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ON THE MAGNITUDE OF THE RENORMALIZATION CONSTANTS IN QUANTUM ELECTRODYNAMICS

BY

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With the aid of an exact formulation of the renormalization method in quantum electrodynamics which has been developed earlier, it is shown that not all of the renormalization constants can be finite quantities. It must be stressed that this statement is here made without any reference to perturbation theory.

Introduction.

In a previous paper[1], the author has given a formulation of quantum electrodynamics in terms of the renormalized Heisenberg operators and the experimental mass and charge of the electron. The consistency of the renormalization method was there shown to depend upon the behaviour of certain functions ($\Pi(p^2)$, $\Sigma_1(p^2)$ and $\Sigma_2(p^2)$) for large, negative values of the argument p^2. If the integrals

$$\int^{\infty}\frac{\Pi(-a)}{a}\,da, \quad \int^{\infty}\frac{\Sigma_i(-a)}{a}\,da \quad (i = 1, 2) \tag{1}$$

converge, quantum electrodynamics is a completely consistent theory, and the renormalization constants themselves are finite quantities. This would seem to contradict what has appeared to be a well-established fact for more than twenty years, but it must be remembered that all calculations of self-energies etc. have been made with the aid of expansions in the coupling constant e. Thus what we know is really only that, for example, the self-energy of the electron, considered as a function of e, is not analytic at the origin. It has even been suggested[2] that a different scheme of approximation may drastically alter the results obtained with the aid of a straightforward application of perturbation theory. It is the aim of the present paper to show—without any attempt at extreme mathematical rigour—that this is actually not the case in present quantum electrodynamics. The best we can

[1] G. Källén, Helv. Phys. Acta **25**, 417 (1952), here quoted as I.
[2] *Cf., e. g.,* W. Thirring, Z. f. Naturf. **6 a** 462 (1951). N. Hu, Phys. Rev. **80**, 1109 (1950).

hope for is that the renormalized theory is finite or, in other words, that the integrals

$$\int^{\infty}\frac{\Pi(-a)}{a^2}\,da,\quad \int^{\infty}\frac{\Sigma_i(-a)}{a^2}\,da, \tag{2}$$

appearing in the renormalized operators, do converge. No discussion of this point, however, will be given here.

General Outline of the Method.

We start our investigation with the assumption that all the quantities K, $(1-L)^{-1}$ and $\frac{1}{N}$ (for notations, cf. I) are finite or that the integrals (1) converge. This will be shown to lead to a lower bound for $\Pi(p^2)$ which has a finite limit for $-p^2 \to \infty$, thus contradicting our assumption. In this way it is proved that not all of the three quantities above can be finite. Our lower bound for $\Pi(p^2)$ is obtained from the formula (cf. I, Eqs. (32) and (32 a))

$$\Pi(p^2) = \frac{V}{-3p^2}\sum_{p^{(z)}=p}{}^{'}\,|\langle 0|j_\nu|z\rangle|^2(-1)^{N_4^{(z)}}.{}^{1)} \tag{3}$$

It was shown in I that, in spite of the signs appearing in (3), the sum for $\Pi(p^2)$ could be written as a sum over only positive terms. Thus we get a lower bound for $\Pi(p^2)$, if we consider the following expression

$$\Pi(p^2) > \frac{V}{-3p^2}\sum_{q+q'=p}{}^{'}\,|\langle 0|j_\nu|q,q'\rangle|^2. \tag{4}$$

In Eq. (4), $\langle 0|j_\nu|q,q'\rangle$ denotes a matrix element of the current (defined in I, Eq. (3)) between the vacuum and a state with one electron-positron pair (for $x_0 \to -\infty$). The energy-momentum vector of the electron is equal to q and of the positron is equal to q'. The sum is to be extended over all states for which $q+q'=p$. We can note here that, if we develop the function $\Pi(p^2)$ in powers of e^2 and consider just the first term in this expansion, only the states included in (4) will give a contribution. For this case, the sum is easily computed, *e. g.* in the following way:

$$\begin{aligned}\Pi^{(0)}(p^2) &= \frac{V}{-3p^2}\sum_{q+q'=p}{}^{'}\,|\langle 0|j_\nu^{(0)}|q,q'\rangle|^2\\ &= \frac{Ve^2}{-3p^2}\sum_{q+q'=p}\langle 0|\overline{\psi}^{(0)}|q'\rangle\gamma_\nu\langle 0|\psi^{(0)}|q\rangle\langle q|\overline{\psi}^{(0)}|0\rangle\gamma_\nu\langle q'|\psi^{(0)}|0\rangle\\ &= \frac{e^2}{12\pi^2}\left(1-\frac{2m^2}{p^2}\right)\sqrt{1+\frac{4m^2}{p^2}}\,\frac{1}{2}\left[1-\frac{p^2+4m^2}{|p^2+4m^2|}\right].\end{aligned} \tag{5}$$

The function $\Pi^{(0)}(p^2)$ has the constant limit $\frac{e^2}{12\pi^2}$ for large values of $-p^2$. This corresponds, of course, to the well-known divergence for the first-order charge-renormalization. We shall see, however, that with the assumptions we have made here the lower bound for the complete $\Pi(p^2)$, obtained from (4), is rather similar to $\Pi^{(0)}(p^2)$.

An Exact Expression for the Matrix Element of the Current.

Our next problem is to obtain a formula for $\langle 0|j_\nu|q,q'\rangle$ with which we can estimate the matrix element for large values of $-(q+q')^2$. For this purpose we first compute

$$\begin{aligned}[j_\mu(x),\psi^{(0)}(x')] &= -N\int_{-\infty}^{x} S(13)\,[j_\mu(x), f(3)]\,dx'''\\ &\quad -iN\int_{x_0'''=x_0} S(13)\,\gamma_4\,[j_\mu(x),\psi(3)]\,d^3x'''.\end{aligned} \tag{6}$$

(Cf. I, Eq. (54).) The last commutator can be computed without difficulty if we introduce the following formula for $j_\mu(x)$

$$j_\mu(x) = \frac{ieN^2}{1-L}\xi_{\mu\lambda}s_\lambda(x) + \frac{L}{1-L}\xi_{\mu\lambda}\frac{\partial^2 A_\nu(x)}{\partial x_\lambda\partial x_\nu} - L\delta_{\mu 4}\Box A_4(x) \tag{7}$$

with

$$\xi_{\mu\lambda} = \delta_{\mu\lambda} - L\delta_{\mu 4}\delta_{\lambda 4} \tag{7 a}$$

and

$$s_\lambda(x) = \frac{1}{2}[\overline{\psi}(x), \gamma_\lambda\psi(x)]. \tag{7 b}$$

The expression (7) is written in such a way that the second time-derivatives of all the A_μ's drop out. With the aid of I, Eqs. (4)—(7) we now get

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[1]) $\Sigma'|\langle 0|j_\nu|z\rangle|^2 = \sum\left(\sum_{k=1}^{3}|\langle 0|j_k|z\rangle|^2 - |\langle 0|j_4|z\rangle|^2\right)$

$$\left.\begin{aligned}[j_\mu(x), \psi(3)]_{x_0'''=x_0} &= \frac{ieN^2}{1-L}\xi_{\mu\lambda}[s_\lambda(x), \psi(3)] \\ &= -\frac{ie}{1-L}\xi_{\mu\lambda}\gamma_4\gamma_\lambda\psi(x)\,\delta(\bar{x}-\bar{x}''').\end{aligned}\right\} \qquad (8)$$

It thus follows that

$$\left.\begin{aligned}[j_\mu(x), \psi^{(0)}(x')] = &-N\int_{-\infty}^{x} S(13)\,[j_\mu(x), f(3)]\,dx''' \\ &-\frac{eN}{1-L}\xi_{\mu\lambda}S(1\,x)\,\gamma_\lambda\,\psi(x).\end{aligned}\right\} \qquad (9)$$

We then proceed by computing

$$\left.\begin{aligned}&\langle 0\,|\{[j_\mu(x), \psi^{(0)}(x')], \bar{\psi}^{(0)}(x'')\}\,|\,0\rangle \\ &= \frac{ieN}{1-L}\xi_{\mu\lambda}S(1x)\,\gamma_\lambda\,S(x\,2) - N\int_{-\infty}^{x} S(13)\,dx''' \\ &\times[\langle 0\,|[j_\mu(x), \{\bar{\psi}^{(0)}(2), f(3)\}]\,|\,0\rangle - \langle 0\,|\{[j_\mu(x), \bar{\psi}^{(0)}(2)], f(3)\}\,|\,0\rangle].\end{aligned}\right\} \qquad (10)$$

If this expression is considered as an identity in x' and x'' it will obviously give us a formula for $\langle 0\,|\,j_\mu\,|\,q, q'\rangle$ and for $\langle q\,|\,j_\mu\,|\,q'\rangle$. (Cf. I, Eqs. (68) and (77).) We transform the right-hand side of (10) in the following way:

$$\{\bar{\psi}^{(0)}(2), f(3)\} = N\int_{-\infty}^{x'''}\{f(3), \bar{f}(4)\}\,S(42)\,dx^{\mathrm{IV}} - \frac{i}{N}[ie\gamma A(3) + K]\,S(32) \qquad (11)$$

and, hence,

$$\left.\begin{aligned}\langle 0\,|[j_\mu(x), \{\bar{\psi}^{(0)}(2), f(3)\}]\,|\,0\rangle &= \frac{e}{N}\gamma_\lambda S(32)\,\langle 0\,|\,[j_\mu(x), A_\lambda(3)]\,|\,0\rangle \\ &+ N\int_{-\infty}^{x'''}dx^{\mathrm{IV}}\,\langle 0\,|\,[j_\mu(x), \{f(3), \bar{f}(4)\}]\,|\,0\rangle\,S(42).\end{aligned}\right\} \qquad (12)$$

The last term in (10) can be treated in a similar way:

$$[j_\mu(x), \bar{\psi}^{(0)}(2)] = N\int_{-\infty}^{x}[j_\mu(x), \bar{f}(4)]\,S(42)\,dx^{\mathrm{IV}} + \frac{eN}{1-L}\bar{\psi}(x)\,\gamma_\lambda S(x\,2)\,\xi_{\lambda\mu} \qquad (13)$$

and

$$\left.\begin{aligned}N\int_{-\infty}^{x}S(13)\,dx'''\,\langle 0\,|\{\bar{\psi}(x), f(3)\}\,|\,0\rangle &= -\langle 0\,|\{\bar{\psi}(x), \psi^{(0)}(x') \\ + iN\int_{x_0'''=x_0}S(13)\,\gamma_4\,\psi(3)\,d^3x'''\}\,|\,0\rangle &= iS(1\,x)\left[1-\frac{1}{N}\right].\end{aligned}\right\} \qquad (14)$$

Collecting (12), (13) and (14) we get

$$\left.\begin{aligned}&\langle 0\,|\{[j_\mu(x), \psi^{(0)}(x')], \bar{\psi}^{(0)}(x'')\}\,|\,0\rangle \\ &= \frac{ie}{1-L}[1+2(N-1)]\,\xi_{\mu\lambda}S(1\,x)\,\gamma_\lambda\,S(x\,2) \\ &- e\int_{-\infty}^{x}S(13)\,\gamma_\lambda\,S(32)\,dx'''\,\langle 0\,|\,[j_\mu(x), A_\lambda(3)]\,|\,0\rangle \\ &- N^2\int_{-\infty}^{x}dx'''\int_{-\infty}^{x'''}dx^{\mathrm{IV}}\,S(13)\,\langle 0\,|\,[j_\mu(x), \{f(3), \bar{f}(4)\}]\,|\,0\rangle\,S(42) \\ &+ N^2\int_{-\infty}^{x}dx'''\int_{-\infty}^{x}dx^{\mathrm{IV}}\,S(13)\,\langle 0\,|\{f(3), [j_\mu(x), \bar{f}(4)]\}\,|\,0\rangle\,S(42).\end{aligned}\right\} \qquad (15)$$

The second term in (15) can be rewritten with the aid of the functions $\Pi(p^2)$ and $\bar{\Pi}(p^2)$.

$$\left.\begin{aligned}\langle 0\,|\,[j_\mu(x), A_\lambda(3)]\,|\,0\rangle &= \int D_R(34)\,\langle 0\,|\,[j_\mu(x), j_\lambda(4)]\,|\,0\rangle\,dx^{\mathrm{IV}} \\ &= \frac{-1}{(2\pi)^3}\int dp\,e^{ip(3x)}\,\varepsilon(p)\,[p_\mu p_\lambda - p^2\,\delta_{\mu\lambda}]\,\frac{\Pi(p^2)}{p^2}.\end{aligned}\right\} \qquad (16)$$

We are, however, more interested in the expression

$$\left.\begin{aligned}\frac{1}{2}[1+\varepsilon(x\,3)]\,\langle 0\,|\,[j_\mu(x), A_\lambda(3)]\,|\,0\rangle &= \frac{i\delta_{\mu\lambda}}{(2\pi)^4}\int dp\,e^{ip(x3)}\,[\bar{\Pi}(p^2) \\ + i\pi\varepsilon(p)\,\Pi(p^2)] &+ \frac{1}{2}[1+\varepsilon(x\,3)]\,\frac{\partial^2\Phi(3\,x)}{\partial x_\mu\,\partial x_\lambda},\end{aligned}\right\} \qquad (17)$$

where

$$\Phi(x) = \frac{1}{(2\pi)^3}\int dp\,e^{ipx}\,\varepsilon(p)\,\frac{\Pi(p^2)}{p^2}. \qquad (17\,\mathrm{a})$$

Obviously, we have

$$\Phi(3\,x) = 0 \qquad (18\,\mathrm{a})$$

$$\frac{\partial\Phi(3\,x)}{\partial x_0'''} = -i\bar{\Pi}(0)\,\delta(\bar{x}-\bar{x}''') \qquad (18\,\mathrm{b})$$

for $x_0''' = x_0$. It thus follows

$$\varepsilon(x\,3)\,\frac{\partial^2\Phi(3\,x)}{\partial x_\mu\,\partial x_\lambda} = \frac{\partial^2}{\partial x_\mu\,\partial x_\lambda}[\varepsilon(x\,3)\,\Phi(3\,x)] + 2\,i\bar{\Pi}(0)\,\delta_{\mu 4}\delta_{\lambda 4}\delta(x\,3). \qquad (19)$$

Using the equation

$$\frac{\partial}{\partial x_\lambda'''} S(13)\,\gamma_\lambda S(32) = 0, \tag{20}$$

we get

$$\left.\begin{aligned} & -e\int_{-\infty}^{+\infty}\frac{1}{2}[1+\varepsilon(x3)]\,S(13)\,\gamma_\lambda S(32)\langle 0|[j_\mu(x), A_\lambda(3)]|0\rangle\,dx''' \\ & = -\frac{ie}{(2\pi)^4}\int dx'''\int dp e^{ip(x3)} S(13)\,\gamma_\mu S(32)\,[\bar{\Pi}(p^2)+i\pi\varepsilon(p)\,\Pi(p^2)] \\ & \qquad + i\delta_{\mu 4}\frac{L}{1-L}S(1x)\,\gamma_4 S(x2). \end{aligned}\right\} \tag{21}$$

Introducing (21) into (15) we obtain

$$\left.\begin{aligned} & \langle 0|\{[j_\mu(x), \psi^{(0)}(x')], \bar{\psi}^{(0)}(x'')\}|0\rangle \\ & = ie\int dx'''\int\frac{dp}{(2\pi)^4}e^{ip(x3)}S(13)\,\gamma_\mu S(32)\,[1-\bar{\Pi}(p^2) \\ & \qquad + \bar{\Pi}(0) - i\pi\varepsilon(p)\,\Pi(p^2)] \\ & - N^2\int_{-\infty}^{x}dx'''\int_{-\infty}^{x'''}dx^{\mathrm{IV}}\,S(13)\langle 0|[j_\mu(x), \{f(3), \bar{f}(4)\}]|0\rangle S(42) \\ & + N^2\int_{-\infty}^{x}dx'''\int_{-\infty}^{x}dx^{\mathrm{IV}}\,S(13)\langle 0|\{f(3), [j_\mu(x), \bar{f}(4)]\}|0\rangle S(42) \\ & \qquad + \frac{2\,ie(N-1)}{1-L}\xi_{\mu\lambda}S(1x)\,\gamma_\lambda S(x2). \end{aligned}\right\} \tag{22}$$

The first term in (22) describes the vacuum polarization and is quite similar to the corresponding expression for a weak external field (cf. I, Appendix). The remaining terms contain the anomalous magnetic moment, the main contribution to the Lamb shift etc. Introducing the notation

$$\left.\begin{aligned} & -N^2\theta(x3)\,\theta(34)\langle 0|[j_\mu(x), \{f(3), \bar{f}(4)\}]|0\rangle \\ & + N^2\theta(x3)\,\theta(x4)\langle 0|\{f(3), [j_\mu(x), \bar{f}(4)]\}|0\rangle \\ & \qquad - \frac{2\,ie(N-1)}{1-L}L\delta_{\mu 4}\gamma_4\delta(x3)\,\delta(34) \\ & \qquad = \frac{ie}{(2\pi)^8}\iint dp\,dp' e^{ip'(3x)+ip(x4)}\Lambda_\mu(p', p) \end{aligned}\right\} \tag{23}$$

$$\theta(x) = \frac{1}{2}[1+\varepsilon(x)], \tag{23 a}$$

we obtain from (22)

$$\left.\begin{aligned} & \langle 0|j_\mu|q, q'\rangle \\ & = \langle 0|j_\mu^{(0)}|q, q'\rangle\Big[1-\bar{\Pi}((q+q')^2)+\bar{\Pi}(0)-i\pi\Pi((q+q')^2) \\ & \qquad + 2\frac{N-1}{1-L}\Big] + ie\langle 0|\bar{\psi}^{(0)}|q'\rangle\Lambda_\mu(-q', q)\langle 0|\psi^{(0)}|q\rangle. \end{aligned}\right\} \tag{24}$$

This is the desired formula for the matrix element of the current.

Analysis of the Function $\Lambda\mu(p', p)$.

We now want to investigate the function $\Lambda\mu(p', p)$ in some detail, especially studying its behaviour for large values of $-(q+q')^2$ in (24). For simplicity, we put $\mu = k \neq 4$ and study

$$\left.\begin{aligned} & ie\Lambda_k(p', p) = \iint dx'''dx^{\mathrm{IV}}\,e^{-ip'(3x)-ip(x4)}N^2\{\theta(x3)\,\theta(x4)\langle 0|\{f(3), \\ & [j_k(x), \bar{f}(4)]\}|0\rangle - \theta(x3)\,\theta(34)\langle 0|[j_k(x), \{f(3), \bar{f}(4)\}]|0\rangle\}. \end{aligned}\right\} \tag{25}$$

We treat the two terms in (25) separately. The first vacuum expectation value can be transformed to momentum space with the aid of the functions

$$A_k^{(+)}(p', p) = V^2\sum_{\substack{p^{(z)} = p \\ p^{(z')} = p'}}\langle 0|f|z'\rangle\langle z'|j_k|z\rangle\langle z|\bar{f}|0\rangle \tag{26}$$

$$A_k^{(-)}(p', p) = V^2\sum_{\cdots}\langle 0|\bar{f}|z'\rangle\langle z'|j_k|z\rangle\langle z|f|0\rangle \tag{27}$$

$$B_k^{(+)}(p', p) = V^2\sum_{\cdots}\langle 0|f|z'\rangle\langle z'|\bar{f}|z\rangle\langle z|j_k|0\rangle \tag{28}$$

$$B_k^{(-)}(p', p) = V^2\sum_{\cdots}\langle 0|j_k|z'\rangle\langle z'|\bar{f}|z\rangle\langle z|f|0\rangle. \tag{29}$$

It then follows that

$$\left.\begin{aligned} & \langle 0|\{f(3), [j_k(x), \bar{f}(4)]\}|0\rangle = \frac{1}{V^2}\sum_{p, p'}\{e^{ip'(3x)+ip(x4)}A_k^{(+)}(p', p) \\ & \quad - e^{ip'(34)+ip(4x)}B_k^{(+)}(p', p) + e^{ip'(x4)+ip(43)}B_k^{(-)}(p', p) \\ & \qquad - e^{ip'(4x)+ip(x3)}A_k^{(-)}(p', p)\}. \end{aligned}\right\} \tag{30}$$

Our discussion started with the assumption that all the renormalization constants and, of course, all the matrix elements of the operators $j_\mu(x)$ and $f(x)$ are finite. As this is a condition on the behaviour of, for example, the function $\Pi(p^2)$ for large values of $-p^2$, and as this function is defined as a sum of matrix elements, it is clear that we also have a condition on the matrix elements themselves, *i. e.* on the functions A and B defined in (26)—(29) for large values of $-p^2, -p'^2$ and $-(p-p')^2$. To get more detailed information on this point we consider the expression

$$\left.\begin{aligned} &\langle z \,|\, [j_\mu(x), A_\nu^{(0)}(x')] \,|\, z\rangle \\ &= -i\frac{L}{1-L}\frac{\partial^2 D(x'-x)}{\partial x_\mu \partial x_\nu} + \int dx'' F_{\mu\nu}(x-x'')\, D(x'-x'') \end{aligned}\right\} \quad (31)$$

with

$$F_{\mu\nu}(x-x'') = \theta(x-x'')\langle z \,|\, [j_\mu(x), j_\nu(x'')] \,|\, z\rangle \quad (32)$$

(cf. I, Eq. (A. 8) and the equation of motion for $A_\mu(x)$). Supposing, for simplicity, that $|z\rangle$ does not contain a photon with energy-momentum vector k, we have

$$\left.\begin{aligned} &\langle z \,|\, j_\mu(x) \,|\, z, k\rangle \\ &= -\frac{L}{1-L}k_\mu k_\nu \langle 0 \,|\, A_\nu^{(0)}(x) \,|\, k\rangle + i\int dx'' F_{\mu\nu}(x-x'')\langle 0 \,|\, A_\nu^{(0)}(x'') \,|\, k\rangle. \end{aligned}\right\} \quad (33)$$

Writing

$$F_{\mu\lambda}(x-x'') = \theta(x-x'')\frac{-1}{(2\pi)^3}\int dp\, e^{ip(x-x'')} F_{\mu\lambda}(p) \quad (34)$$

and using the formula

$$\varepsilon(x-x'') = \frac{1}{i\pi}P\int\frac{d\tau}{\tau}e^{i\tau(x_0-x_0'')} \quad (35)$$

we get

$$iF_{\mu\lambda}(x-x'') = \frac{-1}{(2\pi)^4}\int dp\, e^{ip(x-x'')}\{\overline{F}_{\mu\lambda}(p) + i\pi F_{\mu\lambda}(p)\} \quad (36)$$

with

$$\overline{F}_{\mu\lambda}(p) = P\int\frac{d\tau}{\tau}F_{\mu\lambda}(\overline{p}, p_0+\tau). \quad (37)$$

We further note that from (34) it follows that

$$F_{\mu\lambda}(p) = V\sum_{p^{(z')}=p^{(z)}+p}\langle z \,|\, j_\lambda \,|\, z'\rangle\langle z' \,|\, j_\mu \,|\, z\rangle - V\sum_{p^{(z')}=p^{(z)}-p}\langle z \,|\, j_\mu \,|\, z'\rangle\langle z' \,|\, j_\lambda \,|\, z\rangle. \quad (38)$$

If every expression appearing in our formalism is finite, the integral in (37) must converge. This means that[1]

$$\lim_{p_0\to\pm\infty} F_{\mu\lambda}(\overline{p}, p_0) = 0. \quad (39)$$

Putting $\mu = \lambda = k$ we then get from (38) and (39)

$$\lim_{p_0\to\infty}\sum_{p^{(z')}=p^{(z)}+p}|\langle z \,|\, j_k \,|\, z'\rangle|^2(-1)^{N_4^{(z)}+N_4^{(z')}} = 0 \quad (40\,\text{a})$$

and

$$\lim_{p_0\to-\infty}\sum_{p^{(z')}=p^{(z)}-p}|\langle z \,|\, j_k \,|\, z'\rangle|^2(-1)^{N_4^{(z)}+N_4^{(z')}} = 0. \quad (40\,\text{b})$$

If we first consider a state $|z\rangle$ with no scalar or longitudinal photons, it can be shown with the aid of the gauge-invariance of the current operator (cf. I, p. 426. Eq. (47) there can be verified explicitly with the aid of (32) and (33) above) that only states $|z'\rangle$ with transversal photons will give a non-vanishing contribution to (40 a) and (40 b), and these contributions are all positive. We thus obtain the result

$$\lim_{|p_0^{(z)}-p_0^{(z')}|\to\infty}|\langle z \,|\, j_k \,|\, z'\rangle|^2 = 0 \quad (41)$$

if none of the states $|z\rangle$ and $|z'\rangle$ contains a scalar or a longitudinal photon. Because of Lorentz invariance which requires that Eq. (41) is valid in every coordinate system, it follows, however, that (41) must be valid for all kinds of states. If we make a Lorentz transformation, the "transversal" states in the new coordinate system will in general be a mixture of all kinds of states in the old system. If (41) were not valid also for the scalar and longitudinal states in the old system, it could not hold for the transversal states in the new system.

[1]) The case in which the integrals converge without the functions vanishing will be discussed in the Appendix.

From equation (41) we conclude that

$$\lim_{-(p-p')^2\to\infty} A_k^{(\pm)}(p',p) = 0 \tag{42 a}$$

$$\lim_{-p^2\to\infty} B_k^{(+)}(p',p) = 0 \tag{42 b}$$

$$\lim_{-p'^2\to\infty} B_k^{(-)}(p',p) = 0\,. \tag{42 c}$$

It is, of course, not immediately clear that the sum over all the terms in (26)—(29) must vanish because every term vanishes. What really follows from (40) is, however, that the sum of all the absolute values of $\langle z\,|\,j_\mu\,|\,z'\rangle$ must vanish. If the limits in A and B are then performed in such a way that p^2 and p'^2 are kept fixed for A and $(p-p')^2$ and one of the p^2's are kept fixed for the B's, equations (42) will follow.

To summarize the argument so far, we have shown that if we write

$$\langle 0\,|\{f(3),[j_k(x),\bar f(4)]\}|\,0\rangle = \frac{1}{(2\pi)^6}\iint dp\,dp'\,e^{ip'(3x)+ip(x4)}F_k(p',p) \tag{43}$$

we have

$$\lim_{-(p-p')^2\to\infty} F_k(p',p) = 0\,. \tag{44}$$

Introducing the notations

$$\bar F_k(p',p) = \int\frac{d\tau}{\tau}F_k(p'-\varepsilon\tau,\,p) \tag{45 a}$$

and

$$\tilde F_k(p',p) = \int\frac{d\tau}{\tau}F_k(p',p+\varepsilon\tau) \tag{45 b}$$

(ε is a "vector" with the components $\varepsilon_k = 0$ for $k \neq 4$ and $\varepsilon_0 = 1$) we find from (44) and the assumption that the integrals in (45) converge that

$$\lim_{-(p-p')^2\to\infty}\bar F_k(p',p) = \lim_{-(p-p')^2\to\infty}\tilde F_k(p',p) = 0 \tag{46}$$

(cf. the Appendix). With the aid of the notations (45) we can now write

$$\left.\begin{aligned}&\theta(x3)\,\theta(x4)\langle 0\,|\{f(3),[j_k(x),\bar f(4)]\}|\,0\rangle\\&=\frac{-1}{(2\pi)^8}\iint dp\,dp'\,e^{ip'(3x)+ip(x4)}\,[\tilde{\bar F}_k(p',p)\\&-\pi^2F_k(p',p)+i\pi(\bar F_k(p',p)+\tilde F_k(p',p))]\,.\end{aligned}\right\} \tag{47}$$

In quite a similar way it can be shown that the second term in (25) can be written in a form analogous to (47) with the aid of a function $G_k(p',p)$ which also has the properties (44) and (46). It thus follows

$$\lim_{-(p-p')^2\to\infty} \Lambda_k(p',p) = 0\,. \tag{48}$$

It must be stressed that this property of the function $\Lambda_k(p',p)$ is a consequence of (41) and thus essentially rests on the assumption that all the renormalization constants are finite quantities.

It is clear from (24) that the function Λ_μ transforms as the matrix γ_μ under a Lorentz transformation. The explicit verification of this from (23) is somewhat involved but can be carried through with the aid of the identity

$$\left.\begin{aligned}&\theta(x3)\,\theta(x4)\{f(3),[j_\mu(x),\bar f(4)]\}-\theta(x3)\,\theta(34)\,[j_\mu(x),\{f(3),\bar f(4)\}]\\=\,&\theta(x4)\,\theta(x3)\{\bar f(4),[j_\mu(x),f(3)]\}-\theta(x4)\,\theta(43)\,[j_\mu(x),\{\bar f(4),f(3)\}]\end{aligned}\right\} \tag{49}$$

and the canonical commutators. Eq. (49) can also be used to prove the formula

$$-C^{-1}\Lambda_\mu(-q',q)\,C = \Lambda_\mu^T(-q,q') \tag{50}$$

which is, however, also evident from (24) and the charge invariance of the formalism. From the Lorentz invariance it follows that we can write

$$\Lambda_\mu(p',p) = \sum_{\varrho'=0,1}\sum_{\varrho=0,1}(i\gamma p'+m)^{\varrho'}\,[\gamma_\mu F^{\varrho'\varrho}+p_\mu G^{\varrho'\varrho}+p'_\mu H^{\varrho'\varrho}]\,(i\gamma p+m)^{\varrho} \tag{51}$$

where the functions F, G and H are uniquely defined and depending only on p^2, p'^2, $(p-p')^2$ and the signs $\varepsilon(p)$, $\varepsilon(p')$ and $\varepsilon(p-p')$. From (50) it then follows

$$F^{\varrho\varrho'}(-p, p') = F^{\varrho'\varrho}(-p', p) \tag{52 a}$$

$$G^{\varrho\varrho'}(-p, p') = H^{\varrho'\varrho}(-p', p). \tag{52 b}$$

Utilizing (51) and (52) we get

$$\left.\begin{aligned} ie\langle 0|\overline{\psi}^{(0)}|q'\rangle \Lambda_\mu(-q', q)\langle 0|\psi^{(0)}|q\rangle &= \langle 0|j_\mu^{(0)}|q, q'\rangle R((q+q')^2) \\ &+\frac{e}{2m} S((q+q')^2)(q_\mu - q'_\mu)\langle 0|\overline{\psi}^{(0)}|q'\rangle\langle 0|\psi^{(0)}|q\rangle \end{aligned}\right\} \tag{53}$$

where, in view of (48),

$$\lim_{-(q+q')^2\to\infty} R((q+q')^2) = \lim_{-(q+q')^2\to\infty} S((q+q')^2) = 0. \tag{54}$$

The equations (53) and (54) are the desired result of this paragraph.

Completion of the Proof.

We are now nearly at the end of our discussion. From the assumptions made about $\Pi(p^2)$ (and its consequences for $\overline{\Pi}(p^2)$, cf. the Appendix), Eqs. (53) and (54), the limit of Eq. (24) reduces to

$$\left.\begin{aligned} \lim_{-(q+q')^2\to\infty}\langle 0|j_\mu|q, q'\rangle &= \langle 0|j_\mu^{(0)}|q, q'\rangle\left[1+\overline{\Pi}(0)+2\frac{N-1}{1-L}\right] \\ &= \langle 0|j_\mu^{(0)}|q, q'\rangle\frac{2N-1}{1-L}. \end{aligned}\right\} \tag{55}$$

Our inequality (4) now gives

$$\left.\begin{aligned} \Pi(p^2) &> \frac{V}{-3p^2}\sum_{q+q'=p}{}' |\langle 0|j_\mu|q, q'\rangle|^2 \\ &\to \frac{V}{-3p^2}\sum_{q+q'=p}{}' |\langle 0|j_\mu^{(0)}|q, q'\rangle|^2\left(\frac{2N-1}{1-L}\right)^2 \\ &= \Pi^{(0)}(p^2)\left(\frac{2N-1}{1-L}\right)^2 \to \frac{e^2}{12\pi^2}\left(\frac{2N-1}{1-L}\right)^2. \end{aligned}\right\} \tag{56}$$

Except for the possibility of N being exactly $\frac{1}{2}$ $\left(\text{independent of } e^2 \text{ and } \frac{m^2}{\mu^2}\right)$ we have then proved that, if all the renormalization constants K, $\frac{1}{N}$ and $\frac{1}{(1-L)}$ are finite, the function $\Pi(p^2)$ cannot approach zero for $-p^2 \to \infty$. This is an obvious contradiction and the only remaining possibility is that at least one (and probably all) of the renormalization constants is infinite.

The case $N = \frac{1}{2}$ is rather too special to be considered seriously. We can note, however, that N must approach 1 for $e \to 0$ and that one of the integrals in I Eq. (75) will diverge at the lower limit for $\mu \to 0$, independent of the value of e. The constant N could thus at the utmost be equal to $\frac{1}{2}$ for some special combination (or combinations) of e^2 and $\frac{m^2}{\mu^2}$. As μ is an arbitrarily small quantity it is hardly possible to ascribe any physical significance to such a solution, even if it does exist.

The proof presented here makes no pretence at being satisfactory from a rigorous, mathematical point of view. It contains, for example, a large number of interchanges of orders of integrations, limiting processes and so on. From a strictly logical point of view we cannot exclude the possibility that a more singular solution exists where such formal operations are not allowed. It would, however, be rather hard to understand how the excellent agreement between experimental results and lowest order perturbation theory calculations could be explained on the basis of such a solution.

Appendix.

It has been stated and used above that: if

$$\overline{f}(x) = P\int_0^\infty \frac{f(y)}{y-x}\,dy \quad (f(0) = 0) \tag{A. 1}$$

where $f(x)$ is bounded and continuous for all finite values of x and fulfills

$$|f(x+y) - f(x)| < M|y| \quad \text{for all } x \tag{A. 2}$$

and if the integral converges, both $f(x)$ and $\bar{f}(x)$ will vanish for large values of the argument. This is not strictly true, and in this appendix we will study that point in some detail.

We begin by proving that if the integral in (A. 1) converges absolutely and if

$$\lim_{x\to\infty} \log x\,|f(x)| = 0 \tag{A. 3}$$

it follows that

$$\lim_{x\to\pm\infty} \bar{f}(x) = 0. \tag{A. 4}$$

(Note that the integral $\int^{\infty} \frac{dx}{x\cdot\log x}$ is *not* convergent and that the vanishing of $f(x)$ is already implicit in (A. 3).) To get an upper bound for $\bar{f}(x)$ when $x > 0$ we write

$$\bar{f}(x) = P\int_0^{\infty}\frac{f(y)}{y-x}\,dy = \left(\int_0^{x/2} + P\int_{x/2}^{3x/2} + \int_{3x/2}^{\infty}\right)\frac{f(y)}{y-x}\,dy. \tag{A. 5}$$

(The limit $x\to-\infty$ is simpler and need not be discussed explicitly.) The absolute value of the first term in (A. 5) is obviously less than

$$\frac{2}{x}\int_0^{x/2}|f(y)|\,dy < \text{const.}\,\frac{2}{x}\int_0^{x/2}\frac{dy}{\log y}\to 0. \tag{A. 6}$$

The last term can be treated in a similar way and yields the result

$$\left|\int_{3x/2}^{\infty}\frac{f(y)}{y-x}\,dy\right| \leq \int_{3x/2}^{\infty}\frac{|f(y)|}{y/3}\,dy\to 0. \tag{A. 7}$$

The remaining term can be written

$$\left.\begin{aligned} \left|P\int_{x/2}^{3x/2}\frac{f(y)}{y-x}\,dy\right| &= \left|\int_0^{x/2}\frac{dy}{y}\left[f(x+y)-f(x-y)\right]\right| \\ \leq \int_0^{\varepsilon}\frac{dy}{y}\left|f(x+y)-f(x-y)\right| &+ \int_{\varepsilon}^{x/2}\frac{dy}{y}\left|f(x+y)\right| + \int_{\varepsilon}^{x/2}\frac{dy}{y}\left|f(x-y)\right|. \end{aligned}\right\} \tag{A. 8}$$

In view of (A. 2) and (A. 3), the three terms in (A. 8) vanish separately for large values of x. It thus follows

$$\lim_{x\to\infty}\bar{f}(x) = 0$$

q. e. d.

As the function $\Pi(p^2)$ is positive the condition (A. 3) seems rather reasonable from a physical point of view. On the other hand, the functions F_k in (45) are not necessarily positive. It is, however, also possible to construct a more general argument where (A. 3) is not used, and where even the vanishing of $f(x)$ is not needed. Instead, we then require that from

$$\bar{f}(x) = P\int_0^{\infty}\frac{f(y)}{y-x}\,dy;\; f(y) = 0 \quad\text{for}\quad y\leq 0 \tag{A. 9}$$

will follow

$$f(x) = -\frac{1}{\pi^2}P\int_{-\infty}^{+\infty}\frac{\bar{f}(y)}{y-x}\,dy \tag{A. 10}$$

where both $f(x)$ and $\bar{f}(x)$ are finite.

Note that

$$\left.\begin{aligned} &\frac{1}{\pi^2}P\int_{-\infty}^{+\infty}\frac{dz}{(z-x)(z-y)} \\ &= \frac{-1}{4\pi^2}\iint dw_1\,dw_2\int dz\,e^{i(w_1+w_2)z}\cdot e^{-iw_1x-iw_2y}\,\frac{w_1w_2}{|w_1w_2|} \\ &= \frac{1}{2\pi}\int dw_1\,e^{iw_1(y-x)} = \delta(y-x). \end{aligned}\right\} \tag{A. 11}$$

It then follows that the integral

$$\int^{\infty}\frac{|1+\bar{f}(x)+i\pi f(x)|^2}{x}\,dx > \int^{\infty}\frac{|1+2\bar{f}(x)|}{x}\,dx$$

is divergent, because the second term is convergent in view of (A. 10). This is everything that is needed for the proof.