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*DEDICATED TO PROFESSOR NIELS BOHR ON THE OCCASION OF HIS 70TH BIRTHDAY*

# ON THE POTENTIAL COLLECTIVE FLOW OF A ROTATING NUCLEUS WITH NON-ELLIPSOIDAL BOUNDARY

BY

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## Introduction.

Collective excitation spectra of rotational type are associated with nuclei possessing an equilibrium shape which deviates strongly from spherical symmetry.[1]

The rotational character of the motion is shown by the energy ratios, spins, and intensities, which also give evidence that the nucleus possesses axial symmetry and rotates about axes perpendicular to the symmetry axis.

More detailed information about the collective rotational motion is obtained from the moments of inertia, which can be determined from the observed rotational energy levels. The moments are found to be appreciably smaller than they would be if the nucleus performed a rigid rotation and, in addition, they have a strong dependence on the nuclear deformation. The collective motion of the nucleus has been compared with the hydrodynamical flow, assumed irrotational or potential, of a liquid drop whose boundary is rotating without change of form. The corresponding classical hydrodynamical problem has been studied extensively in connection with the theory of rotating stars (cf. LAMB). An exact solution has been given in the case of a rotating ellipsoid with constant density.[2]

This potential flow for an ellipsoidal boundary has been used by A. BOHR and B. MOTTELSON with a somewhat generalized density distribution such that the surfaces of constant density are similar ellipsoids.

[1] For a survey of the theory of rotational states and of the available experimental evidence, cf. BOHR, 1954; BOHR and MOTTELSON, 1955. Cf. also BOHR, FRÖMAN, and MOTTELSON, 1955; ALAGA, ALDER, BOHR, and MOTTELSON, 1955.

[2] For the rotating ellipsoid, the condition of constant pressure at the surface can be fulfilled, assuming Newtonian attraction, both in the case of irrotational flow and for a flow without internal motion. In the case of a nucleus, the surface condition, of course, has quite another aspect.

For the case of an ellipsoid of revolution of constant density, rotating about an axis perpendicular to its symmetry axis, the moment of inertia is given by

$$\mathfrak{J} = \frac{a^2 e^4}{5(2-e^2)} M, \tag{1}$$

where $a$ is the major semi-axis, $e$ the eccentricity, and $M$ the mass of the nucleus. The nuclear eccentricity may be determined from the quadrupole moment of the nuclear shape

$$Q_0 = \pm \frac{2}{5} a^2 e^2 Z, \tag{2}$$

where $Z$ is the nuclear charge number, and where the positive and negative signs refer to prolate and oblate shape, respectively.

It is found, however, that the moments of inertia, calculated from (1) by means of the observed quadrupole moments, are smaller than the observed moments of inertia by a factor of about three to five. This situation is not appreciably changed by considering the above-mentioned generalized density distribution.

A possible reason for this discrepancy could lie in the assumed density distribution. It has been suggested (Johnson and Teller, 1954) that the protons are more concentrated towards the centre of the nucleus than are the neutrons. Such an effect means a smaller value of "$a$" in (2) than in (1). This increases the moment of inertia calculated from $Q_0$. Since, however, the expected differences in "$a$" are only of the order of 20 %, this effect cannot account for more than a minor part of the discrepancy.

As pointed out by Bohr and Mottelson, another possible way of explaining the discrepancy within the framework of the potential flow model would be to consider nuclear boundaries deviating from the ellipsoidal shape. In order to investigate this point, the potential flow has been calculated for some boundaries of more general form, illustrated in Figs. 1, 2, and 3.

These calculations, reported in the following, show that the moment of inertia as well as the quadrupole moment is quite sensitive to relatively small deviations from ellipsoidal shape, but indicate that the ratio of $\mathfrak{J}/Q^2$, which is the quantity that can be directly compared with the experimental data, is affected to a much lesser extent.

Our results thus indicate that the model considered cannot be expected to account for the observed magnitude of $\mathfrak{J}/Q_0^2$, and, therefore, suggest significant departures of the collective motion from potential flow.

This conclusion is in accord with the recent findings of BOHR and MOTTELSON (1955) who have investigated the validity of the assumption of potential flow for the nuclear collective motion, and who have shown that important deviations from potential flow are to be expected as a consequence of the nuclear shell structure. The effect is to increase the moment of inertia, and estimates indicate that it is possible in this way to account for the magnitude of the observed moments.

## Characteristics of the flow and of the considered nuclei.

Assuming a constant density for the nuclear fluid, the velocity, assumed irrotational, obeys the equations

$$\operatorname{rot} \vec{v} = 0, \quad \operatorname{div} \vec{v} = 0. \tag{3}$$

Introducing $\vec{v} = \operatorname{grad} \Phi$ (for convenience $\Phi$ is chosen as the negative of the velocity potential), we get

$$\Delta \Phi = 0. \tag{4}$$

From the rotation, the boundary obtains a velocity whose normal component shall be equal to the normal component of the potential flow. Thus, the boundary condition is time-dependent. A coordinate system fixed in space is denoted by $(X, Y, Z)$, and the rotating system by $(xyz)$. Then the $(xyz)$ depend on $(X, Y, Z, t)$. In the following, $\Phi$ and other quantities are expressed in $(xyz)$, and are thus time-dependent.

We first consider the case of an ellipsoid of revolution with axes 2a and 2b, the symmetry axis being 2a. The ellipsoid rotates about an axis perpendicular to its symmetry axis with angular velocity $\omega$. In the body-fixed coordinates, the boundary condition is constant and given by

$$\omega\,(a^2 - b^2)\,xy = \frac{\partial \Phi}{\partial x} b^2 x + \frac{\partial \Phi}{\partial y} a^2 y, \tag{5}$$

where the axis of rotation is chosen as the $z$-axis, while the $x$-axis is the nuclear symmetry axis.

The boundary condition is fulfilled by the potential

$$\Phi = \omega \frac{a^2 - b^2}{a^2 + b^2} xy. \tag{6}$$

For this flow, one calculates

$$E_{\text{rot}} = \frac{1}{2}\int \varrho\, \vec{v}^{\,2}\, d\tau = \frac{1}{2} \frac{M a^2 e^4}{5\,(2 - e^2)} \omega^2, \tag{7}$$

corresponding to the value (1) for the moment of inertia.

In elliptic coordinates, the exact solution for the ellipsoid quoted above has the property of being the first term in an expansion in harmonic functions. There is therefore some advantage in using elliptic coordinates. They are given by (see, e. g., LAMB: Hydrodynamics, p. 139)

$$\left.\begin{aligned} x &= k\,\mu\,\zeta, \\ y &= k\sqrt{1-\mu^2}\cdot\sqrt{\zeta^2 - 1}\,\cos\varphi, \\ z &= k\sqrt{1-\mu^2}\cdot\sqrt{\zeta^2 - 1}\,\sin\varphi. \end{aligned}\right\} \tag{8}$$

For constant $\zeta = \zeta_0$, the curve in $\mu$ is an ellipse with $a = k\zeta_0$ and $b = \sqrt{a^2 - k^2}$. Thus, $2\,k$ is the focal distance and $\zeta_0 = e^{-1}$. Constant $\mu = \mu_0$ gives a hyperbola in $\zeta$ with $a = k\mu_0$ and $b = \sqrt{k^2 - a^2}$.

We will now seek other boundaries than the ellipsoidal one. This can be done by giving $\zeta$ of the boundary as a suitable function of $\mu$. In this note, we will use

$$\zeta = f(\mu) = \zeta_0 + C_0 + C_2\mu^2 + C_4\mu^4. \tag{9}$$

An even function of $\mu$ has been chosen in order to describe nuclei with reflection symmetry. For unsymmetric nuclei, the full series in $\mu$ would be needed.

For comparison with the ellipsoidal case, two different shapes of the nucleus will be treated, both converging into an ellipsoid for small parameters.

1. The volume and the major axis remain constant.
2. The volume and the quadrupole moment $Q_0$ remain constant.

*First approximation.* In the constants $C_n$, the change of volume is

$$\Delta V = \text{const.} \left| C_0 + \frac{C_2}{3} + \frac{C_4}{5} - e^2\left(\frac{C_0}{3} + \frac{C_2}{5} + \frac{C_4}{7}\right)\right]. \qquad (10)$$

The change of the major axis is

$$\Delta a = k\,(C_0 + C_2 + C_4). \qquad (11)$$

Further,

$$\varepsilon Q_0 = \int d\tau \varrho_e (3\,x^2 - r^2), \qquad (12)$$

where $\varepsilon$ is the total charge.

For $\varrho_e$ constant, we get in first approximation

$$Q_0 = \frac{2}{5} k^2 \left[1 + C_0 \frac{3\,e - e^3}{1 - e^2} + C_2 \frac{14 - 5\,e^2 - 3\,e^4}{7\,e\,(1 - e^2)} + C_4 \frac{36 - 21\,e^2 - 5\,e^4}{21\,e\,(1 - e^2)}\right]. \qquad (13)$$

In the following we will especially illustrate the calculations for the eccentricity $e = {}^4/_5$, which gives a quadrupole moment of the same order as, though somewhat greater than, is common among the rare earth nuclei. This gives the following relations for the $C_n$ in the two cases under consideration.

$$\left.\begin{array}{lll} 1. & C_0 = -C_2 \cdot 0.1430 & C_4 = -C_2 \cdot 0.8570. \\ 2. & \overline{C}_0 = -\overline{C}_2 \cdot 0.0863 & \overline{C}_4 = -\overline{C}_2 \cdot 1.267. \end{array}\right\} \qquad (14)$$

The coefficients for moderate changes of the surface turn out to be relatively large. In the following we will therefore check the first order approximations of the various physical quantities by numerical calculations for a special surface. We choose case 1 (constant volume and major axis) with $C_2 = 0.4$. To first approximation we get

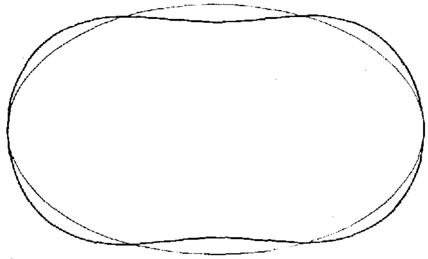

Fig. 1. $V$ and $a$ constant. $C_2 = 0{,}4$

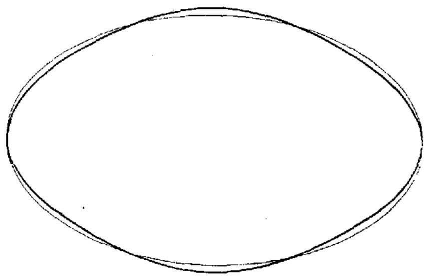

Fig. 2. $V$ and $a$ constant. $C_2 = -0{,}2$

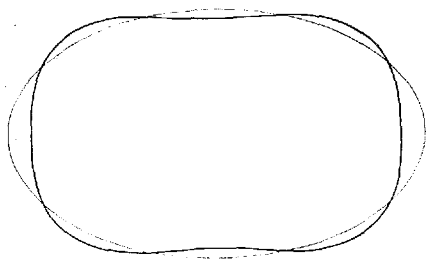

Fig. 3. $V$ and $Q_0$ constant. $\overline{C}_2 = 0{,}4$

$$C_0 = -0.0572, \; C_4 = -0.3428.$$

By numerical calculation,

$$C_0 = -0.0591, \; C_4 = -0.3409.$$

Fig. 1 gives the boundary for these last values. Fig. 2 gives the boundary for $C_2 = -0.2$. Case 2, (constant $V$ and $Q_0$) with $\overline{C}_2 = 0.4$, is illustrated in Fig. 3.

In case 1 we get the following values of $Q_0$.

a) First approximation:

$$Q_0 = \frac{2}{5} k^2 [1 + C_2 \cdot 1.091] \quad (15)$$

or for

$$C_2 = 0.4 \colon Q_0 = k^2 \cdot 0.5746.$$

b) Numerical calculation

$$Q_0 = k^2 \cdot 0.5780. \quad (16)$$

### Calculation of $S_0$.

It is of some interest to calculate also the $2^4$-pole moment of the nuclear shapes considered. This moment may be defined by

$$\varepsilon S_0 = \langle P_4 \rangle = \int d\tau \, \varrho_e (35 \, x^4 - 30 \, x^2 r^2 + 3 \, r^4). \quad (17)$$

For the ellipsoid, this gives

$$S_0^0 = k^4 \cdot 0.6857. \quad (18)$$

In first approximation,

$$S_0 = S_0^0 \left[1 + \frac{1}{99\, e^3 (1-e^2)} \left\{ C_0 \cdot 99\, e^4 (3 - e^2) \right.\right.$$
$$\left.\left. + C_2\, 11\, (28\, e^2 - 15\, e^4 - 3\, e^6) + C_4\, (88 + 144\, e^2 - 147\, e^4 - 15\, e^6) \right\}\right] \quad (19)$$

Case 1. $S_0 = S_0^0 (1 + C_2 \cdot 0.426) = k^4\ 0.8026.$ (20)

The numerically calculated value is: $S_0 = k^4\ \ 0.8032.$

Case 2. $S_0 = S_0^0 (1 - \overline{C}_2\ 1.883) = k^4 \cdot 0.1692.$ (21)

## Determination of the flow.

In elliptical coordinates, the potential equation is as follows:

$$\frac{\partial}{\partial \mu}\left[(1-\mu^2)\frac{\partial \Phi}{\partial \mu}\right] + \frac{1}{1-\mu^2}\frac{\partial^2 \Phi}{\partial \varphi^2} = \frac{\partial}{\partial \zeta}\left[(1-\zeta^2)\frac{\partial \Phi}{\partial \zeta}\right] + \frac{1}{1-\zeta^2}\frac{\partial^2 \Phi}{\partial \varphi^2}\,. \quad (22)$$

The general solution, free from singularities inside the boundary, is

$$\Phi = \sum_{n,\, s} P_n^s(\mu)\, P_n^s(\zeta)\, [A_{ns} \cos s\, \varphi + B_{ns} \sin s\varphi]\,, \quad (23)$$

where $P_n^s$ is the usual spherical harmonic.

The coefficients $A_{ns}$ and $B_{ns}$ can be obtained from the boundary condition

$$\frac{\partial \Phi}{\partial s_n} = V_n^{\omega} \quad (24)$$

which expresses the equality on the boundary of the normal velocities of the potential flow and of the rotation ($\zeta = f(\mu)$).

Here,

$$V_n^{\omega} = \omega k^2 \frac{\sqrt{1-\mu^2}\sqrt{\zeta^2-1}}{B(\mu)} (\mu + \zeta f'(\mu)) \cos \varphi \quad (25)$$

with

$$B(\mu) = k\sqrt{\zeta^2 - \mu^2} \cdot \sqrt{\zeta^2 - 1 + (1-\mu^2)\, f'(\mu)^2}\,. \quad (26)$$

Further:

$$\frac{\partial \Phi}{\partial s_n} = \frac{\partial \Phi}{\partial \zeta}\frac{\zeta^2-1}{B} - \frac{\partial \Phi}{\partial \mu}\frac{(1-\mu^2)f'(\mu)}{B}. \qquad (27)$$

Now, $\frac{\delta \Phi}{\partial s_n} = V_n$ shows that $\Phi$ contains the factor $\cos \varphi$.
Thus, $s = 1$; $B_{ns} = 0$.

We get the following expression for $\Phi$:

$$\Phi = \sum_n A_n \sqrt{1-\mu^2}\, P_n'(\mu) \sqrt{\zeta^2-1}\, P_n'(\zeta) \cos \varphi. \qquad (28)$$

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Before treating the more general case, we give the flow for the ellipsoid treated above, this time in ellipsoidal coordinates.

Then, $f'(\mu) = 0$, and the boundary condition gives

$$\sum_n A_n P_n'(\mu)\,[\zeta P_n'(\zeta) + (\zeta^2-1) P_n''(\zeta)] = \omega k^2 \mu. \qquad (29)$$

For constant $\zeta = \zeta_0$ we get: $P_n'(\mu) = \text{const} \cdot \mu$, which means $n = 2$. Thus, for an ellipsoid, the equation is satisfied by the first term in the expansion, with

$$A_2 = \frac{a^2 e^4}{9\,(2-e^2)}\,\omega. \qquad (30)$$

In the more general case, we will try to satisfy the boundary condition by taking into account further terms in the series for $\Phi$. All calculations can be made explicitly. However, the practical difficulties rise rapidly with the number of terms. In the following, only the three first terms of $\Phi$ have been used, as they give an accuracy that seems satisfactory in the present case.

The three coefficients $A_n$ in

$$\Phi = \sum_{2,4,6} A_n \Phi_n \qquad (31)$$

will be determined by two different methods.

**A.** The expressions in the boundary condition

$$\frac{\partial \Phi}{\partial s_n} = V_n^{\omega} \qquad (32)$$

are developed in power series of $\mu$, and the coefficients for $\mu$, $\mu^3$, and $\mu^5$ are identified. Thus, the condition is best satisfied along the equator of the nuclear drop. The formulae being comparatively long, we only give the results for $e = 4/5$.

$$\left.\begin{aligned} A_2 &= A_2^0(1 - C_0 \cdot 2.354 + C_2 \cdot 1.662 + C_4\, 1.472) \\ A_4 &= - C_2 \cdot 0.00369\, \omega k^2 + C_4 \cdot 0.000733\, \omega k^2 \\ A_6 &= - C_4 \cdot 1.473 \cdot 10^{-4}\, \omega k^2. \end{aligned}\right\} \quad (33)$$

1. Constant $V$ and $a$.

$$\left.\begin{aligned} A_2 &= A_2^0 \cdot (1 + C_2 \cdot 0.736) \\ A_4 &= - C_2 \cdot 4.314 \cdot 10^{-3}\, \omega k^2 \\ A_6 &= C_2 \cdot 1.26 \cdot 10^{-4}\, \omega k^2. \end{aligned}\right\} \quad (34)$$

2. Constant $V$ and $Q$.

$$\left.\begin{aligned} A_2 &= A_2^0, \text{ the coefficient for } \overline{C}_2 \text{ being zero.} \\ A_4 &= - \overline{C}_2 \cdot 0.00462\, \omega k^2 \\ A_6 &= \overline{C}_2 \cdot 1.87 \cdot 10^{-4}\, \omega k^2. \end{aligned}\right\} \quad (35)$$

**B.** Since the accuracy of these coefficients is rather important, we compute them, in case 1, with $C_2 = 0.4$ from an integral condition for the flow over the boundary. The quantity $V_n^\omega - \frac{\partial \Phi}{\partial s_n}$ represents the flow across the boundary owing to the error in the approximation. Now we try to minimize the square error integral.

Putting

$$\left.\begin{aligned} V_n^\omega \sqrt{B(\mu)} &= F(\mu) \cos \varphi \\ \frac{\partial \Phi_m}{\partial s_n} \sqrt{B(\mu)} &= \varphi_m(\mu) \cdot \cos \varphi, \end{aligned}\right\} \quad (36)$$

we have the condition

$$W = \pi k \int_0^1 d\mu \left(F(\mu) - \sum A_n \varphi_n(\mu)\right)^2 = \text{minimum}. \quad (37)$$

With usual notations, we get the following system of equations:

$$(F, \varphi_n) = \sum_m A_m (\varphi_m, \varphi_n). \tag{38}$$

The integrals are calculated numerically for $C_2 = 0.4$. We obtain

$$\left.\begin{aligned} A_2 &= 6.908 \cdot 10^{-2} \omega k^2 \\ A_4 &= -1.606 \cdot 10^{-3} \omega k^2 \\ A_6 &= 6.15 \cdot 10^{-5} \omega k^2. \end{aligned}\right\} \tag{39}$$

(For this value of $C_2$ the numerical coefficients of the first approximation given above are

$$6.768 \cdot 10^{-2}, -1.726 \cdot 10^{-3}, 5.04 \cdot 10^{-5}, \text{ respectively}).$$

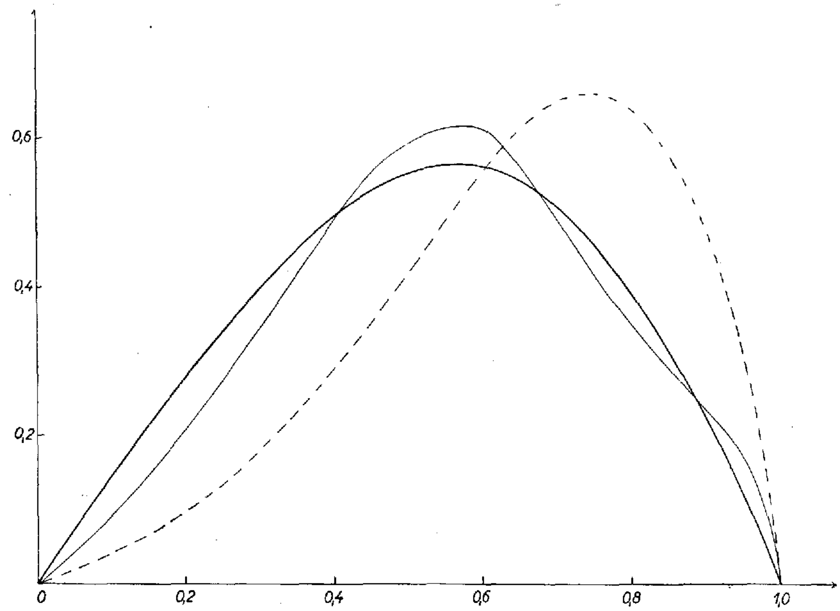


Fig. 4. $F(x)$ solid line thick
$A_2 \psi_2(x)$ dashed line
$\Sigma A_n \psi_n(x)$ solid line thin.

Fig. 4 shows how $F(x) = V_n^{\omega} \sqrt{B}$, where $B$ varies slowly, is approximated by $A_2 \varphi_2 + A_4 \varphi_4 + A_6 \varphi_6$. For comparison, the first term is also inserted, giving the shape of $\varphi_2$.

## Calculation of $E_{rot}$.

We use the expressions above for the collective flow to calculate $E_{rot}$, which is given by the integral over $\frac{1}{2}$ grad$^2\Phi \cdot d\tau$, where $d\tau = k^3 d\mu d\zeta d\varphi(\zeta^2 - \mu^2)$.

We obtain

$$E_{rot} = \frac{\pi \varrho k}{2} \int_{-1}^{+1} d\mu \int_{1}^{\zeta} d\zeta \left(G(\zeta, \mu) - G(\mu, \zeta)\right), \tag{40}$$

where

$$G(\zeta, \mu) = (\zeta^2 - 1)\left[\left(\sum A_n R_n(\mu) P_n'(\zeta)\right)^2 + \left(\sum A_n P_n'(\mu) P_n'(\zeta)\right)^2\right] \tag{41}$$

with

$$R_n(x) = x P_n'(x) + (x^2 - 1) P_n''(x). \tag{42}$$

*First approximation.* The terms containing $A_4^2, A_6^2$, and $A_4 A_6$ can be neglected. In the terms with $A_2 A_4$ and $A_2 A_6$ we can put $\zeta = \zeta_0$. Because of the properties of Legendre functions these integrals are then zero. The only non-zero term is easily calculated.

$$\left.\begin{aligned} E_{rot} = \frac{81\, M(2-e^2)}{10\, k^2 e^2} A_2^2 [1 + \{ C_0(7\, e^5 - 63\, e^3 + 70\, e) + C_2(3\, e^5 \\ -25\, e^3 + 28\, e) + C_4(5\, e^5 - 49\, e^3 + 54\, e)\, 1/3\}\, 1/7 \cdot (2-e^2)^{-1} \cdot (1-e^2)^{-1}]. \end{aligned}\right\} \tag{43}$$

We compute $E_{rot}$ for the two cases, $V$ and $a$ constant, $V$ and $Q_0$ constant.

1. $E_{rot} = E_{rot}^0 \cdot (1 + C_2 \cdot 1.827)$.

For $C_2 = 0.4$, the change is $73\,^0/_0$.

This means a rather strong dependence of $E_{rot}$ on the shape of the nucleus.

2. $E_{rot} = E_{rot}^0\ (1 - \overline{C}_2 \cdot 0.002)$, which means practically no change of $E_{rot}$ for the change of shape characterized by constant $Q_0$.

*Numerical calculation in case 1 for $C_2 = 0{,}4$.*

We will make the calculations for a finite change of the boundary. This, of course, implies long calculations. However, the great change of $E_{\text{rot}}$ from the first approximation makes this test desirable.

We write: $E_{\text{rot}} = \sum_{i,j} A_i A_j K_{ij}$, where $(i,j) = (2, 4, 6)$ and the $K_{ij}$ are defined by formulae (40—42).

All calculations can be made explicitly, but the number of terms increases very rapidly for higher indices. Therefore all terms except the first are calculated by an exact integration in $\zeta$ and a subsequent numerical integration in $\mu$.

We find the following values:

$$\left.\begin{aligned} E_{\text{rot}} &= \frac{\pi \varrho k}{2}\left(A_2^2 \cdot 37.58 + 2\, A_2 A_4 \cdot 81.4 + A_4^2 \cdot 5985 - 2\, A_2 A_6 \cdot 616\right. \\ &\quad \left. + 2\, A_4 A_6 \cdot 1.05 \cdot 10^4 + A_6^2 \cdot 3.8 \cdot 10^5\right) \\ &= \frac{\omega^2 \pi \varrho k^5}{2}(0.1793 - 0.0180 + 0.0154 - 0.0052 - 0.0021 + 0.0014) \\ &= \frac{\omega^2 \pi \varrho k^5}{2} 0.1708. \end{aligned}\right\} \tag{44}$$

Comparing with the first approximation, which gave

$$E_{\text{rot}} = E_{\text{rot}}^0 \cdot 1.73, \tag{45}$$

we find here

$$E_{\text{rot}} = E_{\text{rot}}^0 \cdot 1.93 \tag{46}$$

which shows that the first approximation is qualitatively correct, even if the corrections to the flow are larger than to the static moments.

## Comparison of $Q_0$ and $\mathfrak{J}$.

As mentioned in the Introduction, the quantity which provides the most direct test of the potential flow model is the ratio between $Q_0^2$ and the moment of inertia $\mathfrak{J}$, obtained from $E_{\text{rot}}$.

*Case* 1.

*First approximation.*

$$Q_0^2/\mathfrak{J} = (Q_0^2/\mathfrak{J})_0 \cdot (1 + C_2 \cdot 0.355). \tag{47}$$

The change would go in the direction indicated by the experiments for negative $C_2$, illustrated by Fig. 2. However, in any case the change is small.

*Finite deviation.*

$$\frac{Q_0^2}{\mathfrak{J}} = \left(\frac{Q_0^2}{\mathfrak{J}}\right)_0 \cdot 1.08. \tag{48}$$

This should be compared with the value 1.14 of the first approximation.

*Case* 2.

For constant $V$ and $Q_0$ the change in $E_{\mathrm{rot}}$ is insignificant.

The calculations show that, for comparatively moderate changes of shape of the nucleus, the quantities $\mathfrak{J}$, $Q_0$, and $S_0$ are changed appreciably. The value of $Q_0^2/\mathfrak{J}$, on the contrary, is rather insensitive to such changes in shape which have been considered in these calculations.

On this occasion, I should like to express my deep gratitude to Professor NIELS BOHR for his great interest in my work and enlightening discussions during my many stays in Copenhagen. Further, I want to thank Drs. AAGE BOHR and BEN MOTTELSON for valuable suggestions and for their kind communication of investigations prior to publication. I also thank KERSTIN KJÄLLQUIST, F. M., and ROLF BENGTSSON, F. K., for their helpful aid in carrying through the calculations.

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