Although this representation is irreducible and the representation algebra $\mathfrak{A}^{\sim}$ contains all bounded functions of any finite set of canonical operators, it does not consist of all bounded linear transformations on the representation space. This can be seen from the result(5) that, if $(p(.), q(.))$ denotes the bare-particle system, the systems $(p_\lambda(.), q_\lambda(.))$ defined by the equation

$p_\lambda(.) = \lambda p(.)$, $q_\lambda(.) = \lambda^{-1} q(.)$, are mutually unitarily inequivalent ($0 < \lambda < \infty$). For, by Theorem 1, there must exist an automorphism of $\mathfrak{A}^\sim$ taking the system $(p(.), q(.))$ into the system $(p_\lambda(.), q_\lambda(.))$ for any λ. If $\mathfrak{A}^\sim$ consisted of all bounded linear transformations on the representation space, then every automorphism of it would consist of transformations by a fixed unitary operator; but, by the cited result, no such transformation can actually take the system for $\lambda = 1$ into the system for a different value of λ.

A specific operator in the zero-interaction representation that does not represent an observable is the well-known one called the 'total number of particles', where an unbounded self-adjoint operator on a representation space is said to represent an observable in case every continuous function vanishing outside an interval on the real axis gives, on application to the operator, an element of the representation algebra. (This is in keeping with elementary physical notions; cf. in particular Section 3, below). In purely mathematical terms we may state: *For a standard infinite-dimensional single-particle structure, if N is the self-adjoint operator representing the total number of particles in the zero-interaction system* (cf. below and 8)), *and φ is a bounded Baire function on the reals that is not constant on the non-negative integers, then $\varphi(N)$ is not a field observable* (*i.e. not in A*).

Consider first the case when $\varphi_1(0) \neq \varphi(1)$. Let $x_1, x_2, \ldots$ be a sequence of orthonormal vectors in $\mathfrak{H}$; then there is no difficulty in verifying that, for any field observable R (element of $\mathfrak{A}$), $\lim_n (Rx_n', x_n') = (Ru, u)$, where x_n' is the vector in the bare-particle respresentation space obtained from x_n by injecting $\mathfrak{H}$ in a canonical fashion into the space of covariant symmetric square-integrable tensors over the complex extension of $\mathfrak{H}$, while u is the unit vector (unique within multiplication by a scalar) in the domain of N that is annihilated by N. On the other hand, it is easily seen that $(\varphi(N)x', x') = \varphi(1)$ for every bounded Baire function φ and vector x in $\mathfrak{H}$ (the prime having the same significance as before), while $(\varphi(N)u, u) = 0$. Now more generally, if $\varphi(k) \neq \varphi(k+1)$, then the same argument applies, with u replaced by the symmetric tensor product of $x_1, x_2, \ldots x_k$, and x_n replaced by the symmetric tensor product of u with the x' ($i \leqslant n$).

Remark 1.2. The preceding work shows that the notion of a function of a finite number of canonical variables is independent of the representation of the canonical system, in keeping with its anticipated observable character.

Because of the non-commutativity of the canonical operators, some delicacy is necessary in the formulation of the relevant notion of functions. To clarify this matter we may make the definition: *A function of a collection of closed densely defined operators* on a Hilbert space is a closed densely defined operator that commutes with all unitary operators that commute with every operator in the collection and their adjoints.

In particular, a function of a single normal operator in the foregoing sense can be shown to be precisely a Baire function of the operator in the usual sense when $\mathfrak{H}$ is countable-dimensional, and in a suitably extended sense for inseparable $\mathfrak{H}$ (cf.[6]). Now in the case of the Schrödinger operators $p_1^{(0)}, p_2^{(0)}, \ldots, p_r^{(0)}$ and $q_1^{(0)}, q_2^{(0)}, \ldots, q_r^{(0)}$, on the space $\mathfrak{H}_0$ of square-integrable functions of r real variables, every bounded linear operator on $\mathfrak{H}_0$ is a function of the p's and q's in the foregoing sense, and may be symbolized by $f(p^{(0)}(.), q^{(0)}(.))$. The proof of Theorem 1 shows that *for any other canonical system* $(p(.), q(.))$ over the same single-particle r-dimensional space $\mathfrak{H}_0$, *there exists a unique corresponding function* $f(p(.), q(.))$, the image under the isomorphism given by Theorem 1 of the representation algebra of observables of the first system into that of the second of the original function. This result could be extended to unbounded functions, as well as to the specific functions of the canonical operators defined by Fourier integrals, but these developments are not needed in the present paper.

Remark 1.3. In the treatment of time reversal and other reversal operations, an extension of Theorem 1 to the case of conjugate-linear ring isomorphisms is needed. It may be stated as

COROLLARY 1.1. *For any two canonical systems over the single-particle spaces* $(\mathfrak{H}, \mathfrak{H}', B)$ and $(\mathfrak{H}, \mathfrak{H}', -B)$, *respectively, there exists a unique conjugate-linear ring-isomorphism between the corresponding representation algebras of field observables that exchanges the bounded Baire functions of the canonical* $p(x)$ *and* $q(x')$ *for all* x *in* $\mathfrak{H}$ *and* x' *in* $\mathfrak{H}'$.

Consider first the situation in a finite number of dimensions. As before, a basis may be chosen so that $B(x, y)$ has the form $\Sigma_k x_k y_k$, and the uniqueness theorem in the finite dimensional case reduces the question to that of showing the existence of an appropriate isomorphism for the algebra of the Schrödinger representation. It is easily seen that ordinary complex conjugation in the representation space effects (via the corresponding similarity transformation) the required isomorphism. This isomorphism is unique because the product of any two such conjugate-linear ring isomorphisms is a full algebraic isomorphism leaving fixed the $p(x)$ and $q(x')$, and so the identity. Now in the case of an infinite number of dimensions, the proof given for Theorem 1 applies with obvious changes to the present situation, making use of the foregoing finite-dimensional result.

2. Transformation Properties of the Field Observables

It is axiomatic that a suitable displacement of the single-particle structure should give rise to a corresponding displacement of the field. For example, if L is any Lorentz transformation, it is generally postulated that there is an associated unitary operator $U(L)$ on the state vector space of the field, and that these operators give (within scalar factors at least) a representation of the Lorentz group. This means essentially that any change of frame in ordinary physical space gives a corresponding transformation on the field states. The present section is devoted to the mathematical formulation of this correspondence, i. e., the field kinematics, and to a generalization of it which permits the treatment of field statistics along parallel lines.

We are concerned here with those transformations on the direct sum $\mathfrak{H} \oplus \mathfrak{H}'$ of the real linear space associated with a single-particle structure $(\mathfrak{H}, \mathfrak{H}', B)$, that leave invariant within sign the skew form $[u, v] = B(x, y') - B(y, x')$, where $u = x \oplus x'$ and $v = y \oplus y'$. A transformation leaving the form strictly invariant will be called *symplectic,* while one that reverses its sign will be called *anti-symplectic.*

Theorem 2 establishes, in physical terminology, that any symplectic transformation group (unitary or not) gives a corresponding group of transformations of the associated Bose-Einstein observables. On the other hand, it should be noted that the state vectors, in a particular representation of the field observables, need not be correspondingly transformable. In other terms, if L is any single-particle transformation, then there will exist in a purely formal way a symbolic transformation $U(L)$ on the state vectors, special cases of this correspondence being treated in the field theory literature; but in general, $U(L)$ will not exist in a rigorous analytical sense; and it is here shown that, if T is a function of a finite number of canonical operators (or a uniform limit of such), then nevertheless $U(L)TU(L)^{-1}$ can be given satisfactory mathematical meaning. This gives then a rigorous and covariant scheme for 'quantizing' any given single-particle species in accordance with Bose-Einstein statistics, which uses only the transformation properties of the single particle, and is sufficiently general to be applicable to cases where there exists no invariant positive definite inner product for single-particle wave functions. There is no need for the single-particle space $\mathfrak{H} \oplus \mathfrak{H}'$ to be irreducible under the action of the 'covariance' group, in fact it may decompose continuously, making it possible to quantize simultaneously a continuum of distinct single-particle types.

In particular, whether or not the 'free-field' Hamiltonian H exists as a bona fide operator, the motion $X \to e^{itH} X e^{-itH}$ always exists; the question of the existence of H as an operator has meaning only within a particular representation of the field observables, and can be dealt with by techniques given in[4]. (Conceivably one could ask, independently of the representation, for an H that 'represents an observable' in the sense defined in the last section, but this requirement is so strong that all non-trivial single-particle types are apparently eliminated thereby). The zero-interaction representation has the distinguished feature that any unitary single-particle displacement can be represented by a unitary transformation on the representation space. From this it can be inferred that, if the single-particle Hamiltonian is diagonalizable, then there exists a representation for the field observables in which the corresponding field Hamiltonian has rigorous existence as a self-adjoint operator.

THEOREM 2. *Let $\Sigma = (\mathfrak{H}, \mathfrak{H}', B)$ be a single-particle structure. For any symplectic (resp. anti-symplectic) transformation T on the direct sum $\mathfrak{H} \oplus \mathfrak{H}'$, there is a unique automorphism (resp. conjugate-linear automorphism) $\theta(T)$ of the algebra of field observables over Σ that takes $P(z)$ into $P(Tz)$ for all z in $\mathfrak{H} \oplus \mathfrak{H}'$, where $P(z)$ is the self-adjoint generator of the one-parameter unitary group defined by the equation*

$$U(t) = \exp[itp(x)] \exp[itq(x')] \exp[-\tfrac{1}{2} it^2 B(x,x')] \; (z = x \oplus x'; -\infty < t < \infty).$$

There is no difficulty in showing that $U(.)$ is strongly continuous, and in deriving from the Weyl relations that $U(t+t') = U(t)\, U(t')$ for arbitrary real t and t', so that $P(z)$ is well-defined. As before, the terminology that $P(z)$ is taken into $P(Tz)$ by an automorphism θ is used to mean that $\theta[\varphi(P(z)] = \varphi[P(Tz)]$ for all bounded Baire functions φ and for all canonical systems of operators. It is clear from the definition of $P(z)$ and the spectral theorem (or alternatively, from the irreducibility of the Schrödinger canonical system) that the $\varphi(P(z))$ are actually in the representation algebra $\mathfrak{A}^{\sim}$ of field observables (in every representation).

Next, note the formula

$$\exp [iP(z)] \exp [iP(z')] = \exp [\tfrac{1}{2} i (f'(x) - f(x'))] \exp [iP(z+z')]$$

for arbitrary $z = x \oplus f$ and $z' = x' \oplus f'$ in $\mathfrak{H} \oplus \mathfrak{H}'$, which follows directly from the definition of the $P(z)$ together with the Weyl relations. Conversely, from this formula the original Weyl relations follow on substituting the relevant values for z and z', since $P(.)$ extends both $p(.)$ and $q(.)$. Thus there is a one-to-one correspondence between canonical systems as originally defined and maps $P(.)$ from $\mathfrak{H} \oplus \mathfrak{H}'$ to the self-adjoint operators on a Hilbert space that satisfy the condition given by the preceding formula. Now if $P(.)$ is such a map, and if T is any symplectic transformation on $\mathfrak{H} \oplus \mathfrak{H}'$, it is immediate that the map $P'(.)$, where $P'(z) = P(Tz)$ is of the same type, and so defines a canonical system. The existence and uniqueness of the stated automorphism follow now from Theorem 1. The case of an anti-symplectic transformation follows similarly from Corollary 1.1.

It may be noted that it follows directly from the preceding formula that $P(z) + P(z') \subset P(z+z')$, but the linearity properties of $P(.)$ will not be used in this paper.

Remark 2.1. In the special case of a single-particle structure such as that described in Example 2 of Section 1, creation and annihilation operators such as are often used conventionally may be introduced. While they are not at all essential for foundational purposes, they are helpful in computational questions, their primary advantage being their invariance under complex scalars (that of the canonical variables being restricted to real scalars).

COROLLARY 2.1. *Let Σ be a standard single-particle structure and let $P(.)$ be the map defined in Theorem 2. Then the operator $P(z) - iP(iz)$ has a closure $C(z)$ for all z in $\mathfrak{H} \oplus \mathfrak{H}'$ and $C(\lambda z) = \lambda C(z)$ for all non-vanishing complex numbers λ.*

The formula established in the proof of Theorem 2 shows that the one-parameter groups generated by $P(z)$ and $P(iz)$ satisfy the Weyl relations (unless $z = 0$, in which case the conclusion of the Corollary is trivial). It follows by an argument used in the proof of Theorem 1 that the representation space can be represented as a direct product of two Hilbert spaces in such a manner that $P(z)$ and $P(iz)$ are represented as the direct products of canonical operators over a one-dimensional space with identity operators, on a Hilbert space of possibly infinite dimension. From this it is readily deduced that it suffices, in connection with the existence of the closure, to establish the existence of a closure for $P_1 + ia\,Q_1$, where (P_1, Q_1) is a canonical pair in the Schrödinger representation, and a is real; and this result is well known. To show the homogeneity of $C(.)$ under complex scalars it suffices, because of the obvious homogeneity of $P(.)$ under real non-vanishing scalars, to treat the case when $\lambda = i$, and the result then is immediate.

Other familiar elementary facts about the creation and annihilation operators can similarly be deduced by reduction to the one-dimensional case.

3. The Representation Structure Determined by a State of the Field

The field statistics arise only when additional structure, in the form of a distinguished state of the field, – representing in physical terms the underlying vacuum state, without reference to which the occupation numbers are not fully defined, – is given. The same is true of the field energy as a well-defined operator, rather than as a generator of a transformation on the dynamical variables. The occupation numbers could equally well be defined as such generators, although they do not play this role in as physically natural a way as the energy and momenta. Finally, 'clothing' of the canonical variables arises from the imposition of a definite state.

To deal with these matters, observe that any state E of the algebra $\mathfrak{A}$ of field observables over $(\mathfrak{H}, \mathfrak{H}', B)$ induces a topology on the group of non-singular symplectic or anti-symplectic linear transformations on the single-particle space $\mathfrak{H} \oplus \mathfrak{H}'$, defined as the weakest one for any element A of $E\ (A^{\theta(T)})$ is a continuous function of the transformation T, where $\theta(T)$ is as in Th. 2, and we use, as in the following, exponential notation for automorphisms. An *E-regular transformation* on $\mathfrak{H} \ominus \mathfrak{H}'$ is defined as a non-singular symplectic or anti-symplectic transformation T such that

$$E\,[(A^*A)^{\theta(T)}] \leq c(T)\,E\,[A^*A],$$

where $c(T)$ is a number that is independent of A. A *regular state* E on $\mathfrak{A}$ is one such that $E(Ae^{iP(z)}B)$ is a continuous function of z, relative to any finite-dimensional subspace of $\mathfrak{H} \oplus \mathfrak{H}'$ (in the conventional topology in this subspace, – i. e. the unique one in which it is a real linear topological space), for all pairs of elements A and B of $\mathfrak{A}$. It may be noted that, in case the representation space associated with E is separable, the foregoing functions are automatically continuous if they are Baire functions, as follows from the proof of Theorem 3 together with the fact due to von Neumann that a measurable one-parameter group of unitary transformations in a separable Hilbert space is automatically continuous.

THEOREM 3. *For any state E of the algebra $\mathfrak{A}$ of field observables over a single-particle structure $\Sigma = (\mathfrak{H}, \mathfrak{H}', B)$ there is a corresponding representation structure $(\mathfrak{K}, \varphi, \lambda, \Gamma)$, unique within unitary transformation of the complex Hilbert space $\mathfrak{K}$, such that: 1) φ is a homomorphism of $\mathfrak{A}$ into the algebra of all bounded linear operators on $\mathfrak{K}$; 2) λ is a continuous linear map of $\mathfrak{A}$ into a dense subspace of $\mathfrak{K}$; 3)*

$$[(\varphi(A)\lambda(B), \lambda(C)) = E(C^*AB)]$$

for arbitrary A, B, and C in $\mathfrak{A}$; 4) Γ is a representation of the group $\mathfrak{F}$ of all E-regular transformations on $\mathfrak{H} \oplus \mathfrak{H}'$ by continuous linear or conjugate-linear transformations on $\mathfrak{K}$, such that $\Gamma(T)\lambda(A) = \lambda(A^{\theta(T)})$ for all A in $\mathfrak{A}$ and T in $\mathfrak{F}$.

In case E is regular, there exists also a concrete canonical system $(p^{\sim}(.), q^{\sim}(.))$ over Σ with representation space $\mathfrak{K}$, uniquely determined by the condition that, for every bounded Baire function f, the image under φ of the (abstract) field observables $f(p(x))$ and $f(q(x'))$ is $f(p^{\sim}(x))$ and $f(q^{\sim}(x'))$, for all x and x' in $\mathfrak{H}$ and $\mathfrak{H}'$, respectively. For any E-regular transformation T,

$$\Gamma(T)P^{\sim}(z)\Gamma(T)^{-1} = P^{\sim}(Tz),$$

where $P^{\sim}(.)$ is defined on $\mathfrak{H} \oplus \mathfrak{H}'$ as in Th. 2. In case K is separable, $\Gamma(.)$ has a continuous restriction to any Lie subgroup.

It is well known that with any state of a C^*-algebra is associated the representation structure defined by $(\mathfrak{K}, \varphi, \lambda)$, and that this structure is unique within unitary equivalence (cf.[9]). It may be noted for future use that, automatically, $\varphi(A)\lambda(B) = \lambda(AB)$. To obtain the representation Γ, a representation Γ_0 of $\mathfrak{F}$ on the image $\mathfrak{K}_0$ of $\mathfrak{A}$ under λ may be defined by the equation $\Gamma_0(T)\lambda(A) = \lambda(A^{\theta(T)})$ for all A in $\mathfrak{A}$ and T in $\mathfrak{F}$; that $\Gamma_0(T)$ is thereby well-defined follows directly from the inequality that asserts the E-regularity of T. This regularity shows further that $\Gamma_0(T)$ is a bounded linear transformation on the dense subspace $\mathfrak{K}_0$ of $\mathfrak{K}$, and so may be uniquely extended to a continuous linear transformation $\Gamma(T)$ on all of $\mathfrak{K}$. There is no difficulty in verifying that $\Gamma_0(.)$ is a representation, and in concluding from this that $\Gamma(.)$ is also a representation.

Now let $A(z)$ denote the element of $\mathfrak{A}$, $e^{iP(z)}$, and put $\varphi(A(z)) = A^{\sim}(z)$. It is clear from the fact that φ is a homomorphism and that $A(z)$ satisfies the equation derived in the proof of Th. 2 that

$$A^{\sim}(z)\, A^{\sim}(z') = \exp\,[(i/2)\,[z, z']]\; A\,(z+z').$$

It is readily deduced that the $A^{\sim}(z)$ are unitary. The regularity condition on E means precisely that the inner products $(A^{\sim}(z)\,f, g)$ are continuous functions of z relative to any finite-dimensional subspace of $\mathfrak{H} \oplus \mathfrak{H}'$, for arbitrary fixed f and g in the dense subset $\mathfrak{K}_0$ of $\mathfrak{K}$, which implies, making use of the unitary character of the $A^{\sim}(z)$, that the same is true for all f and g in $\mathfrak{K}$. In particular, $(A^{\sim}(tz)\, f,\, g)$ is for arbitrary fixed f and g in $\mathfrak{K}$ and fixed z in $\mathfrak{H} \oplus \mathfrak{H}'$ a continuous function of t. Thus $[A^{\sim}(tz);\ -\infty < t < \infty]$ is a continuous one-parameter group of unitary operators. It follows that the group has a self-adjoint generator, which will be designated $P^{\sim}(z)$. It is immediate that $P^{\sim}(.)$ satisfies the equation derived in the proof of Th. 2, and so determines a concrete canonical system over Σ with representation space $\mathfrak{K}$, and having the stated property.

Now it is not difficult to show that

$$\Gamma(T)\ \varphi(X)\ \Gamma(T)^{-1} = \varphi(X^{\theta(T)}),$$

making use of 3) and 4). In particular, if $X = f(P(z))$ for some z in $\mathfrak{H} \oplus \mathfrak{H}'$, then it follows that

$$f(P^{\sim}(Tz)) = \Gamma(T)\, f(P^{\sim}(z))\ \Gamma(T)^{-1}.$$

To extend this to the case when f is the unbounded function $f(t) = t$, it suffices to show that W is a bounded linear transformation in a Hilbert space having an inverse of the same type, and if $Wf(A)\ W^{-1} = f(B)$ for two self-adjoint transformations A and B and arbitrary bounded Baire functions f, then $WAW^{-1} = B$. This is equivalent to showing that $WA = BW$. Now B and BW are closed transformations, and taking f to be the characteristic function of the interval $(-n, n)$, and letting $n \to \infty$, it is easy to conclude that $WA \subset BW$. By symmetry, $W^{-1}B \subset AW^{-1}$, and transforming by W gives the reverse inclusion, thereby completing the proof of the stated transformation properties.

In dealing with the last assertion of the theorem, "Lie subgroup" may be taken in the rather general sense of a subgroup in the algebraic sense, bearing a distinguished Lie group structure, together with a continuous algebraic isomorphism of it into the group of E-regular transformations. To prove the assertion, note first that for any vector u in $\mathfrak{K}$ of the form $\lambda(A)$, for some A in $\mathfrak{A}$, $|\Gamma(T)u|$ is a continuous function of the E-regular transformation T, since its square is $E[(A^*A)^{\theta(T)}]$, which is continuous by definition of the topology on $\mathfrak{F}$. By the density of $\lambda(\mathfrak{A})$ in $\mathfrak{K}$, together with the separability of $\mathfrak{K}$, $|\Gamma(T)|$ is the least upper bound of a sequence of these continuous functions, and so is a Baire function. A result due to Hille and Phillips (see the proof of Theorem 9.3.1 in [2A]) shows that the restriction of $\Gamma(.)$ to any continuous one-parameter subgroup is continuous. Using canonical coordinates in a neighbourhood of the identity in the Lie group, the continuity for a Lie group follows from that for the case of a one-parameter group.

Example 3.1. Let Σ be the single-particle structure described in Example 1.2. It is easily seen that every unitary transformation on M is symplectic. Therefore, by Theorem 2, there exists for any unitary transformation U on $\mathfrak{M}$ a (unique) automorphism of the algebra of field observables over $\mathfrak{M}$ taking $P(z)$ into $P(Uz)$, for arbitrary z in $\mathfrak{M}$. The *zero-interaction vacuum* is given by

COROLLARY 3.1. *There exists a unique regular state of the algebra of field observables over Σ that is invariant under the field action of every unitary transformation on the single-particle space $\mathfrak{M}$.*

The existence of at least one such state is clear from the form of the Fock-Cook representation, or alternatively, from the theory of the normal distribution in Hilbert space (cf.[5]). To prove its uniqueness, let E denote any other such state. Applying Theorem 3 and the von Neumann structure theorem for canonical systems over a finite-dimensional space, there exists for any finite-dimensional subspace $\mathfrak{N}$ of $\mathfrak{M}$, a non-negative operator D of absolutely convergent trace, on the representation space $\mathfrak{K}$ of a Schrödinger representation F of the algebra $\mathfrak{A}'$ of field observables over $\mathfrak{N}$, such that

$$E(A) = tr(F(A)D),\ A \text{ in } \mathfrak{A}'.$$

The invariance of E and the known decomposition of $\mathfrak{K}$ under the induced action of the unitary operators on $\mathfrak{N}$ (cf. loc. cit.) imply that D has the form

$$D = \Sigma_{k=0}^{\infty} a_k(n) P_k(\mathfrak{N}),$$

where $P_k(N)$ is the projection of $\mathfrak{A}$ onto the k-particle subspace, and the $a_k(n)$ are non-negative constants which by unitary invariance depend only on the dimension n of $\mathfrak{N}$. In the case of the standard state first described, all of the $a_k(n)$ vanish except when $k = 0$, so that by forming an appropriate linear combination it may be assumed that, for the given state E, $a_0(1) = 0$. Now since E is determined by its values on the $\mathfrak{A}'$ for large $\mathfrak{N}$, $a_k(n)$ must be non-vanishing for some $k > 0$ and $n > 1$. But this gives rise to a contradiction to the vanishing of $a_0(0)$ in the following way.

The $a_k(n)$ determine the $a_k(n-1)$ in an explicit fashion, since if $\mathfrak{N}_0$ is an $(n-1)$-dimensional subspace of $\mathfrak{N}$, then the field observables over $\mathfrak{N}_0$ form a subalgebra of those over $\mathfrak{N}$; it is clear in fact that the $a_k(n-1)$ may be expressed as positive linear combinations of the $a_k(n)$. By direct computation it is found that, if for some fixed n, $a_k(n) = \delta_{kk'}$, then $a_0(n-1) \neq 0$. Hence $a_0(n-1) \neq 0$ whenever $a_k(n) \neq 0$ for some k, and in particular $a_0(1)$ cannot vanish.

4. Covariant Single-Particle Structures

The previous definition of single-particle structure covers phenomenological aspects, but not the kinematical ones, which are naturally essential in treating field kinematics and covariant dynamics, as well as the origin of the state labels used in statistics. Therefore, before treating the clothing of field statistics and kinematics, it is necessary to be explicit concerning the single-particle kinematics and statistics. It is appropriate to define a single-particle structure with distinguished transformation properties, or for short a *covariant particle* (genus) as a single-particle structure $\Sigma = (\mathfrak{H}, \mathfrak{H}', B)$, together with a given *symmetry group* G, and a linear representation of G by symplectic or anti-symplectic transformations on $\mathfrak{H} \oplus \mathfrak{H}'$. On this basis, Th. 2 gives directly the kinematics of the Bose-Einstein field *observables* for a covariant particle; the kinematics of the *states*, which is well-defined only relative to a distinguished vacuum state, will be treated in the next section.

The statistics are somewhat more involved. In an elementary-particle field theory, a distinguished maximal commuting set of diagonalizable single-particle operators play an essential part as state labels for single particles. In a covariant theory these operators are of group-theoretic origin. This may be formalized by defining a 'quantum number algebra' for a covariant particle Σ as a maximal commutative algebra $\mathfrak{C}$ of simultaneously diagonalizable linear transformations on $\mathfrak{H} \oplus \mathfrak{H}'$, contained in the enveloping algebra $\mathfrak{R}$ of the representation U. (For present purposes the topology, if any, in which algebras are closed is irrelevant; for a standard single-particle structure, the weak operator topology may be used, but all that is essential is that the weak closure of $\mathfrak{C}$ be maximal Abelian in that of $\mathfrak{R}$). In the case of the conventional scalar particle of mass m, e.g., Example 2 applies, with $\mathfrak{M}$ taken as the usual space of normalizable wave functions, J as complex conjugation in momentum space; and then $\mathfrak{R}$ consists of all bounded linear and conjugate-linear operators on $\mathfrak{M}$ (if G is the full improper inhomogeneous Lorentz group; if G is the orthochronous group, then the conjugate-linear operations do not arise), while a suitable $\mathfrak{C}$ consists of all multiplications by bounded measureable functions of the momentum-energy (for either G). This choice of $\mathfrak{C}$ corresponds to the conventional usage of momentum-energy as quantum numbers for scalar particles, a particle of definite momentum-energy corresponding to a minimal projection in $\mathfrak{C}$, the non-existence of this in a rigorous mathematical sense (see below, however) corresponding to the fact that physical particles consist of wave packets, and do not have sharp energy-momentum.

It is important to note that all operational aspects of the particle are determined by the algebra $\mathfrak{R}$ and its commutative subalgebra $\mathfrak{C}$, as abstract algebras, independently of their representation by operators in linear spaces, together with the mapping $U(.)$ of G into $\mathfrak{R}$. In conventional theory of the standard relativistic particles, there is available,—and naturally in use, as it maximizes computational facility,—a concrete representation in which $U(.)$ is an irreducible unitary representation in a complex Hilbert space; but such a representation is not always possible, and in any event the final physical results may be obtained without its use. For foundational purposes, a better formulation of a manifold of particles than as a linear subspace of a vector space is as a projection in $\mathfrak{R}$. The particles that the theory contemplates as really observed correspond to the minimal projections in the subring $\mathfrak{C}$, a particle packet corresponding to a non-minimal projection. While minimal projections in the algebraic sense do not always exist (e.g., in standard relativistic theory they do not), effective minimal projections can be introduced mathematically by using the fact that an algebra such as $\mathfrak{C}$ is isomorphic to the bounded measurable functions on a measure space, the point of which play the role of representatives for particles with sharply defined quantum numbers. On the other hand, this particle-phase space is not needed operationally or logically, but only for clarification and interpretation of the concepts of the theory. The commutativity of the projections in $\mathfrak{C}$ implies that the occupation numbers for them will commute, so the corresponding particles may be simultaneously observed.

To give some relevant examples, as well as for correlation with standard relativistic theory and for their intrinsic interest, we now describe some special single-particle structures analogous to those of the conventional theoretical meson and photon. The examples will be based on the action of a Lie group G on a homogeneous space M; the 'relativistic case' refers to that in which G is the improper inhomogeneous Lorentz group, and M is relativistic in space-time.

4.1. Let $\mathfrak{H}$ be the space of real-valued continuous functions f on M that vanish outside compact sets, and let $\mathfrak{H}'$ be the space of real continuous differential forms of maximal dimension on M, and set $B(f, w) = \int_M fw$. Let G act as usual on $\mathfrak{H}$ and $\mathfrak{H}'$. This is a covariant single-particle structure which may be described as the *genus of all scalar particles on* $\mathfrak{M}$ *relative to the covariance group* G; in the relativistic case, it can be identified with the genus of all scalar particles, in the usual sense, of arbitrary mass.

4.2. Suppose M admits an invariant regular measure under G. The actions of G on $\mathfrak{H}$ and $\mathfrak{H}'$ are then equivalent, and a physically equivalent particle structure is obtained by redefining $\mathfrak{H}$ and $\mathfrak{H}'$ as the spaces of real square-integrable functions on M relative to the invariant measure m, and $B(f, g) = \int_M fg$ dm. Since $\mathfrak{H} = \mathfrak{H}'$, it is possible in this case, as in standard relativistic theory, to use only one space.

4.3. The space $\mathfrak{H}$ of the preceding section admitted only real scalars. The action of complex scalars on $\mathfrak{H}$ comes about in the following way. Suppose there exists a transformation W on $L_2(M)$ that preserves B commutes or anti-commutes with all the $U(g)$, and has the property that $W^2 = -I$. In the relativistic case, there is an essentially unique such W on the subspace of wave functions of real mass, i. e., those whose Fourier transforms vanish outside the duals of the light-cones, namely the Hilbert transform relative to the time variable. A complex structure is introduced into $\mathfrak{H}$ by defining multiplication by i as W. A complex Hilbert space structure arises when there is given, in addition, a transformation J on $\mathfrak{H}$, having distinguished commutation relations with the $U(g)$, (i. e., $JU(g)J^{-1} = U(g')$, where the transformation $g \to g'$ is multiplicative), and such that $J^2 = 1$, $WJ = -JW$, and $B(Jf, Jg) = B(g, f,)$. In the relativistic case, it is natural to take J as the operation representing time-reversal. It is easily seen that every element z of $\mathfrak{H}$ can be written uniquely in the form $x + iy$, where x and y are elements left fixed by J, and that $\mathfrak{H}$ attains a complex Hilbert space structure when the inner product is defined as

$$[z', z] = B(x, x') + B(y, y') + i(B(x, y') - B(x', y)),$$

relative to which the $U(g)$ are unitary or antiunitary.

4.4. The elementary-particle species of which a given genus is composed are essentially in one-to-one correspondence with the minimal projections in the center of $\mathfrak{R}$. This center may have no rigorously minimal projections, but by means of the direct integral techniques originated by von Neumann, $\mathfrak{R}$ may always be decomposed as a continuous direct sum of factors $\mathfrak{R}_M$ dependent on a parameter m, and into which G has representations U_M. Taking Example 4.3 in the relativistic case, it is not difficult to show that m can be taken as the mass, and that the elementary constituents are identical (operationally) with the conventional scalar relativistic particles of a given (real or pure imaginary) mass. There is an analytical difference, in that the state vectors of Example 3, and (essentially) also these of the elementary constituents when 'function' is suitably generalized, are real (suitably generalized) functions on M, while the conventional formalism employs positive frequency functions on the dual of M ('momentum space'), but these give abstractly identical representations of the Lorentz group, i. e., they differ only in the labels attached to the particular elementary particles; cf. Section 3 of [9A].

4.5. Let G be the group of real projective 3-space, and M the (four-dimensional) manifold of all projective lines, and let G act as usual on M. There is then no invariant measure on M under G, so that the formulation of Example 1 must be used. If one used the apparently simpler and more conventional type of space, that of all complex-valued continuous functions on M, the action of G would be complex-linear, but would not be symplectic in any natural way, so that no field kinematics would result.

4.6. Let W be a finite-dimensional irreducible representation of G on a real linear space $\mathfrak{L}$, and let $W^{\sim}$ denote the contragredient representation in the dual space $\mathfrak{L}^{\sim}$. In the situation of Example 1, the genus of all particles on M of spin type W may be defined as the covariant particle $(\mathfrak{H} \times \mathfrak{L}, \mathfrak{H}' \times \mathfrak{L}^{\sim}, B \times F; G, U \times (W + W^{\sim}))$, where $F(x, f) = f(x)$ for $x \varepsilon \mathfrak{L}$ and $f \varepsilon \mathfrak{L}^{\sim}$. In case there exists a non-singular bilinear form on $\mathfrak{L}$ that is invariant under W, there is a corresponding canonical correspondence of $\mathfrak{L}$ with $\mathfrak{L}^{\sim}$ which combines with Example 4.2 to make possible the use of only one space as in that example. If the form on $\mathfrak{L}$ is symmetric, the single-particle space will admit an invariant non-singular symmetric, but in general indefinite form. This, e. g., is the situation for the relativistic vector particles, W being here the conventional representation of G on four-vectors. Theorem 2 assures the existence of a convergent and effective field kinematics irrespective of whether there exists an invariant positive-definite inner product.

7. Somewhat more generally than Example 4.6, suppose that for a covariant particle in which $\mathfrak{H}'$ is dual to the topological linear space of Hilbert space structure $\mathfrak{H}$ and G acts contragrediently on $\mathfrak{H}'$ to its action on $\mathfrak{H}$, and in which $B(x, f) = f(x)$ for $x \varepsilon \mathfrak{H}$ and $x' \varepsilon \mathfrak{H}'$, that $\mathfrak{H}$ admits a real symmetric continuous invariant non-singular bilinear form B'. In this event, $\mathfrak{H} \oplus \mathfrak{H}'$ admits a distinguished complex structure and hermitian form; a transformation that is unitary relative to these is symplectic; and the transformations on $\mathfrak{H}$ that leave invariant the given form are represented by complex-linear transformations on $\mathfrak{H} \oplus \mathfrak{H}'$. To see this, let θ denote the map $y \to f$ of $\mathfrak{H}$ onto $\mathfrak{H}'$, where $f(x) = B(x, y)$, it is readily verified that θ is continuous and linear. Now define $i(x+f) = -\theta^{-1} f + \theta x$; then $i^2 = -1$, justifying the notation. If U is a linear transformation on $\mathfrak{H}$ leaving B invariant, and V is the corresponding symplectic transformation

$$x + f \to Ux + U^{*-1} f$$

on $\mathfrak{H} \oplus \mathfrak{H}'$, it is straightforward to verify that $iV = Vi$, so that V is complex-linear relative to the structure derived from i. Defining for $z = x + iy$ and $z' = x' + iy'$ a form

$$[z, z'] = B(x, x') + B(y, y) + i\{B(x, y') - B(x', y)\},$$

there is no difficulty in verifying that this is a hermitian form and that the symplectic transformations are precisely those preserving its imaginary part. It may also be noted that the V's commute with the canonical conjugation $x + iy \to x - iy$, which may be described in physical terms as a type of particle-antiparticle conjugation.

8. In case $\Sigma = (\mathfrak{H}, \mathfrak{H}', B)$ is as in Example 1.2, and in case the action $U(.)$ of the covariance group G is given by unitary or anti-unitary transformations on $\mathfrak{M} = \mathfrak{H} \oplus \mathfrak{H}'$, the particle genus may be called *standard*; the conventional relativistic particles are of this type.

5. Clothed Kinematics and Statistics

It will be convenient to refer to the structure obtained from the algebra of field observables over a particle genus by adding to the system a distinguished state, as 'clothed'. In so doing, we do not wish to suggest that the distinguished state must be the 'physical vacuum', which can only be defined when the dynamics is specified, nor, in general, that the clothed structure has any operational physical meaning. The present discussion is purely mathematical. However, in the special case of the physical vacuum, the 'clothing' can be said to represent the effect of the interaction, and consists, so-to-speak, of a 'cloud' of particles around the original 'bare' one created by self-interactions, in a manner that will be made more explicit in this and the following sections.

The clothed canonical variables were defined and treated in Section 2. The effect of the clothing was to make the elements of the abstract algebra into concrete operators, substantially. This section pursues a similar effect on the kinematics and statistics, and deals in particular with the construction of definite operators in Hilbert space that represent the clothed energy-momenta and occupation numbers of the field.

COROLLARY 3.2. *Let $\Sigma = (\mathfrak{H}, \mathfrak{H}', B;\ G, U(.))$ be a covariant particle genus, let E be a state of the algebra of field observables over Σ, and suppose that $U(.)$ is a continuous representation of the Lie group G by E-regular transformations. Then with the notation of Th. 3, and assuming the separability* of $\mathfrak{K}$,

$$g \to \Gamma(U(g))$$

is a continuous linear representation of G on $\mathfrak{K}$; is unitary on the subgroup G_0 of G consisting of elements leaving E invariant, $E(X^{\theta(U(g))}) = E(X)$ for all X in $\mathfrak{A}$; and for any element L of the Lie algebra of G_0, there is a unique self-adjoint operator $L^\sim$ on $\mathfrak{K}$, determined by the property that

$$\Gamma(U(\exp(tL))) = e^{itL^\sim}.$$

The proof is somewhat similar to that for the existence of clothed canonical variables, and will be omitted.

Remark 5.1. In practice, G_0 will include translation in time, whose generator L is transformed by $U(.)$ (or rather the infinitesimal transformation it induces) into the single-particle energy; and $L^{\sim}$ is then the field energy which is clothed in the sense of being a definite self-adjoint operator arising from the interaction. A variety of elementary and commonly used properties, e. g., the annihilation by $L^{\sim}$ of the vacuum state representative $\lambda(I)$, can be read off immediately from the foregoing results.

To deal with the statistics, it is necessary to use diagonalizable rather than self-adjoint operators, where a *diagonalizable operator* in a complex Hilbert space $\mathfrak{K}$ is an operator T for which there exists a non-singular operator W (both W and W^{-1} being bounded and everywhere defined) such that WTW^{-1} is normal. A collection of such operators is *simultaneously diagonalizable* in case the same W is effective for each of them. If T is diagonalizable and f is a Baire function over the complex numbers, then $f(T)$ is defined as $Wf(T)W^{-1}$; it follows readily from the polar decomposition for non-singular operators that this definition is unique, i. e., $f(T)$ is independent of the transformation W used to effect the diagonalization. The spectrum of T is defined as that of WTW^{-1}, and is similarly unique.

Now the occupation number of a single-particle state is the special case of the notion of occupation number of a linear manifold in which the manifold is the one-dimensional one spanned by the state; and any linear manifold may be correlated with an operator whose range is the manifold, under conditions valid in all interesting concrete cases. As indicated in the preceding section, this operator plays a more fundamental role than does the manifold itself. Therefore it is appropriate to show how suitable occupation numbers may be associated with a given such operator P on the single-particle space. When this space admits a suitable complex structure, e. g., in case the single-particle structure is 'standard', it is appropriate to require that $P^2 = P$, but to cover the more general case in which there is or may not be such a complex structure, it may be assumed that $P^3 = -P$. For reasons indicated in the preceding section, such an operation may be called a *particle manifold,* two such manifolds P_1 and P_2 being *simultaneously observable* in case they commute.

COROLLARY 3.3. *Let P be a particle manifold for the particle genus Σ, and suppose that $F(t) = I + P \sin t + (1 - \cos t) P^2$ is E-regular and depends continuously on t $(-\infty < t < \infty)$, and that K is separable. The one-parameter* *group $\Gamma(F(t))$ then has a densely defined diagonalizable generator N_P, whose proper values are integral, and annihilates the vacuum state representative $\lambda(I)$. The 'occupation numbers' N_P of any finite set of simultaneously observable particle manifolds P are simultaneously diagonalizable.*

By definition, the generator of the group $[\Gamma(F(t)); -\infty < t < \infty]$ has a domain consisting of all vectors u in $\mathfrak{K}$ such that $\lim_{t\to 0}(it)^{-1}[\Gamma(F(t))-I]u$ exists, and transforms u into this limit. Now the mapping $t \to \Gamma(F(t)) = V(t)$, say, is continuous, by virtue of the assumed continuity of $F(.)$, and the continuity of the restriction of $\Gamma(.)$ to any Lie subgroup, when $\mathfrak{K}$ is separable. It is evident that $V(.)$ is a representation of the additive group of the reals by continuous linear transformations on $\mathfrak{K}$. Now $(V2\pi) = I$, so the map $e^{i\theta} \to V(\theta)$ defines a continuous representation of the group of all complex numbers of absolute value one, in the usual topology, on $\mathfrak{K}$. Since this is a compact group, a well-known result due to von Neumann implies that the latter representation is similar, via a non-singular continuous linear transformation of $\mathfrak{K}$ onto $\mathfrak{K}$, to a unitary representation. The existence of a diagonalizable generator N_P now follows from Stone's theorem concerning one-parameter groups of unitary operators. That N_P has in its domain the vacuum state representative $\lambda(I)$ and annihilates it, is clear from the fact that the $V(t)$ leave invariant $\lambda(I)$.

That the proper values of N_P are integral is clear from the same fact for any generator of a continuous unitary representation of the circle group (which can be read off from Stone's theorem). Now let $P_1, \ldots, P_r$ be a finite set of simultaneously observable single-particle manifolds. The map $(e^{i}\theta_1, \ldots, e^{i\theta}r) \to \Pi_k \Gamma(F_k(\theta_k))$, where the subscript k on F indicates the replacement of P by P_k, is a continuous representation of a torus group on $\mathfrak{K}$, and as this group is again compact, the same argument as above shows diagonalizability.

Remark 5.2. There is no difficulty in verifying that, in the case of a standard single-particle structure and the zero-interaction vacuum, the present definition of occupation number gives the same as the conventional definition, according to which the number of particles with wave function x in the field is $C(x)^*C(x)$, where $C(x)$ is the creation operator for an x-particle as defined above. The relevant projection is that on the one-dimensional manifold spanned by x, the P used above being of course the multiple of this projection by i. However, for general states, the conventional definition is physically inappropriate, irrespective of the single-particle definition. This can be seen by comparison with the main physical desiderata for occupation numbers, which are: 1) they should have integral proper values; 2) they should annihilate the vacuum state representative; 3) the total field energy or momentum (etc.) should be (at least formally) representable as the sum of the products of the spectral values of the corresponding single-particle observable with the occupation numbers of the corresponding manifolds; 4) if the vacuum state representative v is in the domain of the clothed creation operator $C^{\sim}(x)$ (here we deal only with the standard single-particle structure case), then the vector $C^{\sim}(x)v$ should represent a state in which exactly one x-particle is present and no y-particle, for any y orthogonal to x.

Condition 1) is satisfied by both definitions. Condition 2) is easily seen to be satisfied by the present definition, but there is no apparent reason why it should be satisfied by the conventional definition, in fact this need not make sense, for in general v need not be in the domain of the $C(x)^*C(x)$. Formally at least, this is the case if and only if $C(x)^*C(x)$ has finite expectation value, and there is no difficulty in establishing that states exist in which $C(x)^*C(x)$ has infinite expectation value in the sense that l.u.b. $[E(T^*T)\colon T\varepsilon\mathfrak{A},\ T^*T \leqslant C(x)^*C(x)] = +\infty$. Condition 3) is easily validated in a formal way for the present definition (this suffices, as the condition is needed only for identification of concepts, and not for any analytical purposes) by taking the case when the single-particle operator H (Hamiltonian, say) is of the form $\Sigma_k \lambda_k P_k$, where the λ_k are real and the P_k as above (thus in the standard case, this H is i times the usual one; this formulation allows the same argument to be applied to both the standard and non-standard cases), then the infinitesimal generator of $[\Gamma(\exp(tH));\ -\infty < t < \infty]$ as a one-parameter group (the corresponding field Hamiltonian) should be $\Sigma_k \lambda_k N_{P_k}$. This is virtually immediate from the fact that $\Gamma(.)$ is a representation. Now if $H = \int \lambda dE_\lambda$ is the spectral resolution of H (the E_λ now being such that $E_\lambda^3 = -E_\lambda$), the integral can be regarded as a generalized sum, and in a formal way it follows that the corresponding field operator is $\int \lambda dN_\lambda$, where N_λ is the number of particles in the manifold E_λ, which is precisely what condition 3) requires. On the other hand, with the conventional definition, no such formal equality holds.

Now consider condition 4), assuming the single-particle structure to be standard. Let P_x denote the projection on the single-particle space whose range is the one-dimensional manifold spanned by x. We proceed quite formally, making no attempt at analytical justification. The vacuum state v is $\lambda(I)$, and is taken by the clothed creation operator $C^{\sim}(x)$ into $\lambda(C(x))$. Now a state vector u in $\mathfrak{K}$ represents a state containing one x-particle provided $\Gamma(e^{itP_x})u = e^{it}u$, and one containing no y-particle in case

$$\Gamma(e^{itP_y})\,u = u.$$

Taking now

$$u = \lambda(C(x)),\ \Gamma(e^{itP_x})\,u = \lambda(C(e^{itP_x}x)) = \lambda(C(e^{it}x)) = e^{it}\lambda(C(x))$$

[by the homogeneity of λ and $C(.)$ under complex scalars] $= e^{it}u$. On the other hand,

$$\Gamma(e^{itP_y})\,u = \lambda(C(e^{itP_y}x)) = \lambda(C(x)) = u.$$

With the conventional definition, no such formal development is possible. On the other hand, there may well be no $x \neq 0$ such that v is in the domain of $P(x)$.

The question arises as to whether an interpretation of the $C(x)^*C(x)$ as particle numbers in some sense is possible. In terms of (physically fictitious, but conceptually graphic) 'pristine' particles, by which we mean the particles constituting the field, in their original state of zero interaction, before the physically real interaction is 'switched on', such an interpretation can be given. These pristine particles are devoid of self-fields and have a certain resemblance to the 'bare' particles introduced in a variety of current theories, but in view of the great analytic complexity and lack of uniformity in these treatments, we shall not attempt a precise comparison between the 'pristine' and 'bare' particles. The 'switching on' of the interaction changes the vacuum state of the field from the zero-interaction vacuum, relative to which the pristine particles proceed independently of each other in accordance with their respective single-particle kinematics, to an equilibrium state for the given interaction, representing the 'physical vacuum'. In this state, no 'physical' particles are present, in the sense that the vacuum representative is annihilated by the occupation numbers as just defined; but a large, possibly infinite, number of pristine particles may be present. The total number of pristine particles will be represented by $\Sigma_k C(e_k)^*C(e_k)$, where the e_k constitute a complete orthonormal set in the single-particle space; this operator will generally have no more than formal existence, i.e., be identically infinite from a physical viewpoint. The total number of physical particles, on the other hand, is the generator of the one-parameter group $\Gamma(e^{itI})$,

$-\infty < t < \infty$ (where I is the identity operator in the single-particle space, assumed standard), which will exist as a finite diagonalizable operator under the stated relatively weak regularity conditions.

Remark 5.3. 'Physical particle' as used here is not the same as the conventional theoretical concept, which is tied to the use of the zero-interaction representation for the incoming and outgoing fields. An 'empirical' physical particle, —a theoretical counterpart to that observed in reality,— may be defined as one whose wave function is in the subspace of $\mathfrak{K}$ in which the total number of particles is unity. There need be no natural way to build up $\mathfrak{K}$ from these empirical particles (if indeed they exist in substantial numbers), and in fact the existence of bound states is a contra-indication for this. The present physical particles may be designated as 'primary' in distinction from the 'empirical' ones, since they evidently constitute the theoretically basic concept. Either type of particle may be defined as 'elementary' in case the transforms of its wave functions under the covariance group span an irreducible subspace (under the group). There is no mathematical reason why the elementary empirical particles should in general transform equivalently to the elementary primary ones which are given independently of the interaction. A heuristic argument suggests that, when the scattering automorphism depends continuously on a coupling constant, the empirical particles should have the same discrete quantum numbers (spin, etc.) as corresponding primary particles (correspondence being in the sense that the empirical particle wave function is in the closure of the image under λ of the subalgebra of bounded functions of the canonical variables for the primary particle), but may well differ in continuous quantum numbers (= the mass, in standard relativistic theory). In principle such mass differences are computable, but it should be noted that there need be no empirical particle corresponding to a given primary particle, as the entire corresponding subspace of $\mathfrak{K}$ may consist of states composed from at least two primary particles, and even when it contains a single-particle state there is no theoretical assurance of uniqueness.

Another point of difference between the representation structure obtained above, and the zero-interaction structure commonly used for the representation of free physical fields, is that,—apart from a heuristic continuity argument,—conceivably, the occupation numbers of a maximal simultaneously observable set of single-particle manifolds do not generate a maximal commuting set of operators in $\mathfrak{K}$. Since $\mathfrak{K}$ is spanned by the transforms of v under the bounded functions of the clothed canonical operators, $\mathfrak{K}$ is spanned by states each of which is built up from a finite number of primary particles. Lack of maximality would mean that distinct states could have identical constitutions in terms of elementary primary particles. This would suggest the existence of a certain type of bound state, i. e., an r-particle state whose transforms under the covariance group span a subspace not transforming according to the direct product of r elementary-particle types. The circumstance cited in the preceding paragraph seems, however, more likely to account for the physical existence of bound states.

It is noteworthy also that, although any *finite* set of occupation numbers of commuting projections is simultaneously diagonalizable, this need not always be the case for all occupation numbers of a maximal simultaneously observable set of single-particle manifolds. The elementary physical interpretation is thereby not materially affected, since virtually any experiment can be interpreted in terms of a large finite set of states, and since, if the physical system under consideration is 'enclosed in a large box', simultaneous diagonalization can be effected under reasonable regularity conditions, despite the infinity of the number of single-particle states, due to the discreteness of the single-particle manifolds which then ensues, together with the compactness of the group of all unitary operators in a maximal Abelian algebra of operators on Hilbert space that is generated by its minimal projections, which renders applicable the argument given above. Nevertheless it is possible that this complication may be significant in relation to convergence questions in a fully covariant theory.

For similar reasons, it is no essential loss of generality, in dealing with a finite set of states, to assume that the clothed particle operators are in the zero-interaction representation; but materially wrong or self-contradictory results may then ensue if it is further assumed that the occupation-number operators may be defined in the conventional way. While this may appear to be a plausible simplifying assumption, its cogent character is indicated by Corollary 3.1, which implies that the only field in which the clothed canonical variables and occupation numbers are unitarily equivalent to those of the zero-interaction theory is the trivial one without interaction. In fact, it asserts even more strongly that the occupation numbers are self-adjoint in the intrinsic Hilbert space metric in $\mathfrak{K}$ (in which the clothed canonical variables are self-adjoint) only if there is no interaction; i. e., roughly speaking, the 'wave' and the 'particle' operators cannot be made simultaneously self-adjoint.

6. Collision Dynamics

In the light of the foregoing and the literature on the relation of abstract C^*-algebras to quantum mechanics, it is clear that the appropriate definition of a dynamical transformation is as an automorphism of the algebra of field observables. This is physically more operational as well as mathematically more natural than the conventional assumption that a dynamical transformation is represented by a unitary operator. In fact, it makes little sense in itself to ask whether a given automorphism is inducible by a unitary transformation, as it may be so induced in certain representations but not in others.

Thus a conventional-type dynamics would be given by the assignment to each ordered pair s, s' of 'space-like' surfaces with s' later than s of an automorphism $a_{s,s'}$, of the algebra of field observables, this automorphism being interpreted as the transformation of a dynamical variable on s into the corresponding variable on s', which interpretation requires the mathematical assumption that $a_{s,s'}\ a_{s',s''} = a_{s,s''}$. Such a dynamics would be (Lorentz-) covariant in case $\theta(g)\ a_{s,s'}\,\theta(g)^{-1} = a_{g(s),\,g(s')}$ for every transformation g of the Lorentz group, where $\theta(g)$ is the automorphism of the algebra of field observables that corresponds to g in accordance with Theorem 2, while $g(s)$ denotes the space-like surface into which s is carried by g. For a collision dynamics, corresponding to the partial simplification of the foregoing in which only the limit of $a_{s,s'}$, as s and s' tend to the infinite past and future respectively, there is a single automorphism a, and the dynamics is covariant when a commutes with all the $\theta\,(g)$. In the following, we restrict attention to the analogue of this type of dynamics, which may quite possibly be all that is observable physically, and in any event plays a fundamental theoretical role.

In principle, the specification of a completely determines the collision dynamics. Given any 'incoming' state E of the algebra of field observables, the corresponding 'outgoing' state is E^a, where for any field observable X, $E^a(X) = E(X^a)$, the action of a and other automorphisms being written exponentially when convenient. But to make any connection with what is physically observed, it is evident that the states must be 'labeled' in identifiable terms, and here the essential and material complication arises that, roughly speaking, it is not the absolute state E that is observed, but rather its deviation in a certain sense from the 'physical' vacuum state.

Conventionally, the physical vacuum state and the related notion of physical particle are introduced, through the use of the total field Hamiltonian, in a fashion that leads rapidly to divergences. In addition to the low mathematical viability of this approach, it is relatively unoperational. A simpler and more direct approach involves rather the definition of the vacuum from the 'scattering' automorphism α, a basic desideratum being invariance under α. In addition in a covariant (e. g. relativistic) theory, it should also be invariant under the covariance group (e. g. the Lorentz group), or at least under the associated transformations used to label the single-particle states (e. g., translations in space-time, whose generators give the energy-momentum operators). Physically these requirements may be expected to determine essentially uniquely the physical vacuum for the automorphisms describing real interactions, as otherwise there should be a hitherto unobserved selection rule.

The next result is to the effect that, for any given covariant automorphism, a physical vacuum exists, together with an associated analytical structure adequate for the fundamental physical interpretations. To fix the ideas, this distinguished automorphism will be called the 'scattering' automorphism. A state invariant under this automorphism and under a distinguished subgroup G_0 of the covariance group, which is not a non-trivial convex linear combination of two other such states, is called a 'physical vacuum' relative to G_0 and the automorphism.

THEOREM 4. *For any covariant scattering automorphism of the algebra* $\mathfrak{A}$ *of field observables over a covariant particle genus, and any maximal Abelian subgroup* G_0 *of the covariance group* G, *there exists a physical vacuum state. The corresponding representation space* $\mathfrak{K}$ *is irreducible under the joint action of the clothed field observables, the scattering automorphism, and the clothed unitary representation of* G_0. *In case the single-particle structure is standard, or more generally if there exists a state of* $\mathfrak{A}$ *invariant under all of* G, *then a physical vacuum relative to all of* G *exists.*

The states of $\mathfrak{A}$ form a compact convex subset Δ of the dual, in its weak topology relative to $\mathfrak{A}$. The scattering automorphism α and the Abelian subgroup G_0 act on the dual to $\mathfrak{A}$ continuously and in such a fashion as to leave invariant Δ. A well-known variant of a fixed-point theorem due originally to BIRKHOFF and KELLOGG allows the conclusion to be drawn that the subcollection of states invariant under α and G_0 is non-empty. It is easily seen that this subcollection is again compact and convex, and so by the Krein-

Milman theorem contains an extreme point, which is a physical vacuum state as defined above. The irreducibility of the joint action on $\mathfrak{K}$ of $\varphi(\mathfrak{A})$, G_0 (via $\Gamma(.)$), and the unitary operator S determined uniquely by the condition $S\lambda(X) = \lambda(X^a)$, for all X in $\mathfrak{A}$, follows by a trivial variation of the proof of Th. 5.3 in [7].

Now suppose there exists a state E_0 of $\mathfrak{A}$ that is invariant under the action of all of G,—such as the zero-interaction vacuum in the case of a standard single-particle structure. Varying the foregoing proof by replacing Δ by the subset consisting of the least convex closed subset of Δ containing E_0 and all of its transforms by positive and negative powers of a, one still obtains a closed convex set of states, each of which is invariant under G, and the totality of which is invariant under a. By the same fixed-point theorem, there exist elements of this subset that are invariant under a. The sub-collection of all such elements is again a compact convex set, and so by the Krein-Milman theorem contains an extreme point, which is then a physical-vacuum state relative to all of G and the given scattering automorphism.

Remark 6.1. By this result, the main burden of obtaining a convergent dynamics is shifted onto the problem of setting up the scattering automorphism for the physical system under consideration. The material simplification brought about by the use of an automorphism rather than a unitary operator is that its form and existence are independent of the employment of special representations, while the S-operator itself will be unitary in general only in one particular representation, which it is part of the theoretical problem* to determine. To illustrate this point, consider a purely hypothetical theory in which S is given in a purely formal manner as exp $[ig\, \Sigma_k\, Q_k^2]$, where the single-particle structure is assumed standard, and $Q_k = q(e_k)$; $e_1, e_2, \ldots$ being a complete orthonormal set in the single-particle space. There is no doubt that $\Sigma_k Q_k^2$ fails to exist as an operator in the zero-interaction representation, nor is there any other apparent canonical system over a single-particle Hilbert space in which this formal expression would appear to define an operator. Nevertheless a corresponding automorphism can be set up fairly briefly, and it follows that S can be represented by a bona fide unitary operator in a certain Hilbert space.

* As here formulated. In certain other current approaches (e.g. those of HAAG, KÄLLEN and WIGHTMAN, and of LEHMANN et al.) it is essentially part of the postulates of the theory that the S-operator is unitary in the zero-interaction representation. This is a substantial implicit restriction on the interaction, and in fact no example of such a theory with real particle creation has yet been constructed.

Remark 6.2. The situation is particularly simple in the case of the single-particle structure of Example 3, Section 1, which is often used in approximate non-covariant theories. The typical interaction Hamiltonian in these theories is a linear form in the P_n and Q_n whose coefficients are bona fide operators in Hilbert space. In certain cases the coefficients of the P_n vanish (or do so after a suitable transformation), and the transformation $Q_n \to a_n Q_n$, $P_n \to a_n^{-1} P_n$, with $a_n \to 0$ sufficiently rapidly will generally convert a divergent such hamiltonian into a hermitian (densely-defined) operator in a Hilbert space. As the time-ordered exponential of the interaction Hamiltonians (in the interaction representation), the formal S-operator induces a (scattering) automorphism which is a Stieltjes product integral of automorphisms generated by such Hamiltonians, and the crucial point in its existence has always been the finiteness of the generator, which follows as indicated. In particular, when the coefficients are operators in Hilbert space such that each is bounded, a transformation of the indicated type exists.

Remark 6.3. For the validity of Theorem 4 and its physical interpretation, it is not essential that the scattering automorphism a be onto. The only difference is that if a is merely into, the emergent S-operator will be merely isometric, and not necessarily unitary. This corresponds to a type of dissipative process for which there is no indication in elementary-particle physics, but which could conceivably be applicable in thermodynamical situations.

It is not even altogether essential that a be into; it would suffice substantially for it to be given as a homomorphism of the algebra $\mathfrak{A}$ of field observables in a concrete representation, into the bounded operators on the representation space. The physical vacuum state E is then defined as satisfying $E(X) = E(X^a)$ in case the field observable X is such that X^a is in $\mathfrak{A}$, together with invariance under G_0 and ergodicity as before, and its existence follows from an extension of the fixed-point theorem mentioned to mappings from points to convex sets (cf. [1 A]). The mapping in question takes a state E_0 invariant under G_0 into the set of all states E_1 of $\mathfrak{A}$ such that $E_1(X) = E_0(X^a)$ whenever X^a is in $\mathfrak{A}$. As an illustration, consider the purely hypothetical theory with standard single-particle structure in which S is given in a formal way as $\exp [i \Sigma_k b_k Q_k^{2r}]$, where $b_1, b_2, \ldots$ is a bounded sequence of real numbers, and r is a positive integer. This is a quite divergent and generally intractable expression, but it may be shown without particular difficulty that, in the zero-interaction representation, it can be formulated as a homomorphism of $\mathfrak{A}$ into the bounded operators. It then follows that there exists a representation in which a may be represented by an isometric transformation S defined in a certain subspace of $\mathfrak{K}$, in the sense that $\lambda(A^a) = S\lambda(A)$ whenever A and A^a are both in $\mathfrak{A}$. Actually, it is plausible that the image under the homomorphism described is $\mathfrak{A}$ itself, in which case S would be unitary and defined on all of $\mathfrak{K}$, but the verification of this would be technically tedious and is not required for the existence of a physical vacuum in the foregoing sense.

Remark 6.4. We are not in a position to treat the uniqueness and the regularity of the physical vacuum, but both seem very plausible for the case of a theoretical description of real particles, and considerably more far-reaching assumptions are commonly made. One such assumption, which we are unable to substantiate theoretically, is that the clothed canonical variables act irreducibly on $\mathfrak{K}$. The justification is that, in a formal way, S and the clothed kinematics are given as functions of the canonical variables, so that from the irreducibility under the action of all of these, as stated in Theorem 4, it is reasonable to conclude the irreducibility under the canonical variables alone. It may well be true that, whenever a and the action of G_0 on the field observables are limits in some suitable sense of inner automorphisms,—which is a way of formulating the plausible requirement that the theory be obtainable as a limit of cut-off theories,—then this irreducibility follows. However, not enough is known at present concerning the approximation of the scattering automorphism by inner ones either to make this line of attack effective, or to demonstrate its insufficiency.

Remark 6.5. It should be noted that even when the physical vacuum state is not invariant under all of G (this is state in the sense of expectation value; the corresponding state vector in $\mathfrak{K}$ is always invariant under all of G), the physical results of the theory, i. e., the S-matrix elements between finite-particle states labeled by the given maximal Abelian family of quantum numbers, are nevertheless fully invariant. For although a transformation g of G will in general change the physicel vacuum state E into a new state E^g, defined by the equation $E^g(X) = E(X^{\theta(U(g))})$, a similarity transformation on the quantum numbers, by $U(g)$, is required, resulting in a new maximal Abelian Algebra conjugate to the original one; and due to the covariance of the scattering automorphism, the two effects cancel.

Remark 6.6. It may be noted that there are a variety of algebras similar to the algebra of field observables as defined above that could be used in place of it without any essential change in the foregoing. Theorems 1—4 would be substantially unaffected if $\mathfrak{A}$ were replaced by the C^*-algebras generated by all bounded Baire (or continuous, or uniformly continuous) functions of the canonical variables; and would be valid in modified form for the C^*-algebra generated by the continuous functions vanishing at infinity of the canonical variables. Which of these algebras to use would appear to be mainly a matter of technical convenience. On physical grounds, it may be expected that the resulting representation spaces relative to the physical vacuum (and accompanying structures) would be identical; this is quite parallel to the circumstance that the L_2-completions of the Baire functions, continuous functions, and polynomials in [0,1] are identical. There is no difficulty in formulating in precise mathematical terms the relatively weak regularity assumptions on the physical vacuum under which the S-matrix finally obtained will be independent of the type of algebra employed. It is important to note also that the foregoing work applies to the still more general formulation of dynamics in terms of transformation of the linear forms that define expectation values in states (in mathematical terms, only the dual of the scattering automorphism is really needed). In this situation, the unitary *S-operator* would have to be replaced by a bilinear form on $\mathfrak{K}$ (the *S-matrix*), but otherwise Theorems 3 and 4 are substantially unaffected. A dynamics of this type is essentially completely determined by the knowledge of the vacuum expectation values $E_{vac}[e^{iP}(out)^{(z)}\, e^{iP}(in)^{(z')}]$ as a function of the two single-particle wave functions z and z'.

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