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STELLAR MODELS

BASED ON THE PROTON-PROTON REACTION

BY

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1. Introduction.

With the recognition that the proton-proton chain reaction may provide the greater part of the energy production of dwarf stars[1] a type of stellar model, which has not so far been studied in any great detail, becomes of interest. Indeed, since the temperature enters into the rate of the proton-proton reaction only with a power of about four, the energy production will take place in an extended region around the center of the star. Consequently, the existence of the convective core, which is a pronounced feature of point-source models and carbon-cycle models, is by no means certain. And even in cases where the convective core exists an appreciable fraction of the energy is likely to be produced outside the core, and it is necessary to take the variation of the flux of energy through the star into account.

Previous investigations, which are important in this connexion, include papers by I. EPSTEIN[2], and by I. EPSTEIN and L. MOTZ[3]. These papers give models for the Sun, in which the proton-proton reaction is taken into account. A paper by OSTERBROCK[4] gives models for red dwarf stars, calculated on the assumption that convective layers, extending downwards from the surface, exist. A. REIZ[5] has calculated a model which is applicable to stars composed entirely of hydrogen and helium. It is a special case of the type of model considered in the present paper.

The aim of the present investigation is to answer the question: Given an energy production law of the form

$$\varepsilon = \varepsilon_0 \varrho T^4 \tag{1}$$

how do the properties of the star vary with the opacity law?

In answering this question the methods for integrating the equations of the equilibrium of the star will first be discussed (section 2). Section 3 gives the main results of using the punched

card equipment of the IBM Watson Scientific Computing Laboratory, New York City, for solving the differential equations. Altogether twenty different opacity laws have been considered. Detailed tables are given for eleven models. Finally, in section 4, the models are used to construct the Hertzsprung-Russell diagram for stars composed entirely of hydrogen and helium. It will be found that the results confirm those obtained independently by REIZ[5].

In appendix 1 the power expansions for the behaviour of the solutions near the center of the star are developed, while appendix 2 gives tables for the eleven models discussed in section 3.

2. A method for the integration of the equations of stellar equilibrium.

2.1. *The fundamental equations.* The stellar models to be considered in the present paper are specified in the following way: 1) The chemical composition is uniform throughout the star. 2) The star is in radiative equilibrium except for a possible convective core around the center. Convective zones near the surface are not considered. 3) The radiation pressure can be neglected. 4) The energy production is given by a law of the form

$$\varepsilon = \varepsilon_0 \varrho^{\delta} T^{\nu} \tag{2}$$

where ε is the production of subatomic energy per gram per second, ϱ the density, T the temperature, and ε_0, δ, and ν, constants. 5) The opacity is given by a law of the form

$$\varkappa = \varkappa_0 \varrho^{1-\alpha} T^{-3-s} \tag{3}$$

where $\varkappa$ is the mass opacity of the stellar material and $\varkappa_0$, α, and s, are constants. 6) The stellar material behaves like an ideal gas.

The fundamental equations governing the structure of a star of these properties are well known. In the present section they shall be discussed with special attention to the fact that the energy production takes place in an extended region around the center of the star. Also, a set of variables, which is particularly suited for solution by means of automatic computing machines, shall be introduced.

The starting point is the four standard equations of a star in equilibrium:[6]

$$dP/dr = -GM_r \varrho\, r^{-2} \tag{4}$$

$$dM_r/dr = 4\pi \varrho\, r^2 \tag{5}$$

$$dL_r/dr = 4\pi \varrho\, r^2\, \varepsilon \tag{6}$$

$$dT/dr = \begin{cases} -3\,(16\pi ac)^{-1}\, \varkappa \varrho\, L_r r^{-2}\, T^{-3} & \text{rad. eq. (7a)} \\ 0.4\; TP^{-1}\, dP/dr & \text{conv. eq. (7b)} \end{cases}$$

Here r denotes the distance from the center of the star, P the pressure, M_r the mass contained within the sphere of radius r, concentric with the star, L_r the flux of energy across this sphere, G the constant of gravitation, a the Stefan-Boltzmann constant, and c the velocity of light.

The physical contents of these equations can be stated as follows. The first equation is the condition that the star is in mechanical equilibrium in its own gravitational field, *i. e.* that the gravitational attraction on any element of matter will be compensated by the pressure gradient. Equation (7) is the equation governing the transport of energy from the center towards the surface. It assumes one of two forms, depending on whether the main agent of transport of energy is electro-magnetic radiation or convective currents, or, in other words, whether the point in question is in radiative or convective equilibrium. In radiative equilibrium the gradient of the radiation pressure, $aT^4/3$, becomes proportional to the flux of energy, $L_r r^{-2}$, and the opacity per unit volume, $\varkappa\varrho$. Where convective currents are present the matter will be in adiabatic equilibrium, with the ratio of the specific heats equal to 5/3, valid for monatomic gases, and the temperature gradient is independent of the flux of energy. Equations (5) and (6) express the relation between the microscopic quantities, ϱ and $\varrho\varepsilon$, and the macroscopic quantities, M_r and L_r. To these equations we must add the equation of state of the stellar material, in our case of a perfect gas,

$$P = \Re\, \mu^{-1}\, \varrho\, T \tag{8}$$

where μ is the mean molecular weight, and $\Re$ is the gas constant.

The quantities $\varkappa$ and ε describe the physical behaviour of the stellar material, $\varkappa$ measuring the interaction of the radiation and the matter, ε giving the output of subatomic energy. Thus it is clear that they will depend on the physical parameters of the matter and the chemical composition, *i. e.* we can write

$$\varkappa = \varkappa\,(\varrho,\ T,\ \text{chemical composition})$$
$$\varepsilon = \varepsilon\,(\varrho,\ T,\ \text{chemical composition}).$$

The problem of computing the structure of a star with a given radius R, mass M, and total energy output L, is now equivalent to finding a solution of the eq. (4) to (8) which satisfies the boundary conditions

$$\left.\begin{aligned} M_r &= M \\ L_r &= L \\ P &= 0 \\ T &= \text{surface temperature} \end{aligned}\right\} \text{for } r = R \qquad (9)$$

$$\left.\begin{aligned} L_r &= 0 \\ M_r &= 0 \end{aligned}\right\} \text{for } r = 0. \qquad (10)$$

The surface temperature can be put equal to zero without any appreciable error being introduced. The problem is thus one of four simultaneous differential equations with two point boundary conditions—the fundamental problem of all such work as the present.

In order to illustrate the character of the problem we will now briefly discuss two different, though mathematically equivalent, methods for solving the problem by means of stepwise numerical integrations, namely a) by integrating from the surface and b) by integrating from the center of the star.

a) Suppose R, M, and L, to be given. If we then *assume* a chemical composition we can, by stepwise integration, calculate the run of the quantities P, T, L_r, and M_r, as functions of r, going from the surface towards the center. In general we will find, however, that the two conditions $L_r = M_r = 0$ for $r = 0$ are not satisfied. In order to get the proper solution we must, therefore, carry out a number of integration runs, varying systematically *two* chemical parameters.

b) In order to start a numerical calculation from the center

we must assume given the chemical composition and the central values of the density and the temperature, ϱ_c, and T_c. As the condition that our model is physically possible we have the one condition that ϱ and T must vanish for the same value of r. Such a model can be found by carrying out integrations for systematically varied ϱ_c, say. Thus, for given composition and T_c we will, in general, determine one definite star with certain values of R, L, and M. By also varying T_c we can find solutions with prefixed values of, for instance, M. In this way we have arrived at the celebrated theorem of VOGT and RUSSELL: Given the chemical composition and the mass of the star, the radius and luminosity follow. Finally we arrive at the same conclusion as when discussing a), that in order to fit the solutions to given values of R and L, as well as M, we must vary *two* chemical parameters.

Although the conclusions of the two discussions a) and b) are equivalent, the two procedures are still quite different, in that a) requires two parameters to be varied in order to find the solution with given R, L, and M, while in b) four parameters must be varied in order to obtain the same result. This is the reason why calculations of the structure and composition of given definite stars, as for instance EPSTEIN's work on solar models, is carried out in the manner described as a). Even then a considerable amount of work is required before the eigensolution is found, and it is highly desirable to reduce the number of parameters to be varied to one, when a more extensive program of calculations of stellar models is undertaken, even at the cost of some accuracy. This is accomplished by the application of homology transformations.

2.2. *Homology transformations.* We speak of two stellar models being homologous when values of the physical variables describing one of them can be obtained by multiplying the corresponding values for the other model by definite scale factors. Denoting by T_0, L_{r0}, ϱ_0, M_{r0}, the variables at the point r_0 in one model, we get for the point $r = r_1$ in the homologous model

$$\left.\begin{aligned} r_1 &= C_r r_0 \\ T_1 &= C_T T_0 \\ L_{r1} &= C_L L_{r0} \\ \varrho_1 &= C_\varrho \varrho_0 \\ M_{r1} &= C_M M_{r0} \end{aligned}\right\} \qquad (11)$$

The existence of models described by quantities with suffix 1 is established only if these variables satisfy the equations (4) to (8). In the case of the equations (4), (5), and (8), this is apparently so, provided the scale factors satisfy suitable conditions. Equations (6) and (7), which contain the, as yet, unspecified functions $\varkappa$ and ε, must, however, be considered in some detail. First we have, by assumption,

$$dT_0/dr_0 = -3\,(16\,\pi\,ac)^{-1}\,\varkappa_0\,(\varrho_0 T_0)\,\varrho_0 L_{r0}\,r_0^{-2}\,T_0^{-3} \tag{12}$$

The condition that the configuration (11) does, in fact, satisfy eq. (7a) is

$$dT_1/dr_1 = -3\,(16\,\pi\,ac)^{-1}\,\varkappa_1\,(\varrho_1 T_1)\,\varrho_1 L_{r1} r_1^{-2}\,T_1^{-3} \tag{13}$$

or, using (11),

$$\left.\begin{aligned} &C_T C_r^{-1}\,dT_0/dr_0 = \\ &-3\,(16\,\pi\,ac)^{-1}\,C_\varrho C_L C_r^{-2} C_T^{-3}\,\varkappa_1\,(\varrho_1 T_1)\,\varrho_0 L_{r0} r_0^{-2}\,T_0^{-3}. \end{aligned}\right\} \tag{14}$$

Comparing (12) and (14) we find that two models are homologous if their laws of opacity satisfy the functional equation

$$\varkappa_0\,(\varrho_0 T_0) = C_\varrho C_L C_r^{-1} C_T^{-4}\,\varkappa_1\,(\varrho_0 C_\varrho,\ T_0 C_T) \tag{15}$$

This will always be the case if the opacity can be written on the form

$$\varkappa = \varkappa_0\,\varrho^{1-\alpha}\,T^{-3-s} \tag{16}$$

where α and s denote constants, while $\varkappa_0$ is a quantity which varies from one model of the homologous family to the next. In quite a similar manner we deduce from eq. (6) that in order to make the homologous transformation valid we must have

$$\varepsilon = \varepsilon_0\,\varrho^{\delta}\,T^{\nu}. \tag{17}$$

An application of this result will introduce an important simplification in the problem if the opacity and energy production can be written as (16) and (17), where the chemical composition enters only through the factors $\varkappa_0$ and ε_0. In that case the change of the chemical composition will only cause the model to vary within the same homologous family of solutions. Consequently, once a single member of the family has been found, it will be a

simple matter to discuss the relation between the chemical composition and the parameters of the stellar model, R, M, and L.

We shall now proceed on the assumption of the validity of the expressions (16) and (17), postponing the discussion of their physical applicability. Then the problem is solved as soon as one solution with the proper boundary conditions is known. Adopting the method b) we can now chose arbitrary values for $\varkappa_0$, ε_0, and T_c. By varying ϱ_c we find the solution which satisfies the condition $\varrho = 0$ and $T = 0$ simultaneously for some value of r, R. This will give us a stellar model with definite values of R, L, M, T_c, ϱ_c, $\varkappa_0$, ε_0, and μ. Of these quantities $\varkappa_0$, ε_0, and μ, are assumed to be functions of the chemical composition. If the structure for some other values of R, L, and M, is wanted we only have to use scale factors. Of the three conditions to be satisfied, one determines the central temperature. The others impose two conditions on the chemical parameters. One of these is Eddington's mass-luminosity relation, the other one is the condition that the total energy released by nuclear processes equals the luminosity.

It should be mentioned that a simplification of the integration procedure does not appear in the approach described in a), and it is quite obvious that the method b) should be used.

2.3. *Homology invariant variables.* The method for finding the eigensolutions of the fundamental equations outlined above could probably be used for the actual numerical procedure. Additional simplifications may, however, be introduced by using different variables, with the further important advantage that the equations become far better suited for solution with the aid of automatic computing machinery. Indeed, as will be demonstrated presently, it will be a great advantage to use as variables the homology invariant quantities

$$V = -d\log P/d\log r = GM_r\varrho\, r^{-1}P^{-1} \tag{18}$$

$$U = d\log M_r/d\log r = 4\pi\varrho\, r^3 M_r^{-1} \tag{19}$$

$$W = d\log L_r/d\log r = 4\pi\varepsilon_0\, r^3\varrho^{1+\delta}\, T^{\nu}L_r^{-1} \tag{20}$$

$$\left.\begin{aligned} H = V/(n+1) &= -d\log T/d\log r \\ &= 3\varkappa_0\,(16\pi ac)^{-1}\varrho^{2-\alpha}L_r r^{-1}T^{-7-s}\end{aligned}\right\} \tag{21}$$

Of these variables V and U are well known, H is closely related to the equally well known polytropic index, n, while W has been introduced by OSTERBROCK and the present author[7]. As can readily be shown, the boundary conditions for these variables are

$$U = W = 3, \qquad V = H = 0 \qquad \text{for } r = 0 \tag{22}$$

$$U = W = 0, \qquad V \to \infty, \; H \to \infty \qquad \text{for } R = r. \tag{23}$$

The differential equations satisfied by these variables are deduced by logarithmic differentiation of the eq. (18) to (21), making use of eq. (4) to (8) and also of the equations themselves. We get

$$dV/V = (U + H - 1)\, dr/r \tag{24}$$

$$dU/U = (3 - V + H - U)\, dr/r \tag{25}$$

$$dW/W = (3 - (1 + \delta)\, V - (\nu - 1 - \delta)\, H - W)\, dr/r \tag{26}$$

$$dH/H = ((9 + s - \alpha)\, H - (2 - \alpha)\, V + W - 1)\, dr/r. \tag{27}$$

It is now apparent that we can eliminate the last physical variable, r, simply by choosing the independent variable among the four homology invariants. The most convenient variable for this purpose appears to be V* and we are then left with the equations

$$dU/dV = U\,(3 - V + H - U)\, V^{-1}\,(U + H - 1)^{-1} \tag{28}$$

$$\left.\begin{aligned} dW/dV = W\,(3 &- (1 + \delta)\, V \\ &- (\nu - 1 - \delta)\, H - W)\, V^{-1}\,(U + H - 1)^{-1} \end{aligned}\right\} \tag{29}$$

$$\left.\begin{aligned} dH/dV = H\,((9 &+ s - \alpha)\, H \\ &- (2 - \alpha)\, V + W - 1)\, V^{-1}\,(U + H - 1)^{-1}. \end{aligned}\right\} \tag{30}$$

The great advantages of using the variables V, U, H, and W, now become apparent. In fact, expressed in these variables, the four fundamental differential equations are reduced to three differential equations and a quadrature. For, in order to return to the physical variables from a solution expressed in the homology invariant variables, we only have to perform a quadrature, *e. g.*

* A similar method has been used by LEVEE[8], who choses W as his independent variable.

$$\log r/r_0 = \int_{V_0}^{V} V^{-1} (U + H - 1)^{-1} \, dV \tag{31}$$

and then use the equations (18) to (21). Also, the differential equations are very convenient for treatment by means of automatic computing machinery, because they do not involve exponentials.

Having now demonstrated the advantage of using the variables (18) to (21) we only have to understand their behaviour at the boundaries before we can use them for actual computations. We have already given the boundary conditions for all our variables at the center and the surface of the star, eq. (22) and (23). Integrating, as we intend to do, from the center towards the surface, the new independent variable, V, varies from zero to infinity. In practise one must, of course, break off at some suitably large value of V. As to the conditions at the center we find, by inserting the values of the variables at the center in the eq. (28) to (30), that $V = 0$ is a singularity, so that a parameter is necessary to label a solution starting at the center. This is not surprising, when compared with the procedure for solving the problem in physical variables discussed above. It is quite clear that, also when using the new variables, it will be necessary to carry out trial computations, varying one parameter, before the solution satisfying the boundary conditions both at the center and at the surface is found. As the parameter labeling the trial solutions it has been found convenient to use

$$H_0' \equiv (dH/dV)_{V=0} = (n_c + 1)^{-1} \tag{32}$$

where n_c denotes the polytropic index at the center of the star. The numerical solution cannot be started from the center where all derivates become indeterminate. We have, therefore, expanded H, U, and W, in powers of V, the coefficients of the series being functions of H_0'. The evaluation of the power series is elementary, but rather lengthy, and has been given in appendix 1.

Suppose now that a value of H_0' is chosen. Using the expansions of appendix 1 we can then compute H, W, and U, for a value of V close to zero, *e. g.* $V = 0.2$. From here we can continue the solution of the equations (28) to (30) to some large value

of V ($V = 15$ is convenient) by step-by-step numerical integration. The question is now, what is the criterion that H_0' is chosen in such a way that the solution corresponds to a configuration in which P and T simultaneously tend to zero? In order to find this condition we observe that near the surface of the star we must have an approximate relation of the kind

$$P \propto T^q \tag{33}$$

where q is some positive number which is left undetermined for the moment. But from this relation it follows that near the surface we have

$$n + 1 = V/H = d(\log P)/d(\log T) = q \tag{34}$$

i. e. near the surface the polytropic index must tend to a finite positive limit. The actual value of this can now easily be found from the eq. (30). Near the surface we can neglect the constants and the functions U and W in comparison with H and V, which increase beyond any limits. Writing n_0 for the value of n at the surface, we have then

$$H = V/(n_0 + 1) \tag{35}$$

and we find

$$n_0 + 1 = (8 + s - \alpha)/(2 - \alpha). \tag{36}$$

The required criterion is that the quantity V/H tends to this limit for large V.

It is of considerable interest to know what happens if the parameter H_0' is not chosen to be equal to the eigenvalue. The numerical work shows that the solutions are extremely sensitive to variations of this parameter. In fact, if H_0' is chosen only slightly below the eigenvalue, H will reach a maximum and the denominator $U + H - 1$ will become zero for some finite value of V. If, on the other hand, H_0' is chosen larger than the eigenvalue, H will increase so as to make $n + 1 = V/H < 2.5$ at some value of V. At this point the equation of radiative equilibrium will cease to be valid. Only if H_0' is chosen quite close to the eigenvalue will the solution ever reach $V = 15$. Generally, the sensitivity of the solutions can be understood from the presence of the rather large coefficient $9 + s - \alpha$ in eq. (30). In the eigen-

solution the quantity $(9 + s - \alpha) H - 1 + W - (2 - \alpha) V$ will remain small only because the first and the last term nearly cancel. Any deviation from this solution will quickly be amplified, when the solution is followed towards larger V.

Once the eigensolution, expressed in homology invariant variables, have been found, there remains the problem of calculating the solutions expressed in physical variables. This calculation necessitates one further integration, *e. g.* the quadrature (31). The variables M_r, L_r, P, and T, could then be found by means of the eq. (18) to (21). The automatic computing machinery being available it was, however, more convenient to evaluate all of the physical variables by means of quadratures. From the eq. (18) to (21) and (24) to (27) we find

$$\left.\begin{aligned} \log P &= -\int_{V_0}^{V} dV/(U + H - 1) + \text{constant} \\ \log T &= -\int_{V_0}^{V} H dV/V\,(U + H - 1) + \text{constant} \\ \log M_r &= \int_{V_0}^{V} U dV/V\,(U + H - 1) + \text{constant} \\ \log L_r &= \int_{V_0}^{V} W dV/V\,(U + H - 1) + \text{constant} \end{aligned}\right\} \quad (37)$$

The constants of integration were chosen so that the functions $\log r/R$, $\log P/P_c$, $\log T/T_c$, $\log M_r/M$, and $\log L_r/L$, resulted. For $\log r$ and $\log M_r$ this made an analytic approximation of the solutions beyond $V = 15$ necessary. This was derived in the following manner.

2.4. *Expansions valid near the surface of the star.* Let us, following C. M. and H. Bondi[9], introduce the three homology invariant variables

$$Q = -\, d \log r/d \log P = V^{-1} \quad (38)$$

$$S = -\, d \log M_r/d \log P = U/V \quad (39)$$

$$N = \quad d \log T/d \log P = (n + 1)^{-1} = H/V. \quad (40)$$

These variables are convenient near the surface where V and H tend to infinity. Using eq. (18) to (21) and remembering that $L_r = L$ near the surface we now get

$$dQ/Q = dr/r - dM_r/M_r + dT/T = (-Q + S + N)\, dP/P \tag{41}$$

$$dS/S = 4\, dr/r - 2\, dM_r/M_r + dP/P = (1 - 4\,Q + 2\,S)\, dP/P \tag{42}$$

$$\left.\begin{aligned} dN/N &= (2-\alpha)\, dP/P - (8 + s - \alpha)\, dT/T - dM_r/M_r \\ &= \left[(2-\alpha) - (8 + s - \alpha)\, N + S\right] dP/P. \end{aligned}\right\} \tag{43}$$

Owing to its close relation to V we shall find it convenient to use Q as the independent variable, rather than S as used by Bondi and Bondi. We then get the differential equations

$$dS/dQ = S(1 - 4\,Q + 2\,S)\, Q^{-1} (S + N - Q)^{-1} \tag{44}$$

$$dN/dQ = N((2-\alpha) - (8 + s - \alpha)\, N + S)\, Q^{-1} (S + N - Q)^{-1}. \tag{45}$$

We intend to use these only for $V \geqslant 15$, i. e. for $0 < Q \leqslant 1/15$. Also, S is small near the surface, and thus we have approximately

$$dS/dQ = S Q^{-1} N_0^{-1} \tag{46}$$

whence

$$S = A^{-1} Q^{1/N_0} \tag{47}$$

where A is a constant. This approximation is better than might at first be expected. This is due to the fact that N, in the applications, usually is close to $^1/_4$, so that $N(1 - 4\,Q + 2\,S)(S + N - Q)^{-1}$ remains close to unity even for rather large values of Q.

The variable N will be nearly constant equal to

$$N_0 = (1 + n_0)^{-1} = (2 - \alpha)/(8 + s - \alpha) \tag{48}$$

near the surface. A better approximation can be found if eq. (45) is analysed with respect to the importance of the various terms. It becomes apparent that for small variations of N it makes sense to regard $N/(N + S - Q)$ as a constant at the same time as the variation of $(2 - \alpha) - (8 + s - \alpha)\, N + S$ is taken into account. In fact, this latter quantity can be written

$$-(8 + s - \alpha)\ (N - N_0) + S.$$

At the surface we have $S = 0$, and near the surface the two terms are comparable. This suggests that it would be a good approximation to write

$$N - N_0 = BS \tag{49}$$

where B is a suitable constant. For its determination we get from (45)

$$BdS/dQ = N_0 (N_0 + S - Q)^{-1} SQ^{-1} (1 - (8 + s - \alpha) B)$$

or, using (46),

$$B = N_0 (3 - \alpha - \eta (n_0 + 1) Q)^{-1}. \tag{50}$$

Strictly, S and Q are zero where the approximations are valid. The form given, eq. (50), suggests that slightly better results would be obtained for finite values of Q if a coefficient B, which is slowly increasing with Q, is used, η being a factor less than, but of the order of, unity.

We now get from eq. (38) and (41), corresponding to (31),

$$\log r/R = -\int_0^Q (N + S - Q)^{-1} dQ. \tag{51}$$

Beginning with the most important, the order of magnitude of the quantities is, N_0, Q, S, and $N - N_0$. We can therefore expand the integrand

$$(N + S - Q)^{-1} = (N_0 - Q)^{-1} - (1 + B)(N_0 - Q)^{-2} S \ldots \tag{52}$$

where we have used (49). The first term can be integrated exactly. In the second term we use the first two terms of the expansion for $(N_0 - Q)^{-2}$. In this way we get

$$\left.\begin{aligned} \log_{10} r/R = \log_{10} (1 - QN_0^{-1}) \\ + \frac{(n_0 + 1)^2}{n_0 + 2} \left(1 + \frac{2 (n_0 + 1)(n_0 + 2)}{n_0 + 3} Q\right) SQ \log e \, (1 + B) \end{aligned}\right\} \tag{53}$$

where we have used eq. (48).

The approximation for $\log M_r/M$ is derived in the following way. Using eq. (39) and (41) we find

$$\log M_r/M = -\int_0^Q SQ^{-1}\,(N - Q + S)^{-1}\,dQ \tag{54}$$

Inserting the expansion (52) we find that none of the terms can be integrated exactly, and we have to expand $(N_0 - Q)^{-1}$ and $(N_0 - Q)^{-2}$ in power series in QN_0^{-1}. Using (47) we can integrate term by term, and get, after some reduction,

$$\left.\begin{aligned}\log_{10} M_r/M = &- U \log e\left[\frac{1}{n_0+1}\frac{Q}{N_0} + \frac{1}{n_0+2}\left(\frac{Q}{N_0}\right)^2 + \frac{1}{n_0+3}\left(\frac{Q}{N_0}\right)^3 \ldots\right] \\ &+ \left(\frac{Q}{N_0}\right)^2 U^2\,(1+B)\,\log e\left[\frac{1}{2\,n_0+2} + \frac{2}{2\,n_0+3}\frac{Q}{N_0} \ldots\right].\end{aligned}\right\} \tag{55}$$

As an illustration of the use of these relations we take the following values which have been obtained from one of the integrations described in section 3. The constants of the model are

$$\alpha = 0.5$$
$$s = -2.1.$$

We then find from (36)

$$n_0 + 1 = N_0^{-1} = 3.6.$$

The integration from the center gives for $V = 15$:

$$U = 0.2162.$$

Then, from (38) and (39),

$$S = 0.0144$$
$$Q = 0.0667.$$

From (50) we find

$$B = 0.12,$$

and from (53) and (55)

$$\log_{10} r/R = 9.8826 - 10$$
$$\log_{10} M_r/M = 9.9926 - 10.$$

The expansions are equally useful for starting integrations from the surface. In this case each solution will be specified by the value of the parameter A (eq. (47)). Choosing a starting value for V the expansions will provide values of N, r/R, and M_r/M.

2.5. *Invariant parameters of the models.* In addition to the functions (37), the invariant parameters, which specify the models, must be found. Corresponding to the homology transformations each model can be characterized by four parameters. In an obvious extension of the convention adopted by CHANDRASEKHAR[10] we choose the parameters to be the following:

The ratio of central density to mean density,

$$F = \varrho_c/\bar{\varrho}, \tag{56}$$

the central temperature constant,

$$E = T_c R/\mu M, \tag{57}$$

the constant in the mass-luminosity relation,

$$C = \frac{LR^{3\alpha+s}\varkappa_0}{M^{5+s+\alpha}\mu^{7+s}}\left(\frac{k}{Gm_H}\right)^{7+s}\frac{3}{4(4\pi)^{3-\alpha}ac}, \tag{58}$$

which, expressing L, R, and M, in solar units, becomes

$$\log C = -27.0448 + 0.3274\,\alpha - 7.3638\,s + \log\frac{LR^{3\alpha+s}\varkappa_0}{M^{5+s+\alpha}\mu^{7+s}},$$

and the ratio of central energy production rate to mean energy production rate,

$$D = \varepsilon_c M/L. \tag{59}$$

Using the eq. (18) to (21) it can be shown that these parameters satisfy the relations

$$F = (T/T_c)\,(M_r/M)\,(P/P_c)^{-1}\,(r/R)^{-3}\,(U/3) \tag{60}$$

$$E = (M_r/M)\,(r/R)^{-1}\,(T/T_c)^{-1}\,V^{-1}\,G\,(k/m_H)^{-1} \tag{61}$$

$$C = (M_r/M)^{5+s+\alpha}\,(r/R)^{-3\alpha-s}\,(L_r/L)^{-1}\,HU^{-2+\alpha}\,V^{-7-s} \tag{62}$$

$$D = (L_r/L)\,(P/P_c)^{-\delta}\,(M_r/M)^{-1}\,(T/T_c)^{\delta-\nu}WU^{-1}. \tag{63}$$

It should be noted that the quantities on the right hand side are independent of the point in the star which is used in their determination. This constancy can serve as a check on the last stage of the calculation of the solutions.

2.6. *Convective cores.* Up to this point the question of convective cores has been ignored. It is, however, very easy to extend the already developed procedure for integrating the equations of equilibrium of a star to the case of a star with a convective core. As is well known, the structure of a convective core is described by

$$T = T_c\,\theta\,(\xi) \tag{64}$$

$$\varrho = \varrho_c\,\theta\,(\xi)^{3/2} \tag{65}$$

where θ is the Emden function for the polytropic index $n = 3/2$, and ξ is proportional to r. The function θ, together with V and U expressed as functions of ξ, have been tabulated[11]. Furthermore we have

$$H = V/(n+1) = 2\,V/5. \tag{66}$$

Suppose now that the core extends to a value of $V = V_{core}$. Outside this point eq. (30) replaces (66). At V_{core} all our variables, including V, U, H, and W, must be continuous, and we can find the proper starting point for the numerical integrations from their values on the boundary of the core. Of these V and U are known from the tables quoted above, and H is found from eq. (66). The variable W, finally, can be found for any point in the core, using (20), (6), (17), (64), and (65), which give

$$W = \xi^3\,\theta^{\nu+3\,\delta/2}\Big/\int_0^{\xi}\theta^{\nu+3\delta/2}\,\xi^2\,d\,\xi\,. \tag{67}$$

This quantity is a function of ξ and $\nu + 3\,\delta/2$ only and has been tabulated by the present author[12].

Having thus determined V, U, W, and H, on the boundary of the core we can carry out the stepwise integration of eq. (28) to (30) to see whether the condition (36) is satisfied for large V. If not, it is a sign that the core has not been assigned the right extent.

2.7. *Summary of the method.* As a summary of the present section, here is a short directory in the use of the method:

Given the four exponents, α, s, δ, and ν, find the series expansions valid near the center, using the formulae of appendix 1. It is most convenient to choose a suitable small value of V, *e.g.*

0.2, and then, by using the Taylor series, to find U, W, and H, as polynomials in H_0'. (If it is known already from other evidence that the model possesses a convective core, this calculation and the following one can, of course, be omitted.)

In order to determine whether a convective core is present or not, compute a trial solution, starting with $n_c = 3/2$, *i.e.* $H_0' = 0.4$, and using (28) to (30) for a step-by-step integration. If H or $U + H - 1$ become zero, a convective core is actually present. If, on the other hand, H increases so rapidly as to make $n + 1 = V/H$ smaller than 2.5 at some point no convective core is present.

Trial solutions corresponding to varying initial conditions must now be calculated until a solution is found for which the polytropic index $n + 1 = V/H$ approaches the proper surface value (36) for large values of V. The parameter to be varied is H_0' in the cases of no convective core, and V_{core} when a core is present. In the latter case the initial values are taken from the tables of the Emden functions, as described in section 2.6.

The run of the physical variables can now be found, r following from (31), and the other variables from eliminations among the eq. (18) to (21), or from the quadratures (37). If the five physical variables are expressed in units of R, P_c, T_c, M, and L, the series expansions (53) and (55) will be useful.

With the complete solution thus computed the constants of the model follow from (60) to (63).

3. Numerical results.

3.1. *The use of the punched card equipment.* In the preceding section it has been shown that the calculation of the structure of a star with the opacity given by $\varkappa = \varkappa_0 \varrho^{1-\alpha} T^{-3-s}$ and the energy production given by $\varepsilon = \varepsilon_0 \varrho^{\delta} T^{\nu}$ can be reduced to the stepwise integration of the eq. (28) to (30). In this section we shall describe how the calculations for the case $\varepsilon = \varepsilon_0 \varrho T^4$ have been carried out by means of the IBM punched card equipment at the Watson Laboratory, New York City, and the results obtained will be given.

Before any numerical calculations can be made, the differential equations must be approximated so as to permit a solution

in a finite number of algebraic operations, which is, essentially, a problem of replacing the integrals by suitable summations. For this purpose a process of successive approximations was used. In the first approximation the solution was computed by repeated expansions, using only the first term of the Taylor series, according to the formulae

$$\left.\begin{aligned} U_{m+1} &= U_m + \Delta V\,(dU/dV)_m \\ W_{m+1} &= W_m + \Delta V\,(dW/dV)_m \\ H_{m+1} &= H_m + \Delta V\,(dH/dV)_m \end{aligned}\right\} \qquad (68)$$

where ΔV stands for the constant steplength of the independent variable, and we have used the subscript m to denote the value of the variables at the point $V_m = V_0 + m\,\Delta V$. Thus, in the first approximation, we get tables of the three variables U, W, and H, which, however, do not exactly satisfy the differential equations. These tables can then be used to calculate good approximations for second order terms in the expansions, *e.g.*

$$^1/_2\,(\Delta V)^2\,(d^2U/dV^2)_m \approx U_{m+1} - 2\,U_m + U_{m-1} \qquad (69)$$

and a second order run can then be computed using

$$U_{m+1} = U_m + \Delta V\,(dU/dV)_m + {}^1/_2\,(\Delta V)^2\,(d^2U/dV^2)_m \qquad (70)$$

and similarly for W and H.

The efficiency of this method depends strongly on the steplength, ΔV, which must be chosen small in order to make the process converge rapidly. A value of ΔV of 0.1 was found suitable when four decimal places were carried. In fact, no higher approximations than the second were needed.

In the course of the calculations extensive use was made of the excellent collection of computing machines at the IBM Watson Laboratory. However, the only particular technique worth mentioning in the present connexion is the one used for solving the eq. (28) to (30) by means of the model 604 electronic calculating punch. This machine has a rather small capacity for numbers, a total of only 50 digits. It was found possible to solve the differential equations only by using two punched cards to advance the variables by one step. In this way numbers may be stored provisionally on a card while it passes from the punch station to the

second reading station. Use of this technique, and of the selective circuits of the machine, made it possible to fit the variables into the capacity of the machine. The speed of the process was 50 integration steps per minute. During the search for the eigensolutions several hundred integration runs were performed.

The other problem of the numerical solution of the equations is the sensitivity of the solutions against variations in the initial values. Owing to this sensitivity any deviation of the trial value of H_0' from the eigenvalue will cause the solution to end at a physically impossible point, either $n = \infty$ or $n = 0$. The sensitivity is so strong that a solution which is started from the point $V = 0.2$ and calculated with four decimal places will rarely go beyond $V = 6$ before an impossible point is reached. Thus one can have two starting values of H_0' at $V = 0.2$, differing by one unit in the fourth decimal, one of which will cause n to vanish at $V = 6$, while the other will send n off to infinity before $V = 6$. One way of overcoming this difficulty would be to carry more decimals. This was, however, not possible with the 604. Another method is suggested by the following table, which shows some results of two runs:

	$H_0' = 0.3605$			$H_0' = 0.3610$		
V	U	W	H	U	W	H
0.2	2.9232	2.6846	0.0708	2.9233	2.6844	0.0709
3.0	1.8551	0.3200	0.8340	1.8662	0.3213	0.8682
5.4						∞
5.8			0.0000			

It is apparent that the two solutions, which differ widely at $V = 5.5$ are still close together at $V = 3$. It therefore suggests itself to start further runs from $V = 3$, interpolating the initial values between the two solutions:

$$\left.\begin{aligned} V &= 3 \\ U &= 1.8551 + 0.0111\,\gamma \\ W &= 0.3200 + 0.0013\,\gamma \\ H &= 0.8340 + 0.0342\,\gamma \end{aligned}\right\} \qquad (71)$$

where γ is a parameter to be determined by further integrations. This method proved to be quite satisfactory, but had to be used

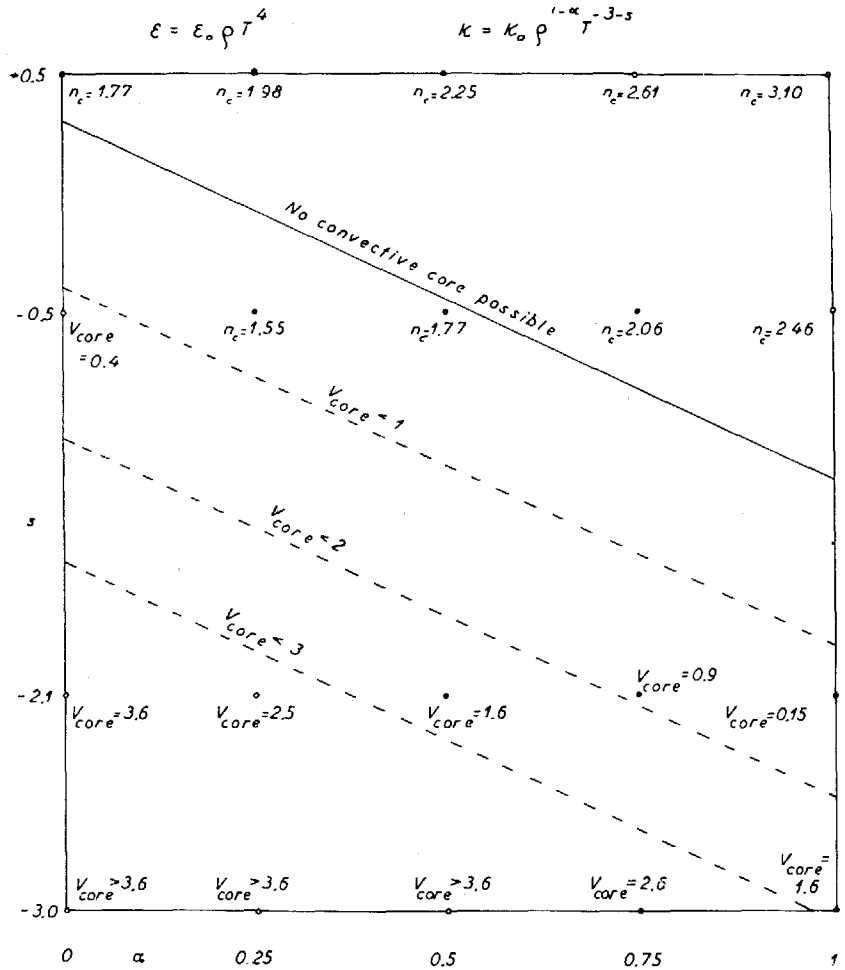

Fig. 1.

several times for increasing values of V, usually at V equal to 3, 6, and 10. Only the solutions started at $V = 10$ could be followed as far as $V = 15$. The result of this procedure was a pair of solutions lying closely on either side of the eigensolution for each of the intervals in V between 0, 3, 6, and 10. The finally adopted solution was found by linear interpolation between these pairs, use being made of the interpolation factors γ as defined above. As a check, the final solution was compared with the differential equations. Usually the error in the increase of the

solution for one step was found to be within one unit in the fourth decimal, and only in a single case a deviation of as much as four units was encountered. This fully justifies the procedure. It is interesting to note that if all runs had been calculated all the way from $V = 0.2$, at least ten decimals would have been necessary in the calculations.

3.2. *The behaviour of the convective core.* During the initial stages of the work twenty different opacity laws were considered, *viz.* all combinations of $\alpha = 0.0$, 0.25, 0.5, 0.75, and 1.00, and $s = +0.5, -0.5, -2.1$, and -3.0. In this way the two important cases of constant opacity ($\alpha = 1$, $s = -3.0$) and Kramers opacity ($\alpha = 0$, $s = +0.5$) and a number of intermediate cases were covered. At the later stages only eleven of the cases were investigated. However, the material gives information concerning the extent of the convective core for all of the twenty models. This information is represented in figure 1. This diagram also shows the results of applying the criterion for the existence of a convective core to the model in question[7], [12]. It is apparent that this criterion alone is sufficient for a reasonably good first indication as to whether a core may be expected or not.

3.3. *Invariant parameters of eleven models.* Table 1 gives the invariant parameters of the eleven models which have been considered in detail, calculated according to eq. (60) to (63). For R, M, and L, solar units have been used.

The principal results of an inspection of this table are the following:

a) For given R, M, and μ, T_c increases for increasing α and s. Speaking in terms of figure 1, T_c increases towards the upper right of the diagram. For models along the line connecting Kramers and constant opacity the central temperature is nearly constant, decreasing slightly towards the latter.

b) Qualitatively, the ratio $\varrho_c/\bar{\varrho}$ varies in the same way as the central temperature. However, the drop of the density concentration towards the constant opacity end of the diagram is more pronounced than is the drop of central temperature.

c) The ratio $\varepsilon_c M/L$ is nearly constant, independent of the opacity law.

TABLE 1.

$\varepsilon = \varepsilon_0 \varrho T^4 \qquad \varkappa = \varkappa_0 \varrho^{1-\alpha} T^{-3-s}$

Model no.	α	s	log $RT_c/\mu M$ = log E	log $\varrho_c/\bar{\varrho}$ = log F	log $C + 10$	log $\varepsilon_c M/L$ = log D
1	0.00	+ 0.5	7.317	1.642	4.076	0.942
2	0.25	+ 0.5	7.416	2.005	4.567	0.951
3	0.50	+ 0.5	7.604	2.658	5.385	0.964
4	0.25	— 0.5	7.295	1.529	4.761	0.942
5	0.50	— 0.5	7.397	1.913	5.206	0.951
6	0.75	— 0.5	7.615	2.664	6.002	0.965
7	0.50	— 2.1	7.204	1.153	5.761	0.925
8	0.75	— 2.1	7.280	1.431	6.022	0.942
9	1.00	— 2.1	7.434	1.982	6.465	0.963
10	0.75	— 3.0	7.174	1.040	6.508	0.915
11	1.00	— 3.0	7.253	1.316	6.716	0.933

3.4. *The structure of the models.* In appendix 2 the variation of the physical parameters through the eleven models is given. In the calculations four decimals were carried throughout. The figures given have been rounded to three decimals.

The figures termed *variations* give variations of the fundamental variables U, W, and H, for variations of the initial values. They correspond to the coefficients of γ as used in eq. (71). The corresponding variations of the initial values could not be determined with any accuracy.

4. Applications of the models.

4.1. *The Sun.* The integrations described in section 3 have been used for the construction of two different models for the Sun. During this work the integrations were only used to describe the central parts of the Sun, use being made of the variations explained in section 3.4, while the exterior regions were convered to a large extent by the series expansions of section 2.4. It was found possible to fit the model to a physically given opacity, which contained contributions from the heavy elements, free-free transitions in hydrogen and helium, and scattering on free electrons. Details of these models have been published elsewhere[13].

4.2. *The hydrogen-helium star.* The most natural application of the models is the construction of the Hertzsprung-Russell diagram for stars composed entirely of hydrogen and helium, since the models are based on an energy production law of the type (1). Indeed, the energy production by the proton-proton reaction is given by[14]

$$\varepsilon = 10^{-29.0054} X^2 \varrho T^4.$$

Here X is the abundance of hydrogen, by mass. However, in such a star the opacity will be due to scattering on free electron and free-free transitions in hydrogen and helium, and cannot directly be expressed in the form (16). The principal question is thus how to apply our models to stars in which these two agents both contribute to the opacity. In this question we choose the following approach.

According to the theorem of VOGT and RUSSELL the structure of a star is uniquely determined by the mass and the chemical composition. In a mixture of hydrogen and helium there is only one chemical parameter. Consequently our stars form a two-parametrical sequence, the parameters being the mass, M, and the hydrogen abundance, X. Consider now the two contributions to the opacity. In general it will, of course, be necessary to take both of them into account. It seems likely, however, that in a certain region of the (M, X)-diagram the electron scattering will be negligible. The behaviour of our stars in this region will then be given by the model based on an opacity law of the form

$$\varkappa = 2.74 \cdot 10^{22} (1 + X) \varrho T^{-3.5} \tag{72}$$

where the constant is the one used in the construction of the solar models[13].

Correspondingly, we expect to find, in another region of the (M, X)-diagram, that the free-free transitions can be neglected in the opacity. In this region we can then use the model based on

$$\varkappa = 0.2 (1 + X). \tag{73}$$

In the remaining part of the (M, X)-diagram we must take both the free-free transitions and the scattering into account. Here we may hope that one or more of the models, based on

opacity laws intermediate between the two already quoted will be of use. As explained below this is indeed the case.

Model 1.

Free-free transitions predominate.

Our first task, when using this model, is to determine the region in the (M, X)-diagram in which the scattering is negligible. We therefore first investigate how the opacity varies through this model, using the table of appendix 2,

V	r/R	$\log(\varrho/\varrho_c)(T/T_c)^{-3.5}$
0	0.00	0.00
6	0.41	0.45
15	0.72	0.63

We see that the Kramers opacity factor increases outwards in the star. We therefore only have to know that the scattering is relatively unimportant at the center to conclude that it is so through the whole star. On the basis of the four invariant constants of the model and the homology transformations we can convert the condition that the ratio of scattering opacity to free-free opacity has a definite value at the center into a relation between M and X. Indeed, using eq. (56) to (59) and demanding that the scattering opacity of eq. (73) is less than 10 % of the free-free opacity of eq. (72) we get

$$\log M \leqslant -1.0597 + \frac{1}{32}\log(1+X)\,X^2\,\mu^{-46}. \qquad (74)$$

Table 2 gives this function together with some more data for the corresponding stars.

Table 2.

Model 1. The sequence for which $\varkappa_{\text{scattering}}/\varkappa_{\text{free-free}} = 0.1$ at the center of the star.

X	Max $\log M$	$\log R$	$\log T_c$	$\log \varrho_c$	$\log L$	$\frac{1}{4}\log LR^{-2}$
1.00	9.38	9.85	6.55	0.85	6.49	9.20
0.75	9.27	9.75	6.60	1.03	6.51	9.25
0.50	9.13	9.63	6.67	1.26	6.53	9.32
0.25	8.94	9.45	6.78	1.61	6.53	9.41
0.10	8.80	9.26	6.91	2.03	6.50	9.49

The model does not possess a convective core.

It is apparent that the present model is valid only for very red dwarfs.

MODEL 8.

Free-free opacity and scattering opacity compete.

We shall now try to find a model which can be made to represent the case that the opacities from the two sources, eq. (72) and (73), are of the same order of magnitude. It is therefore necessary to adopt a method for combining the two contributions. In the present survey it was judged sufficiently accurate simply to add them together. We thus represent our physical opacity by the expression.

$$\varkappa_{\text{physical}} = 2.74 \cdot 10^{22} (1 + X) \varrho T^{-3.5} + 0.3 (1 + X) \qquad (75)$$

where the customary factor of 3/2 has been applied to the scattering. We now want to find a model in which the opacity, given by (75), runs closely to a function of the form (16). For this purpose we proceed as follows: Using the tables of appendix 2 we can compute the run of the actual opacity through each of our models. We have, in fact,

$$\varkappa/\varkappa_{\text{center}} = (\varrho/\varrho_c)^{1-\alpha} (T/T_c)^{-3-s}. \qquad (76)$$

Also, we can compute the run of the free-free opacity from

$$\varkappa_{\text{free-free}}/\varkappa_{\text{free-free, center}} = (\varrho/\varrho_c) (T/T_c)^{-3.5} \qquad (77)$$

The condition that the particular model is useful in the present context then becomes that there exists a relation of the kind

$$(\varrho/\varrho_c)^{1-\alpha} (T/T_c)^{-3-s} = x (\varrho/\varrho_c) (T/T_c)^{-3.5} + y \qquad (78)$$

where x and y are positive.

This test was carried out for the nine models available with the result that it was found that in model 8 we have

$$(\varrho/\varrho_c)^{0.25} (T/T_c)^{-0.9} = 0.110 (\varrho/\varrho_c) (T/T_c)^{-3.5} + 0.954 \qquad (79)$$

within an accuracy of 6 % throughout the star.

The two terms on the right hand side of (79) represent the contributions from the free-free transitions and the scattering.

At the center we thus have a contribution from the free-free transitions of $10\,^0/_0$. At $r/R = 0.73$ they contribute by $44\,^0/_0$.

The present model places a strict condition on the value of the opacity at the center of the star. In fact, for the terms of eq. (75) and (79) to be proportional at the center we must have

$$\varrho_c T_c^{-3.5} = \frac{0.3 \cdot 0.110}{0.954 \cdot 2.74 \cdot 10^{22}} = 10^{-23.90}. \tag{80}$$

Further, the constant of the opacity law becomes

$$\varkappa_0 = 0.3145\,(1 + X). \tag{81}$$

The invariant constants of the model now give

$$32 \log M = 16.666 + \log\,(1 + X)\,\varepsilon_0\,\mu^{-49}. \tag{82}$$

This gives a one-dimensional sequence of stars. It has been tabulated in Table 3.

TABLE 3.

Model 8. The sequence of stars in which the free-free transitions contribute $10\,^0/_0$ of the opacity at the center and $44\,^0/_0$ at $r/R = 0.73$.

X	$\log M$	$\log R$	$\log T_c$	$\log \varrho_c$	$\log L$	$^1/_4 \log LR^{-2}$
1.00	0.08	9.86	7.20	1.31	0.26	0.13
0.75	9.96	9.76	7.25	1.48	0.24	0.18
0.50	9.81	9.64	7.31	1.70	0.22	0.23
0.25	9.62	9.46	7.41	2.05	0.17	0.31
0.10	9.46	9.27	7.53	2.46	0.09	0.39

The convective core extends in this model to $r/R = 0.16$, includes $10\,^0/_0$ of the mass and $51\,^0/_0$ of the energy production.

MODEL 9.

Electron scattering predominates.

This model was computed on the assumption that $\varkappa$ is constant. The condition that it is useful in the present investigation is that the opacity due to the free-free transitions is small. We therefore

first consider the variation of the corresponding factor through the star:

V	$\log\ (\varrho/\varrho_c)\,(T/T_c)^{-3.5}$
0	0.00
5	0.65
10	1.03

It is apparent that the importance of the free-free transitions increases as one moves outwards in the star, like in model 1 and 8. This increase continues even to the surface. Here we have, in fact, $P \propto T^4$ so

$$\varrho\ T^{-3.5} \propto T^{-0.5} \qquad \text{(near the surface).} \quad (83)$$

Since there exists no region where this model is strictly applicable we must contend ourselves with some reasonable condition for the unimportance of the free-free transitions. As such we adopt that they must contribute by $10\,^0/_0$ or more only at points exterior to $V = 15$, $r/R = 0.74$. Table 4 has been calculated on this assumption.

TABLE 4.

Model 11. In the sequence of stars given the free-free transitions contribute by $10\,^0/_0$ to the opacity at $r/R = 0.74$.

X	$\log M$	$\log R$	$\log T_c$	$\log \varrho_c$	$\log L$	$^1/_4 \log LR^{-2}$	P_r/P_c
1.00	0.60	9.98	7.58	1.37	2.35	0.60	0.035
0.75	0.48	9.88	7.63	1.53	2.33	0.64	
0.50	0.33	9.76	7.69	1.76	2.31	0.70	
0.25	0.14	9.58	7.79	2.10	2.26	0.78	
0.10	9.98	9.39	7.91	2.51	2.18	0.85	0.056

The convective core in this model includes $24\,^0/_0$ of the radius, $22\,^0/_0$ of the mass, and $78\,^0/_0$ of the energy production. P_r is the radiation pressure at the center.

The Hertzsprung-Russell diagram.

The results quoted have been collected in the accompanying HR-diagram. Additional results, for stars with $\log M = 0.5$, have been plotted. With this value of the mass the radiation pressure

Hertzsprung-Russell diagram for the homogeneous stars, composed entirely of hydrogen and helium

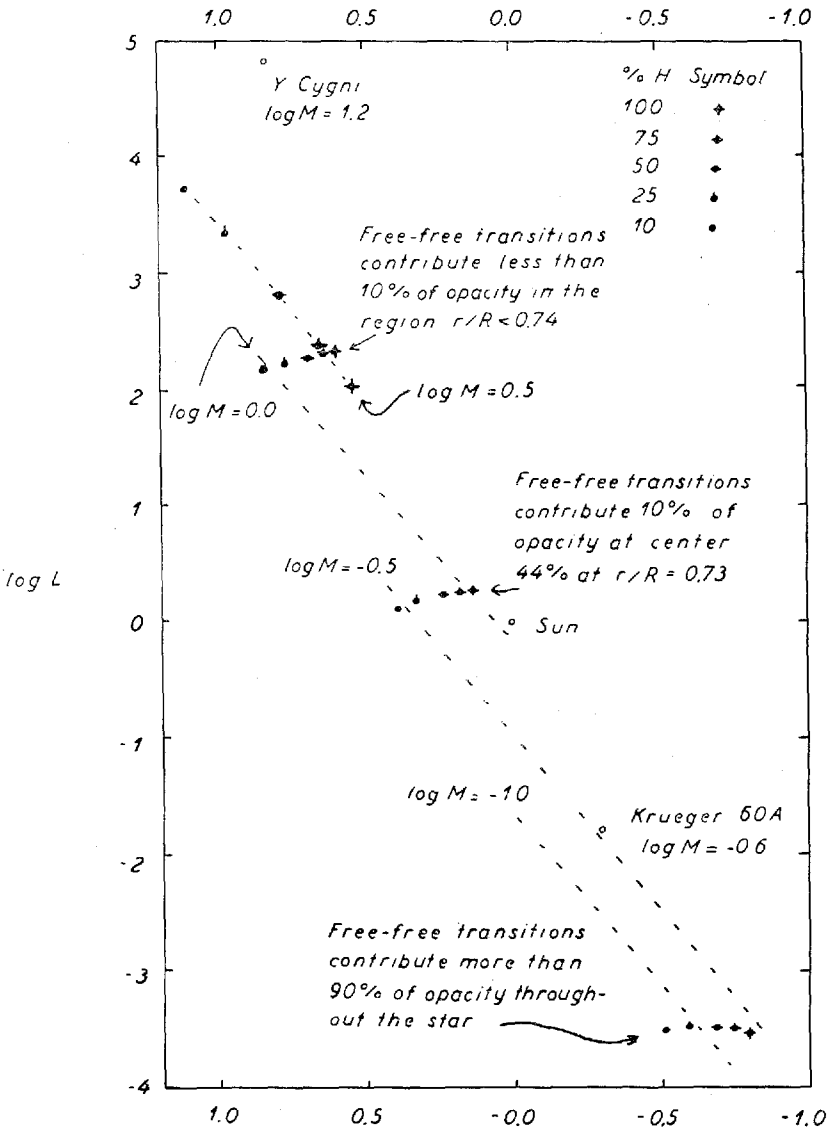

Fig. 2.

becomes appreciable, and the basis of the model breaks down. For comparison the points corresponding to the data for the Sun, Y Cygni and Krueger 60 A, have also been plotted (fig. 2).

The results of the present section may be compared with those obtained by A. REIZ[5]. REIZ has calculated a model of the type considered in the present paper, based on the opacity law

$$\varkappa = \varkappa_0 \varrho^{0.5} T^{-1.75}. \tag{84}$$

It is found that the present results agree well with those found by REIZ.

The writer is grateful to Dr. W. J. ECKERT for placing the facilities of the Watson Scientific Computing Laboratory at his disposal. He also wishes to thank the staff of this laboratory for their generous help during the work with the punched card machines.

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Appendix 1.

Expansions for the central region of the star.

In this appendix the series expansions valid at the center of the star will be derived. V will be taken as the independent variable throughout. Thus, dashes denote derivatives with respect to V, and subscript zero values of the functions at $V = 0$, *i.e.* at the center of the of the star. The formulae will be derived such as to be equally useful for a star in convective and radiative equilibrium at the center. Expressions whose validity is confined to one of these cases will be distinguished by *rad. eq.* or *conv. eq.* written in the bracket together with its number.

It will, in this section, be convenient to introduce symbols for some particular functions. Thus we define

$$T = H/V \tag{85}$$

$$g = 9 + s - \alpha \tag{86}$$

$$S = (g H - (2 - \alpha) V + W - 1)/(U + H - 1) \tag{87}$$

$$P = U/(U + H - 1) \tag{88}$$

$$Q = (3 - V + H - U)/V \tag{89}$$

$$A = W/(U + H - 1) \tag{90}$$

$$b = \nu - \delta - 1 \tag{91}$$

$$B = (3 - (1 + \delta)\, V - bH - W)/V \tag{92}$$

Then the differential equations (28) to (30) can be written

$$H' = ST \qquad \text{(93 rad. eq.)}$$

$$U' = PQ \tag{94}$$

$$W' = AB \tag{95}$$

In convective equilibrium eq. (93 rad. eq.) is replaced by

$$H_0'' = H_0''' = H_0^{\text{iv}} \text{ etc.} = 0 \qquad \text{(96 conv. eq.)}$$

On the basis of these expressions we want to find the first and higher derivatives of U and W, and the second and higher derivatives of H, at $V = 0$, subject to the conditions

$$U_0 = W_0 = 3 \tag{97}$$

$$H_0 = 0 \tag{98}$$

In all these derivatives H_0' will enter as a parameter (cfr. section 2.3).

First differentiation.

Using (98) we have the Taylor expansion

$$H = H_0' V + {}^1/_2\, H_0'' V^2 + \frac{1}{6} H_0''' V^3 \ldots \tag{99}$$

Then from (85)

$$T = H_0' + {}^1/_2\, H_0'' V + \frac{1}{6} H_0''' V^2 \ldots \tag{100}$$

$$T_0 = H_0'. \tag{101}$$

From (87) and (88) we get

$$S_0 = 1 \tag{102}$$

$$P_0 = 3/2. \tag{103}$$

At $V=0$, Q becomes indeterminate. We can find the limiting value by using the standard rule of differentiating the numerator and denominator seperately (this rule will be used frequently in what follows) and obtain

$$Q_0 = -1 + H_0' - U_0'. \tag{104}$$

Further, using (94), we get

$$U_0' = P_0 Q_0 = -\frac{3}{2}(1 - H_0' + U_0') \tag{105}$$

and solving for U_0'

$$U_0' = -\frac{3}{5}(1 - H_0'). \tag{106}$$

Then (104) becomes

$$Q_0 = -\frac{2}{5}(1 - H_0'). \tag{107}$$

The following quantity will also be useful

$$U_0' + H_0' = (8\,H_0' - 3)/5. \tag{108}$$

From (90) and (92) we now get

$$A_0 = 3/2 \tag{109}$$

$$B_0 = -(1+\delta) - bH_0' - W_0' \tag{110}$$

and from (95)

$$W_0' = A_0\,B_0 = -3\,(1+\delta)/2 - 3\,bH_0'/2 - 3\,W_0'/2. \tag{111}$$

Solving for W_0':

$$W_0' = -3\,((1+\delta) + bH_0')/5 \tag{112}$$

and inserting in (110)

$$B_0 = 2\,W_0'/3 = -2\,((1+\delta) + bH_0')/5. \tag{113}$$

Second differentiation.

Differentiating (100) we get

$$T' = {}^1/_2\, H_0'' + H_0''' \, V/3 \tag{114}$$

and

$$T_0' = {}^1/_2\, H_0''. \tag{115}$$

Further, from (87),

$$\left.\begin{aligned} S' &= (gH' - (2-\alpha) + W')/(U + H - 1) \\ &\quad -(gH-(2-\alpha)\, V + W - 1)(U' + H')(U + H - 1)^{-2}. \end{aligned}\right\} \tag{116}$$

Using (97), (108), and (112), we obtain

$$S_0' = ((5\,g - 3\,b - 8)\, H_0' - 10 + 5\,\alpha - 3\,\delta)/10. \tag{117}$$

Now, from (93), (115), and (101)

$$H_0'' = S_0\, T_0' + S_0'\, T_0 = {}^1/_2\, H_0'' + H_0' S_0' \qquad \text{(118 rad. eq.)}$$

and solving for H_0'', using (117),

$$H_0'' = H_0'\,((5\,g - 3\,b - 8)\, H_0' - 10 + 5\,\alpha - 3\,\delta)/5. \qquad \text{(119 rad. eq.)}$$

Differentiation of (88) yields

$$P' = U'/(U + H - 1) - U\,(U' + H')\,(U + H - 1)^{-2} \tag{120}$$

and inserting (106)

$$P_0' = 3\,(1 - 6\, H_0')/20. \tag{121}$$

Differentiation of (89) gives

$$\left.\begin{aligned} Q' &= (-1 + H' - U')/V - (3 - V + H - U)/V^2 = \\ &\quad (-1 + H' - U' - Q)/V. \end{aligned}\right\} \tag{122}$$

By the standard limiting rule we get

$$Q_0' = H_0'' - U_0'' - Q_0' \tag{123}$$

or

$$Q_0' = {}^1/_2\, H_0'' - {}^1/_2\, U_0''. \tag{124}$$

We can now calculate

$$\left.\begin{aligned} U_0'' &= P_0 Q_0' + P_0' Q_0 \\ &= 3\,H_0''/4 - 3\,U_0''/4 - 3\,(6\,H_0'^2 - 7\,H_0' + 1)/\,50 \end{aligned}\right\} \quad (125)$$

and, solving for U_0'', obtain the results

$$U_0'' = 3\,H_0''/7 - 6\,(6\,H_0'^2 - 7\,H_0' + 1)/175 \quad (126)$$

$$Q_0' = 2\,H_0''/7 + 3\,(6\,H_0'^2 - 7\,H_0' + 1)/175. \quad (127)$$

Inserting (119) in (126) we get

$$\left.\begin{aligned} U_0'' &= 3\,((25\,g - 15\,b - 52)\,H_0'^2 \\ &\quad + (25\,\alpha - 15\,\delta - 36)\,H_0' - 2)/175. \end{aligned}\right\} \quad (128 \text{ rad. eq.})$$

By differentiation of (90) we get

$$\left.\begin{aligned} A' &= W'\,(U + H - 1)^{-1} - W\,(U' + H')\,(H + U - 1)^{-2} \\ &= W'\,(U + H - 1)^{-1} - A\,(U' + H')\,(U + H - 1)^{-1} \end{aligned}\right\} \quad (129)$$

and, using (108) and (112), the limit

$$A_0' = -3\,((8 + 2\,b)\,H_0' + 2\,\delta - 1)/20. \quad (130)$$

For B' we get, from (92),

$$\left.\begin{aligned} B' &= (-(1+\delta) - bH' - W')/V - (3 - (1+\delta)\,V \\ &\quad - bH - W)/V^2 = (-(1+\delta) - bH' - W' - B)/V \end{aligned}\right\} \quad (131)$$

which gives the limit

$$B_0' = -\tfrac{1}{2}\,bH_0'' - \tfrac{1}{2}\,W_0''. \quad (132)$$

Now the equation for W'' can be derived from (95), (130), and (132). Solving this equation we get

$$\left.\begin{aligned} W_0'' = 6\,((8\,b + 2\,b^2)\,H_0'^2 + (8 + b + 8\,\delta + 4\,\delta\,b)\,H_0' \\ - 1 + \delta + 2\,\delta^2)/175 - 3\,bH_0''/7. \end{aligned}\right\} \quad (133)$$

Then B_0' can be found:

$$\left.\begin{aligned} B_0' = -3\,((8\,b + 2\,b^2)\,H_0'^2 + (8 + b + 8\,\delta + 4\,\delta\,b)\,H_0' \\ - 1 + \delta + 2\,\delta^2)/175 - 2\,bH_0''/7. \end{aligned}\right\} \quad (134)$$

Inserting (119) into (133) we get

$$\left.\begin{aligned} W_0'' = 3\,((56\,b + 19\,b^3 - 25\,bg)\,H_0'^2 \\ + (16 + 52\,b + 16\,\delta - 25\,\alpha\,b + \\ 23\,\delta\,b)\,H_0' - 2 + 2\,\delta + 4\,\delta^2)/175. \end{aligned}\right\} \quad (135 \text{ rad. eq.})$$

Third differentiation.

The general procedure for deriving the derivatives having been made clear during the two first differentiations, we need only give the principal results of the third differentiation.

$$\left.\begin{aligned} S_0'' = H_0''(-10 + 7\,g - 3\,b)/14 + [H_0'^2(484 + 216\,b - 280\,g + \\ 12\,b^2) + H_0'(398 - 57\,b + 216\,\delta + 105\,g - 280\,\alpha + 24\,\delta\,b) \\ - 210 - 57\,\delta + 105\,\alpha + 12\,\delta^2]/350. \end{aligned}\right\} \quad (136)$$

$$\left.\begin{aligned} H_0''' = 3\,[H_0'^3(350\,g^2 + 120\,b^2 - 390\,g\,b - 1370\,g + 822\,b + 1332) \\ + H_0^2(525\,\alpha\,g - 285\,b\,\alpha - 315\,g\,\delta + 195\,\delta\,b - 1090\,\alpha - 945\,g + \\ 513\,b + 702\,\delta + 2018) \\ + H_0'(175\,\alpha^2 + 75\,\delta^2 - 210\,\alpha\,\delta - 595\,\alpha + 363\,\delta + 490)]/700. \end{aligned}\right\} \quad (137)$$

$$P_0'' = -6\,H_0''/7 + 3\,(348\,H_0'^2 - 196\,H_0' + 23)/700. \quad (138)$$

$$\left.\begin{aligned} U_0''' = H_0'''/3 + 2\,(1 - 2\,H_0')\,H_0''/7 \\ + 4\,(12\,H_0'^3 - 20\,H_0'^2 + 9\,H_0' - 1)/175. \end{aligned}\right\} \quad (139)$$

$$\left.\begin{aligned} U_0''' = [H_0'^3(350\,g^2 + 120\,b^2 - 390\,g\,b - 1770\,g + 1062\,b + \\ 2164) + H_0'^2(525\,\alpha\,g - 285\,b\,\alpha - 315\,g\,\delta + 195\,\delta\,b - 745\,g - \\ 1490\,\alpha + 393\,b + 942\,\delta + 2178) + H_0'(175\,\alpha^2 + 75\,\delta^2 - \\ 210\,\alpha\,\delta - 395\,\alpha + 243\,\delta + 234) - 16]/700. \end{aligned}\right\} \quad (140 \text{ rad. eq.})$$

$$\left.\begin{aligned} A_0'' = 3\,H_0''(-5 - b)/14 + 3\,[H_0'^2(8\,b^2 + 144\,b + 484) + \\ H_0'(16\,\delta\,b + 144\,\delta - 38\,b - 234) + 8\,\delta^2 - 38\,\delta + 23]/700. \end{aligned}\right\} \quad (141)$$

$$\left.\begin{aligned} W_0''' = &-bH_0'''/3 + 2H_0''[H_0'(13b + 3b^2) + 3\delta b + 5\delta + 5]/35 \\ &+ 2[H_0'^3(2b^3 - 24b^2 - 146b) + H_0'^2(6\delta b^2 + 15b^2 - 48\delta b \\ &+ 69b - 146\delta - 146) + H_0'(6\delta^2 b + 30\delta b + 3b - 24\delta^2 \\ &+ 69\delta + 93) + 2\delta^3 + 15\delta^2 + 3\delta - 10]/875. \end{aligned}\right\} \quad (142)$$

$$\left.\begin{aligned} W_0''' = &[H_0'^3(-1750bg^2 - 944b^3 + 2550b^2 g + 9450bg \\ &- 6822b^2 - 11988b) + H_0'^2(-2625b\alpha g + 2175bg\delta \\ &+ 2025b^2\alpha - 1647\delta b^2 + 8050\alpha b + 4725bg - 3645b^2 - \\ &7014b\delta + 1000g + 1000g\delta - 15338b - 2768\delta - 2768) + \\ &H_0'(-875b\alpha^2 - 687b\delta^2 + 1650\alpha\delta b + 2975b\alpha - 2775\delta b \\ &+ 1000\alpha\delta - 792\delta^2 - 2426b - 2048\delta + 1000\alpha - 1256) \\ &+ 16\delta^3 + 120\delta^2 + 24\delta - 80]/3500. \end{aligned}\right\} \quad (143 \text{ rad. eq.})$$

Summary of the equations.

The definitions which will be needed when using the equations of the present appendix are given in eq. (2), (3), (86), and (91). The general form of the results is described in section 2.3.

The developments are arranged so as to be equally useful for convective and radiative equilibrium. The particular formulae to be used in the two cases are the following:

Function	Convective equilibrium	Radiative equilibrium
H_0'	$H_0' = 0.4$	H_0' is the fundamental parameter
U_0'	eq. (106)	eq. (106)
W_0'	(112)	(112)
H_0''	$H_0'' = 0$	(119)
U_0''	(126)	(126) or (128)
W_0''	(133)	(133) or (135)
H_0'''	$H_0''' = 0$	(137)
U_0'''	(139)	(139) or (140)
W_0'''	(142)	(142) or (143)

Appendix 2.

The run of the physical variables through 11 stellar models.

MODEL 1.

$$\varkappa = \varkappa_0 \varrho\, T^{-3.5}.$$

V	U	W	H	$\log r/R$	$\log P/P_c$	$\log T/T_c$	$\log M_r/M$	$\log L_r/L$
0.0	3.000	3.000	0.000	$-\infty$	0.000	0.000	$-\infty$	$-\infty$
0.5	2.808	2.253	0.172	9.007	9.891	9.962	8.621	9.431
1.0	2.616	1.640	0.330	9.160	9.780	9.924	9.037	9.733
1.5	2.424	1.155	0.474	9.251	9.668	9.888	9.268	9.861
2.0	2.234	0.784	0.607	9.318	9.552	9.852	9.424	9.926
2.5	2.045	0.512	0.731	9.372	9.432	9.816	9.539	9.960
3.0	1.860	0.320	0.848	9.407	9.307	9.780	9.628	9.979
3.5	1.678	0.193	0.960	9.457	9.177	9.744	9.698	9.989
4.0	1.502	0.111	1.068	9.493	9.042	9.707	9.756	9.995
4.5	1.334	0.061	1.175	9.526	8.901	9.670	9.803	9.998
5.0	1.175	0.031	1.281	9.557	8.754	9.632	9.842	9.999
5.5	1.027	0.015	1.386	9.586	8.603	9.594	9.874	0.000
6.0	0.891	0.007	1.492	9.613	8.447	9.554	9.900	0.000
6.5	0.767	0.003	1.599	9.638	8.289	9.515	9.920	0.000
7.0	0.658	0.001	1.706	9.662	8.130	9.476	9.937	0.000
7.5	0.562	0.001	1.815	9.684	7.972	9.438	9.951	0.000
8.0	0.478	0.000	1.925	9.704	7.815	9.400	9.961	0.000
8.5	0.408	0.000	2.037	9.722	7.663	9.364	9.969	0.000
9.0	0.347	0.000	2.149	9.739	7.515	9.328	9.976	0.000
9.5	0.297	0.000	2.262	9.755	7.373	9.294	9.981	0.000
10.0	0.254	0.000	2.376	9.769	7.237	9.262	9.984	0.000
10.5	0.219	0.000	2.490	9.781	7.107	9.231	9.987	0.000
11.0	0.189	0.000	2.604	9.793	6.983	9.201	9.990	0.000
11.5	0.164	0.000	2.719	9.804	6.864	9.173	9.992	0.000
12.0	0.143	0.000	2.834	9.813	6.752	9.147	9.993	0.000
12.5	0.125	0.000	2.949	9.822	6.645	9.122	9.994	0.000
13.0	0.110	0.000	3.063	9.830	6.543	9.097	9.995	0.000
13.5	0.098	0.000	3.176	9.837	6.445	9.074	9.996	0.000
14.0	0.086	0.000	3.288	9.844	6.352	9.052	9.997	0.000
14.5	0.077	0.000	3.398	9.850	6.262	9.032	9.997	0.000
15.0	0.068	0.000	3.504	9.856	6.176	9.011	9.998	0.000

Variations

V	ΔU	ΔW	ΔH
3	+111	+13	+342
6	39	1	229
10	27	0	519

Model 2.

$$\varkappa = \varkappa_0 \varrho^{0.75} T^{-3.5}.$$

V	U	W	H	log r/R	log P/P_c	log T/T_c	log M_r/M	log L_r/L
0.0	3.000	3.000	0.000	$-\infty$	0.000	0.000	$-\infty$	$-\infty$
0.5	2.800	2.263	0.160	8.876	9.891	9.964	8.591	9.414
1.0	2.600	1.652	0.306	9.032	9.778	9.929	9.012	9.722
1.5	2.399	1.164	0.438	9.126	9.662	9.894	9.247	9.855
2.0	2.199	0.787	0.559	9.195	9.542	9.860	9.407	9.922
2.5	2.000	0.509	0.671	9.251	9.415	9.825	9.526	9.959
3.0	1.803	0.313	0.776	9.300	9.282	9.790	9.619	9.979
3.5	1.609	0.183	0.875	9.344	9.140	9.754	9.693	9.989
4.0	1.420	0.101	0.971	9.384	8.989	9.716	9.754	9.995
4.5	1.238	0.052	1.064	9.422	8.827	9.678	9.805	9.998
5.0	1.066	0.024	1.157	9.458	8.655	9.637	9.847	9.999
5.5	0.904	0.010	1.249	9.493	8.472	9.595	9.881	0.000
6.0	0.757	0.004	1.342	9.527	8.279	9.552	9.909	0.000
6.5	0.625	0.002	1.436	9.559	8.078	9.507	9.931	0.000
7.0	0.510	0.000	1.532	9.590	7.871	9.462	9.948	0.000
7.5	0.413	0.000	1.629	9.618	7.662	9.416	9.962	0.000
8.0	0.333	0.000	1.727	9.645	7.456	9.371	9.972	0.000
8.5	0.268	0.000	1.827	9.670	7.254	9.328	9.979	0.000
9.0	0.216	0.000	1.929	9.692	7.060	9.286	9.984	0.000
9.5	0.175	0.000	2.031	9.712	6.875	9.246	9.988	0.000
10.0	0.143	0.000	2.134	9.730	6.700	9.209	9.991	0.000
10.5	0.117	0.000	2.238	9.746	6.535	9.174	9.993	0.000
11.0	0.097	0.000	2.342	9.760	6.380	9.141	9.995	0.000
11.5	0.081	0.000	2.446	9.773	6.233	9.110	9.996	0.000
12.0	0.068	0.000	2.551	9.785	6.095	9.080	9.997	0.000
12.5	0.057	0.000	2.655	9.796	5.965	9.052	9.997	0.000
13.0	0.049	0.000	2.760	9.805	5.841	9.026	9.998	0.000
13.5	0.042	0.000	2.864	9.814	5.724	9.001	9.998	0.000
14.0	0.036	0.000	2.968	9.822	5.614	8.978	9.999	0.000
14.5	0.031	0.000	3.070	9.830	5.508	8.955	9.999	0.000
15.0	0.027	0.000	3.172	9.836	5.407	8.934	9.999	0.000

Variations

V	ΔU	ΔW	ΔH
3	+32	+4	+93
6	61	1	298
10	32	0	737

Model 3.

$$\varkappa = \varkappa_0 \varrho^{0.5} T^{-3.5}.$$

V	U	W	H	$\log r/R$	$\log P/P_c$	$\log T/T_c$	$\log M_r/M$	$\log L_r/L$
0.0	3.000	3.000	0.000	$-\infty$	0.000	0.000	$-\infty$	$-\infty$
0.5	2.792	2.275	0.146	8.647	9.890	9.967	8.553	9.394
1.0	2.582	1.668	0.279	8.805	9.776	9.934	8.979	9.709
1.5	2.372	1.176	0.399	8.901	9.656	9.902	9.219	9.847
2.0	2.161	0.792	0.508	8.974	9.530	9.869	9.384	9.918
2.5	1.950	0.506	0.607	9.034	9.396	9.835	9.508	9.957
3.0	1.739	0.304	0.698	9.087	9.250	9.801	9.605	9.978
3.5	1.530	0.170	0.784	9.135	9.092	9.765	9.685	9.990
4.0	1.325	0.087	0.866	9.182	8.919	9.726	9.751	9.995
4.5	1.124	0.040	0.945	9.227	8.727	9.686	9.806	9.998
5.0	0.932	0.016	1.023	9.272	8.512	9.641	9.853	9.999
5.5	0.751	0.005	1.101	9.318	8.271	9.592	9.891	0.000
6.0	0.586	0.001	1.180	9.365	8.002	9.539	9.922	0.000
6.5	0.442	0.000	1.260	9.412	7.705	9.481	9.947	0.000
7.0	0.322	0.000	1.342	9.460	7.386	9.419	9.965	0.000
7.5	0.230	0.000	1.427	9.505	7.056	9.356	9.977	0.000
8.0	0.162	0.000	1.515	9.547	6.730	9.294	9.985	0.000
8.5	0.115	0.000	1.604	9.585	6.418	9.235	9.990	0.000
9.0	0.082	0.000	1.695	9.618	6.128	9.180	9.994	0.000
9.5	0.060	0.000	1.787	9.647	5.860	9.130	9.996	0.000
10.0	0.044	0.000	1.879	9.672	5.614	9.084	9.997	0.000
10.5	0.034	0.000	1.972	9.694	5.389	9.041	9.998	0.000
11.0	0.026	0.000	2.065	9.714	5.182	9.002	9.998	0.000
11.5	0.020	0.000	2.159	9.731	4.990	8.966	9.999	0.000
12.0	0.016	0.000	2.253	9.746	4.813	8.933	9.999	0.000
12.5	0.013	0.000	2.348	9.759	4.648	8.902	9.999	0.000
13.0	0.010	0.000	2.444	9.771	4.493	8.873	0.000	0.000
13.5	0.008	0.000	2.542	9.782	4.349	8.846	0.000	0.000
14.0	0.007	0.000	2.642	9.792	4.213	8.820	0.000	0.000
14.5	0.006	0.000	2.745	9.801	4.085	8.796	0.000	0.000
15.0	0.005	0.000	2.853	9.809	3.965	8.773	0.000	0.000

Variations

V	ΔU	ΔW	ΔH
3	+132	+19	+340
6	37	0	150
10	1	0	55

Model 4.

$$\varkappa = \varkappa_0 \varrho^{0.75} T^{-2.5}.$$

V	U	W	H	$\log r/R$	$\log P/P_c$	$\log T/T_c$	$\log M_r/M$	$\log L_r/L$
0.0	3.000	3.000	0.000	$-\infty$	0.000	0.000	$-\infty$	$-\infty$
0.5	2.817	2.242	0.185	9.049	9.892	9.959	8.638	9.446
1.0	2.632	1.628	0.352	9.200	9.783	9.919	9.051	9.741
1.5	2.447	1.146	0.503	9.290	9.673	9.881	9.278	9.866
2.0	2.262	0.780	0.641	9.354	9.560	9.844	9.431	9.928
2.5	2.078	0.513	0.769	9.406	9.445	9.808	9.543	9.961
3.0	1.898	0.325	0.888	9.450	9.325	9.772	9.630	9.979
3.5	1.721	0.198	1.003	9.488	9.202	9.736	9.699	9.989
4.0	1.550	0.116	1.114	9.522	9.073	9.700	9.755	9.994
4.5	1.386	0.066	1.223	9.553	8.941	9.663	9.801	9.997
5.0	1.231	0.036	1.330	9.582	8.804	9.626	9.839	9.999
5.5	1.085	0.019	1.437	9.609	8.663	9.589	9.870	9.999
6.0	0.951	0.010	1.545	9.634	8.519	9.552	9.895	0.000
6.5	0.828	0.005	1.654	9.657	8.373	9.514	9.916	0.000
7.0	0.718	0.002	1.763	9.679	8.226	9.477	9.933	0.000
7.5	0.621	0.001	1.874	9.699	8.080	9.441	9.946	0.000
8.0	0.536	0.001	1.986	9.718	7.936	9.405	9.957	0.000
8.5	0.462	0.000	2.099	9.735	7.795	9.370	9.966	0.000
9.0	0.399	0.000	2.213	9.750	7.658	9.336	9.972	0.000
9.5	0.345	0.000	2.328	9.765	7.526	9.304	9.978	0.000
10.0	0.299	0.000	2.444	9.778	7.399	9.272	9.982	0.000
10.5	0.260	0.000	2.560	9.790	7.277	9.243	9.985	0.000
11.0	0.227	0.000	2.677	9.800	7.160	9.214	9.988	0.000
11.5	0.199	0.000	2.795	9.810	7.049	9.187	9.990	0.000
12.0	0.175	0.000	2.912	9.819	6.942	9.161	9.992	0.000
12.5	0.154	0.000	3.030	9.828	6.841	9.136	9.993	0.000
13.0	0.137	0.000	3.148	9.835	6.744	9.113	9.994	0.000
13.5	0.122	0.000	3.265	9.842	6.651	9.090	9.995	0.000
14.0	0.108	0.000	3.381	9.849	6.562	9.069	9.996	0.000
14.5	0.097	0.000	3.497	9.855	6.476	9.048	9.996	0.000
15.0	0.087	0.000	3.610	9.860	6.394	9.019	9.997	0.000

Variations

V	ΔU	ΔW	ΔH
3	+91	+9	+260
6	18	0	83
10	23	0	301

Model 5.

$$\varkappa = \varkappa_0 \varrho^{0.5} T^{-2.5}.$$

V	U	W	H	$\log r/R$	$\log P/P_c$	$\log T/T_c$	$\log M_r/M$	$\log L_r/L$
0.0	3.000	3.000	0.000	$-\infty$	0.000	0.000	$-\infty$	$-\infty$
0.5	2.808	2.254	0.171	8.911	9.891	9.962	8.605	9.426
1.0	2.613	1.643	0.323	9.065	9.780	9.925	9.023	9.729
1.5	2.418	1.158	0.460	9.157	9.667	9.889	9.255	9.858
2.0	2.222	0.785	0.584	9.225	9.549	9.854	9.413	9.924
2.5	2.026	0.511	0.698	9.280	9.426	9.818	9.530	9.960
3.0	1.833	0.317	0.803	9.327	9.297	9.783	9.620	9.979
3.5	1.642	0.188	0.903	9.369	9.160	9.747	9.694	9.989
4.0	1.456	0.105	0.999	9.407	9.016	9.710	9.753	9.995
4.5	1.277	0.055	1.092	9.444	8.862	9.672	9.803	9.998
5.0	1.106	0.026	1.184	9.478	8.698	9.633	9.844	9.999
5.5	0.946	0.012	1.276	9.511	8.525	9.593	9.878	0.000
6.0	0.799	0.005	1.368	9.542	8.344	9.551	9.905	0.000
6.5	0.667	0.002	1.462	9.573	8.154	9.508	9.928	0.000
7.0	0.550	0.001	1.557	9.602	7.960	9.464	9.945	0.000
7.5	0.451	0.000	1.654	9.629	7.763	9.421	9.959	0.000
8.0	0.368	0.000	1.752	9.654	7.568	9.378	9.969	0.000
8.5	0.299	0.000	1.852	9.677	7.376	9.336	9.977	0.000
9.0	0.244	0.000	1.953	9.698	7.191	9.296	9.982	0.000
9.5	0.199	0.000	2.056	9.717	7.014	9.257	9.987	0.000
10.0	0.164	0.000	2.160	9.735	6.845	9.221	9.990	0.000
10.5	0.136	0.000	2.264	9.750	6.686	9.186	9.992	0.000
11.0	0.113	0.000	2.369	9.764	6.535	9.154	9.994	0.000
11.5	0.095	0.000	2.474	9.777	6.393	9.123	9.995	0.000
12.0	0.080	0.000	2.580	9.788	6.258	9.094	9.996	0.000
12.5	0.068	0.000	2.687	9.799	6.131	9.067	9.997	0.000
13.0	0.058	0.000	2.793	9.808	6.010	9.041	9.997	0.000
13.5	0.050	0.000	2.900	9.817	5.896	9.016	9.998	0.000
14.0	0.043	0.000	3.008	9.825	5.788	8.993	9.998	0.000
14.5	0.038	0.000	3.116	9.832	5.684	8.971	9.999	0.000
15.0	0.033	0.000	3.225	9.839	5.586	8.950	9.999	0.000

Variations

V	ΔU	ΔW	ΔH
3	+50	+6	+126
6	30	0	115
10	22	0	383

Model 6.

$$\varkappa = \varkappa_0 \varrho^{0.25} T^{-2.5}.$$

V	U	W	H	$\log r/R$	$\log P/P_c$	$\log T/T_c$	$\log M_r/M$	$\log L_r/L$
0.0	3.000	3.000	0.000	$-\infty$	0.000	0.000	$-\infty$	$-\infty$
0.5	2.797	2.268	0.154	8.647	9.891	9.965	8.562	9.403
1.0	2.592	1.660	0.292	8.804	9.778	9.931	8.986	9.714
1.5	2.385	1.170	0.414	8.899	9.660	9.898	9.224	9.850
2.0	2.176	0.790	0.523	8.970	9.536	9.864	9.387	9.920
2.5	1.967	0.507	0.621	9.029	9.404	9.830	9.509	9.958
3.0	1.758	0.307	0.711	9.081	9.262	9.796	9.605	9.978
3.5	1.550	0.173	0.795	9.129	9.107	9.760	9.684	9.990
4.0	1.345	0.090	0.874	9.174	8.938	9.722	9.749	9.995
4.5	1.144	0.042	0.950	9.218	8.750	9.682	9.804	9.998
5.0	0.951	0.017	1.025	9.262	8.539	9.638	9.851	9.999
5.5	0.768	0.006	1.099	9.307	8.303	9.590	9.889	0.000
6.0	0.600	0.002	1.174	9.354	8.038	9.538	9.921	0.000
6.5	0.452	0.000	1.251	9.401	7.743	9.481	9.946	0.000
7.0	0.329	0.000	1.331	9.448	7.423	9.419	9.964	0.000
7.5	0.233	0.000	1.413	9.494	7.090	9.356	9.977	0.000
8.0	0.163	0.000	1.498	9.537	6.757	9.294	9.985	0.000
8.5	0.114	0.000	1.586	9.576	6.437	9.234	9.991	0.000
9.0	0.081	0.000	1.675	9.610	6.138	9.178	9.994	0.000
9.5	0.058	0.000	1.765	9.640	5.863	9.127	9.996	0.000
10.0	0.043	0.000	1.856	9.666	5.610	9.080	9.997	0.000
10.5	0.032	0.000	1.948	9.688	5.379	9.037	9.998	0.000
11.0	0.024	0.000	2.040	9.708	5.166	8.998	9.999	0.000
11.5	0.019	0.000	2.132	9.726	4.970	8.961	9.999	0.000
12.0	0.015	0.000	2.224	9.741	4.788	8.927	9.999	0.000
12.5	0.012	0.000	2.316	9.755	4.619	8.896	0.000	0.000
13.0	0.010	0.000	2.409	9.767	4.461	8.866	0.000	0.000
13.5	0.008	0.000	2.502	9.778	4.312	8.839	0.000	0.000
14.0	0.006	0.000	2.594	9.789	4.173	8.813	0.000	0.000
14.5	0.005	0.000	2.687	9.798	4.041	8.789	0.000	0.000
15.0	0.004	0.000	2.780	9.806	3.916	8.766	0.000	0.000

Variations

V	ΔU	ΔW	ΔH
3	+78	+12	+178
6	19	0	66
10	11	0	282

Model 7.

$$\varkappa = \varkappa_0 \varrho^{0.5} T^{-0.9}.$$

V	U	W	H	$\log r/R$	$\log P/P_c$	$\log T/T_c$	$\log M_r/M$	$\log L_r/L$
0.0	3.000	3.000	0.000	$-\infty$	0.000	0.000	$-\infty$	$-\infty$
0.5	2.824	2.231	0.200	9.191	9.892	9.957	8.686	9.475
1.0	2.655	1.608	0.400	9.338	9.786	9.914	9.093	9.762
1.5	2.494	1.126	0.600	9.423	9.681	9.872	9.312	9.879
2.0	2.338	0.769	0.769	9.483	9.578	9.832	9.455	9.935
2.5	2.180	0.514	0.918	9.529	9.475	9.793	9.560	9.964
3.0	2.022	0.336	1.057	9.567	9.371	9.755	9.639	9.980
3.5	1.867	0.215	1.190	9.599	9.266	9.719	9.702	9.989
4.0	1.717	0.135	1.318	9.627	9.160	9.684	9.753	9.994
4.5	1.573	0.083	1.444	9.652	9.052	9.649	9.795	9.997
5.0	1.435	0.050	1.569	9.675	8.944	9.614	9.829	9.998
5.5	1.306	0.030	1.693	9.696	8.836	9.580	9.857	9.999
6.0	1.185	0.018	1.818	9.715	8.727	9.547	9.881	0.000
6.5	1.073	0.010	1.943	9.732	8.619	9.515	9.900	0.000
7.0	0.970	0.006	2.068	9.748	8.512	9.483	9.916	0.000
7.5	0.876	0.004	2.195	9.763	8.406	9.452	9.930	0.000
8.0	0.790	0.002	2.323	9.776	8.303	9.422	9.941	0.000
8.5	0.714	0.001	2.452	9.788	8.201	9.392	9.950	0.000
9.0	0.645	0.001	2.582	9.800	8.102	9.364	9.958	0.000
9.5	0.583	0.000	2.712	9.810	8.006	9.336	9.964	0.000
10.0	0.528	0.000	2.843	9.820	7.913	9.310	9.970	0.000
10.5	0.479	0.000	2.975	9.828	7.823	9.284	9.974	0.000
11.0	0.435	0.000	3.107	9.836	7.736	9.260	9.978	0.000
11.5	0.396	0.000	3.239	9.844	7.652	9.236	9.981	0.000
12.0	0.361	0.000	3.372	9.851	7.572	9.213	9.983	0.000
12.5	0.330	0.000	3.504	9.857	7.494	9.191	9.986	0.000
13.0	0.302	0.000	3.636	9.863	7.418	9.170	9.988	0.000
13.5	0.277	0.000	3.767	9.868	7.346	9.150	9.989	0.000
14.0	0.255	0.000	3.897	9.873	7.276	9.130	9.990	0.000
14.5	0.234	0.000	4.026	9.878	7.208	9.112	9.992	0.000
15.0	0.216	0.000	4.153	9.883	7.142	9.093	9.993	0.000

Variations

V	ΔU	ΔW	ΔH
6	+18	0	+108

Model 8.

$$\varkappa = \varkappa_0 \varrho^{0.25} T^{-0.9}.$$

V	U	W	H	$\log r/R$	$\log P/P_c$	$\log T/T_c$	$\log M_r/M$	$\log L_r/L$
0.0	3.000	3.000	0.000	$-\infty$	0.000	0.000	$-\infty$	$-\infty$
0.5	2.824	2.231	0.200	9.089	9.892	9.957	8.661	9.467
1.0	2.655	1.609	0.394	9.237	9.786	9.914	9.068	9.754
1.5	2.483	1.131	0.554	9.323	9.679	9.874	9.289	9.872
2.0	2.308	0.774	0.697	9.385	9.572	9.835	9.438	9.931
2.5	2.132	0.514	0.827	9.434	9.463	9.798	9.546	9.962
3.0	1.958	0.331	0.947	9.475	9.350	9.762	9.630	9.980
3.5	1.787	0.206	1.060	9.511	9.234	9.726	9.697	9.989
4.0	1.620	0.124	1.168	9.542	9.115	9.690	9.751	9.994
4.5	1.459	0.073	1.275	9.572	8.992	9.655	9.796	9.997
5.0	1.305	0.041	1.380	9.598	8.865	9.619	9.833	9.998
5.5	1.161	0.023	1.484	9.623	8.734	9.584	9.864	9.999
6.0	1.027	0.012	1.589	9.646	8.601	9.548	9.889	0.000
6.5	0.903	0.006	1.695	9.668	8.466	9.512	9.910	0.000
7.0	0.791	0.003	1.802	9.688	8.330	9.477	9.927	0.000
7.5	0.691	0.002	1.910	9.707	8.194	9.442	9.941	0.000
8.0	0.602	0.001	2.019	9.724	8.059	9.408	9.952	0.000
8.5	0.525	0.000	2.131	9.740	7.926	9.374	9.961	0.000
9.0	0.457	0.000	2.243	9.755	7.797	9.342	9.968	0.000
9.5	0.399	0.000	2.357	9.769	7.671	9.311	9.974	0.000
10.0	0.348	0.000	2.471	9.781	7.550	9.281	9.979	0.000
10.5	0.305	0.000	2.587	9.793	7.433	9.252	9.982	0.000
11.0	0.268	0.000	2.703	9.803	7.320	9.224	9.986	0.000
11.5	0.236	0.000	2.820	9.813	7.212	9.198	9.988	0.000
12.0	0.209	0.000	2.936	9.822	7.109	9.172	9.990	0.000
12.5	0.185	0.000	3.054	9.830	7.010	9.148	9.992	0.000
13.0	0.165	0.000	3.171	9.837	6.915	9.125	9.993	0.000
13.5	0.147	0.000	3.288	9.844	6.824	9.103	9.994	0.000
14.0	0.132	0.000	3.405	9.850	6.737	9.081	9.995	0.000
14.5	0.118	0.000	3.522	9.856	6.653	9.061	9.996	0.000
15.0	0.107	0.000	3.638	9.862	6.572	9.041	9.996	0.000

Variations

V	ΔU	ΔW	ΔH
5	+25	+2	+81
10	7	0	106

Model 9.

$$\varkappa = \varkappa_0 T^{-0.9}.$$

V	U	W	H	$\log r/R$	$\log P/P_c$	$\log T/T_c$	$\log M_r/M$	$\log L_r/L$
0.0	3.000	3.000	0.000	$-\infty$	0.000	0.000	$-\infty$	$-\infty$
0.5	2.820	2.237	0.189	8.892	9.892	9.957	8.618	9.444
1.0	2.635	1.625	0.351	9.042	9.784	9.918	9.030	9.738
1.5	2.445	1.146	0.491	9.132	9.673	9.880	9.258	9.863
2.0	2.254	0.780	0.614	9.197	9.559	9.844	9.412	9.926
2.5	2.060	0.512	0.723	9.250	9.440	9.809	9.527	9.960
3.0	1.867	0.321	0.822	9.296	9.315	9.773	9.616	9.979
3.5	1.676	0.192	0.913	9.336	9.183	9.738	9.689	9.989
4.0	1.488	0.109	1.000	9.374	9.042	9.702	9.748	9.994
4.5	1.305	0.058	1.084	9.410	8.890	9.665	9.798	9.997
5.0	1.129	0.029	1.166	9.444	8.729	9.626	9.840	9.999
5.5	0.963	0.014	1.248	9.477	8.555	9.586	9.874	0.000
6.0	0.810	0.006	1.330	9.509	8.370	9.545	9.903	0.000
6.5	0.671	0.002	1.414	9.540	8.175	9.502	9.926	0.000
7.0	0.548	0.001	1.500	9.570	7.971	9.458	9.944	0.000
7.5	0.443	0.000	1.588	9.599	7.761	9.413	9.958	0.000
8.0	0.356	0.000	1.678	9.626	7.551	9.369	9.969	0.000
8.5	0.285	0.000	1.771	9.652	7.343	9.325	9.977	0.000
9.0	0.228	0.000	1.866	9.675	7.140	9.283	9.983	0.000
9.5	0.183	0.000	1.962	9.696	6.946	9.243	9.988	0.000
10.0	0.148	0.000	2.058	9.715	6.761	9.205	9.991	0.000
10.5	0.120	0.000	2.156	9.732	6.586	9.169	9.993	0.000
11.0	0.098	0.000	2.255	9.747	6.421	9.135	9.995	0.000
11.5	0.081	0.000	2.354	9.761	6.266	9.103	9.996	0.000
12.0	0.067	0.000	2.453	9.774	6.118	9.073	9.997	0.000
12.5	0.057	0.000	2.552	9.785	5.980	9.044	9.997	0.000
13.0	0.048	0.000	2.650	9.795	5.848	9.018	9.998	0.000
13.5	0.040	0.000	2.748	9.805	5.724	8.992	9.998	0.000
14.0	0.034	0.000	2.845	9.813	5.605	8.968	9.999	0.000
14.5	0.030	0.000	2.942	9.821	5.493	8.945	9.999	0.000
15.0	0.025	0.000	3.036	9.828	5.385	8.923	9.999	0.000

Variations

V	ΔU	ΔW	ΔH
3	+53	+5	+112
6	19	1	59

Model 10.

$\varkappa = \varkappa_0 \varrho^{0.25}$.

V	U	W	H	$\log r/R$	$\log P/P_c$	$\log T/T_c$	$\log M_r/M$	$\log L_r/L$
0.0	3.000	3.000	0.000	$-\infty$	0.000	0.000	$-\infty$	$-\infty$
0.5	2.824	2.231	0.200	9.232	9.892	9.957	8.699	9.478
1.0	2.655	1.608	0.400	9.380	9.786	9.914	9.105	9.765
1.5	2.494	1.126	0.600	9.465	9.681	9.872	9.324	9.881
2.0	2.342	0.768	0.800	9.524	9.578	9.831	9.467	9.937
2.5	2.198	0.514	1.000	9.569	9.478	9.791	9.569	9.966
3.0	2.059	0.338	1.156	9.605	9.380	9.753	9.645	9.981
3.5	1.919	0.220	1.296	9.635	9.282	9.716	9.705	9.989
4.0	1.782	0.141	1.432	9.661	9.184	9.680	9.754	9.994
4.5	1.648	0.089	1.565	9.684	9.086	9.645	9.793	9.997
5.0	1.520	0.056	1.697	9.705	8.988	9.612	9.826	9.998
5.5	1.398	0.035	1.827	9.724	8.890	9.579	9.853	9.999
6.0	1.284	0.022	1.958	9.740	8.793	9.547	9.876	0.000
6.5	1.176	0.013	2.090	9.756	8.697	9.515	9.895	0.000
7.0	1.077	0.008	2.222	9.770	8.602	9.485	9.910	0.000
7.5	0.985	0.005	2.354	9.783	8.508	9.455	9.924	0.000
8.0	0.901	0.003	2.487	9.795	8.416	9.427	9.935	0.000
8.5	0.824	0.002	2.621	9.806	8.326	9.399	9.944	0.000
9.0	0.754	0.001	2.756	9.816	8.239	9.372	9.952	0.000
9.5	0.691	0.001	2.890	9.825	8.154	9.346	9.959	0.000
10.0	0.633	0.000	3.026	9.833	8.071	9.321	9.964	0.000
10.5	0.581	0.000	3.162	9.841	7.990	9.296	9.969	0.000
11.0	0.534	0.000	3.297	9.848	7.912	9.273	9.973	0.000
11.5	0.491	0.000	3.432	9.855	7.837	9.250	9.977	0.000
12.0	0.452	0.000	3.567	9.861	7.764	9.228	9.980	0.000
12.5	0.418	0.000	3.700	9.867	7.693	9.207	9.892	0.000
13.0	0.386	0.000	3.832	9.872	7.625	9.187	9.984	0.000
13.5	0.357	0.000	3.962	9.877	7.558	9.168	9.986	0.000
14.0	0.330	0.000	4.091	9.882	7.494	9.149	9.988	0.000
14.5	0.307	0.000	4.216	9.886	7.431	9.131	9.989	0.000
15.0	0.285	0.000	4.339	9.891	7.371	9.113	9.990	0.000

Model 11.

$\varkappa = \varkappa_0$.

V	U	W	H	log r/R	log P/P_c	log T/T_c	log M_r/M	log L_r/L
0.0	3.000	3.000	0.000	$-\infty$	0.000	0.000	$-\infty$	$-\infty$
0.5	2.824	2.231	0.200	9.134	9.892	9.957	8.678	9.474
1.0	2.655	1.608	0.400	9.281	9.786	9.914	9.084	9.761
1.5	2.494	1.126	0.600	9.366	9.681	9.872	9.303	9.878
2.0	2.336	0.770	0.759	9.426	9.577	9.832	9.447	9.934
2.5	2.174	0.514	0.894	9.472	9.473	9.793	9.552	9.964
3.0	2.010	0.336	1.018	9.511	9.367	9.756	9.633	9.980
3.5	1.847	0.213	1.133	9.544	9.259	9.720	9.698	9.989
4.0	1.687	0.132	1.244	9.574	9.148	9.685	9.750	9.994
4.5	1.532	0.080	1.351	9.601	9.034	9.650	9.793	9.997
5.0	1.384	0.047	1.457	9.625	8.918	9.616	9.829	9.998
5.5	1.243	0.027	1.563	9.648	8.798	9.581	9.859	9.999
6.0	1.111	0.015	1.669	9.669	8.677	9.547	9.884	0.000
6.5	0.989	0.008	1.775	9.689	8.555	9.514	9.904	0.000
7.0	0.877	0.005	1.883	9.707	8.432	9.480	9.921	0.000
7.5	0.776	0.003	1.992	9.724	8.308	9.447	9.935	0.000
8.0	0.685	0.002	2.103	9.740	8.186	9.415	9.947	0.000
8.5	0.604	0.001	2.216	9.754	8.066	9.383	9.956	0.000
9.0	0.533	0.000	2.330	9.768	7.948	9.353	9.964	0.000
9.5	0.470	0.000	2.445	9.780	7.833	9.323	9.970	0.000
10.0	0.416	0.000	2.561	9.792	7.722	9.294	9.975	0.000
10.5	0.368	0.000	2.678	9.802	7.614	9.267	9.979	0.000
11.0	0.327	0.000	2.797	9.812	7.509	9.240	9.983	0.000
11.5	0.291	0.000	2.916	9.821	7.409	9.215	9.985	0.000
12.0	0.259	0.000	3.035	9.829	7.313	9.190	9.988	0.000
12.5	0.232	0.000	3.155	9.836	7.220	9.167	9.990	0.000
13.0	0.208	0.000	3.275	9.844	7.131	9.144	9.991	0.000
13.5	0.187	0.000	3.396	9.850	7.045	9.123	9.992	0.000
14.0	0.169	0.000	3.516	9.856	6.963	9.102	9.993	0.000
14.5	0.153	0.000	3.636	9.862	6.883	9.082	9.994	0.000
15.0	0.139	0.000	3.757	9.867	6.807	9.063	9.995	0.000

Variations

V	ΔU	ΔW	ΔH
5	+14	+1	+44

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