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# ON THE RESTRICTED PROBLEM OF THREE BODIES

BY

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### Synopsis.

Three bodies are assumed to move in a plane, subject to Newton's law of gravitation, one of the bodies being infinitely small, and the two others moving in circles around their common centre of gravitation. To expand the coordinates of the small body in powers of the time is generally assumed to be impractical, but it is shown here that by introducing certain auxiliary dependent variables, the equations of motion are transformed into a differential system of the *second degree*, permitting to calculate the coefficients of the series by a set of recurrence formulas particularly adapted to the modern calculating machines. Sufficient conditions for the convergence of the resulting series are obtained, and a simple numerical example is given.

1. We have in view the well-known particular case of the Problem of Three Bodies where the movement takes place in a plane, and two of the masses describe circles about their common centre of gravity, while the mass of the third body is infinitely small. Expansion of the coordinates in powers of the time  $t$  can be obtained by successive differentiations of the equations of motion, but this way of calculating the coefficients of the powers of  $t$  has been given up as too tedious<sup>1</sup>. We intend to show here that the calculation of the coefficients can be carried out with comparative ease when the equations of motion are transformed into a differential system of the *second degree*, permitting to calculate the coefficients of  $t^p$  by a set of recurrence formulas, particularly adapted to the modern calculating machines. The process is closely related to that employed in one of my papers on the differential equations of G.W. HILL<sup>2</sup>.

The equations of motion are given in Darwin's paper, p. 103. We write them, with a change of notation<sup>3</sup>,

$$\left. \begin{aligned} \frac{d^2 p}{dt^2} - 2N \frac{dq}{dt} + Mp(r^{-3} - 1) + (p - 1)(s^{-3} - 1) &= 0, \\ \frac{d^2 q}{dt^2} + 2N \frac{dp}{dt} + Mq(r^{-3} - 1) + q(s^{-3} - 1) &= 0, \end{aligned} \right\} \quad (1)$$

where

$$r^2 = p^2 + q^2, \quad s^2 = r^2 + 1 - 2p, \quad (2)$$

$$N^2 = M + 1. \quad (3)$$

<sup>1</sup> G. H. DARWIN: "Periodic Orbits". Acta mathematica, 21 (1897), 129—132.

<sup>2</sup> J. F. STEFFENSEN: "On the Differential Equations of Hill in the Theory of the Motion of the Moon (II)". Acta mathematica, 95 (1956), 25—37.

<sup>3</sup> Darwin's  $x, y, n, p, q, C$  have in succession been replaced by  $p, q, N, M, s, K$ .

Jacobi's integral is

$$M(r^2 + 2r^{-1}) + (s^2 + 2s^{-1}) - \left(\frac{dp}{dt}\right)^2 - \left(\frac{dq}{dt}\right)^2 = K. \quad (4)$$

In these equations  $p$  and  $q$  are the coordinates of the infinitesimal body, the masses of the finite bodies are  $M$  and  $1$ , their distance from each other  $1$ , and the angular velocity of the system  $N$ .

Referring for further particulars to Darwin's paper we put

$$X = r^{-3} - 1, \quad Y = s^{-3} - 1, \quad (5)$$

so that

$$\left. \begin{aligned} r \frac{dX}{dt} + 3(X+1) \frac{dr}{dt} &= 0, \\ s \frac{dY}{dt} + 3(Y+1) \frac{ds}{dt} &= 0. \end{aligned} \right\} \quad (6)$$

while (1) can be written

$$\left. \begin{aligned} \frac{d^2p}{dt^2} - 2N \frac{dq}{dt} + MpX + pY - Y &= 0, \\ \frac{d^2q}{dt^2} + 2N \frac{dp}{dt} + MqX + qY &= 0. \end{aligned} \right\} \quad (7)$$

For the determination of  $p, q, r, s, X, Y$  we have the 6 equations (7), (6) and (2) which we propose to satisfy by power series in  $t$  without making use of Jacobi's integral. We put

$$p = \sum_{\nu=0}^{\infty} a_{\nu} t^{\nu}, \quad q = \sum_{\nu=0}^{\infty} b_{\nu} t^{\nu}, \quad (8)$$

$$r = \sum_{\nu=0}^{\infty} c_{\nu} t^{\nu}, \quad s = \sum_{\nu=0}^{\infty} d_{\nu} t^{\nu}, \quad (9)$$

$$X = \sum_{\nu=0}^{\infty} e_{\nu} t^{\nu}, \quad Y = \sum_{\nu=0}^{\infty} f_{\nu} t^{\nu}. \quad (10)$$

Inserting these series in the 6 equations and demanding that the coefficient of  $t^n$  shall vanish, we find by (7)

$$\begin{aligned}
 (n+1)(n+2)a_{n+2} - 2N(n+1)b_{n+1} + M \sum_{\nu=0}^n a_{\nu} e_{n-\nu} \\
 + \sum_{\nu=0}^n a_{\nu} f_{n-\nu} - f_n = 0, \\
 (n+1)(n+2)b_{n+2} + 2N(n+1)a_{n+1} + M \sum_{\nu=0}^n b_{\nu} e_{n-\nu} \\
 + \sum_{\nu=0}^n b_{\nu} f_{n-\nu} = 0,
 \end{aligned}
 \tag{11}$$

by (6)

$$\begin{aligned}
 \sum_{\nu=0}^n (\nu+1) e_{\nu+1} c_{n-\nu} + 3 \sum_{\nu=0}^n (\nu+1) c_{\nu+1} e_{n-\nu} \\
 + 3(n+1)c_{n+1} = 0, \\
 \sum_{\nu=0}^n (\nu+1) f_{\nu+1} d_{n-\nu} + 3 \sum_{\nu=0}^n (\nu+1) d_{\nu+1} f_{n-\nu} \\
 + 3(n+1)d_{n+1} = 0,
 \end{aligned}
 \tag{12}$$

and by (2)

$$\begin{aligned}
 \sum_{\nu=0}^n c_{\nu} c_{n-\nu} &= \sum_{\nu=0}^n a_{\nu} a_{n-\nu} + \sum_{\nu=0}^n b_{\nu} b_{n-\nu}, \\
 \sum_{\nu=0}^n d_{\nu} d_{n-\nu} &= \sum_{\nu=0}^n c_{\nu} c_{n-\nu} - 2a_n \quad (n > 0), \\
 d_0^2 &= c_0^2 + 1 - 2a_0.
 \end{aligned}
 \tag{13}$$

As initial values (constants of integration) we choose the coordinates and components of velocity of the infinitely small body at the time  $t = 0$ , that is  $a_0, a_1, b_0, b_1$ . Hence we find by (13),  $c_0$  and  $d_0$  being positive (since  $r$  and  $s$  represent distances from the finite masses)

$$c_0 = \sqrt{a_0^2 + b_0^2}, \quad d_0 = \sqrt{c_0^2 + 1 - 2a_0}, \tag{14}$$

whereafter by (10) and (5)

$$e_0 = c_0^{-3} - 1, \quad f_0 = d_0^{-3} - 1. \tag{15}$$

The remaining constants are calculated by the recurrence formulas (11) — (13). We state these in the form and order in which they are to be employed.

$$2 c_0 c_n = \sum_{\nu=0}^n a_\nu a_{n-\nu} + \sum_{\nu=0}^n b_\nu b_{n-\nu} - \sum_{\nu=1}^{n-1} c_\nu c_{n-\nu}. \quad (16)$$

$$2 d_0 d_n = \sum_{\nu=0}^n c_\nu c_{n-\nu} - \sum_{\nu=1}^{n-1} d_\nu d_{n-\nu} - 2 a_n. \quad (17)$$

$$- n c_0 e_n = 3 \sum_{\nu=1}^n \nu c_\nu e_{n-\nu} + \sum_{\nu=1}^{n-1} \nu e_\nu c_{n-\nu} + 3 n c_n. \quad (18)$$

$$- n d_0 f_n = 3 \sum_{\nu=1}^n \nu d_\nu f_{n-\nu} + \sum_{\nu=1}^{n-1} \nu f_\nu d_{n-\nu} + 3 n d_n. \quad (19)$$

$$\left. \begin{aligned} - n(n+1) a_{n+1} &= M \sum_{\nu=0}^{n-1} a_\nu e_{n-\nu-1} + \sum_{\nu=0}^{n-1} a_\nu f_{n-\nu-1} \\ &\quad - 2 N n b_n - f_{n-1}. \end{aligned} \right\} (20)$$

$$- n(n+1) b_{n+1} = M \sum_{\nu=0}^{n-1} b_\nu e_{n-\nu-1} + \sum_{\nu=0}^{n-1} b_\nu f_{n-\nu-1} + 2 N n a_n. \quad (21)$$

We give below the first few of these recurrence formulas.

$$\left. \begin{aligned} c_0 c_1 &= a_0 a_1 + b_0 b_1. \\ d_0 d_1 &= c_0 c_1 - a_1. \\ - c_0 e_1 &= 3 c_1 (e_0 + 1). \\ - d_0 f_1 &= 3 d_1 (f_0 + 1). \end{aligned} \right\} (22)$$

$$\left. \begin{aligned} - 2 a_2 &= M a_0 e_0 + f_0 (a_0 - 1) - 2 N b_1, \\ - 2 b_2 &= M b_0 e_0 + f_0 b_0 + 2 N a_1. \\ 2 c_0 c_2 &= 2 a_0 a_2 + a_1^2 + 2 b_0 b_2 + b_1^2 - c_1^2. \\ 2 d_0 d_2 &= 2 c_0 c_2 + c_1^2 - d_1^2 - 2 a_2. \\ - c_0 e_2 &= 2 c_1 e_1 + 3 c_2 (e_0 + 1). \\ - d_0 f_2 &= 2 d_1 f_1 + 3 d_2 (f_0 + 1). \end{aligned} \right\} (23)$$

$$\left. \begin{aligned} - 6 a_3 &= M (a_0 e_1 + a_1 e_0) + f_0 a_1 + f_1 (a_0 - 1) - 4 N b_2. \\ - 6 b_3 &= M (b_0 e_1 + b_1 e_0) + b_0 f_1 + b_1 f_0 + 4 N a_2. \end{aligned} \right\} (24)$$

$$\left. \begin{aligned} c_0 c_3 &= a_0 a_3 + a_1 a_2 + b_0 b_3 + b_1 b_2 - c_1 c_2. \\ d_0 d_3 &= c_0 c_3 + c_1 c_2 - d_1 d_2 - a_3. \\ -3 c_0 e_3 &= 5 c_1 e_2 + 7 c_2 e_1 + 9 c_3 (e_0 + 1). \\ -3 d_0 f_3 &= 5 d_1 f_2 + 7 d_2 f_1 + 9 d_3 (f_0 + 1). \end{aligned} \right\} (24)$$

It is seen that these forms lend themselves easily to the calculating machine.

2. In order to examine the convergence we write (16)—(21) in the following form where the constants of integration and those of zero order have been isolated. In (25), (29) and (30) we assume  $n \geq 3$ , in (26)—(28)  $n \geq 2$ .

$$\left. \begin{aligned} c_0 c_n &= a_0 a_n + a_1 a_{n-1} + b_0 b_n + b_1 b_{n-1} - c_1 c_{n-1} \\ &+ \frac{1}{2} \sum_{\nu=2}^{n-2} (a_\nu a_{n-\nu} + b_\nu b_{n-\nu} - c_\nu c_{n-\nu}). \end{aligned} \right\} (25)$$

$$d_0 d_n = c_0 c_n - a_n + \frac{1}{2} \sum_{\nu=1}^{n-1} (c_\nu c_{n-\nu} - d_\nu d_{n-\nu}). \quad (26)$$

$$-nc_0 e_n = 3nc_n(e_0 + 1) + 2 \sum_{\nu=1}^{n-1} \nu c_\nu e_{n-\nu} + n \sum_{\nu=1}^{n-1} c_\nu e_{n-\nu}. \quad (27)$$

$$-nd_0 f_n = 3nd_n(f_0 + 1) + 2 \sum_{\nu=1}^{n-1} \nu d_\nu f_{n-\nu} + n \sum_{\nu=1}^{n-1} d_\nu f_{n-\nu}. \quad (28)$$

$$\left. \begin{aligned} -n(n+1)a_{n+1} &= a_0(f_{n-1} + Me_{n-1}) + a_1(f_{n-2} + Me_{n-2}) + \\ &\sum_{\nu=2}^{n-2} a_\nu(f_{n-\nu-1} + Me_{n-\nu-1}) + a_{n-1}(f_0 + Me_0) - f_{n-1} - 2Nn b_n. \end{aligned} \right\} (29)$$

$$\left. \begin{aligned} -n(n+1)b_{n+1} &= b_0(f_{n-1} + Me_{n-1}) + b_1(f_{n-2} + Me_{n-2}) \\ &+ \sum_{\nu=2}^{n-2} b_\nu(f_{n-\nu-1} + Me_{n-\nu-1}) + b_{n-1}(f_0 + Me_0) + 2Nna_n. \end{aligned} \right\} (30)$$

We now put, as in an earlier paper<sup>1</sup>,

$$K_\nu = \frac{\lambda^\nu}{\nu(\nu+1)} \quad (\lambda > 0) \quad (31)$$

<sup>1</sup> Acta mathematica, 93 (1955), 173.

and assume that it has been proved for  $2 \leq v \leq n$  that

$$|a_v| \leq AK_v, \quad |b_v| \leq BK_v \quad (32)$$

and for  $1 \leq v \leq n-1$  that

$$|c_v| \leq CK_v, \quad |d_v| \leq DK_v, \quad |e_v| \leq EK_v, \quad |f_v| \leq FK_v. \quad (33)$$

We then find sufficient conditions, by (25) — (28) for the validity of (33) in the case  $v = n$ , and by (29) — (30) for (32) in the case  $v = n+1$ , so that the inequalities (32) and (33) are valid for all  $v$  under consideration. We proceed as in the paper quoted, making use of the identity

$$\left. \begin{aligned} K_v K_{m-v} &= \lambda^m \left[ \left( \frac{1}{v} + \frac{1}{m-v} \right) \frac{1}{m(m+1)} - \right. \\ &\left. \left( \frac{1}{v+1} + \frac{1}{m-v+1} \right) \frac{1}{(m+1)(m+2)} \right]. \end{aligned} \right\} \quad (34)$$

From this, writing for abbreviation

$$s_n = \sum_{v=1}^n \frac{1}{v} \quad (35)$$

we obtain the sums<sup>1</sup>

$$\sum_{v=1}^{n-1} K_v K_{n-v} = 2 \frac{n-1 + 2s_{n-1}}{n(n+1)(n+2)} \lambda^n, \quad (36)$$

$$\sum_{v=2}^{n-2} K_v K_{n-v} = \left( 2 \frac{n-1 + 2s_{n-1}}{(n+1)(n+2)} - \frac{1}{n-1} \right) \frac{\lambda^n}{n}, \quad (37)$$

$$\sum_{v=2}^{n-2} K_v K_{n-v-1} = \left( 2 \frac{n-2 + 2s_{n-2}}{n(n+1)} - \frac{1}{2(n-2)} \right) \frac{\lambda^{n-1}}{n-1}, \quad (38)$$

$$\sum_{v=1}^{n-1} v K_v K_{n-v} = \frac{n-1 + 2s_{n-1}}{(n+1)(n+2)} \lambda^n. \quad (39)$$

<sup>1</sup> Interpreted as zero, if the upper limit of summation is less than the lower.



3. Dealing first with (25), we obtain,  $c_0$  being positive, by (32) and (33)

$$\left. \begin{aligned}
 & c_0 |c_n| \leq (A |a_0| + B |b_0|) K_n \\
 & + (A |a_1| + B |b_1| + C |c_1|) K_{n-1} \\
 & + \frac{1}{2} (A^2 + B^2 + C^2) \sum_{\nu=2}^{n-2} K_\nu K_{n-\nu}
 \end{aligned} \right\} \text{or}$$

$$\left. \begin{aligned}
 & c_0 |c_n| \leq (A |a_0| + B |b_0|) \frac{\lambda^n}{n(n+1)} \\
 & + (A |a_1| + B |b_1| + C |c_1|) \frac{\lambda^{n-1}}{(n-1)n} \\
 & + \frac{1}{2} (A^2 + B^2 + C^2) \left( 2 \frac{n-1+2s_{n-1}}{(n+1)(n+2)} - \frac{1}{n-1} \right) \frac{\lambda^n}{n}.
 \end{aligned} \right\} (40)$$

If, now, we demand that the right-hand side of this inequality shall be  $\leq c_0 CK_n = c_0 C \frac{\lambda^n}{n(n+1)}$ , we obtain after multiplication by  $n(n+1)\lambda^{-n}$  as a sufficient condition for the validity of  $|c_\nu| \leq CK_\nu$  in all cases under consideration

$$\left. \begin{aligned}
 & A |a_0| + B |b_0| + (A |a_1| + B |b_1| + C |c_1|) \frac{n+1}{n-1} \cdot \frac{1}{\lambda} \\
 & + \frac{1}{2} (A^2 + B^2 + C^2) \left( 2 \frac{n-1+2s_{n-1}}{n+2} - \frac{n+1}{n-1} \right) \leq Cc_0.
 \end{aligned} \right\} (41)$$

We replace this condition by a simpler but more rigid condition obtained by replacing the factors depending on  $n$  by absolute numbers which are at least as large.

Since  $\frac{n+1}{n-1} = 1 + \frac{2}{n-1}$ , this factor is constantly decreasing and may for  $n \geq 2$  be replaced by 3.

Putting next

$$S_n = 2 \frac{n-1+2s_{n-1}}{n+2} - \frac{n+1}{n-1}, \tag{42}$$

and observing that

$$s_n \leq 1 + \frac{1}{2} + \frac{1}{3} + \frac{n-3}{4} = \frac{13}{12} + \frac{n}{4} \quad (43)$$

so that

$$s_{n-1} \leq \frac{5}{6} + \frac{n}{4}$$

or  $4s_{n-1} \leq n + \frac{10}{3}$ , we find by inserting this in (42)

$$S_n \leq 2 - \frac{14}{3n+6} - \frac{2}{n-1}.$$

Hence

$$S_n < 2. \quad (44)$$

Observing finally that  $\frac{n+1}{n-1} \leq 3$  for  $n \geq 2$ , we may replace (41) by the more rigid condition

$$\left. \begin{aligned} A|a_0| + B|b_0| + (A|a_1| + B|b_1| + C|c_1|)\frac{3}{\lambda} \\ + A^2 + B^2 + C^2 \leq c_0 C \end{aligned} \right\} \quad (45)$$

which is independent of  $n$ .

4. Next, as regards (26), we find by (36), corresponding to (41), the sufficient condition

$$Cc_0 + A + (C^2 + D^2)\frac{n-1+2s_{n-1}}{n+2} \leq Dd_0. \quad (46)$$

The condition that the factor depending on  $n$  shall be steadily decreasing may be written in the form

$$s_{n-1} > \frac{5}{2} + \frac{2}{n} \quad (47)$$

which is satisfied for  $n \geq 10$ . We therefore have in this region<sup>1</sup>

$$\frac{n-1+2s_{n-1}}{n+2} < \frac{3}{4} + \frac{1}{6}s_9 < \frac{5}{4} \quad (48)$$

<sup>1</sup> A table of  $s_n$  is found in S. SPITZER: Tabellen für die Zinseszinsen und Renten-Rechnung, Wien 1897, 369—370.

which is also valid for  $n < 10$ . We may therefore replace (46) by the simpler but more rigid condition

$$Cc_0 + A + \frac{5}{4}(C^2 + D^2) \leq Dd_0. \quad (49)$$

5. From (27) we obtain by (39) and (36) as a sufficient condition

$$3 \mid e_0 + 1 \mid C + 4 CE \frac{n-1 + 2s_{n-1}}{n+2} \leq Ec_0, \quad (50)$$

and from this, by (48), the more rigid sufficient condition

$$3 \mid e_0 + 1 \mid C + 5 CE \leq Ec_0. \quad (51)$$

Since (28) is obtained from (27) by a simple exchange of letters we may at once by (51) write down the following sufficient condition, resulting from (28)

$$3 \mid f_0 + 1 \mid D + 5 DF \leq Fd_0. \quad (52)$$

6. As regards (29), we have, by (31)–(33) and (38),

$$\left. \begin{aligned} n(n+1) \mid a_{n+1} \mid &\leq \mid a_0 \mid (F + ME) \frac{\lambda^{n-1}}{(n-1)n} \\ &+ \mid a_1 \mid (F + ME) \frac{\lambda^{n-2}}{(n-2)(n-1)} \\ &+ A(F + ME) \left( 2 \frac{n-2 + 2s_{n-2}}{n(n+1)} - \frac{1}{2(n-2)} \right) \frac{\lambda^{n-1}}{n-1} \\ &+ A \mid f_0 + Me_0 \mid \frac{\lambda^{n-1}}{(n-1)n} + F \frac{\lambda^{n-1}}{(n-1)n} + 2NB \frac{\lambda^n}{n+1}. \end{aligned} \right\} (53)$$

If we demand that the right-hand side of this shall be  $\leq n(n+1)AK_{n+1} = A \frac{n}{n+2} \lambda^{n+1}$ , we obtain after multiplication by  $\frac{n+2}{n} \lambda^{1-n}$  the condition

$$\left. \begin{aligned} & [| a_0 | (F + ME) + A | f_0 + Me_0 | + F] \frac{n+2}{(n-1)n^2} \\ & + 2NB \frac{n+2}{n(n+1)} \lambda + | a_1 | (F + ME) \frac{n+2}{n(n-1)(n-2)} \frac{1}{\lambda} \\ & \quad + A (F + ME) R_n \leq A \lambda^2 \end{aligned} \right\} \quad (54)$$

where

$$R_n = \frac{2(n+2)}{n(n-1)} \left( \frac{n-2+2s_{n-2}}{n(n+1)} - \frac{1}{4(n-2)} \right). \quad (55)$$

We proceed to show that

$$R_n \leq \frac{1}{8} \quad (n \geq 3). \quad (56)$$

We write (55) in the form

$$R_n = \left( \frac{2}{n-1} + \frac{4}{n(n-1)} \right) \left( 2 \frac{s_{n-2}-1}{n(n+1)} + \frac{3}{4} \frac{n-3}{(n+1)(n-2)} \right)$$

where we may assume  $n \geq 5$ , since  $R_3 = 0$ ,  $R_4 = \frac{1}{8}$ . Now the first factor in  $R_n$  is evidently decreasing, and the second factor is the sum of two decreasing expressions, since

$$\frac{s_{n-2}-1}{n(n+1)} > \frac{s_{n-1}-1}{(n+1)(n+2)} \quad (n \geq 5)$$

which can be written

$$2(s_{n-2}-1) > 1 + \frac{1}{n-1},$$

and

$$\frac{n-3}{(n+1)(n-2)} > \frac{n-2}{(n+2)(n-1)} \quad (n \geq 5)$$

which can be written

$$n(n-5) + 2 > 0.$$

The remaining factors depending on  $n$  in (54) are steadily decreasing, and we find for  $n \geq 3$

$$\frac{n+2}{(n-1)n^2} = \frac{1}{(n-1)n} + \frac{2}{(n-1)n^2} \leq \frac{5}{18}, \quad (57)$$

$$\frac{n+2}{n(n+1)} = \frac{1}{n+1} + \frac{2}{n(n+1)} \leq \frac{5}{12}, \quad (58)$$

$$\left. \begin{aligned} \frac{n+2}{n(n-1)(n-2)} &= \frac{1}{(n-1)(n-2)} \\ &+ \frac{2}{n(n-1)(n-2)} \leq \frac{5}{6}. \end{aligned} \right\} (59)$$

Inserting finally the limits (56) — (59) in (54), we obtain the more rigid, but of  $n$  independent, sufficient condition

$$\left. \begin{aligned} \frac{5}{18} [ |a_0| (F + ME) + A |f_0 + Me_0| + F ] + \frac{5}{6} NB\lambda \\ + \frac{5}{6} |a_1| (F + ME) \frac{1}{\lambda} + \frac{1}{8} A (F + ME) \leq A\lambda^2. \end{aligned} \right\} (60)$$

7. As regards finally (30), a comparison with (29) shows that we obtain the same form as (53), the only difference being that  $a$  and  $b$ ,  $A$  and  $B$  have been exchanged and the term  $F \frac{\lambda^{n-1}}{(n-1)n}$  left out. We may therefore immediately write down the sufficient condition corresponding to (60)

$$\left. \begin{aligned} \frac{5}{18} [ |b_0| (F + ME) + B |f_0 + Me_0| ] + \frac{5}{6} NA\lambda \\ + \frac{5}{6} |b_1| (F + ME) \frac{1}{\lambda} + \frac{1}{8} B (F + ME) \leq B\lambda^2. \end{aligned} \right\} (61)$$

8. The result of the preceding investigation is that, if for a certain  $n \geq 3$  it has been proved that (32) is satisfied for  $2 \leq \nu \leq n$  and (33) for  $1 \leq \nu \leq n-1$ , and if, besides, the inequalities

(45), (49), (51), (52), (60) and (61) are all satisfied, then the expansions (8) — (10) are convergent, provided that  $\sum K_p |t|^p$  converges, that is, for  $|t| \leq \frac{1}{\lambda}$ .

The question arises whether, when the constants of integration are arbitrarily given, it is always possible to find such values of  $\lambda, A, B, C, D, E, F$  that the six inequalities are all satisfied. We proceed to prove that this is really so.

To begin with,  $\lambda$  can always be chosen so large that (60) and (61) are satisfied, no matter what values the other constants possess, and (45) can for sufficiently large  $\lambda$  be replaced by

$$A |a_0| + B |b_0| + \frac{5}{4}(A^2 + B^2) < C \left( c_0 - \frac{5}{4}C \right) \quad (62)$$

while the three remaining inequalities which we write in the form

$$A + C \left( c_0 + \frac{5}{4}C \right) \leq D \left( d_0 - \frac{5}{4}D \right), \quad (63)$$

$$3 |e_0 + 1| C \leq E (c_0 - 5C), \quad (64)$$

$$3 |f_0 + 1| D \leq F (d_0 - 5D), \quad (65)$$

are unchanged. Now it follows from (64) and (65) that we must choose

$$C < \frac{1}{5}c_0, \quad D < \frac{1}{5}d_0, \quad (66)$$

after which (64) and (65) are satisfied, provided that we choose  $E$  and  $F$  sufficiently large. After this, (62) will be satisfied, if we choose  $A$  and  $B$  sufficiently small in comparison with  $C$ , and (63) if  $A$  and  $C$  are sufficiently small in comparison with  $D$ . In thus choosing small values for  $A, B, C$  and  $D$  we do not run into difficulties, because (31) — (33) show that small values of these constants can be compensated by choosing  $\lambda$  sufficiently large.

There is, thus, always a solution for sufficiently small values of  $|t|$ , if  $c_0 > 0, d_0 > 0$  as assumed in (14).

9. If at the time  $t = 0$  we have  $q = 0$ ,  $\frac{dp}{dt} = 0$ , that is  $b_0 = 0$ ,  $a_1 = 0$ , certain simplifications occur. In that case there are only the two arbitrary constants  $a_0$  and  $b_1$  left, and we find first by (14) and (15), if  $a_0 \neq 0$  and  $a_0 \neq 1$ ,

$$\left. \begin{aligned} c_0 &= |a_0|, \quad d_0 = |a_0 - 1|, \quad e_0 = |a_0|^{-3} - 1, \\ f_0 &= |a_0 - 1|^{-3} - 1. \end{aligned} \right\} \quad (67)$$

The recurrence formulas now show that  $b_\nu$  vanishes when  $\nu$  is an even number, and the other coefficients when  $\nu$  is odd. Under these circumstances the working formulas (16) — (21) are best written thus

$$\left. \begin{aligned} -(2n-1)2na_{2n} &= M \sum_{\nu=0}^{n-1} a_{2\nu} e_{2n-2\nu-2} + \sum_{\nu=0}^{n-1} a_{2\nu} f_{2n-2\nu-2} \\ &- 2N(2n-1)b_{2n-1} - f_{2n-2}. \end{aligned} \right\} \quad (68)$$

$$\left. \begin{aligned} 2c_0c_{2n} &= \sum_{\nu=0}^n a_{2\nu} a_{2n-2\nu} + \sum_{\nu=1}^n b_{2\nu-1} b_{2n-2\nu+1} \\ &- \sum_{\nu=1}^{n-1} c_{2\nu} c_{2n-2\nu}. \end{aligned} \right\} \quad (69)$$

$$2d_0d_{2n} = \sum_{\nu=0}^n c_{2\nu} c_{2n-2\nu} - \sum_{\nu=1}^{n-1} d_{2\nu} d_{2n-2\nu} - 2a_{2n}. \quad (70)$$

$$-nc_0e_{2n} = 3 \sum_{\nu=1}^n \nu c_{2\nu} e_{2n-2\nu} + \sum_{\nu=1}^{n-1} \nu e_{2\nu} c_{2n-2\nu} + 3nc_{2n}. \quad (71)$$

$$-nd_0f_{2n} = 3 \sum_{\nu=1}^n \nu d_{2\nu} f_{2n-2\nu} + \sum_{\nu=1}^{n-1} \nu f_{2\nu} d_{2n-2\nu} + 3nd_{2n}. \quad (72)$$

$$\left. \begin{aligned} -2n(2n+1)b_{2n+1} &= M \sum_{\nu=1}^n b_{2\nu-1} e_{2n-2\nu} \\ &+ \sum_{\nu=1}^n b_{2\nu-1} f_{2n-2\nu} + 4Nn a_{2n}. \end{aligned} \right\} \quad (73)$$

The first few of these formulas are

$$\left. \begin{aligned} -2 a_2 &= M a_0 e_0 + f_0 (a_0 - 1) - 2 N b_1. \\ c_0 c_2 &= a_0 a_2 + \frac{1}{2} b_1^2. \\ d_0 d_2 &= c_0 c_2 - a_2. \\ -c_0 e_2 &= 3 c_2 (e_0 + 1). \\ -d_0 f_2 &= 3 d_2 (f_0 + 1). \end{aligned} \right\} (74)$$

$$-6 b_3 = M b_1 e_0 + b_1 f_0 + 4 N a_2. \quad (75)$$

$$\left. \begin{aligned} -12 a_4 &= M (a_0 e_2 + a_2 e_0) + f_2 (a_0 - 1) + a_2 f_0 - 6 N b_3. \\ 2 c_0 c_4 &= 2 a_0 a_4 + a_2^2 + 2 b_1 b_3 - c_2^2. \\ 2 d_0 d_4 &= 2 c_0 c_4 + c_2^2 - d_2^2 - 2 a_4. \\ -c_0 e_4 &= 2 c_2 e_2 + 3 c_4 (e_0 + 1). \\ -d_0 f_4 &= 2 d_2 f_2 + 3 d_4 (f_0 + 1). \end{aligned} \right\} (76)$$

$$-20 b_5 = M (b_1 e_2 + b_3 e_0) + b_1 f_2 + b_3 f_0 + 8 N a_4. \quad (77)$$

$$\left. \begin{aligned} -30 a_6 &= M (a_0 e_4 + a_2 e_2 + a_4 e_0) + f_4 (a_0 - 1) + a_2 f_2 \\ &\quad + a_4 f_0 - 10 N b_5. \\ c_0 c_6 &= a_0 a_6 + a_2 a_4 + b_1 b_5 + \frac{1}{2} b_3^2 - c_2 c_4. \\ d_0 d_6 &= c_0 c_6 + c_2 c_4 - d_2 d_4 - a_6. \\ -3 c_0 e_6 &= 3 (c_2 e_4 + 2 c_4 e_2 + 3 c_6 e_0) + e_2 c_4 + 2 e_4 c_2 + 9 c_6. \\ -3 d_0 f_6 &= 3 (d_2 f_4 + 2 d_4 f_2 + 3 d_6 f_0) + f_2 d_4 + 2 f_4 d_2 + 9 d_6. \end{aligned} \right\} (78)$$

10. As a simple numerical example of the application of (74) — (78) we choose  $a_0 = \frac{1}{2}$ ,  $b_1 = -1$  besides the already assumed  $b_0 = 0$ ,  $a_1 = 0$  leading to (67). For  $N$  and  $M$  we choose the values  $N = 1.1$ ,  $M = .21$  which satisfy (3). The results are given in the table below.



$\nu$	$a_\nu$	$c_\nu$	$d_\nu$
0	.5	.5	.5
2	.2825	1.2825	.7175
4	-.4332729	-4.407273	-2.4107271
6	1.3130591	19.199425	8.728045

  

$\nu$	$e_\nu$	$f_\nu$	$\nu$	$b_\nu$
0	7.	7.	1	-1.
2	-61.56	-34.44	3	1.2045
4	527.3519	214.5577	5	-2.687845
6	-4442.1231	-1319.5487		

A partial check on these calculations is obtained by calculating the value of Jacobi's constant  $K$  by (4) for various values of  $t$ . I have found

$$\begin{aligned}
 t = 0, & & K = 4.1425 \\
 t = .03, & & K = 4.1424999
 \end{aligned}$$

which seems satisfactory.

As regards the convergence, (32) and (33) are satisfied by the coefficients given in the table if, for instance, we choose  $\lambda = 20$ ,  $A = .005$ ,  $B = .002$ ,  $C = .02$ ,  $D = .04$ ,  $E = 1.2$ ,  $F = 3.2$ , and since these values also satisfy all the six inequalities (45), (49), (51), (52), (60) and (61), the expansions (8) -- (10) are at least convergent for  $|t| \leq \frac{1}{20}$ .

This space of time may at first appear to be small, but the expansion for  $q$  shows that it corresponds to a movement in the vertical direction of nearly one tenth of the original distance of the infinitesimal body from either of the two finite bodies.

