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MOMENTS OF INERTIA OF ROTATING NUCLEI

BY

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I. Introduction.

Collective nuclear excitations of rotational and vibrational character have been observed to occur systematically throughout most of the periodic table. Such states are populated in radioactive decay processes, and are also produced in inelastic scattering reactions. In particular, the Coulomb excitation process, which has been developed in recent years, has proved a powerful tool in the study of low-lying collective excitations in nuclei.

It has been possible to interpret many of the observed features of the collective spectra by comparing the collective modes of motion of the nucleus with the oscillations of an irrotational fluid* (A. Bohr, 1952; K. Ford, 1953; A. Bohr and B. R. Mottelson, 1953). In such a model, the excitation spectrum depends essentially on the nuclear equilibrium shape; it is thus of decisive importance that, in contrast to the case of an amorphous liquid drop, nuclei may acquire large equilibrium deformations as a consequence of their shell structure (Rainwater, 1950).

The nuclear shape depends on the configuration of the nucleons. In the vicinity of closed shells, the equilibrium shape is approximately spherical, and the expected collective spectrum corresponds to a set of normal vibrations, of which the lowest energy modes will be of quadrupole type.

In regions far removed from closed shells, the nuclear equilibrium shape deviates strongly from spherical symmetry, and the oscillation spectrum can be separated into shape oscillations and a rotational type of motion. In such a description the rotational motion is of wave-like character with the moment of inertia depending essentially on the deformation.

* Collective nuclear excitations similar to the vibrations of a liquid drop were first considered by N. Bohr and F. Kalckar (1937).

The observed nuclear collective spectra are found to follow such a general pattern. Thus, rotational spectra, characterized by their numerous regularities regarding energy ratios, spin sequences, and transition intensities, are associated with nuclei which exhibit large quadrupole moments and have especially been observed in the regions $150 < A < 190$ and $A > 225$. The moments of inertia are found to be appreciably smaller than corresponding to rigid rotation and to increase markedly with the deformation.*

In other regions of elements where the nuclear equilibrium shape, especially in even-even nuclei, is more nearly spherical, the collective excitations have been found to have many of the characteristics of quadrupole vibrations about a spherical equilibrium (SCHARFF-GOLDHABER and WENESER, 1955).

In the more detailed analysis of the nuclear collective spectra, it is found that the shell structure not only determines the nuclear equilibrium shape, but also has an important influence on other aspects of the nuclear potential energy surface as well as on the character of the collective flow.

Thus, the restoring force for the vibrational motion is expected to decrease rather rapidly as one moves away from closed-shell configurations; indeed, such an effect is observed in the trends of the vibrational frequencies.

The structure of the collective flow manifests itself in the mass transport associated with this motion, which can be determined from the observed excitation energies. It is found that the rotational moments of inertia as well as the inertial parameters for the vibrational motion are considerably larger than corresponding to the model of irrotational flow.**

In the present paper, we consider the analysis of the moments of inertia for rotating nuclei in terms of the motion of the nucleons.

* For a discussion of rotational spectra and a survey of empirical data, cf., e. g., BOHR and MOTTELSON (1955); A. BOHR (1954). Cf. also ALAGA, ALDER, BOHR, and MOTTELSON (1955) and BOHR, FRÖMAN, and MOTTELSON (1955) for the intensity rules, and the forthcoming review article on Coulomb excitation by ALDER, BOHR, HUUS, MOTTELSON, WINTHER, and ZUPANČIČ.

** The detailed estimate of the moment of inertia for irrotational flow is somewhat uncertain due to the possible difference between the density distribution of neutrons and protons, as well as to the influence of higher multipoles in the nuclear shape. Estimates of these effects indicate, however (cf. GUSTAFSON, 1955), that they are too small to account for the magnitude of the observed moments.

The treatment follows the method discussed by INGLIS (1954), in which the kinetic energy of rotation is obtained by considering the motion of the nucleons in the rotating self-consistent field.

The corresponding collective Hamiltonian for a system of interacting nucleons is discussed in § 2, while the evaluation of the moments of inertia is treated in § 3. For independent particle motion in an average nuclear field, the rotational moments of inertia are found to be approximately those corresponding to rigid rotation. However, the correlations in the nucleonic motion arising from residual interactions modify this result in an essential manner, and give rise, for small deformations, to a wave-like rotational motion. The absolute value of the moments of inertia depends inversely on the strength of the residual interactions, and the moments corresponding to irrotational flow are only approached when the interactions become comparable to the effect of the average field and so destroy the entire shell structure.

The observed moments, discussed in § 4, indicate a strength of interaction about three times smaller than corresponding to this strong interaction limit. Such an estimate of the interactions appears to be consistent with that obtained from other evidence. The residual interactions are also found to be responsible for the transition from rotational to vibrational collective spectra in the even-even nuclei with the approach to closed-shell regions.

II. Relation between Collective Hamiltonian and Nucleonic Motion.

Collective nuclear excitation spectra of vibrational or rotational type are expected to occur when the corresponding collective mode of motion is slow compared to the intrinsic motion of the nucleons. When this adiabatic condition is fulfilled, the nucleus will possess, for each state of the intrinsic structure, a spectrum of collective excitations.

The collective motion is described in terms of a set of co-ordinates α which, in the case of rotations, represent the angles of orientation of the nucleus; for vibrations, the collective co-ordinates may be chosen to represent the amplitudes of normal oscillations.

The Hamiltonian for the collective motion may be obtained by considering the nucleonic motion for slowly varying α. Expanding the energy of the nucleons in powers of the time derivative $\dot{\alpha}$, one obtains to a first approximation an expression of the type

$$H_{\mathrm{coll}} = E(\alpha) + \frac{1}{2} B(\alpha)\, \dot{\alpha}^2, \tag{1}$$

which thus represents the collective Hamiltonian. The first term in this expression, which is the nucleonic energy for fixed α, gives the potential energy for the collective motion, while the second term, involving an inertial coefficient $B(\alpha)$, gives the collective kinetic energy. Both the functions $E(\alpha)$ and $B(\alpha)$ may depend on the intrinsic state of the nucleonic motion.

The problem of obtaining the collective nuclear Hamiltonian is similar to the adiabatic derivation of the rotation-vibration Hamiltonian for molecules. In the molecular case, however, the inertial parameter B is to a good approximation given by the nuclear motion, while the electronic contribution to the mass transport constitutes only a small correction.

The collective nuclear co-ordinates are themselves functions of the nucleonic variables (cf. below), and the nucleonic motion for prescribed α is therefore a constrained motion. The constraints express the condition that the shape and orientation of the nucleonic system as a whole have the prescribed values. Thus, if the major part of the interactions can be represented by a self-consistent field, the constraints are approximately satisfied if one considers the motion of the nucleons in a field of the prescribed shape and orientation.

We may thus find the Hamiltonian (1) by treating the nucleonic motion in the time-dependent potential $V(\alpha(t))$ (Inglis, 1954, 1955). This motion is described by a Hamiltonian of the form

$$H = \sum_p T_p + \sum_p V(x_p, \alpha(t)) + U, \tag{2}$$

where x_p represents the co-ordinates of the p^{th} nucleon. The first term in (2) is the nucleonic kinetic energy, the second term represents the average potential which is here a function of t, while the last term represents residual effects of the nucleonic interactions not included in the average field.

For fixed α, we denote by ψ_i and E_i the proper functions and energies of (2) obeying

$$H(\alpha)\,\psi_i = E_i(\alpha)\,\psi_i. \tag{3}$$

These energy values E_i give the potential energy functions in (1).

For slowly varying α, the solution to the Hamiltonian problem (2) may be obtained by means of a time-dependent perturbation calculation. If there is no degeneracy in the static problem, the energy increase of the system resulting from the motion of the field is proportional to $\dot{\alpha}^2$ to leading order, and for the inertial parameter in (1), appropriate to the state ψ_0, one finds (Inglis, 1955)

$$B(\alpha) = 2\,\hbar^2 \sum_{i \neq 0} \frac{\left| <0 \left| \frac{\partial}{\partial \alpha} \right| i> \right|^2}{E_i - E_0}. \tag{4}$$

In the special case of rotations of axially symmetric nuclei, the mass parameter (4) gives the moment of inertia $\mathfrak{J}$ if $\frac{\partial}{\partial \alpha}$ generates a rotation about an axis perpendicular to the nuclear symmetry axis.

One thus obtains

$$\mathfrak{J} = 2\,\hbar^2 \sum_i \frac{| <0 \,|\, J_x \,|\, i> |^2}{E_i - E_0}, \tag{5}$$

where J_x is the total angular momentum of the particles about the intrinsic x-axis, which has been chosen perpendicular to the nuclear symmetry axis z.

The solution of the time-dependent problem (2) also determines other collective properties of the system. Thus, for the gyromagnetic ratio of the rotational motion, one obtains

$$g_R = \frac{\hbar^2}{\mathfrak{J}} \sum_i \frac{1}{E_i - E_0} \left(<0 \,|\, \mu_x \,|\, i> <i \,|\, J_x \,|\, 0> + \text{compl. conj.} \right), \tag{6}$$

where the magnetic moment operator is given by

$$\mu_x = \sum_p' (g_l \, l_x + g_s \, s_x)_p \tag{7}$$

in terms of the orbital and spin contributions of the individual particles.

The simple separation between collective and intrinsic motion corresponding to the Hamiltonian (1) is possible when the time-dependence of the nuclear field implies only a small modification of the motion of the individual nucleons with respect to the field. The adiabatic treatment employed above is then appropriate, and the dynamic effect of the motion of the field can be represented by the collective kinetic energy in (1).

If, however, there are near-lying intrinsic states which are strongly coupled by the motion of the field, the perturbation treatment may break down. The nucleus must then be treated in terms of a coupled system of collective motion and the intrinsic degrees of freedom involved. This situation is, for instance, met with in the partial decoupling between the rotational motion and the spin of the last odd nucleon in rotational spectra with an angular momentum component of $K = 1/2$ along the symmetry axis (cf. references in footnote on p. 4). Indeed, the level structure in odd-A nuclei is such that the motion of the last odd nucleon may quite frequently be somewhat perturbed by the rotational motion (Kerman, 1955; cf. also the odd-even moments of inertia differences discussed below (p. 22)).

The simple derivation of the collective Hamiltonian considered above exhibits the main physical conditions underlying the separation between collective and intrinsic motion. A more detailed treatment may be obtained in terms of a canonical transformation of the equations of motion which describe the system of interacting nucleons. In such a way one may introduce partly a set of collective co-ordinates α, and partly a set of coordinates q describing the intrinsic motion.

Various aspects of such a transformation have been considered in a number of recent papers (A. Bohr, 1954; Süssmann, 1954; Tolhoek, 1955; Tomonaga, 1955; Coester, 1955; Nataf, 1955; Marumori, Yukawa, and Tanaka, 1955; Villars, 1955; Lipkin, De Shalit, and Talmi, 1955). Without intering into a detailed discussion of this approach, we shall attempt, with the following general remarks, to indicate its relationship to the above derivation of the collective Hamiltonian.

The transformed Hamiltonian may be written in the form

$$H = H_{\text{int}}(q, \dot{q}, \alpha) + T_{\text{coll}}(\alpha, \dot{\alpha}) + H_{\text{coupl}}(\dot{q}, q, \dot{\alpha}, \alpha), \tag{8}$$

where the first term describes the intrinsic motion for fixed α. The second term represents the collective kinetic energy (cf. the second term in (1)), while the last term in (8) contains the couplings between the intrinsic and collective motion. These couplings partly describe the effect on the nucleonic motion of the time-dependence of the collective field, as contained in (2). In addition, the transformation introduces a second type of coupling terms associated with the fact that part of the inertial effect implied by the first type of coupling is already contained in T_{coll}. The second type of coupling thus tends to screen off the first type, and the problem is to choose the collective co-ordinates α in such a way that these two contributions approximately cancel; the major part of the dynamic effects associated with the motion of the nuclear field is then contained in T_{coll}, and the inertial parameter for the collective motion is thus expected to be given by (4).

If one can in such a manner obtain a Hamiltonian in which H_{coupl} is small, one gets approximate solutions to the wave equation of the adiabatic form

$$\Psi = \Phi_{\nu}(\alpha)\, \psi_i(q, \alpha), \tag{9}$$

where $\psi_i(q, \alpha)$ represents the intrinsic motion for fixed α, while $\Phi_{\nu}(\alpha)$ gives the collective motion specified by the quantum numbers ν.

An especially simple class of transformations is that which introduces a collective motion of irrotational character. If we further assume incompressible flow, the collective co-ordinates are given by (cf. Bohr and Mottelson, 1953, p. 10; A. Bohr, 1954)

$$\alpha_{\lambda\mu} = \sum_{p} \frac{4\pi}{3A} \left(\frac{r_p}{R_0}\right)^{\lambda} Y^{*}_{\lambda\mu}(\vartheta_p, \varphi_p), \tag{10}$$

which represent the mass multipole moments.

For a system such that a transformation of the type (10) leads to separation of the motion, the collective motion can be described in terms of an irrotational flow obeying hydrodynamical equations (A. Bohr, 1954). The moment of inertia is then directly related to the density distribution; thus, for a spheroid of constant density one obtains, for small deformations,

$$\mathfrak{J}_{\text{irrot}} = \frac{2}{5} AM(\Delta R)^2 \tag{11}$$

in terms of the difference ΔR between the major and minor semi-axes. The nuclear mass number and the nucleonic mass are denoted by A and M, respectively.

A closed-shell configuration in an anisotropic harmonic oscillator field would provide a very special case in which a separation between intrinsic and rotational motion is obtained by a transformation of the type (10)*. The appropriate collective angles are then defined in terms of the principal axes of the quadrupole mass tensor. It has also been verified that, in this case, the expression (5) yields the irrotational moment (11) (Inglis, 1954; cf. also p. 11 below).

For most systems, however, a transformation of the type (10) leaves important residual coupling terms, which imply a very intricate interweaving between the intrinsic motion and the collective motion associated with these particular collective co-ordinates. Still, provided the adiabatic condition is fulfilled, the system will possess simple collective modes of excitation, since the couplings may be incorporated in a modified collective motion. In order to exhibit the corresponding separation of the Hamiltonian, a co-ordinate transformation of a more general type than (10) is needed, and the collective flow is no longer of irrotational character.

III. Estimates of Rotational Moments of Inertia for the Nuclear Shell Structure.

The expression (5) for the moment of inertia depends quite sensitively on the character of the nucleonic motion.

* This case has also been noted by Lipkin, de Shalit, and Talmi (1955).

We first consider the limiting situation in which the intrinsic nuclear structure may be described in terms of the independent motion of the nucleons in the average potential. A closed expression for the sum in (5) may be obtained in the case of an anisotropic harmonic oscillator potential with no spin-orbit coupling. For a rotation about one of the principal axes, x, one obtains

$$\left.\begin{aligned}\mathfrak{J}_x = \frac{\hbar}{2}\frac{1}{\omega_y\,\omega_z}\left[\frac{(\omega_y-\omega_z)^2}{\omega_y+\omega_z}\sum_p (n_y+n_z+1)_p \right.\\ \left. + \frac{(\omega_y+\omega_z)^2}{\omega_y-\omega_z}\sum (n_z-n_y)_p\right],\end{aligned}\right\} \quad (12)$$

where ω_y and ω_z are the oscillator frequencies along the y- and z-axis, while n_y and n_z are the corresponding oscillation quantum numbers.

In the case of a single particle in the lowest state ($n_x = n_y = n_z = 0$), the moment (12) is just that corresponding to irrotational flow of the average density distribution of the particle. Indeed, this result is valid for the ground state in an arbitrary potential (Wick, 1947). Again for many-particle configurations consisting entirely of closed shells (occupation a function only of $N = n_x + n_y + n_z$), the last term in (12) vanishes and the moment has the irrotational value (11) with its characteristic dependence on the square of the eccentricity (Inglis, 1954).

For a closed-shell configuration, however, the nuclear equilibrium shape is spherical and the moment of inertia vanishes. The strongly deformed nuclei, which possess rotational spectra, have configurations deviating essentially from closed shells. The last term in (12) then gives important contributions implying considerable deviation from irrotational flow in the collective motion of the particles.*

Instead, in the limit of many nucleons, the moment of inertia tends towards that corresponding to rigid rotation of the average density distribution. Thus, the expression (12) approaches the value

* Such additional terms in the moment of inertia have also been considered by R. J. Blin-Stoyle and V. F. Weisskopf (private communication), who have treated nuclear potentials other than those of harmonic oscillator type. For such potentials, even closed-shell configurations may give moments exceeding the irrotational value.

$$\mathfrak{J}_{\text{rig}} = \sum_p M\overline{(y^2+z^2)}_p = \hbar \sum_p \left(\frac{n_y + 1/2}{\omega_y} + \frac{n_z + 1/2}{\omega_z} \right)_p \tag{13}$$

in the case of the ground state configuration.

This approach to the rigid moment is independent of the potential in which the particles move, as can be seen by employing the statistical approximation. The problem is considered most simply by going over to the rotating co-ordinate system where the potential is independent of time, but where the Coriolis and centrifugal forces must be added to the kinetic energy. In the absence of rotation, the velocity distribution is isotropic at each point, and the Coriolis forces cannot alter this situation to first order in the rotational frequency. Therefore, to this order, there is no net current in the rotating co-ordinate system, and the average flow is like that of a rigid body.*

Since the first-order effects of the rotation are equivalent to the effect of a magnetic field, the absence of an induced flow in the rotating co-ordinate system corresponds to the absence of diamagnetic effects in a classical electron gas (N. Bohr, 1911).

For a finite number of independent nucleons in an average potential, there may be rather large fluctuations of the moment of inertia (5) about the value $\mathfrak{J}_{\text{rig}}$. Thus, if the sum (12) is evaluated for a fixed deformation as a function of the number of nucleons, one finds quite violent fluctuations even for $A \sim 250$ and deformations of the observed order of magnitude. However, the fluctuations are much smaller if one considers, for each configuration, the self-consistent deformation, obtained by minimizing the total energy as a function of the deformation subject to the constraint of constant volume. In the harmonic oscillator case, the fluctuations then disappear, and one obtains just the rigid moment independent of configuration.** For other potentials in which the level structure is less regular, there may still remain some fluctuations in the moment associated with the binding of the last few particles.

* Sessler and Foley (1954) have considered a problem which in certain respects is similar to that discussed here. They find that a Thomas-Fermi treatment of an atom with a net angular momentum leads to a collective flow corresponding to rigid rotation.

** The closed-shell configurations form a singular exception to this result, since they have spherical equilibrium shape and a vanishing moment of inertia.

Thus, if the intrinsic nuclear structure could be described in terms of undisturbed independent particle motion, one would expect essentially the rigid moment of inertia. However, the inclusion of correlations in the nucleonic motion, arising from even relatively weak interactions, has an important influence on the collective motion and the resulting moment of inertia.*

The coupling scheme for a nuclear shell structure with the inclusion of particle interactions depends on the competition between the coupling effect of these interactions and the coupling of the particles to the nuclear deformation (cf. BOHR and MOTTELSON, 1953, § II c, and especially fig. 6). For small deformations, where the former effect dominates, the particle angular momenta are coupled together to a resultant J; for large deformations, the latter effect is dominant, and the particles are coupled independently to the nuclear axes.

For an even-even nucleus, short-range attractive forces favour a state of $J = 0$ (MAYER, 1950; EDMONDS and FLOWERS, 1952; RACAH, 1952). For small deformations, for which the ground state wave function may be expanded in powers of the deformation, one thus has

$$\psi = \psi(J = 0) + \beta\psi(J \neq 0) + \cdots, \tag{14}$$

where β is the conventional deformation parameter for ellipsoidal shapes defined by

$$\beta = \frac{4}{3}\sqrt{\frac{\pi}{5}}\frac{\Delta R}{R_0} \approx 1.06\frac{\Delta R}{R_0} \tag{15}$$

in terms of the mean nuclear radius R_0 and the difference ΔR between major and minor semi-axes. The first term in the wave function (14) does not contribute to the moment of inertia (5) and one therefore obtains

$$\mathfrak{J} = \text{const}\ \beta^2. \tag{16}$$

* The possible significance of the residual interactions for the nuclear moments of inertia has been suggested in a somewhat different context by FORD (1954) and INGLIS (1954). These authors anticipate an effect opposite to that obtained below, since they assume the independent particle approximation to give irrotational flow.

The proportionality constant in (16) depends inversely on the excitation energies for the admixed states in (14), and thus on the strength of the interactions.

In the other limit of large deformations, one approaches the independent particle situation with the resulting rigid moment of inertia.

Some insight into the gradual transition between the two coupling schemes may be obtained by considering a greatly simplified model, in which the whole effect of nucleons outside of closed shells is represented by two interacting nucleons in p-states. Choosing the field to be of harmonic oscillator type, the closed shells may be treated collectively in terms of their resistance to deformation and their irrotational contribution to the moment of inertia. By varying the effective number of nucleons in closed shells, one obtains a sequence of configurations with varying equilibrium deformations, for which the moment of inertia may be evaluated by means of (5). The strength of the interaction between the nucleons outside closed shells may be characterized by a parameter which measures the ratio of the interaction energy to the configuration spacing $\hbar\omega$. This interaction parameter may be taken as

$$v = \frac{U}{\hbar\omega}, \tag{17}$$

where U is the energy difference between the $J = 0$ and $J = 2$ states of the two nucleons.

Corresponding to the different values of v, one obtains from this model a family of curves for $\mathfrak{J}$ as a function of the equilibrium deformation (cf. Fig. 1). These curves show the qualitative features discussed above, varying rapidly for small deformations and approaching $\mathfrak{J}_{\text{rig}}$ for $\beta \gg v$. In the limit of $v \sim 1$, in which the shell structure is destroyed by the interaction, one approaches the irrotational flow.

The curves in Fig. 1 only cover values of β larger than about 0.6 v. For configurations nearer to closed shells, the model considered gives no stable equilibrium deformation, and instead yields a collective spectrum corresponding to vibrations about a spherical equilibrium shape.

Such a general behaviour is expected to be characteristic of nuclear configurations which couple to $J = 0$ or $1/2$ in the absence of deformations, and thus in particular of the ground state configurations of even-even nuclei. In fact, for such configurations, the nuclear potential energy of deformation is proportional to β^2 for small deformations. The absence of a linear term, which is

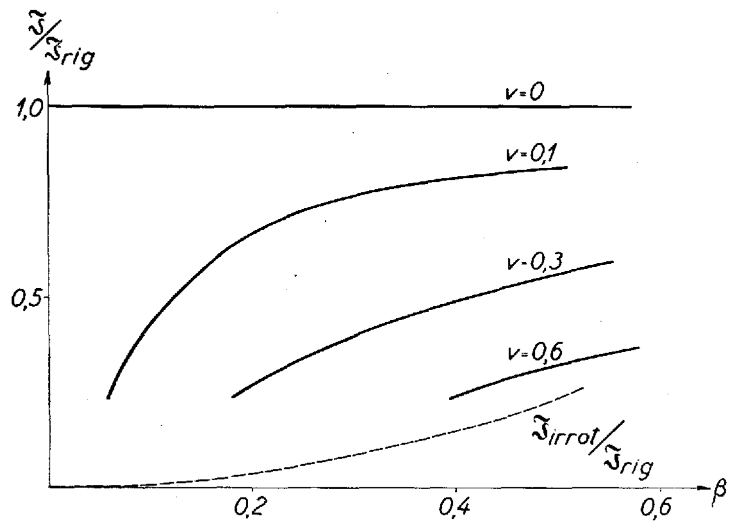

Fig. 1. Moments of Inertia for the Two-Nucleon Model.
The figure shows the dependence of the moment of inertia $\mathfrak{J}$ on the nuclear deformation β (cf. (15)), as estimated from the simplified two-nucleon model discussed in the text. The different curves correspond to different values of the residual interactions, as specified by the interaction parameter v (cf. (17)). For $v = 0$, the moment of inertia is equal to the value corresponding to rigid rotation, $\mathfrak{J}_{\text{rig}}$ (cf. (18)), independent of deformation. For $v \sim 1$, the moment approaches the value for irrotational flow, given by (11), and indicated by the dotted curve.

a consequence of the residual interactions, implies that, as one moves away from closed-shell regions, the deforming tendency of the particles in unfilled shells results at first merely in a decrease of the effective surface tension. Thus, nuclei possessing equilibrium deformations are expected to occur only in regions sufficiently far removed from closed shells, where the tendency towards deformation may overcome the effect of the interactions.

An estimate of the relative importance of the residual interactions may be obtained from the observed nuclear coupling schemes. Thus, the very occurrence of even-even nuclei with stable equilibrium deformations, as revealed by the existence of rotational spectra, indicates that, for these nuclei, the coupling scheme is approaching that of independent particles. This conclusion is further supported by the analysis of the ground state spins and intrinsic excitation spectra for the strongly deformed nuclei (MOTTELSON and NILSSON, 1955).

Even for the largest observed deformations, there remain, however, significant effects of the interactions as revealed especially by the systematic difference in the binding energy of even-even and odd-A nuclei, amounting to about 1 MeV in the heavy nuclei (cf., e. g., MAYER and JENSEN, 1955, p. 9). A similar effect is revealed in the conspicuously different intrinsic excitation spectra exhibited by odd-A and even-even nuclei. While, in the former, the observed level spacing is a few hundred keV, corresponding to the expected spacing between single-particle levels, the first intrinsic excitation in the even-even nuclei is rarely observed to lie below an MeV.*

These differences can be interpreted in terms of a pairing effect similar to the one discussed previously (MAYER, 1950) for the coupling scheme in spherical nuclei. In deformed axially symmetric nuclei, where the particles are filled pairwise in degenerate orbits distinguished only by their sense of precession about the nuclear symmetry axis, the pairing effect can be simply accounted for in terms of the especially strong interaction between paired nucleons associated with their similar wave functions.

Such a pairing energy has the effect of increasing the energy denominators in (5), except in the contribution due to unpaired particles, and thus reducing the moment of inertia below the value for rigid rotation. In order to obtain an estimate of this effect, we have evaluated the sum (5), employing single-particle wave functions appropriate to a deformed potential with spin-orbit coupling (NILSSON, 1955). When one includes in the energy denominators a pairing energy estimated to be on the average 1.5 MeV for $A \sim 150$, the moment of inertia, for a deformation of $\beta = 0.3$, is reduced by a factor of about two. From a comparison

* A striking example of this odd-even difference is provided by the comparison between the level spectra of W^{182} and W^{183}, recently measured by MURRAY, BOEHM, MARMIER, and DUMOND (1955).

with Fig. 1 the observed pairing energies are thus seen to imply an interaction parameter v of about 0.3*.

One may employ similar methods as used in the calculation of $\mathfrak{J}$ to evaluate the expression (6) for the gyromagnetic ratio for the collective motion. Using the wave functions of NILSSON (1955), one obtains for even-even nuclei values for g_R which fluctuate rather little about the average value Z/A and are relatively insensitive to the strength of the pairing interaction.

IV. Discussion of Empirical Data.

The systematically occurring rotational spectra in the region $150 \leqslant A \leqslant 188$ have been especially well studied. The moments of inertia for the even-even nuclei in this region, determined from the observed rotational level spacings, are plotted in Fig. 2 as a function of the nuclear deformation. The moments are given in units of the value

$$\mathfrak{J}_{\text{rig}} = \frac{2}{5} MAR_0^2 \, (1 + 0.31\,\beta + 0.44\,\beta^2 \ldots) \qquad (18)$$

associated with a rigid rotation of an ellipsoid of constant density.

The nuclear deformation is estimated from the observed electric quadrupole moment Q_0 of the nuclear shape which, for an ellipsoidal nucleus, is related to β by

$$Q_0 = \frac{3}{\sqrt{5\,\pi}} Z R_0^2 \, \beta \, (1 + 0.16\,\beta \ldots), \qquad (19)$$

where Z is the nuclear charge number.

The Q_0-values are determined from the electric quadrupole transition probabilities between two members of a rotational band. The reduced transition probability for such a transition from a state I_i to a state I_f is given by

$$B\,(E2) = \frac{5}{16\,\pi} e^2 \, Q_0^2 < I_i \, 2 \, K \, 0 \mid I_i \, 2 \, I_f \, K >^2. \qquad (20)$$

* This estimate of the residual interactions also appears compatible with the analysis of the low energy neutron scattering data in terms of the optical model (FESHBACH, PORTER, and WEISSKOPF, 1954) which yields a mean free path for nucleonic motion in the nuclear field a few times longer than the nuclear radius.

TABLE I.

Even-even nuclei				Odd-A nuclei				
Isotope	$\frac{3\hbar^2}{\mathfrak{J}}$ (keV)	Q_0 (10^{-24} cm^2)	Ref.	Isotope	I_0	$\frac{3\hbar^2}{\mathfrak{J}}$ (keV)	Q_0 (10^{-24} cm^2)	Ref.
${}_{60}\mathrm{Nd}^{150}$	131	4.8	a, b, c					
${}_{62}\mathrm{Sm}^{152}$	121	5.6	b, c, d					
Sm^{154}	82	7.5	b, c					
				${}_{63}\mathrm{Eu}^{153}$	5/2	72	7.7	b, c
${}_{64}\mathrm{Gd}^{154}$	123	6.6	b, c	Gd^{155}	(3/2)	62		b
Gd^{156}	89	8.4	b, c	Gd^{157}	(3/2)	56		b
Gd^{158}	79	9.3	b. c					
Gd^{160}	76	9.8	b. c					
				${}_{65}\mathrm{Tb}^{159}$	3/2	58	8.4	b, c
${}_{66}\mathrm{Dy}^{160}$	86	7.1	e	Dy^{161}	(5/2)	62		b
Dy^{162}	82	7.9	b, c	Dy^{163}	(5/2)			
Dy^{164}	74	9.2	b, c					
				${}_{67}\mathrm{Ho}^{165}$	7/2	63	8.6	b, c
${}_{68}\mathrm{Er}^{164}$	90	7.3	f					
Er^{166}								
Er^{168}	80	7.5	b, c, i	Er^{167}	7/2	52		b
Er^{170}								
				${}_{69}\mathrm{Tm}^{169}$	1/2	76	8.1	b, c
${}_{70}\mathrm{Yb}^{170}$	84	7.1	e, g					
Yb^{172}								
Yb^{174}	78	9.2	b, c	Yb^{173}	5/2	68	8.8	b
Yb^{176}								
				${}_{71}\mathrm{Lu}^{175}$	7/2	76	8.0	b, c, h
${}_{72}\mathrm{Hf}^{176}$	89	7.1	b, h, m	Hf^{177}	(7/2)	75	8.5	b, c, h
Hf^{178}	91	7.7	b, c, h, j	Hf^{179}	(9/2)	67		b, c, h
Hf^{180}	93	7.2	b, c, d, h, j					
				${}_{73}\mathrm{Ta}^{181}$	7/2	91	7.1	b, c, h, j, k
${}_{74}\mathrm{W}^{182}$	100	6.9	c, d, h	W^{183}	1/2	72	6.5	c, h
W^{184}	112	6.2	c, h					
W^{186}	124	6.1	c, h					
				${}_{75}\mathrm{Re}^{185}$	5/2	108		c, h, l
				Re^{187}	5/2	115		c, h, l
${}_{76}\mathrm{Os}^{186}$	137	5.6	n					
Os^{188}	155	5.2	d					

Moments of Inertia and Quadrupole Moments for Nuclei in the Region $150 \leqslant A \leqslant 188$.

References and Text to Table I.

a) SIMMONS et al. (1955).
b) HEYDENBURG and TEMMER (1955).
c) HUUS et al. (1955).
d) SUNYAR (1955).
e) McGOWAN (1952a).
f) BROWN and BECKER (1954).
g) GRAHAM et al. (1952).
h) McCLELLAND et al. (1955).
i) McGOWAN (1950).
j) STELSON and McGOWAN (1955).
k) HUUS and ZUPANČIČ (1953).
l) FAGG and WALICKI (1955).
m) McGOWAN (1952b).
n) McGOWAN (1951).

The table lists the available evidence on the shape and moment of inertia of nuclei in the region $150 \leqslant A \leqslant 188$. Only those nuclei have been included which appear to exhibit collective excitations of rotational character. Thus, Sm^{150} and Gd^{152} have been omitted since their low-lying collective excitations are of vibrational type, as are also observed in the even-even nuclei just outside the considered region of A.

For the even-even nuclei, column two lists the energies of the first excited $(2+)$ rotational states, while the third column gives the Q_0-values deduced from the electric quadrupole transitions between this $(2+)$ state and the $(0+)$ ground state, by means of (20). The data are obtained from Coulomb excitation experiments and lifetime measurements. The Q_0-values represent a weighted average of the available determinations. For the even isotopes of Er and Yb, only a single transition has been observed in the Coulomb excitation of the natural element. This transition is tentatively assigned to all the abundant even isotopes.

For the odd-A nuclei, the determination of the moment of inertia and the quadrupole moment depends on the ground state spin I_0. The table lists I_0-values determined from spectroscopic evidence (cf., e. g., HOLLANDER, PERLMAN, and SEABORG (1953)) and, in parenthesis, the more tentative values derived from rotational level spacings and radioactive decay schemes.

The quantum number K appearing in the vector addition coefficient represents the component of angular momentum along the nuclear axis, and is a constant for a given rotational band.

Estimates of Q_0 can also be obtained from spectroscopic determinations of the ground state quadrupole moment Q of odd-A nuclei, using the relation

$$Q = Q_0 \frac{I_0}{I_0+1} \frac{2\,I_0-1}{2\,I_0+3}, \qquad (21)$$

where I_0 is the ground state spin. The Q_0-values obtained in this manner are consistent with those derived from the transition probabilities by means of (20). (Cf. Bohr and Mottelson, 1955; Kamei, 1955).

The dependence of the observed moments of inertia on the nuclear deformation, illustrated in Fig. 2, is seen to correspond to the behaviour expected for a shell structure with some residual interaction (cf. § III). The full-drawn curve in Fig. 2 which follows

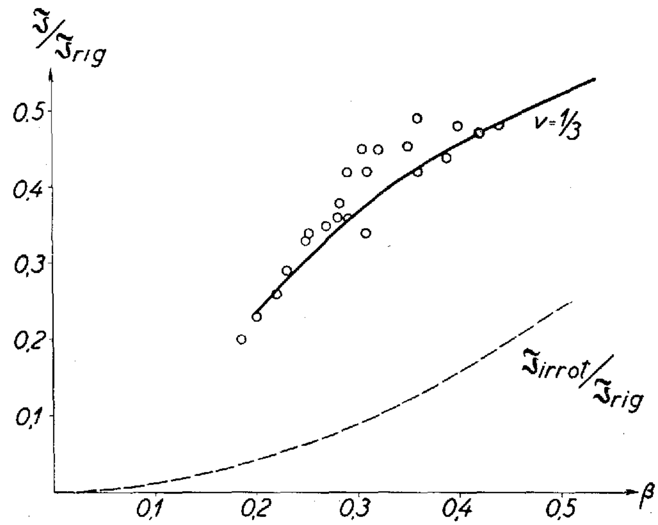

Fig. 2. Dependence of Nuclear Moments of Inertia on the Nuclear Deformation. The empirical moments of inertia for even-even nuclei in the region $150 \leq A \leq 188$ are plotted as a function of the nuclear deformation. The moments of inertia, obtained from the data in Table I, are given in units of the rigid moment (18), while the deformation parameters β are obtained from the Q_0-values in Table I by means of (19). The nuclear radius has been taken to be $R_0 = 1.2\ A^{1/3}\ 10^{-13}$ cm. The full-drawn curve represents a theoretical estimate, based on the two-nucleon model with an interaction parameter $v = 1/3$ (cf. Fig. 1). For comparison, the moment of inertia corresponding to irrotational flow is shown by the dotted curve.

the main trend of the experimental points is obtained from the simplified two-nucleon model and corresponds to an interaction parameter $v = 0.33$ (cf. (17) and Fig. 1). The scatter of the experimental points about this curve is of the order of magnitude of the estimated experimental uncertainties. However, some fluctuations about a smooth curve are to be expected, associated with specific differences of the individual nuclear configurations.

The strength of interaction ($v \sim 0.33$), revealed by the empirical moments of inertia, is just of the magnitude estimated from the observed nuclear pairing energies (cf. pp. 16—17 above). Such residual interactions, while of major significance for the collective flow, are still a factor of about three smaller than those which would destroy the basic nuclear shell structure.

The estimated value of v refers to the nuclei in the region $150 \leqslant A \leqslant 188$; the observed variation with A of the nuclear pairing energy suggests that v varies approximately as $A^{-1/3}$.

As discussed in the previous section, one expects with the approach to closed-shell configurations a transition from rotational to vibrational collective spectra, especially in the even-even nuclei. For the two-nucleon model (cf. p. 14 above), the transition occurs when the deformation becomes comparable to $0.6\, v$. Such transitions are in fact observed to occur in the neighbourhood of Sm and Os, where the deformations are about $\beta = 0.2$.

Since the transition from vibrational to rotational spectra takes place when the nuclear coupling scheme is approaching that of independent particle motion, the transition region may be characterized, approximately independently of v, by a moment of inertia equal to a certain fraction of $\mathfrak{J}_{\mathrm{rig}}$. A tentative estimate for this fraction may be obtained from the two-nucleon model, which yields $\mathfrak{J}_{\mathrm{min}} = 0.23\, \mathfrak{J}_{\mathrm{rig}}$. This would imply that rotational spectra should occur in even-even nuclei only when the energy E_2 of the first excited $(2+)$ state satisfies the relation

$$E_2 < \frac{3\, \hbar^2}{0.23\, \mathfrak{J}_{\mathrm{rig}}} \approx \frac{32\, \hbar^2}{MAR_0^2}. \tag{22}$$

Excitation energies appreciably smaller than this limit have been observed only in the heavy element regions ($A > 225$) and ($150 < A < 190$) and in the relatively light elements around $A = 24$ and $A = 8$ (cf., e. g., Scharff-Goldhaber, 1953). The systematic occurrence of rotational spectra in the former regions is well established, and tentative evidence for a rotational spectrum in Mg^{24} is provided by the observed $4+$ state with an energy about three times that of the $2+$ state.*

* Also in Be^8 there is tentative evidence for a $4+$ state, whose energy is about 3.7 times that of the $2+$ state (cf., e. g., Ajzenberg and Lauritsen, 1955); for this nucleus the large deformation indicated by the collective excitations may also be described as a tendency towards α-particle formation (cf. Wheeler, 1937).

In other regions of elements the condition (22) is not satisfied, and the observed collective excitation spectra in even-even nuclei exhibit the expected vibrational character (Scharff-Goldhaber and Weneser, 1955).

Another feature of the nuclear moments of inertia, which may be understood in terms of the residual nucleonic interactions, is the observed difference between the moments of even-even and neighbouring odd-A nuclei. It is found that the latter are systematically larger than the former, by an amount varying from a few per cent up to as much as 40 per cent, while there appear to be no corresponding differences in the deformations (cf. Table I).* This increase in the moments of inertia for the odd-A nuclei may represent the especially large contribution to (5) of the last odd particle which, in general, possesses low-lying states of excitation. Similar odd-even differences in the gyromagnetic ratio g_R are thus also to be expected.

In such cases where an appreciable fraction of the rotational angular momentum is associated with the motion of a single nucleon, one expects significant higher-order corrections to the adiabatic treatment, implying small deviations from the simple rotational energy spectrum (cf. p. 8 above).

Perturbations of this type are revealed in the very accurately determined energy spectrum of W^{183} (Murray et al., 1955), and have been accounted for in terms of the non-adiabatic coupling between the two lowest intrinsic configurations (Kerman, 1955). The detailed analysis of these perturbations permits a determination of the corresponding matrix element in (5), and it is found that the resulting contribution to $\mathfrak{J}$ is just of the magnitude of the difference between the moments of inertia for W^{183} and W^{182}.

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* The similarity of the quadrupole deformations in the even-even and odd-A nuclei has also been noted by Heydenburg and Temmer (1955). Evidence for odd-even differences in the moments of inertia in the region $A > 225$ has been discussed by Bohr, Fröman, and Mottelson (1955).

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