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ON QUANTUM FIELD THEORIES

BY

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CONTENTS

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In the present paper[1], the general conditions which a relativistic quantum theory of interacting particles must satisfy are investigated and brought into mathematical form. Some difficulties connected with the infinite number of degrees of freedom are pointed out. Especially the fact that the canonical commutation relations no longer have unique solutions must be taken into account in all discussions of field theory. It is shown that the "free field vacuum" of the Tamm-Dancoff method and Dyson's matrix $U(t_1, t_2)$ for finite t_1 or t_2 cannot exist. The possibility of a conventional field theory in which the fields commute for equal times is investigated.

Introduction.

In the past few years, considerable effort has been devoted to the question whether a field theory of conventional type plus renormalization may be regarded as a well defined mathematical scheme which is only formulated in a somewhat awkward way. An affirmative answer to this question would mean that it is possible—at least in a certain idealization—to separate the problem of the interaction of elementary particles from that of their existence and constitution. Of course it is quite clear that such a separation cannot ultimately be satisfactory[2]. From a certain point of view, it might even be correct to say that it is just this

[1] The major part of this work has been done during the author's stay at the Institute for Theoretical Physics in Copenhagen as a member of the CERN Theoretical Study Group.

[2] To mention only one of the many arguments which have been put forward in this connection: the presence of electromagnetic interaction is always accompanied by a mass contribution of the order of magnitude of the electron mass (mass difference proton-neutron, charged and neutral meson). This seems to suggest that, in a more satisfactory formulation of electrodynamics, one should not have the "mechanical mass" of the electron as the basic constant, but something independent of the particle type, say a cut-off length l. Ordinary electrodynamics corresponds to the idealization $l = 0$, in which case the mass becomes infinite.

undue separation which causes the divergence difficulties in field theory. On the other hand, we must keep in mind that, in essence, quantum field theory is (or rather should be) nothing but an extension of quantum mechanics to a system with infinitely many degrees of freedom in such a way that, instead of invariance under Galilei transformations, we have Lorentz invariance. Now it is evident[3] that there is no contradiction between the general framework of quantum physics, the relativity principle, and the fact that we want to deal with a system of arbitrarily many particles. Therefore a mathematically consistent formulation of such a theory should certainly be possible without introducing essentially new physical ideas to explain the constitution of elementary particles. This statement contrasts strangely with the fact that it is extremely easy to write down any number of formalisms which may be regarded as the quantum mechanics of some hypothetical system, whereas up to now not a single model of a field theory has been given which meets all consistency requirements. The aim of the present paper is to work out some criteria of consistency to which the formalism must be subjected, to point out a few of the mathematical difficulties connected with the infinite number of degrees of freedom, and to apply the results to a discussion of some of the basic assumptions and calculating methods of a field theory of conventional type. In this way we come back to the question mentioned in the beginning.

In Chapter I, the general assumptions are formulated and combined in a form suitable for later application. They include the principles of quantum physics, relativistic invariance, and the "particle postulates", i. e. those requirements which ensure that the theory can be interpreted as describing the interaction processes of particles. Chapter II deals with some mathematical questions. A few simple consequences, which may be drawn without recourse to perturbation calculations, are given in the first paragraphs of Chapter III. It is shown that general field theories exist, i. e. theories in which the commutation relations of the field operators for equal times are not fixed a priori, and that they are just as wide in scope as are S-matrix theories. For the case of conventional theories in which the field operators

[3] This will also be discussed in Chapter I.

are required to commute for equal times, a very simple argument shows that the so-called "free field vacuum" does not exist. This implies that in the (old) Tamm-Dancoff method, in which this concept plays an important role, additional difficulties are introduced, and also that Dyson's matrix $U(t_1, t_2)$ cannot exist if one or both of the times t_1, t_2 are finite. The question as to whether a conventional theory is possible at all is investigated in the last paragraph by a perturbation calculation. It appears that the requirement of commutability for equal times leaves only a few types of models, and also that the appearance of highly singular functions, which need careful definition and handling, is an inevitable consequence at least in perturbation calculations.

In order to eliminate all difficulties or complications which are not essential to the main question, the discussion has been limited, in Chapter III, to the simplest conceivable physical situation, the case in which only one type of particle exists. It is supposed to have spin zero and mass m.

I. General assumptions and immediate consequences.

§ 1. The postulates.

A. Quantum Physics.

We deal with a Hilbert space $\mathfrak{h}$. To every "state" of the system corresponds a normalized vector Ψ of $\mathfrak{h}$. If we know that the system is in the state Ψ_1, the probability of observing a state Ψ_2 is given by the absolute square of the scalar product

$$w = |(\Psi_2, \Psi_1)|^2.$$

To connect the abstract scheme to physical reality it is necessary to give the coordination between the "states" experimentally defined in terms of Geiger counters etc., and their corresponding elements Ψ of $\mathfrak{h}$. This is called "the physical interpretation of the theory".

B. Relativistic Invariance.

The term Lorentz transformation (L. T.) will be understood to include also translation in space and time, but not reflections; that is to say, we deal with the 10-parametric inhomogeneous Lorentz group where the elements may be characterized by the coefficients $a_{\mu\nu}$, b_μ of the transformation

$$x'_\mu = a_{\mu\nu}\, x_\nu - b_\mu \qquad (\mu, \nu = 0,\ 1,\ 2,\ 3).$$

The index 0 refers to the (real) time coordinate. As the scalar product between two 4-vectors p and x we shall write

$$px = p_i\, x_i - p_0 x_0 \qquad (i = 1, 2, 3).$$

We also consider the reflection elements s_r, s_t and $s = s_r s_t$ giving the transformations

$$\begin{aligned} s_r: &\quad x'_i = -x_i, \quad & x'_0 = x_0 \\ s_t: &\quad x'_i = x_i, \quad & x'_0 = -x_0 \\ s: &\quad x'_\mu = -x_\mu. & \end{aligned}$$

The invariance of the theory with respect to these reflections will not be relevant in the context of this paper but, for the sake of completeness, a short discussion is included.

It is convenient to state the invariance requirements in connection with A in a form introduced by WIGNER[4]: To every element L of the Lorentz group there is attached a unitary operator $D(L)$ in $\mathfrak{h}$ so that

$$D(L_1)\, D(L_2) = \pm D(L_1 L_2).$$

Mathematically speaking: in $\mathfrak{h}$ we must have a representation of the Lorentz group. The physical significance of the operators $D(L)$ is the following. Let M_Ψ be the apparatus needed for the experimental realization of the state Ψ. We can envisage another apparatus $M_{\Psi'}$ which—if viewed from a Lorentz frame Σ'—would be described in exactly the same way as M_Ψ is described

[4] E. WIGNER, Ann. of Math. 40, 149 (1939). See also V. BARGMAN and E. WIGNER, Proc. Nat. Acad. Sci. 34, 211 (1948).

by an observer in Σ, i. e. $M_{\Psi'}$ differs from M_Ψ only in space-time orientation. Then, if L is the L. T. which carries Σ' over into Σ, we have

$$\Psi' = D(L)\,\Psi$$

for every state Ψ.

The invariance against reflections may be formulated in a similar way. One difference is that the operators T, C, and J, which correspond to the elements s_r, s_t and s, respectively, could also be "anti-unitary"[5]. In fact, it appears that, for the cases of physical interest, C and J are anti-unitary whereas T is unitary. The physical significance of T is straightforward. We have to consider two experimental arrangements which are related like a right- and left-hand glove. In the case of C, a similar general definition does not seem possible since we cannot interchange past and future. However, for the special case of a state which is localized around a point $x = (\mathbf{x}, t)$, one may take the time-inverted state as the state of localization around $x' = (\mathbf{x}, -t)$.

It is important to keep in mind that we can speak of the invariance of the theory with respect to these transformations only if operators T, C, J exist which have both the correct structure relations with each other and with the $D(L)$,[6] and also the physical significance mentioned. One can show, for instance, that in all representations of the Lorentz group, which correspond to particle systems, there exist operators which obey the structure relations prescribed for T or C.[7] It is not true, however, that every Lorentz invariant theory is automatically invariant under space or time inversion, since the formal operators T, C need not have the specified observational significance if more than one particle is involved[8].

C. Particle Postulates.

We shall not here distinguish between "elementary" and "compound" particles. This is a matter of convenience rather than of principle. As characteristic of a particle we regard the localization of the events which may be caused by it. Thus, we

[5] An anti-unitary operator is an operator which transforms a linear relation $\Sigma a_k \Psi_k = 0$ into the linear relation between the image vectors $\Sigma a_k^* \Psi'_k = 0$ (complex conjugate coefficients!). The scalar product of the image vectors is the complex conjugate of the scalar product of the original vectors $(\Psi'_1 \Psi'_2) = (\Psi_1 \Psi_2)^*$.

[6] For instance, $J\,D(b) = D(-b)\,J$, $J\,D(\Lambda) = D(\Lambda)\,J$, $J^2 = \pm 1$, where b is a translation, Λ a homogeneous L. T.

[7] The case of particles with mass zero is an exception. The statement continues to hold, however, if $m = 0$ is regarded as the limiting case of a small mass.

[8] I had overlooked this fact in the manuscripts mentioned in ref. 23. The point was cleared up in a discussion with G. LÜDERS and L. L. FOLDY.

could use, for instance, a coincidence arrangement of two Geiger counters separated by a distance d with a resolving time $\delta t \ll \frac{d}{c}$ to single out experimentally the states containing only one particle. If we consider all such coincidence arrangements for arbitrary position, orientation, and velocity of the apparatus and arbitrary distance d (which is only supposed to be larger than some $d_{\min}$), then the one-particle states are those which give a negative result in all these measurements[9]. The magnitude of $d_{\min}$ is irrelevant. It could be taken as large as one pleases but, for practical reasons, it will be convenient to set an upper limit to the size of the object which we choose to call a particle.

For a theoretical analysis, it is important that the manifold of one-particle states is thus defined as a relativistically invariant subspace $\mathfrak{h}^{(1)}$ of $\mathfrak{h}$. We can now split $\mathfrak{h}^{(1)}$ further into invariant subspaces $\mathfrak{h}_j^{(1)}$ and imagine this process to be carried through as far as possible, that is to say, so far that each $\mathfrak{h}_j^{(1)}$ belongs to an *irreducible* representation of the Lorentz group. The different $\mathfrak{h}_j^{(1)}$ can be distinguished by attributes which have a relativistically invariant meaning. These attributes can be considered as the attributes of the "particle species j". It is therefore just a definition of the concept "particle species" if we say

C_1. *The manifold of states of one particle of type j belongs to an irreducible representation $D_j(L)$ of the Lorentz group.*

The possible irreducible representations have been classified by Wigner[4]. Each of them is characterized by two numbers m and s. These have the physical meaning of the mass and the spin of the particle, respectively.

The representations belonging to imaginary values of m can obviously not be attached to particles and, therefore, the first action of the particle postulates must be to exclude these representations. Then, for $m > 0$, the spin can take only integer or half integer values and the representations may be described within the well-known formalism of the wave equations of spin particles. For $m = 0$ there exist also representations corresponding to a continuous spin variable. Whether or not these representations have physical importance is yet an open question. They will not be considered here.

[9] The vacuum state is of course also admitted, but can be excluded in a trivial way by single counter measurements.

The decomposition of $\mathfrak{h}^{(1)}$ into the $\mathfrak{h}_j^{(1)}$ becomes ambiguous if $\mathfrak{h}^{(1)}$ contains several equivalent irreducible representations, i. e. if we have several different particles with equal mass and spin. This is analogous to the appearance of degenerate eigenfunctions in an eigenvalue problem. Incidentally, it may be noted that the eigenvalue problem for the Hamiltonian in conventional quantum mechanics is in field theory replaced by the mathematical problem of reducing the representation $D(L)$ into its irreducible components. In the degenerate case, it may be more appropriate to consider the total space $\Sigma\mathfrak{h}_j^{(1)}$ spanned by all the representations which belong to the same mass and spin as the manifold of states for *one* particle. This would, for instance, mean that instead of protons and neutrons we speak of nucleons, etc.

Now going over to situations in which more than one particle is involved, it is convenient to introduce the concept of "partial state around a point x". By this we understand those properties of the system which may be measured by any experimental set-up within a large but finite space-time volume around x. Then, we formulate the following fundamental assumption:

C_2. *Whatever total state of the system we consider, the partial state around $x = (\mathbf{x}, t)$ approaches a one-particle state for $|t| \to \infty$.*

For this assumption three conditions are necessary. First, we exclude from consideration situations in which the particles fill the total space with a finite density. Secondly, there are (per definitionem) no "bound states". Thirdly, in this general discussion we allow only normalizable state vectors (proper elements of $\mathfrak{h}$). This ensures that we have only a finite interaction region outside which the particles are separated so much that the attraction between them can be neglected. It does not mean of course that in practical calculations we could not use plane waves as well, if only they are handled with proper care.

Now, the experimental characterization of an arbitrary state Ψ may be given in terms of measuring results obtained in the distant past, for short at $t = -\infty$. As the particles may then be considered as isolated, these results can be described by means of single-particle states. If we introduce in $\mathfrak{h}_j^{(1)}$ an arbitrary basis system $f_k^{(j)}$, a complete basis in the total $\mathfrak{h}$ is given by the vectors $\bar{\Phi}^{(N)}_{\substack{jj'\cdots\\kl\cdots}}$. This symbol describes a state which asymptotically at $t = -\infty$ (expressed by the minus sign above Φ) has N particles of types $j, j' \ldots$ which are in the single-particle states $k, l \ldots$ etc.

If in $\Phi^{(N)}$ one particle species occurs several times, we must take the symmetry principle into account.

C_3. *Each particle obeys either Bose- or Fermi statistics.*

$$\bar{\Phi}_{\substack{jj\\kl}} = \begin{cases} +\Phi_{\substack{jj\\lk}} & \text{for Bose particles} \\ -\bar{\Phi}_{\substack{jj\\lk}} & \text{for Fermi particles} \end{cases}$$

The operators $D(L)$ take a simple form in this basis system. Each single particle state $f_k^{(j)}$ entering in $\Phi^{(N)}$ is transformed independently. Thus, within the manifold of asymptotic two-particle states $\bar{\Phi}^{(2)}_{(jj')}$, the effective part of $D(L)$ is just the Kronecker product $D_j(L) \times D_{j'}(L)$, where $D_j(L)$ and $D_{j'}(L)$ are the *irreducible* representations attached to the particle types j and j', respectively. If $j = j'$, we must again take account of the symmetry principle. We indicate the effect of symmetrization by brackets and in this case write $\{D_j(L) \times D_j(L)\}$. The structure of the total representation in $\mathfrak{h}$ may then be written

$$\left.\begin{aligned} D(L) = 1 + \sum_j D_j(L) + \sum_j \{D_j(L) \times D_j(L)\} + \cdots \\ + \sum_{j,j'} D_j(L) \times D_{j'}(L) + \cdots. \end{aligned}\right\} \quad (1)$$

The meaning of the $+$ and Σ signs is that $\mathfrak{h}$ may be decomposed into orthogonal subspaces (namely the asymptotic $0, 1, 2 \ldots$ particle states) in each of which one of the terms is operating. The identical transformation (1) refers to the vacuum state. We have thus already incorporated the further postulate

C_4. *There is one state in $\mathfrak{h}$ which is invariant under all L. T., the vacuum state Φ_0.*

Equation (1) is important because its meaning is not restricted to the special basis system $\bar{\Phi}^{(N)}$ from which we started. The (direct) sum and direct product of representations have an invariant group theoretical meaning. Thus, the particle postulates fix the structure of the representation $D(L)$ completely. To give an example: if we work in the center of mass system (spatial momentum zero), then the energy spectrum must have a characteristic structure: On top of a number of discrete eigenvalues $0, m_j$, we have a continuous spectrum starting at $2\,m_1$ and having a branch point at each value $\Sigma n_j\, m_j$, where the n_j are an arbitrary set of integer numbers. Similar considerations can be made for the angular momentum. Generally speaking, if in a theory the operators P_μ and $M_{\mu\nu}$ of linear and angular momentum (infinitesimal translations and rotations) are given, then (1) affords a mathematically clean-cut criterion as to whether the theory is acceptable on the basis of the postulates C. Since this decision is essentially a problem of group theory, it can be hoped that some more powerful methods of attack will be developed than those available to physicists at present.

It may be instructive to illustrate the argument leading to (1) by an example from wave mechanics. This may make the significance of the basis states $\Phi^{(N)}$ somewhat clearer and also give an exact meaning to statements like "at $t = \infty$ the interaction vanishes". Let us consider a non-relativistic two-body problem without bound states. The Hamiltonian shall be

$$H = -\frac{1}{2\,m}(\nabla_1^2 + \nabla_2^2) + V(|\vec{x}_1 - \vec{x}_2|) = H_0 + V.$$

Ψ is an arbitrary state vector. As always in this paper, the Heisenberg picture will be used. Ψ is represented by a function $\psi(\vec{x}_1, \vec{x}_2)$ which is interpreted as the probability amplitude for the positions $\vec{x}_1, \vec{x}_2$ of the particles at $t = 0$. Now we consider the sequence of unitary operators

$$U(t) = e^{iH_0 t}\, e^{-iHt}. \quad (2)$$

It may be shown that, if V decreases more strongly than $\frac{1}{r}$ with the separation of the particles, then this sequence of operators converges strongly for $|t| \to \infty$ [10,11]. The limit for $t = -\infty$ we call R. From the definition (2) it follows that

$$U(t)\, e^{iHt'}\Psi = e^{iH_0 t'}\, U(t-t')\, \Psi. \tag{3}$$

For $t = -\infty$

$$R e^{iHt'}\Psi = e^{iH_0 t'} R\Psi. \tag{4}$$

If we now introduce a new basis system in such a way that Ψ is represented by the wave function $\varphi(\vec{x}_1, \vec{x}_2)$ which belonged to the state R, Ψ in the old basis, then the (total) Hamiltonian takes the simple form $H = -\frac{1}{2m}(\nabla_1^2 + \nabla_2^2)$. $\varphi(\vec{x}_1, \vec{x}_2)$ is now the probability amplitude for the positions $\vec{x}_1, \vec{x}_2$ at $t = 0$ as it would be calculated from the results of asymptotic observations under the assumption that the particles had moved without interaction until $t = 0$. This new basis system is the analogue to the $\Phi^{(N)}$, apart from the fact that there we have expanded according to a discrete set of eigenfunctions. In the new basis, H is separable into one-particle components, which corresponds to the independent transformation of the $f_k^{(j)}$ in $\Phi^{(N)}$. In this example, H is regarded as the prototype for all Lorentz transformations. The "vanishing of the interaction" at $|t| = \infty$ is to be understood in the sense of a strong operator convergence. This means that the operator of the interaction energy *does not itself vanish*, but that, applied to a state $e^{iHt}\Psi$ (where Ψ is arbitrary, but fixed), it gives an image vector which has zero length in the limit $|t| \to \infty$. These remarks may serve to handle the limits $|t| \to \infty$ in the time-dependent formulation of scattering problems in a way which is both nearer to the physical meaning and simpler in mathematics than the introduction of a convergence factor $e^{-\alpha|t|}$ to "switch off the interaction". One must only keep in mind that all the asymptotic relations are to be regarded in the sense of a strong operator convergence.

[10] This theorem provides the basis for the treatment of scattering problems by Dirac's time-dependent perturbation method or, in more modern language, by means of the interaction representation.

[11] A sequence of operators U_k is said to converge strongly towards a limit U if the application of $(U_k - U)$ on an *arbitrary* but fixed state Ψ produces a sequence of image vectors Ψ_k' which decrease in length towards zero for $k \to \infty$;

$$\lim_{k\to\infty} \| (U_k - U), \Psi \| = 0.$$

§ 2. The *S*-matrix.

The same arguments which have been used in § 1 to define the basis $\bar{\Phi}^{(N)}$ can be carried through if everywhere we put $t = +\infty$ instead of $t = -\infty$. Then we obtain a second complete orthogonal basis $\overset{+}{\Phi}{}^{(N)}$. The unitary operator which connects both of them is the S-matrix

$$\bar{\Phi}_\xi = S\overset{+}{\Phi}_\xi. \tag{5}$$

Here, ξ is the array of indices (identical on both sides of the equation) which characterizes the results of the asymptotic observations. We simply call ξ the "configuration".

$$S_{\xi'\xi} = \langle \overset{+}{\Phi}_{\xi'} | S | \overset{+}{\Phi}_\xi \rangle = \langle \bar{\Phi}_{\xi'} | S | \bar{\Phi}_\xi \rangle \tag{6}$$

is the transition amplitude from the asymptotic configuration ξ at $t = -\infty$ to ξ' at $t = +\infty$. These matrix elements are therefore related in the well-known way to the cross sections for various processes. Because an L. T. changes the configuration at $t = +\infty$ and $t = -\infty$ in the same way, we have

$$[D(L), S] = 0 \qquad \text{for every } L \tag{7}$$

and similarly

$$[T, S] = 0. \tag{7a}$$

In the case of time inversion, we first define the operators $\bar{C}$ and $\overset{+}{C}$ by

$$\bar{C}\bar{\Phi}_\xi = \bar{\Phi}_{\check{\xi}}; \quad \overset{+}{C}\overset{+}{\Phi}_\xi = \overset{+}{\Phi}_{\check{\xi}}, \tag{8}$$

where $\check{\xi}$ is obtained from ξ by applying the single-particle time inversion operators C_j to all the one-particle states $f_k^{(j)}$ entering into the configuration. The actual time inversion operator C must, however, not only change ξ into $\check{\xi}$, but also interchange $t = +\infty$ and $t = -\infty$, i. e.

$$C\bar{\Phi}_\xi = \overset{+}{\Phi}_{\check{\xi}} \tag{9}$$

or

$$CS = \overset{+}{C};\ SC = \overline{C}. \tag{9a}$$

Now

$$C^2 = \overline{C}^2 = \overset{+}{C}{}^2 = \pm 1.$$

Therefore

$$S^{\dagger}\overline{C} = \overline{C}S^{\dagger} \tag{10}$$

and, similarly,

$$S^{\dagger}\overline{J} = \overline{J}S. \tag{10a}$$

This is a general formulation of the principle of detailed balancing[12].

The S-matrix defined by (5) would perhaps better be called the complete S-matrix. In the majority of the work on S-matrix theory, a distinction between elementary particles and bound states is made, and the term S-matrix refers to the submatrix of our S between the scattering states of the elementary particles.

§ 3. The formalism of field theory.

A complete operator system[13] in ζ is given by the creation and destruction operators of the various particles in the different quantum states. They will be called $\bar{u}_k^{(j)}$ and $\bar{u}_k^{(j)\dagger}$, respectively, if they refer to the asymptotic configuration at $t = -\infty$. $\bar{u}_k^{(j)\dagger}$ acts on a basis vector by adding to the configuration ξ one particle of type j in the state k:

$$\bar{u}_k^{(j)\dagger}\overline{\Phi}_{\xi} = \lambda\overline{\Phi}_{\xi + \binom{j}{k}}. \tag{11}$$

[12] It is interesting to see at what point of the argument leading to (10) the requirement of invariance under time inversion is used. $\overline{C}$ and $\overset{+}{C}$ may always be defined in a Lorentz invariant theory and they have the correct structure relations. The definition (9a) is adapted to give the correct physical significance to C. Thus, if the theory is invariant under time inversion, C must also have the correct structure relations, and vice versa. Now the relations with $D(L)$ are correct, because S commutes with all $D(L)$. However, $C^2 = \pm 1$ is not obvious and this imposes some restrictions on S (detailed balancing).

[13] "Complete" means that every operator in ζ may be approximated by polynomials in these basic operators.

The corresponding destruction operator is the hermitian conjugate. It is convenient to fix the factor λ so that the operators obey the canonical commutation relations[14]

$$\left.\begin{aligned} [\bar{u}_k^{(j)}\,\bar{u}_l^{(j')}] &= [\bar{u}_k^{(j)\dagger}\,\bar{u}_l^{(j')\dagger}] = 0 \\ [\bar{u}_k^{(j)}\,\bar{u}_l^{(j')\dagger}] &= \delta_{kl}\delta_{jj'} \end{aligned}\right\}\ \text{for Bose particles,} \tag{12}$$

$$\left.\begin{aligned} \{\bar{u}_k^{(j)}\,\bar{u}_l^{(j')}\} &= \{\bar{u}_k^{(j)\dagger}\,\bar{u}_l^{(j')\dagger}\} = 0 \\ \{\bar{u}_k^{(j)}\,u_l^{(j')\dagger}\} &= \delta_{kl}\delta_{jj'}. \end{aligned}\right\}\ \text{for Fermi particles.} \tag{13}$$

If we were dealing with a system with a finite number of degrees of freedom these commutation relations alone would fix the operators $u_k^{(j)}, u_k^{(j)\dagger}$ uniquely up to a unitary transformation. In the following chapter it will be discussed that this is no longer true in our case. In order to achieve a unique specification (apart from equivalence) one must require, in addition to (12) or (13), that there exists one state (the vacuum) for which

$$\bar{u}_k^{(j)}\Phi_0 = 0 \tag{14}$$

for all k and j.

In mathematical language: $\mathfrak{h}$ is the space of an irreducible representation of the commutator ring (12), (13), with the auxiliary condition (14); irreducible, because the operators in question form a complete system.

Again we can replace the minus signs above Φ and $u^{\dagger}$ in (11) by plus signs and obtain a second set of creation and destruction operators referring now to the asymptotic configurations at $t = +\infty$. The two sets are of course connected by the S-matrix.

The creation and destruction operators $\bar{u}_k^{(j)\dagger}, \bar{u}_k^{(j)}$ can be combined in the well-known way to a continuous manifold of operators $\bar{\varphi}^{(j)}(x)$ (see, for instance, Chapter III, § 1). These operator fields are the "incoming fields" of Yang-Feldman and Källén[15]; similarly, the $\overset{+}{\varphi}{}^{(j)}(x)$ are the "outgoing fields". The essential

[14] The deeper significance of this choice is that the creation operators in different basis systems of one-particle states are simply related. Thus, if $f_k = \Sigma C_{k\alpha} f_{\alpha}$, then $u_k^{\dagger} = \Sigma C_{k\alpha} u_{\alpha}^{\dagger}$.

[15] C. N. Yang and D. Feldman, Phys. Rev. **79**, 972 (1950); G. Källén, Ark. f. Fys. **2**, 187 (1951).

feature of field theory is that these operators $\overset{-}{\varphi}{}^{(j)}(x)$ and $\overset{+}{\varphi}{}^{(j)}(x)$, which have a simple physical interpretation in terms of asymptotic observations, are regarded as the asymptotic limits for $t = \pm\infty$ of other operators, the "actual fields" $\psi^{(j)}(x)$. In symbols

$$\psi^{(j)}(x) \to \overset{\pm}{\varphi}_j(x) \quad \text{for} \quad t \to \pm\infty, \tag{15}$$

where the arrow again indicates strong operator convergence. These $\psi^j(x)$ are regarded as the basic quantities of the theory. They are defined by their commutation relations (at least in theories of conventional type), and the infinitesimal Lorentz operators P_μ and $M_{\mu\nu}$ are given as functions of them. This procedure is in complete analogy to that in quantum mechanics. There is only one additional requirement in field theory, namely, that the $\psi^j(x)$ should have the simple relativistic transformation properties of a field. The effect is to fix the $M_{\mu\nu}$ as soon as the P_μ are given. This restriction is, however, not very serious in itself and in fact it will be explicitly demonstrated later that theories exist which are built on the concept of covariant operator fields $\psi_j(x)$ satisfying the "asymptotic condition" (15) and which are able to yield any S-matrix. The central question is then by what relations the $\psi_j(x)$ may be defined in the basic equations of the theory. Is it possible, for instance, to require that the field operators at points separated by a space-like distance commute? Although this mathematical question will not be decided in this paper, we can at least exhibit some of the pitfalls which tend to make rather inconclusive many general statements reached by the standard methods.

In the scheme of field theory outlined above we have one field for every existing particle species. The extension to the case in which one introduces a smaller number of fields and thereby a distinction between elementary and compound particles is of course more interesting for practical applications (meson theory), but will not be considered here.

II. Mathematical Considerations.

§ 1. Inequivalent representations of the canonical commutator ring.

This section refers both to the commutation relations (12) and the anticommutation relations (13); nevertheless the discussion will be restricted to the former. Combining the indices k and j to a single index k, the relations are brought into the canonical form by substituting

$$q_k = \frac{1}{\sqrt{2}}(u_k + u_k^\dagger), \quad p_k = \frac{1}{i\sqrt{2}}(u_k - u_k^\dagger); \tag{16}$$

q_k and p_k are then hermitian operators satisfying

$$[q_k q_l] = [p_k p_l] = 0, \quad [p_k q_l] = -i\delta_{kl}. \tag{17}$$

In the conventional field theories we have commutation relations of the form

$$\left.\begin{aligned} [\psi(\vec{x})\,\psi(\vec{x}')] &= [\pi(\vec{x})\,\pi(\vec{x}')] = 0, \\ [\pi(\vec{x})\,\psi(\vec{x}')] &= -i\delta(\vec{x}-\vec{x}'). \end{aligned}\right\} \tag{18}$$

These are reduced to (17), for instance, by putting

$$q_k = \int f_k(\vec{x})\,\psi(\vec{x})\,d\vec{x}, \quad p_k = \int f_k(\vec{x})\,\pi(\vec{x})\,d\vec{x}, \tag{19}$$

where the $f_k(\vec{x})$ are an arbitrary complete system of real, orthogonal, normalized functions.

It is a well known fact that for a *finite number* N of degrees of freedom there is only one (irreducible) representation of the operators p, q of (17) (apart from equivalences). This may be obtained by considering the q_k as multiplication operators, the p_k as differentiation operators. It is more convenient for our purpose to work with the u_k, $u_k^\dagger$ and the "occupation numbers"

$$\nu_k = u_k^\dagger u_k \tag{20}$$

which have integer eigenvalues ranging from 0 to ∞. A complete orthogonal system is given by the simultaneous eigenfunctions of the ν_k. Each basis vector is then specified by an array of N occupation numbers $(\nu_1 \cdots \nu_N)$. u_k acts as a destruction operator, $u_k^\dagger$ as creation operator for the k^{th} "oscillator", i. e. they decrease or increase ν_k by 1. The "vacuum state" $\Phi_0 = (0,0 \cdots 0)$ may be defined abstractly by

$$u_k \Phi_0 = 0 \qquad \text{for all } k. \tag{21}$$

If we pass now to the limit $N \to \infty$ one new feature appears. A possible basis vector results from any distribution of integer numbers ν_k over the infinitely many oscillators. The "number" of these possibilities is *no longer countable*. It is given by $\aleph_0^{\aleph_0} = \aleph_1$ ($\aleph_0$ representing a countable infinite set, $\aleph_1$ the continuum). Thus, the straightforward extension of the method used for finite N leads to a vector space $\tilde{\mathfrak{h}}$ with a continuum of orthonormal *basis* vectors. This is no longer a Hilbert space in the ordinary sense of the word, though the term "non-separable Hilbert space" is used for it in mathematics.

In this connection, we must remember that, for the description of the physical situation, there is no need for such a large space. It is also well known that even for $N = \infty$ there is a representation of (12) within an ordinary Hilbert space. In fact we have used this representation in the previous discussion. It follows from the assumption that there is one vacuum state satisfying (14). Starting from this assumption, the argument can be carried through in essentially the same way as for finite N. The point is, however, that, for infinite N, (14) is no longer a consequence of (12). In other words, there will be different irreducible representations of (12).

One might perhaps be tempted to think that the ambiguity left by the relations (12) or (17) is a matter of mathematical sophistication without relevance to field theory. However, the following examples will show that, starting from the "standard representation" of (12) in $\mathfrak{h}$, we obtain, by the very simplest substitutions, operators in the same space which belong to inequivalent representations. If one disregards the inequivalence and tries to calculate a unitary matrix which connects the two operator systems, one obtains infinite results.

Let us start from operators u_k, $u_k^\dagger$ obeying (12) and (14),

i. e. belonging to the "standard representation" (12). Then, introduce the linear combinations

$$\left.\begin{aligned} v_k &= \cosh \varepsilon \cdot u_k + \sinh \varepsilon \cdot u_k^\dagger, \\ v_k^\dagger &= \sinh \varepsilon \cdot u_k + \cosh \varepsilon \cdot u_k. \end{aligned}\right\} \tag{22}$$

The commutation relations are unchanged. If we take ε to be infinitesimal, we have

$$\left.\begin{aligned} \delta u_k &= v_k - u_k = \varepsilon u_k^\dagger, \\ \delta u_k^\dagger &= v_k^\dagger - u_k^\dagger = \varepsilon u_k. \end{aligned}\right\} \tag{23}$$

Writing this formally as an infinitesimal unitary transformation, the generating operator is

$$T = \frac{i}{2} \Sigma (u_k^\dagger u_k^\dagger - u_k u_k), \tag{24}$$

i. e.

$$i\varepsilon [T u_k] = \varepsilon u_k^\dagger = \delta u_k, \quad i\varepsilon [T u_k^\dagger] = \varepsilon u_k = \delta u_k^\dagger.$$

It is easily seen that T is not a proper operator, but transforms every vector of $\mathfrak{h}$ into one with infinite length. However, we can follow the matter a little further by going over to the non-separable space $\tilde{\mathfrak{h}}$ in which a vector is represented by a function $\psi(q_1, q_2 \ldots)$ of infinitely many variables and

$$u_k = \frac{1}{\sqrt{2}}\left(q_k + \frac{\partial}{\partial q_k}\right); \quad u_k^\dagger = \frac{1}{\sqrt{2}}\left(q_k - \frac{\partial}{\partial q_k}\right).$$

One can then work out the effect of the operation $e^{i\varepsilon T}$ and try to recognize its implications for the "physically interesting states" which are a subset of $\tilde{\mathfrak{h}}$, namely the Hilbert space $\mathfrak{h}$ generated by the $u_k^\dagger$ from Φ_0. The result is that $e^{i\varepsilon T}$ transforms every vector of $\mathfrak{h}$ into one which has a zero scalar product with any second vector from $\mathfrak{h}$.

$$(\Phi_2, e^{i\varepsilon T}\Phi_1) = 0 \text{ for every } \Phi_1, \Phi_2 \text{ from } \mathfrak{h}. \tag{25}$$

Nevertheless, $\Phi_1' = e^{i\varepsilon T}\Phi_1$ is a vector with finite norm in $\tilde{\mathfrak{h}}$. In order to understand how this may happen we consider the case of a very large but finite N. Then there is no distinction between $\tilde{\mathfrak{h}}$ and $\mathfrak{h}$. The expressions (25) are slightly different from zero, and of course $\| \Phi_1' \| = \| \Phi_1 \|$. As N becomes larger, the projection of Φ_1' into the subspace belonging to a total "particle number" $\nu = \Sigma \nu_k < n$ (n fixed) becomes smaller and is compensated by a growth of the total probability for $\nu < n$. In this way Φ_1' moves out of $\mathfrak{h}$ in the limit, because there are no states in $\mathfrak{h}$ belonging to an "actually" infinite particle number, though the particle number may be arbitrarily large.

To sum up: $v_k, v_k^\dagger$ are proper operators in the ordinary Hilbert space $\mathfrak{h}$ (according to (22)), obeying the same commutation relations as $u_k, u_k^\dagger$, but there is no proper unitary transformation connecting the two operator systems, i. e. these belong to inequivalent representations of (12). For the representation defined by the $v_k, v_k^\dagger$, there is no "vacuum state" satisfying

$$v_k \Phi_0' = 0.$$

A similar example which is closer to practical calculations in field theory is the following:

We take two free fields which obey the field equations

$$\left.\begin{aligned} (\Box - m_1^2)\, \psi_1(x) &= 0, \\ (\Box - m_2^2)\, \psi_2(x) &= 0, \end{aligned}\right\} \quad (26)$$

and which coincide (including their first time derivatives) for $t = 0$;

$$\left.\begin{aligned} \psi_1(\vec{x}, 0) &= \psi_2(\vec{x}, 0) = \psi(\vec{x}, 0), \\ \dot{\psi}_1(\vec{x}, 0) &= \dot{\psi}_2(\vec{x}, 0) = \pi(\vec{x}, 0), \end{aligned}\right\} \quad (27)$$

and define in the usual way for each field a splitting into a creation and an annihilation part, for instance for ψ_1

$$\left.\begin{aligned} u_1(\vec{p}) &= (2\pi)^{-\frac{3}{2}} \int (E_1 \psi(\vec{x}) + i\pi(\vec{x}))\, e^{i\vec{p}\vec{x}} d\vec{x}, \\ u_1^\dagger(\vec{p}) &= (2\pi)^{-\frac{3}{2}} \int (E_1 \psi(\vec{x}) - i\pi(\vec{x}))\, e^{i\vec{p}\vec{x}} d\vec{x}, \\ E_1 &= \sqrt{\vec{p}^2 + m_1^2}. \end{aligned}\right\} \quad (28)$$

Then the operators $u_1(\vec{p}), u_1^\dagger(\vec{p})$ are connected to the $u_2(\vec{p}), u_2^\dagger(\vec{p})$ by a transformation like (22), namely

$$u_2(\vec{p}) = \frac{E_1 + E_2}{2E_1} u_1(\vec{p}) + \frac{E_1 - E_2}{2E_1} u_1^\dagger(\vec{p}). \quad (29)$$

Now, if there is any state for which

$$u_1(\vec{p}),\, \Phi_0 = 0 \qquad \text{for all } \vec{p},$$

then there is no Φ_0' which satisfies

$$u_2(\vec{p}),\, \Phi_0' = 0 \qquad \text{for all } \vec{p},$$

and vice versa. This may show that the "strange representations" of (12) will almost inevitably turn up in any discussion in field theory.

We have already mentioned that similar considerations apply to Fermi particles. The effect of the Pauli principle is to "reduce" the number of basis vectors in $\tilde{\mathfrak{h}}$ to $2^{\aleph_0}$, which is still the continuum.

The existence of different representations of (12), (13) was discovered some time ago[16], but has not entered into the consciousness of physicists until very recently[17]. A systematic study and classification has been made by WIGHTMAN and GÅRDING[18].

§ 2. Functions of the field operators.

After the basic operators have been defined by commutation relations of the form (18), the conventional field theories proceed to give the Lorentz operators $P_\mu, M_{\mu\nu}$ (energy-momentum, angular momentum) as functions of the basic operators. As we have seen, the definition (18) is not complete, but we suppose now that it has been augmented by some auxiliary condition which fixes the representation. Then we meet with a second

[16] J. v. NEUMANN, Composition Math. 6, 1 (1938); K. O. FRIEDRICHS, Math. Aspects of the Quantum Theory of Fields, Interscience Publishers, New York 1953, Chapter on "Myriotic Fields".
[17] VAN HOVE, Physica 18, 145 (1952); WIGHTMAN and SCHWEBER, Phys Rev., in print. I am indebted to Prof. WIGHTMAN for a preprint of this paper.
[18] GÅRDING and WIGHTMAN, Proc. Nat. Acad. Sci. 40, 617 (1954).

problem. Almost all simple-looking formal expressions in the $\psi(\vec{x})$, $\pi(\vec{x})$, which we may think of writing down, are actually not proper operators. In most cases, they will even have infinite matrix elements between any two states of $\mathfrak{h}$. What one would like to have, then, is a simple criterion for the class of "sensible functions" of the field operators, allowing to decide immediately whether an expression is acceptable or not. We shall illustrate the problem for the "standard representation" (12) with which we have to deal in the case of the asymptotic fields $\overset{=}{\varphi}(x)$. Here, a criterion which satisfies practical purposes can indeed be given easily. The method is well known. Nevertheless, some of the arguments may be recalled. The points I want to emphasize are: 1) the characterization of the class of sensible functions of the operators (18) depends only on the type of representation for these operators, not on their physical meaning; 2) the task is solved for the "standard representation" and may be extended to others as soon as their relation to the "standard representation" is known. This is, for instance, the case for the two examples given in the preceding section.

Let us start from the equations (16)—(19) and assume (21)[19]. The inversion of (19) is

$$\psi(\vec{x}) = \Sigma q_k f_k(\vec{x}), \quad \pi(\vec{x}) = \Sigma p_k f_k(\vec{x}). \tag{30}$$

Now it is clear that all polynomials in the p_k, q_k (or u_k, $u_k^\dagger$), which involve a finite number of additions and multiplications, are well defined operators. We can apply them, for instance, to any basis vector and obtain again a normalizable state. This is not true for the continuous manifold of operators $\psi(\vec{x})$ which are infinite sums of the u_k, $u_k^\dagger$. In fact one checks easily that $\psi(\vec{x})$ transforms every basis vector $\Phi_{kl\ldots}^{(N)}$ into a vector with infinite length. However, $\psi(\vec{x})$ has at least finite matrix elements, and expressions like $\int f(\vec{x})\,\psi(\vec{x})\,d\vec{x}$ are well defined operators if f is square integrable. Hence, $\psi(\vec{x})$ may be regarded as an improper operator in the same way in which one can regard eigenfunctions in the continuous spectrum as improper state vectors. The situation is worse for $\psi^2(\vec{x})$ which has only infinite matrix elements. The remedy may here be found in the following observation. If we have a power series in the u_k, $u_k^\dagger$ *in which each term is arranged in S-product order*[20] (i. e. all destruction operators stand to the right), then only a finite number of terms contribute to a matrix element between two of the basis vectors $\Phi_{kl\ldots}^{(N)}$. Therefore, any expression which is in S-product order has at least finite matrix elements and may then be regarded in general as an improper operator in a similar way to $\psi(\vec{x})$. The simple way of putting two dots around an expression is a safeguard against infinite matrix elements. If X is supposed to be a proper operator, then $X^\dagger X$ must have finite matrix elements, and vice versa. Thus one has the simple criterion:

If, in the process of rearranging an expression $X(\psi(\vec{x}), \pi(\vec{x}))$ in S-product order, no explicit infinities occur (i. e. if the contractions are finite), X has finite matrix elements. If the same is true for $X^\dagger X$, then X is a proper operator.

The criterion can be extended to other "discrete" representations[21] as they can be related to the standard representation by substitutions.

[19] If we identify $\psi(\vec{x})$, $\pi(\vec{x})$ with $\overline{\varphi}(\vec{x})$, $\dot{\overline{\varphi}}(\vec{x})$, respectively, the equations (16) and (19) should be replaced by the somewhat more complicated ones which give the splitting into creation and destruction parts (equ. (28)).

III. Applications.

In the following chapters, the specialization to the case mentioned in the introduction, in which we have to deal with only one type of particle, will be made. For convenience, a short description of the formal apparatus is given first.

§ 1. Notations.

The manifold of states of a single spinless particle of mass m is most easily described in momentum space. An arbitrary state is then represented by a function $f(p)$ (p the momentum 4-vector) which needs to be defined only for those values of p corre-

[20] We use the notations of G. C. Wick, Phys. Rev. 80, 268. The S-product between A and B is indicated by double dots: $:AB:$.

[21] For the definition of discrete representation, see the papers mentioned in ref. 18.

sponding to a possible momentum vector of the particle, i. e. for p lying on the positive-energy shell of the hyperboloid $p^2 + m^2 = 0$. However, it is more convenient to regard $f(p)$ as defined in the whole *positive* cone and reject the irrelevant points of momentum space by a factor $\delta(p^2 + m^2)$ which must appear in all relations of physical significance. Thus the scalar product of two wave functions f_1 and f_2 is defined by

$$(f_2 f_1) = \int f_2^*(p) f_1(p) \delta(p^2 + m^2) dp, \tag{31}$$

where it must always be kept in mind that the integration is essentially one over the positive-energy cone of p only. The transformation properties of the wave functions are

$$D(b) f = f': \quad f'(p) = e^{ipb} f(p) \quad \text{(translations)}.$$
$$D(\Lambda) f = f'': \quad f''(p) = f(\Lambda^{-1}, p) \quad \text{(homogeneous L. T.)}.$$

A description in ordinary space coordinates is obtained by the Fourier transformation

$$f(x) = (2\pi)^{-\frac{3}{2}} \int f(p) e^{ipx} \delta(p^2 + m^2) dp. \tag{32}$$

This function satisfies the Klein-Gordon equation

$$(\Box - m^2) f(x) = 0, \tag{33}$$

but contains only positive-energy Fourier components so that the initial condition $f(\vec{x}, 0)$ is sufficient to determine $f(\vec{x}, t)$ for all times. The scalar product in this formulation has the more complicated form

$$(f_2, f_1) = 2i \int f_2^*(\vec{x}, 0) \frac{\partial f_1}{\partial t}(\vec{x}, 0) d\vec{x}. \tag{34}$$

Therefore, $f(\vec{x})$ may not be directly interpreted as the probability amplitude for the position $\vec{x}$.

Let $\bar{u}_k$ be the destruction operator for the state f_k (wave function $f_k(p)$) in the asymptotic configuration at $t = -\infty$ (see Chapter I). Then it is convenient to define the continuous manifold of destruction operators $\bar{u}(p)$ by

$$\bar{u}_k = \int \bar{u}(p) f_k^*(p) \delta(p^2 + m^2) dp, \tag{35 a}$$

or conversely

$$\bar{u}(p) = \Sigma f_k(p) \bar{u}_k, \tag{35 b}$$

where it has been assumed that the $f_k(p)$ form a complete system, orthogonal and normalized according to (31). The creation operators $u^\dagger(p)$ are the hermitian adjoints and the relations (35) apply of course equally to the operators $\overset{+}{u}_k$, $\overset{+}{u}(p)$ for the asymptotic configuration at $t = +\infty$.

The commutation relations are most conveniently expressed in the symbolic form

$$\left.\begin{aligned}&[\bar{u}(p) \bar{u}^\dagger(p')] \delta(p^2 + m^2) \delta(p'^2 + m^2) \\ &\quad = \delta^4(p - p') \delta(p^2 + m^2).\end{aligned}\right\} \tag{36}$$

The transformation properties are:

$$\left.\begin{aligned}&D(b) \bar{u}(p) D^\dagger(b) = e^{-ipb} \bar{u}(p) \quad \text{(translation)}, \\ &D(\Lambda) \bar{u}(p) D^\dagger(\Lambda) = \bar{u}(\Lambda, p) \quad \text{(homogeneous L. T.)}.\end{aligned}\right\} \tag{37}$$

In analogy to (32) we define

$$\bar{u}(x) = (2\pi)^{-\frac{3}{2}} \int \bar{u}(p) e^{ipx} \delta(p^2 + m^2) dp. \tag{38}$$

The creation and destruction operators are combined to the "incoming field"

$$\bar{\varphi}(x) = \bar{u}(x) + \bar{u}^\dagger(x). \tag{39}$$

§ 2. The existence of general field theories.

As the $\bar{u}(p)$, $\bar{u}^\dagger(p)$ form a complete operator system, an arbitrary operator may be expanded in a series of S-products of the $\bar{u}(p)$, $\bar{u}^\dagger(p)$,

$$X = \Sigma X^{nm}, \tag{40}$$

where X^{nm} contains n creation and m destruction operators,

$$X^{nm} = \int F^{nm}(p';p)\,\bar{u}^{\dagger}(p')\,\bar{u}(p)\,\delta(p'^2+m^2)\,\delta(p^2+m^2)\,dp'\,dp. \qquad (40a)$$

Here p' is an abbreviation of the n arguments $p_1'\cdots p_n'$; similarly p stands for $p_1\cdots p_m$, $\bar{u}(p)$ for the product of the m destruction factors, etc. The integration is extended over the positive-energy cones only. In order that X be invariant under homogeneous L. T., each F^{nm} must be a Lorentz invariant function of its arguments. Invariance of X under translations means that F^{nm} contains a factor $\delta(\Sigma p' - \Sigma p)$. The S-matrix must fulfil both conditions, the operator $\psi(0)$ ("actual field" at the origin) the first.

$$\psi^{nm}(0) = (2\pi)^{-\frac{3}{2}}\int f^{nm}(p';p)\,\bar{u}^{\dagger}(p')\,\bar{u}(p)\,\delta(p'^2+m^2)\,\delta(p^2+m^2)\,dp'\,dp. \qquad (41)$$

f^{nm} is an invariant function of n vectors p' and m vectors p, symmetric against any permutation of the variables on either side of the semicolon. Also

$$f^{nm*}(p';p) = f^{mn}(p;p'). \qquad (42)$$

From (41) we get $\psi(x)$, replacing f by $fe^{i(\Sigma p-\Sigma p')x}$, and

$$\left.\begin{aligned} \frac{1}{2E_P}\psi^{nm}(\vec{P},t) &\equiv (2\pi)^{-\frac{3}{2}}\int \psi^{nm}(\vec{x},t)\,e^{-i\vec{P}\vec{x}}\,d\vec{x} \\ = \int f^{nm}(p';p)\,\delta(\Sigma\vec{p}-\Sigma\vec{p}'-\vec{P})\,&e^{-i(\Sigma E-\Sigma E')t}\,\bar{u}^{\dagger}(p')\,\bar{u}(p)\,\delta(p'^2+m^2) \\ &\times\delta(p^2+m^2)\,dp'\,dp. \end{aligned}\right\} \qquad (43)$$

The letters E mean the energies belonging to the respective momentum vectors, e. g. $E_P = \sqrt{\vec{P}^2+m^2}$. To fulfil the asymptotic condition (15) we must choose f so that

$$\lim_{t\to-\infty}\psi^{nm}(\vec{P},t)\to 0 \qquad \text{(except for } \psi^{01} \text{ and } \psi^{10}\text{)},$$

$$\lim_{t\to+\infty}\psi^{nm}(\vec{P},t)\neq 0 \qquad \left\{\begin{array}{l}\text{(for at least some index pairs } n, m \text{ in} \\ \text{order to obtain an } S\text{-matrix different} \\ \text{from the identity).}\end{array}\right.$$

This can only be so if f contains a factor $\delta_+(\Sigma E - \Sigma E' - \alpha)$ which in turn can only arise from one of the invariant functions

$$h_1 = \delta_+\left[(\Sigma p - \Sigma p')^2 + m^2\right]$$

or

$$h_2 = \delta_-\left[(\Sigma p - \Sigma p')^2 + m^2\right].$$

We write

$$f = g_1 h_1 + g_2 h_2. \qquad (44)$$

This also gives a separation of ψ into two terms ψ_1 and ψ_2. Taking into account the δ-factor in the spatial momentum components, we can write

$$h_1 = \frac{1}{2E_P}\left[\delta_+(E_P+\varepsilon) + \delta_+(E_P-\varepsilon)\right],$$

where

$$\varepsilon = \Sigma E - \Sigma E'.$$

In the limits $|t|\to\infty$ the factors $he^{-i\varepsilon t}$ will become

	$h_1 e^{-i\varepsilon t}$	$h_2 e^{-i\varepsilon t}$
$t\to-\infty$	$\frac{1}{2E_P}e^{iE_Pt}\delta(E_P+\varepsilon)$	$\frac{1}{2E_P}e^{-iE_Pt}\delta(E_P-\varepsilon)$
$t\to+\infty$	$\frac{1}{2E_P}e^{-iE_Pt}\delta(E_P-\varepsilon)$	$\frac{1}{2E_P}e^{iE_Pt}\delta(E_P+\varepsilon)$

To satisfy the asymptotic condition at $t=-\infty$ the function g_1 must vanish in a region around $\varepsilon = -E_P$ or, in other words, the variability domain of the 4-vector

$$q = \Sigma p - \Sigma p' \qquad (45)$$

must be so restricted by g_1 that in the backward cone values giving $q^2+m^2=0$ are excluded. Similarly, g_2 must exclude values $q^2+m^2=0$ in the forward cone. If this is satisfied, then

$$\left.\begin{aligned} \overset{+}{u}(P) &= \sum_{n,m}\int g_1^{nm}(p';p)\,\bar{u}^{\dagger}(p')\,\bar{u}(p)\,\delta(\Sigma p-\Sigma p'-P) \\ &\qquad\delta(p'^2+m^2)\,\delta(p^2+m^2)\,dp'\,dp \\ \overset{+}{u}{}^{\dagger}(P) &= \sum_{n,m}\int g_2^{nm}(p';p)\,\bar{u}^{\dagger}(p')\,\bar{u}(p)\,\delta(\Sigma p-\Sigma p'+P) \\ &\qquad\delta(p'^2+m^2)\,\delta(p^2+m^2)\,dp'\,dp. \end{aligned}\right\} \qquad (46)$$

As g_1 and g_2 are arbitrary for the relevant values of their arguments it may be expected that the frame is wide enough to give any S-matrix. Indeed this can be checked easily in the limit of very weak interaction.

§ 3. Commutation relations of the field variables.

The representation of the field as

$$\left.\begin{aligned} \psi(x) = (2\pi)^{-\frac{3}{2}} \Sigma \int f^{nm}(p'; p)\, e^{i(\Sigma p - \Sigma p')x}\, \bar{u}^{\dagger}(p')\, \bar{u}(p) \\ \delta(p'^2 + m^2)\, \delta(p^2 + m^2)\, dp'\, dp \end{aligned}\right\} \quad (47)$$

and the knowledge we have obtained about the structure of the f^{nm} allow a few statements about the commutation relations. If we form the product $\psi(x')\,\psi(x)$ and rearrange it in S-product order we get

$$\left.\begin{aligned} \psi(x')\,\psi(x) = (2\pi)^{-3}\, \Sigma s! \binom{m'}{s}\binom{n}{s} \int d\bar{p}'\, d\bar{p}\, dp'\, dp \\ \overbrace{\bar{u}^{\dagger}(\bar{p}')}^{n'}\, \overbrace{\bar{u}^{\dagger}(p')}^{n-s}\, \overbrace{\bar{u}(\bar{p})}^{m'-s}\, \overbrace{\bar{u}(p)}^{m} \times e^{i(\Sigma\bar{p} - \Sigma\bar{p}')x' + i(\Sigma p - \Sigma p')x} \\ \int f^{n'm'}(\bar{p}'; \bar{p}, q)\, f^{nm}(p', q; p)\, e^{i\Sigma q(x'-x)}\, \delta(\varkappa^2)\, dq, \end{aligned}\right\} \quad (48)$$

where $\delta(\varkappa^2)$ stands for the product of the $\delta(p^2 + m^2)$ for all the momentum vectors which appear. There are s momentum vectors q in the last integral. The commutator $[\psi(x')\,\psi(x)]$ is then obtained by subtracting the same term with x and x' interchanged. By virtue of the asymptotic condition we must have

$$f^{10}(q) = f^{01}(q) = 1. \quad (49)$$

We first look at the *vacuum expectation value* of (48). Here only the terms with $m = 0;\ m' = s = n;\ n' = 0$ contribute.

$$\left.\begin{aligned} \langle 0 \,|\, \psi(x')\,\psi(x) \,|\, 0 \rangle \\ = (2\pi)^{-3}\, \Sigma s! \int |\, f^{s0}(q)\,|^2\, e^{i(\Sigma q)(x'-x)}\, \delta(q^2 + m^2)\, dq. \end{aligned}\right\} \quad (50)$$

Because of the relations

$$i\Delta^{+}(\xi; a) = (2\pi)^{-3} \int_{Q_0 > 0} e^{iQ\xi}\, \delta(Q^2 + a^2)\, dQ, \quad (51)$$

$$\Delta^{+}(\xi; a) - \Delta^{+}(-\xi; a) = \Delta(\xi; a) \quad (52)$$

we can write

$$\left.\begin{aligned} -i \langle 0 \,|\, [\psi(x')\,\psi(x)] \,|\, 0 \rangle \\ = \Delta(x' - x; m) + \sum_{s=2}^{\infty} \int F^{(s)}(a)\, \Delta(x' - x; a)\, da^2 \end{aligned}\right\} \quad (53)$$

with

$$F^{(s)}\left(\sqrt{-Q^2}\right) = s! \int |\, f^{s0}(q)\,|^2\, \delta(\Sigma q - Q)\, \delta(q^2 + m^2)\, dq. \quad (54)$$

For equal times ($t' = t$)

$$\left.\begin{aligned} \langle 0 \,|\, [\psi(x')\,\psi(x)] \,|\, 0 \rangle &= 0 \\ \langle 0 \,|\, [\dot{\psi}(x')\,\psi(x)] \,|\, 0 \rangle &= -i\delta(\vec{x} - \vec{x}')\left[1 + \sum_{2}^{\infty} \int F^{(s)}(a)\, da^2\right] \\ &= -i\,\delta(\vec{x} - \vec{x}').\, C. \end{aligned}\right\} \quad (55)$$

$F^{(s)}(a)$ is a positive, real function and hence $C > 1$, unless all f^{n0} vanish for $n \neq 1$.[22]

In a conventional theory (in which $[\psi(x)\,\psi(x')] = 0$ for equal times) the latter alternative may be excluded. One can, for instance, calculate the matrix element of the commutator $[\psi(x')\,\psi(x)]$ between the vacuum and a two-particle state. Under the assumption $f^{n0} = 0$ for $n \neq 1$, this gives

$$\begin{aligned} &\langle p_1, p_2 \,|\, [\psi(x')\,\psi(x)] \,|\, 0 \rangle \\ &= e^{-i(p_1 + p_2)x'} \int f^{21}(p_1, p_2; q)\, e^{iq(x'-x)}\, \delta(q^2 + m^2)\, dq \\ &\quad - e^{-i(p_1 + p_2)x} \int f^{21}(p_1, p_2; q)\, e^{-iq(x'-x)}\, \delta(q^2 + m^2)\, dq. \end{aligned}$$

This expression should then vanish for arbitrary $p_1, p_2, \vec{x}'$ and $\vec{x}$ as soon as $t' = t$. This can only be true if f^{21} is identically zero and this, in turn, would mean that there is no elastic scattering.

Thus, in a conventional theory, it is necessary that f^{n0} differs from zero at least for some value of $n \neq 1$. According to (55), this then implies that $\dot{\psi}(x)$ cannot be identified with the canon-

[22] The formulae (53)—(55) have been given previously by H. LEHMANN, Nuovo Cimento **11**, 342 (1954). The derivation here is essentially LEHMANN's.

ically conjugate momentum of the field, though one could perhaps have[23]

$$\dot{\psi}(\vec{x}) = C\,\pi(\vec{x}) \quad \text{with} \quad C > 1. \tag{56}$$

§ 4. Conventional theory.

We suppose the theory to be defined in terms of operators $\psi(\vec{x}), \pi(\vec{x})$ obeying (18). The commutation relations are the same as those for the $\overline{\varphi}(\vec{x}), \dot{\overline{\varphi}}(\vec{x})$ and it appears therefore natural to ask whether a unitary operator R exists which transforms the one set into the other.

$$\psi(\vec{x}) = R\,\overline{\varphi}(\vec{x})\,R^{\dagger}, \quad \pi(\vec{x}) = R\,\dot{\overline{\varphi}}(\vec{x})\,R^{\dagger}. \tag{57}$$

In other words: can $\psi(\vec{x}), \pi(\vec{x})$ belong to the same representation of the canonical commutator ring as $\overline{\varphi}(\vec{x}), \dot{\overline{\varphi}}(\vec{x})$? This is usually assumed and, in fact, $R^{\dagger}$ is Dyson's matrix $U(0; -\infty)$.

The canonical momenta $\pi(\vec{x})$ need not have simple relativistic transformation properties. But, with respect to translations in space, we shall also require

$$D(\vec{b})\,\pi(\vec{x})\,D^{\dagger}(\vec{b}) = \pi(\vec{x} - \vec{b}). \tag{58}$$

(58) holds in all customary theories (for instance if $\pi(\vec{x})$ is proportional to $\dot{\psi}(\vec{x})$) and, indeed, any other assumption but (58) would appear extremely unnatural. From (58) and the analogous equation for $\psi(\vec{x})$ it follows that R must commute with the space translations

$$[R\,\vec{P}] = 0. \tag{59}$$

This is also a well known fact in conventional theories (conservation of spatial momentum in all virtual processes). Now the spectrum of $\vec{P}$ is continuous, apart from a single discrete eigenvalue 0 which belongs to the vacuum state Φ_0. If we apply (59) to Φ_0 we obtain

$$\vec{P} R\,\Phi_0 = \vec{P}\Phi_0' = 0. \tag{60}$$

If R is a unitary operator, then Φ_0' is again a normalized state and (60) indicates that

[23] If one drops the factor C in (55), (56), one has to take it up in the asymptotic condition. This is only a different way of expressing the same thing. Cf. ref. 21 and G. Källén, Helv. Phys. Acta **25**, 417 (1952).

$$\Phi_0' = \Phi_0, \tag{61}$$

because there are no other discrete eigenstates of $\vec{P}$. In all theories considered so far, (61) is contradicted immediately by the form of the Hamiltonian. However, we can also disprove (61) without reference to any particular form for H. This equation would imply

$$v(\vec{P})\,\Phi_0 = 0 \qquad \text{for all } \vec{P},$$

with

$$v(\vec{P}) = R\,\pi(\vec{P})\,R^{\dagger} = E_P\,\tilde{\psi}(\vec{P}) + i\tilde{\pi}(\vec{P}).$$

Here $\tilde{\psi}(\vec{P}), \tilde{\pi}(\vec{P})$ are the Fourier transforms of $\psi(\vec{x})$ and $\pi(\vec{x})$. Therefore

$$\langle a \,|\, \pi(\vec{P}) \,|\, 0 \rangle = iE_P \langle a \,|\, \tilde{\psi}(\vec{P}) \,|\, 0 \rangle,$$
$$\langle 0 \,|\, \pi(\vec{P}) \,|\, a \rangle = -iE_P \langle 0 \,|\, \tilde{\psi}(\vec{P}) \,|\, a \rangle$$

and, thus,

$$\left.\begin{aligned} &\langle 0 \,|\, [\pi(\vec{P}')\,\psi(\vec{P})] \,|\, 0 \rangle \\ &= -iE' \langle 0 \,|\, \tilde{\psi}(\vec{P}')\,\tilde{\psi}(\vec{P}) + \tilde{\psi}(\vec{P})\,\tilde{\psi}(\vec{P}') \,|\, 0 \rangle \\ &= \delta(\vec{P}' - \vec{P})\left(1 + \sum_{s=2}^{\infty}\int \frac{\sqrt{\vec{P}^2 + m^2}}{\sqrt{\vec{P}^2 + a^2}} F^{(s)}(a)\,da^2\right) = C'\,\delta(P' - P) \end{aligned}\right\} \tag{62}$$

with $C' > 1$ which contradicts the commutation relations. We conclude:

The unitary matrix R of (57) cannot exist, and the same applies to the "free field vacuum" of the Tamm—Dancoff method.[24]

[24] The question may be raised as to whether the non-existence of Φ_0' represents a serious obstacle against the use of the Tamm-Dancoff method in practical calulations. One may argue that, loosely speaking, Φ_0' is a state which has zero expansion coefficients with respect to any orthogonal system of physically interesting states. Nevertheless, the *ratio* of these coefficients is finite and mathematically definable; for instance, we can regard Φ_0' as an eigenfunction of the momentum belonging to the continuous spectrum. It may be hoped then that the normalization factor zero will not enter into the final expressions for physical quantities. Now it appears probable that this is indeed true as long as we deal with collision problems in which Φ_0' is only used in an intermediate stage of the calculations and we get back to the physical vacuum by passing to the limits $|t| \to \infty$. On the other hand, if the method is used for the determination of bound states, the situation is worse, because for these problems the difference between continuous and discrete spectrum is essential and the relation of the final result to the basis system built up from Φ_0' cannot be eliminated. The failure to renormalize the Tamm-Dancoff method in these cases is probably intimately connected to the non-existence of Φ_0'.

Thus, if a scheme on the lines of (18) is possible at all, one must be careful about the type of representation for the operators. The "obvious" choice (57) is excluded.

§ 5. Study of the commutation relations by perturbation calculation.

Do fields exist which satisfy

$$[\psi(x')\,\psi(x)] = 0 \qquad \text{for } (x'-x)^2 > 0\,? \tag{63}$$

This is a much more general question since we leave it open how the conjugate momenta $\pi(\vec{x})$ may be defined and what type of representation is meant. It can be studied by going back to (48). (63) gives some integral equations for the f^{nm} and the question is whether these have solutions. Of course, a rigorous discussion appears hopeless, but a perturbation calculation is possible. We put

$$f^{nm} = {}^{(0)}f^{nm} + \varepsilon\;{}^{(1)}f^{nm} + \varepsilon^2\;{}^{(2)}f^{nm} + \cdots \tag{64}$$

with the zero approximation

$$\left.\begin{aligned} {}^{(0)}f^{nm} &= 0 \qquad \text{for } n, m \neq 0{,}1 \text{ or } 1{,}0,\\ {}^{(0)}f^{01} &= {}^{(0)}f^{10} = 1, \end{aligned}\right\} \tag{65}$$

corresponding to ${}^{(0)}\psi(x) = \overline{\varphi}(x)$. Then we obtain in first order, setting $x' = 0$,

$$\int \delta(q^2+m^2)\,dq\,[(l+1)\,{}^{(1)}f^{k,l+1}(p',p;q)\,(e^{-ipx} - e^{i(\Sigma p - \Sigma p' + q)x}) + (k+1)\,{}^{(1)}f^{k+1,l}(p',q;p)\,(e^{i(\Sigma p-\Sigma p'-q)x} - e^{iqx}) = 0$$

for $x^2 > 0$.

For $t = 0$ the equation must hold for all $\vec{x}$. Taking the three-dimensional Fourier transform, we have the following functional equation for the f:

$$\left.\begin{aligned} &\frac{1}{2E_q}[(l+1)\,{}^{(1)}f^{k,l+1}(p';p,q) - (k+1)\,{}^{(1)}f^{k+1,l}(p',\overline{q};p)]\\ &= \frac{1}{2E_P}[(l+1)\,{}^{(1)}f^{k,l+1}(p';p,P) - (k+1)\,{}^{(1)}f^{k+1,l}(p',\overline{P};p). \end{aligned}\right\} \tag{66}$$

Here $\overline{q}$ stands for the four-vector $(-\vec{q} + E_q)$; similarly $\overline{P} = (-\vec{P} + E)$ and $\vec{P} = \Sigma\vec{p}' - \Sigma\vec{p} - \vec{q}$.

The f-functions are Lorentz invariant functions if their arguments are regarded as 4-vectors with square length m^2. It is seen that by (66) only those f^{nm}-functions are coupled which belong to the same value of $\nu = n + m$ so that each value of ν may be treated independently. We discuss the first non-trivial case ($\nu = 2$). Here, because of the Lorentz invariance, we are dealing with functions of only one argument. It is convenient to put

$$\begin{aligned} {}^{(1)}f^{11}(p';p) &= F[-(p'-p)^2 - m^2],\\ 2\,{}^{(1)}f^{20}(p_1',p_2') &= G[-(p_1'+p_2')^2 - m^2]. \end{aligned}$$

Then (66) with $k = 1$, $l = 0$ gives a functional equation for $F(x)$ and $G(x)$ which may be solved. One finds the general solution[25]

$$F(x) = G(x) = \frac{A}{x} + B + Cx. \tag{67}$$

The case $\nu = 2$ which we have considered corresponds to an interaction Hamiltonian which involves (at least in the first order) only terms containing three factors of the field variables (three-coupling). Thus (67) says that, in first approximation, there are only three types of three-coupling, compatible with our general assumptions. The A-coupling gives rise to the field equation

$$(\Box - m^2)\,\psi(x) = \frac{A}{2}\,\psi^2(x) \tag{68}$$

and the interaction Hamiltonian

$$V = \int \psi^3(\vec{x})\,d\vec{x}.$$

B and C produce derivative couplings. We single out the case A for further discussion. Taking advantage of the x-representation in which the formulae are more compact, we can write

$${}^{(1)}\psi(x) = -\int \Delta_R(x,1)\!:\varphi^2(1)\!:dx_1. \tag{69}$$

[25] The calculation is given in the CERN manuscript T/RH-1, Copenhagen, March 1954.

The extension of the argument to the second order is straightforward. One finds

$$^{(2)}\psi(x) = 2\int \Delta_R(x,1)\,\Delta_R(1,2)\colon \varphi^2(2)\,\varphi(1)\colon dx_1\,dx_2. \tag{70}$$

This agrees with the formal solution of (68) by the method of YANG-FELDMAN and KÄLLÉN apart from the fact that there we would obtain the additional term

$$\int \Delta_R(x,1)\,\Delta_R(1,2)\,\Delta^{(1)}(1,2)\,\varphi(2)\,dx_1\,dx_2.$$

Now, because of (38), (39), an operator of the form $\int f(x-x_1)\,\varphi(x_1)\,dx_1$ can only be a multiple of $\varphi(x)$ (if it is supposed to have well defined matrix elements). Such a term would be without physical interest and is furthermore excluded here by the asymptotic condition.

In the third order we meet with the well known difficulties from products of singular functions. It is interesting to note that, in a perturbation calculation, this is an inevitable consequence of (63), irrespective of the definition of the conjugate momenta or of the form of the Hamilton function. The mathematical reason is that (63) implies in the lowest order a "local" interaction, i. e. an expression for $^{(1)}\psi(x)$ involving products of $\varphi(x')$ at the same point. In the higher orders we obtain the contractions of these powers of $\varphi(x')$, that is to say products of Δ-functions of the "closed loop" type. It is not the object of this paper to discuss whether these difficulties are really serious or whether they could be overcome in a satisfactory manner by a careful definition of the limiting processes. We shall here merely indicate the extension of (69), (70) to the third order.

The troublesome terms are those which involve two contractions in the commutator $[{}^{(2)}\psi(x)\,{}^{(1)}\psi(x)]$. They are

$$\alpha = \int \Delta_R(\bar{x},1)\,\Delta_R(x,3)\,\Delta_R(1,2)\,\Delta(2,3)\,\Delta^{(1)}(2,3)\,\varphi(1)\,dx_1\,dx_2\,dx_3,$$

$$\beta = \int \Delta_R(\bar{x},1)\,\Delta_R(x,3)\,\Delta_R(1,2)\,\Delta(1,3)\,\Delta^{(1)}(2,3)\,\varphi(2)\,dx_1\,dx_2\,dx_3,$$

$$\gamma = \int \Delta_R(\bar{x},1)\,\Delta_R(x,3)\,\Delta_R(1,2)\,\Delta(2,3)\,\Delta^{(1)}(1,3)\,\varphi(2)\,dx_1\,dx_2\,dx_3.$$

There is some ambiguity in these expressions, most evident in the case of α which contains $\Delta(\xi)\,\Delta^{(1)}(\xi)$. A calculation in momentum space with a suitable order of integration gives the convergent result

$$\alpha \sim \int \Delta_R(\bar{x}-x_1)\int_{2m}^{\infty} da\,\frac{\sqrt{a^2-4m^2}}{(a^2-m^2)^2}\,\Delta(x_1-x;a)\,\varphi(x_1)\,dx_1.$$

If we put

$$F(x-x_1) = \int_{2m}^{\infty} da\,\frac{\sqrt{a^2-4m^2}}{(a^2-m^2)^2}\,\Delta_R(x-x_1;a),$$

then we can write for equal times of $\bar{x}$ and x

$$\alpha \sim \int F(x-x_1)\,\Delta(\bar{x}-x_1)\,\varphi(x_1)\,dx_1,$$

and this may be compensated by a term from $[{}^{(3)}\psi(x)\,{}^{(0)}\psi(\bar{x})]$ if we put

$$^{(3)}\psi'(x) \sim \int F(x-x_1)\colon \varphi^2(x_1)\colon dx_1.$$

The same technique of shifting the retardation sign from one of the functions $\Delta_R(\bar{x},1)$, $\Delta_R(x,3)$ may be applied in the case β and γ. If we add to β and γ the antisymmetric supplement arising from $[{}^{(2)}\psi(x)\,{}^{(1)}\psi(\bar{x})]$, we have

$$\int \Delta_R(\bar{x},1)\,\Delta_R(x,3)\,\varphi(2)\{[\Delta_R(1,2)\,\Delta(1,3)\,\Delta^{(1)}(2,3) - \Delta_R(3,2)\,\Delta(3,1)\,\Delta^{(1)}(2,1)] + [\Delta_R(1,2)\,\Delta(2,3)\,\Delta^{(1)}(1,3) - \Delta_R(3,2)\,\Delta(2,1)\,\Delta^{(1)}(1,3)]\}\,dx_1\,dx_2\,dx_3.$$

The two square brackets we call K_1 and K_2, respectively, and

$$\int K_1\,\varphi(2)\,dx_2 = \int \Phi_1(x_1,x_3,P)\,\varphi(P)\,\delta(P^2+m^2)\,dP,$$

$$\int K_2\,\varphi(2)\,dx_2 = \int \Phi_2(x_1,x_3,P)\,\varphi(P)\,\delta(P^2+m^2)\,dP.$$

Similar to α, Φ_1 and Φ_2 may be defined as convergent expressions which vanish for space-like (x_1-x_3) and which are antisymmetrical in x_1, x_3. We can split them like $\Delta(x_1-x_3)$

into a retarded and an advanced part. For equal times of $\bar{x}$ and x we can use

$$\Delta_R(\bar{x}, 1)\, \Delta_R(x, 3)\, \Phi_R(1, 3) = \Delta_R(\bar{x}, 1)\, \Delta(x, 3)\, \Phi_R(1, 3),$$

$$\Delta_R(\bar{x}, 1)\, \Delta_R(x, 3)\, \Phi_A(1, 3) = \Delta(\bar{x}, 1)\, \Delta_R(x, 3)\, \Phi_A(1, 3).$$

Such expressions may be compensated by a suitable term in ${}^{(3)}\psi$.

These remarks seem to indicate that it is possible to carry on the expansion (69), (70) to higher orders, but a more thorough investigation of this point is necessary.

Summary.

An investigation is made of the possibility of defining a theory which is in accord with the principles of quantum physics and special relativity and which describes the interaction processes of particles. There is no contradiction between these three requirements, and a simple mathematical expression for the combination of them is given in equation (1). One may then regard field theory as an extension of quantum mechanics to a system with infinitely many degrees of freedom in such a way that (1) is satisfied. The fact that we are dealing with infinitely many degrees of freedom gives rise to some mathematical problems which—though not generally recognized—have been solved to a large extent. They involve the ambiguity of the canonical commutation relations

$$[p_k, q_l] = -i\delta_{kl}; \qquad k, l = 1, 2 \cdots \infty$$

and the question how to recognize whether some function $F(p, q)$ is a proper operator, an improper operator or a senseless expression.

In comparison with the general set up of a relativistic quantum theory of particles the conventional field theories introduce some additional requirements. The basic variables are there supposed to have 1) the simple relativistic transformation properties of a field, 2) vanishing commutators for equal times, and 3) they should satisfy the asymptotic condition (15). Taken together these requirements imply a very strong restriction and it is not clear whether they are not actually incompatible if equ. (1) is taken into account[26]. We have proved that if 2) is dropped we have again the wide class of possible theories allowed by the general considerations of Chapter I. The same holds of course if both 1) and 3) are dropped. It does not seem to help, however, to leave out only 1) as long as we still want relativistic invariance of the S-matrix.

In the lowest orders of a perturbation expansion the assumptions are compatible and have physical significance as borne out by the experience from quantum electrodynamics and β-decay.

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[26] Of course the trivial solution of uncoupled fields is always possible.

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