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THE COUPLING OF NUCLEAR SURFACE OSCILLATIONS TO THE MOTION OF INDIVIDUAL NUCLEONS

BY

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I. Introduction.

The close packing of the particles in the nucleus and the existence of a relatively sharp nuclear boundary have led to the comparison of the nucleus with a liquid drop[1), 2)]. This model has found numerous applications in the theory of nuclear reactions, and also accounts for certain static properties of the nucleus. Thus, main features of the empirical binding energies can be interpreted in a simple manner if the energy of the nuclear droplet is expressed as a sum of surface energy, volume energy, and electrostatic energy. The treatment of the nucleus as a deformable body has met with considerable success in the theory of nuclear fission[3)].

According to the liquid drop model, the fundamental modes of nuclear excitation correspond to collective types of motion, such as surface oscillations and elastic vibrations. Even if it has not been possible, with certainty, to associate observed nuclear levels with particular modes of oscillation, the model gives an immediate explanation of the rapid increase of the level density with increasing excitation of the nucleus.

In recent years, new progress in the theory of nuclear structure has been obtained through the development of the so-called single particle model[4)–7)]. This model assumes that nuclear stationary states, like electron configurations in atoms, can be approximately described in terms of the motion of the individual particles in an average field of force.

The single particle model explains the pronounced stability of certain nuclear species, those which possess closed shells of protons or neutrons, and has been highly successful in accounting for the spins of nuclear ground states. Nuclear magnetic moments and electric quadrupole moments also give strong evidence of

shell structure. Among the many other applications of the model it may be mentioned that it has proved valuable in the interpretation of β-transitions and isomeric states.

The liquid drop model and the single particle model represent opposite approaches to the problem of nuclear structure. Each refers to essential aspects of nuclear structure, and it is to be expected that features of both models must be taken into account simultaneously in a detailed description of nuclear properties.

The necessity of combining the two models is clearly indicated by the observed behaviour of nuclear quadrupole moments. On the one hand, as already mentioned, the quadrupole moments give definite evidence of shell structure; in particular they change sign on the passing of the magic numbers, as predicted by the single particle model. On the other hand, for many nuclei, the magnitude of the quadrupole moments is too large to be accounted for in terms of individual nucleons and suggests that the equilibrium shape of the nucleus itself deviates from spherical symmetry.

The behaviour of the quadrupole moments finds a simple explanation[8] if one considers the motion of the individual particles in a deformable nucleus. Due to the centrifugal pressure exerted by the particles on the nuclear walls, the nucleus may acquire a considerable deformation. The quadrupole moments thus induced have the same sign as those observed and appear also to have the right order of magnitude.

The coupling between the single particle motion and the nuclear deformation gives rise to a certain sharing of angular momentum between the particle and the surface. The quantization of angular momenta may therefore deviate essentially from the case of the pure single particle model. While the latter model may be termed a quasi-atomic model, the combined model bears many analogies to molecular structures[10], where we have to do with the interplay between electronic and nuclear motion.

The modification in the angular momentum coupling has a direct influence on the magnetic moment of the nucleus. It may perhaps be possible, along these lines, to explain the fact that

the magnetic moments, although strongly correlated to the single particle values, still deviate considerably from these values[9), 10)].

It is the purpose of the present and following investigations to consider various properties of a nucleus described in terms of a deformable surface coupled to the motion of individual nucleons. This combined model may be referred to as the quasi-molecular model.

Apart from the problem of nuclear moments, mentioned above, the model may find application in the analysis of the energy spectrum of the nucleus and of transitions between nuclear states. Such transitions not only involve a change of state of the individual nucleons, but must be expected to be accompanied, in general, by changes in the vibrational and rotational state of the nucleus.

The nucleus may possess additional degrees of freedom associated with simple types of motion, such as elastic vibrations, connected with the compressibility of the nucleus[2), 11)], and the so-called dipole vibrations[12)–14)], connected with the polarizability of the nucleus. These types of motion, however, are expected to have considerably greater frequencies than the surface vibrations and should therefore in general be of smaller influence on the nuclear ground state and low-lying excited levels.

The degree of accuracy obtainable by the model is difficult to estimate on a theoretical basis at the present state of knowledge regarding nuclear forces. For the ground state of the nucleus, empirical evidence appears to indicate that the model may be adequate for many purposes. With increasing excitation of the nucleus, and decreasing spacing between the energy levels, however, the effect of configuration perturbations becomes of growing importance.

For high excitation energies, it must be expected that the stationary states of the nucleus can, in general, no longer be described in terms of the motion of individual nucleons and simple oscillatory modes of the nucleus. The coupling between the various possible types of motion will imply a complexity in the state of the nucleus which suggests the application of statistical or thermodynamic methods. It is in this energy region, which in general is

reached by the capture of a particle into the nucleus, that the concept of the compound nucleus has proved fruitful in accounting for nuclear reactions*.

In the present paper, we are considering some general features of the quasi-molecular model. In particular, we shall discuss the classification of nuclear states and the adequacy of simple coupling schemes. For simplicity, we confine ourselves to the case of a single nucleon interacting with the nuclear surface. In actual applications of the model, it will be necessary to take into account all degrees of freedom of the particle structure which may be excited by the surface oscillations. The special case considered here is illustrative, however, of the general procedure which may be applied in problems of this type.

Section II is concerned with the classical theory of nuclear surface vibrations. The surface is described by the five expansion parameters corresponding to the spherical harmonics of order 2; higher orders are of minor importance here. It is convenient to divide the kinetic energy of oscillation into a vibrational and a rotational part. The quantum theory of the surface oscillations is discussed in Section III.

Section IV deals with the motion of a nucleon in a deformed nucleus and, finally, in Section V, we consider the coupling of the single particle motion to the oscillations of the nuclear surface. A general solution of this problem would be rather complicated, but, in the limits of weak and strong coupling between the particle and the surface, simple approximate solutions can be obtained.

Of particular interest is the case of strong coupling, where the single particle produces a large deformation of the nucleus. In this limiting case, which may be approximately realized for certain nuclear states, the system is, dynamically, closely analogous to a linear molecule. For such states, the individual particles may be considered as moving in an average field corresponding to a cylindrically symmetric equilibrium shape of the nucleus. The nuclear surface performs small vibrations about the equilibrium shape, while the symmetry axis precesses around the total angular momentum vector of the nucleus.

* For a closer discussion of the relationship of the single particle model to the compound nucleus, cf. V. WEISSKOPF[15].

II. Classical Theory of Nuclear Surface Oscillations.

1) Surface vibrations as harmonic oscillators.

The theory of nuclear surface oscillations has been developed by several authors[3), 16), 17)]. In the following paragraphs, we shall attempt to present it in a form especially suited for the treatment of surface oscillations coupled to the motion of individual nucleons.

Let the surface of the nucleus, in polar coordinates, be given by $R(\vartheta, \varphi)$. We expand R in spherical harmonics, writing

$$R(\vartheta, \varphi) = R_0 (1 + \sum_{\lambda, \mu} \alpha_{\lambda, \mu} Y_{\lambda, \mu}(\vartheta, \varphi)), \tag{1}$$

where R_0 is the radius of the nucleus in its spherical equilibrium shape. The function $Y_{\lambda, \mu}$ is the normalized spherical harmonic of order λ, μ; the phase factor is that used by CONDON and SHORTLEY[18)]. The expansion parameters $\alpha_{\lambda, \mu}$ are the coordinates which describe the deformation of the nuclear surface. Since R is real, we have $\alpha_{\lambda, \mu} = (-)^{\mu} \alpha^{*}_{\lambda, -\mu}$.

The idea of a continuous nuclear surface does not apply if we consider surface elements of linear dimensions comparable with, or smaller than, the distance between the nucleons. The quantities $\alpha_{\lambda, \mu}$ therefore lose their meaning if λ becomes of the order of, or larger than, $A^{1/3}$.

If the coefficients $\alpha_{\lambda, \mu}$ are small, the potential energy of deformation takes the form

$$V = \frac{1}{2} \sum_{\lambda, \mu} C_{\lambda} |\alpha_{\lambda, \mu}|^2, \tag{2}$$

while the associated kinetic energy is given by

$$T = \frac{1}{2} \sum_{\lambda, \mu} B_{\lambda} |\dot{\alpha}_{\lambda, \mu}|^2. \tag{3}$$

The quantities B_{λ} and C_{λ} depend on more detailed assumptions regarding the properties of nuclear matter.

For an incompressible nucleus of constant density ϱ_0, one finds

$$B_\lambda = \frac{1}{\lambda} \varrho_0 R_0^5, \tag{4}$$

assuming nuclear matter to have irrotational flow. If, moreover, the charge of the nucleus Ze is uniformly distributed over its volume, one obtains

$$C_\lambda = (\lambda - 1)(\lambda + 2) R_0^2 S - \frac{3}{2\pi} \frac{\lambda - 1}{2\lambda + 1} \frac{Z^2 e^2}{R_0}, \tag{5}$$

where S is the surface tension. As an approximate estimate of S we may use the average value

$$4\pi R_0^2 S = 15.4 A^{2/3} \text{MeV}, \tag{6}$$

deduced from nuclear binding energies[19].

It is to be emphasized that the above expressions refer to a greatly simplified nuclear model. The nuclear deformability may vary considerably from nucleus to nucleus and may be strongly influenced by shell structures in the nucleon binding*.

In order to solve the dynamical problem given by equations (2) and (3) we introduce the momentum conjugate to $\alpha_{\lambda,\mu}$ defined by

$$\pi_{\lambda,\mu} = \frac{\partial T}{\partial \dot{\alpha}_{\lambda,\mu}} = B_\lambda \dot{\alpha}^*_{\lambda,\mu}. \tag{7}$$

The Hamiltonian of the nuclear surface now takes the form

$$H_S = T + V = \sum_{\lambda,\mu} \left\{ \frac{1}{2 B_\lambda} |\pi_{\lambda,\mu}|^2 + \frac{C_\lambda}{2} |\alpha_{\lambda,\mu}|^2 \right\} \tag{8}$$

and the surface oscillations may thus be considered as a system of harmonic oscillators with frequencies

$$\omega_\lambda = \sqrt{\frac{C_\lambda}{B_\lambda}} \tag{9}$$

and mass coefficients B_λ.

* The writer is indebted to Drs. B. MOTTELSON and W. J. SWIATECKI for illuminating comments on this point.

For the problem of the interaction of the surface oscillations with the single particle motion, it is convenient to make a change of coordinates in such a way that the kinetic energy of the surface separates into a vibrational and a rotational part. This coordinate transformation will be considered in the following paragraphs for deformations of order $\lambda = 2$. A particle moving in the nuclear field interacts, to a first approximation, only with deformations of even order, and of these the lowest order $\lambda = 2$ is of special interest. Already deformations of order $\lambda = 4$ can be shown to be of much less importance in the present connection.

2) Deformations of order two.

The general deformation of order 2 represents, for small values of the α-coefficients, an ellipsoid oriented at random in space. Instead of characterizing this deformation by the five coordinates α_μ (we drop the index $\lambda = 2$), it may be described by three angular coordinates specifying the orientation of the ellipsoid and two internal parameters determining its shape. (The sum of the three principal axes of the ellipsoid, to a first approximation, remains constant during the deformation).

Consider a coordinate system K' whose axes coincide with the principal axes of the ellipsoid. The orientation of K' with respect to a fixed frame of reference K will be characterized by three Eulerian angles θ, φ, ψ of which θ, φ represent the polar angles of the z'-axis in the K-system, while $\theta, \pi-\psi$ denote the polar angles of the z-axis in the K'-system. It is often convenient to write $(\theta_i) = (\theta_1, \theta_2, \theta_3) = (\theta, \varphi, \psi)$.

The deformation defined by the α_μ in K is in K' given by the coefficients

$$a_\nu = \sum_{\mu=-2}^{2} \alpha_\mu D_{\mu\nu}(\theta_i), \tag{10}$$

where $D_{\mu\nu}(\theta_i)$ are the transformation functions for the spherical harmonics of order 2.

For our special choice of K', we have $a_2 = a_{-2}$ and $a_1 = a_{-1} = 0$. Taking the inverse of (10) we get, since $D_{\mu\nu}$ is a unitary matrix,

$$\alpha_\mu = \sum_\nu a_\nu D^*_{\mu\nu}(\theta_i). \tag{11}$$

Equations (10) and (11) define a coordinate transformation from the five α_μ to the new coordinates a_0, a_2, θ_i.

These equations do not define the new coordinates uniquely. If we restrict ourselves to right-handed coordinate systems, there are in fact 24 ways of choosing K' with axes along the ellipsoidal axes. To each choice corresponds a particular set of coordinates a_ν, θ_i. If the deformation possesses rotational symmetry, there are even an infinite number of ways of choosing K'. The ambiguity in the coordinates is of significance for the symmetry properties of the wave function and will be considered later (§ III,3).

It is convenient to make the further substitution

$$\left.\begin{aligned} a_0 &= \beta \cos\gamma \\ a_2 &= a_{-2} = \frac{1}{\sqrt{2}}\beta \sin\gamma. \end{aligned}\right\} \tag{12}$$

The set of coordinates β, γ, θ_i will be referred to as β_μ. Due to the unitary character of $D_{\mu\nu}$ we have

$$\sum_\mu |\alpha_\mu|^2 = \sum_\nu a_\nu^2 = a_0^2 + 2\,a_2^2 = \beta^2 \tag{13}$$

and β is thus a measure of the total deformation of the nucleus. From (2) and (13) we have

$$V = \frac{1}{2} C \beta^2 \tag{14}$$

for the potential energy of deformation.

The increments of the three axes of the ellipsoid, for a deformation β, γ, is given by

$$\delta R_\varkappa = \sqrt{\frac{5}{4\pi}}\beta R_0 \cos\left(\gamma - \varkappa\frac{2\pi}{3}\right), \tag{15}$$

where $\varkappa = 1, 2, 3$ for the x', y', z'-axes, respectively.

If we define the eccentricities by

$$e_3 = 1 - \frac{R_2^2}{R_1^2} \approx \frac{2}{R_0}(\delta R_1 - \delta R_2) \tag{16}$$

and the cyclic permutations, we find

$$e_\varkappa = \sqrt{\frac{15}{\pi}}\,\beta \sin\left(\gamma - \varkappa\frac{2\pi}{3}\right). \tag{17}$$

The coordinate γ is a shape parameter which describes the deviation from rotational symmetry. For $\gamma = 0 + p\frac{\pi}{3}$, where p

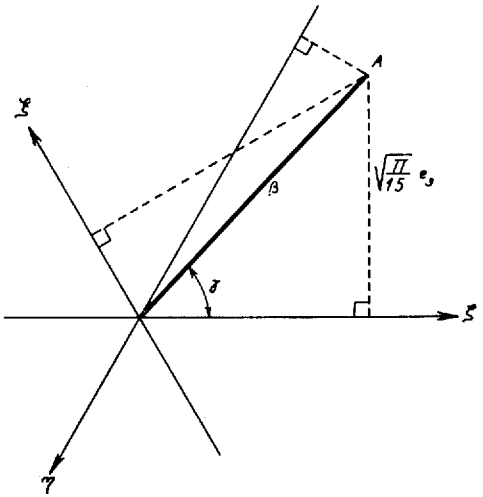

Fig. 1. The nuclear deformation is characterized by the point A in the polar diagram. The radius vector equals the total deformation parameter β, while the polar angle, measured from the ζ-axis, is given by the shape parameter γ. The eccentricities e_1, e_2, and e_3 are equal to $\sqrt{\frac{\pi}{15}}$ times the distance of A from the ξ-, η-, and ζ-axes, respectively. If A falls on the ξ-, η-, or ζ-axis, the nucleus possesses rotational symmetry with respect to its x'-, y'- or z'-axis, respectively.

is an integer, two of the axes are equal and the ellipsoid becomes a spheroid. A diagram of nuclear deformations is shown in Fig. 1.

3) Vibrational and rotational energy.

In order to express the kinetic energy of the oscillating nucleus in the β_μ-coordinates, we derive from (11)

$$\dot{\alpha}_\mu = \sum_\nu \dot{a}_\nu D^*_{\mu\nu}(\theta_i) + \sum_{\nu,j} a_\nu \dot{\theta}_j \frac{\partial}{\partial\theta_j} D^*_{\mu\nu}(\theta_i). \tag{18}$$

If we introduce this value for $\dot{\alpha}_\mu$ in (3), the kinetic energy splits into three terms. The first term is quadratic in $\dot{\alpha}_\nu$ and represents vibrations by which the ellipsoid changes its shape, but retains its orientation. The second term, quadratic in $\dot{\theta}_i$, represents a rotation of the ellipsoid without change of shape. The third term, which contains mixed time derivatives $\dot{\alpha}_\nu \dot{\theta}_i$, vanishes, as can be shown from simple properties of the $D_{\mu\nu}$-coefficients and their derivatives.

We may thus write

$$T = T_{\text{vib}} + T_{\text{rot}}. \tag{19}$$

For the vibrational energy, one gets immediately

$$T_{\text{vib}} = \frac{1}{2} B \sum_\nu |\dot{\alpha}_\nu|^2 = \frac{1}{2} B (\dot{\beta}^2 + \beta^2 \dot{\gamma}^2) \tag{20}$$

by means of (12).

To obtain a convenient form for T_{rot}, we use the relation

$$\sum_j \dot{\theta}_j \frac{\partial}{\partial \theta_j} D_{\mu\nu}(\theta_i) = i \sum_{\varkappa, m} q_\varkappa D_{\mu m}(\theta_i) (M_\varkappa)_{m\nu}, \tag{21}$$

where the $M_\varkappa$ are five-dimensional matrices well known from the quantum mechanical representation of an angular momentum of two units. They obey the commutation rules

$$M_\varkappa M_\lambda - M_\lambda M_\varkappa = -i M_{\varkappa \times \lambda}, \tag{22}$$

where the index $\varkappa \times \lambda$ refers to the axis formed by a vector product of the $\varkappa$- and λ-axes. A representation is used in which $(M_3)_{m\nu} = m \delta_{m\nu}$. The quantities $q_\varkappa$ denote the components of angular velocity of the ellipsoid along its axes, and may be written

$$q_\varkappa = \sum_j q_{\varkappa j} \dot{\theta}_j. \tag{23}$$

The coefficient matrix $q_{\varkappa j}$ has been given by CASIMIR[20] (p. 15)*.

By means of (18) and (21) one now obtains from (3)

$$T_{\text{rot}} = \frac{1}{2} B \sum_{\nu, \nu'} a_\nu a_{\nu'} \sum_{\varkappa, \varkappa'} q_\varkappa q_{\varkappa'} (M_\varkappa M_{\varkappa'})_{\nu\nu'} \tag{24}$$

* The angles here denoted by θ, φ, ψ are equal to CASIMIR's ϑ, $\psi - \frac{\pi}{2}$, $\varphi + \frac{\pi}{2}$.

by making use of the unitary character of $D_{\mu\nu}$. In this expression, the terms with $\varkappa \neq \varkappa'$ vanish, since ν takes only even values.

We therefore get the familiar type of expression

$$T_{\text{rot}} = \frac{1}{2} \sum_{\varkappa} q_{\varkappa}^2 \mathcal{J}_{\varkappa}, \tag{25}$$

where the moments of inertia $\mathcal{J}_{\varkappa}$ are given by

$$\mathcal{J}_{\varkappa} = B \sum_{\nu\nu'} a_{\nu} a_{\nu'} (M_{\varkappa}^2)_{\nu\nu'} \tag{26}$$

or

$$\mathcal{J}_{\varkappa} = 4 B \beta^2 \sin^2 \left\{ \gamma - \varkappa \frac{2\pi}{3} \right\} \tag{27}$$

by means of (12).

The moments of inertia are proportional to the square of the deformation parameter β. In terms of the eccentricities (17), one may write

$$\mathcal{J}_{\varkappa} = \frac{4\pi}{15} B e_{\varkappa}^2 = \frac{1}{4} \mathcal{J}_0 e_{\varkappa}^2, \tag{28}$$

where $\mathcal{J}_0$ denotes the moment of inertia of the entire nucleus with respect to its center. The circumstance that, for small deformations, $\mathcal{J}_{\varkappa} \ll \mathcal{J}_0$, shows that only a small fraction of the nuclear matter is effectively involved in the rotational motion.

4) Angular momentum of the nucleus.

The angular momentum $\vec{\mathfrak{M}}$ of the oscillating nucleus may be determined directly from the expression

$$\vec{\mathfrak{M}} = \int \varrho \, (\vec{r} \times \vec{v}) \, d\tau, \tag{29}$$

where the integral is extended over the volume of the nucleus.

For irrotational flow, the velocity $\vec{v}$ of the nuclear fluid may be written

$$\vec{v} = -\operatorname{grad} \chi, \tag{30}$$

where, for small deformations, the velocity potential is given by

$$\chi = -\sum_{\mu} \frac{1}{2} r^2 \dot{\alpha}_{\mu} Y_{\mu}(\vartheta, \varphi). \tag{31}$$

One thus finds

$$\vec{\mathcal{M}} = \frac{1}{2}\varrho_0 R_0^5 \sum_{\mu,\mu'} \alpha_\mu^* \dot{\alpha}_{\mu'} \int Y_\mu^* \left\{\vec{r}\times\mathrm{grad}\right\} Y_{\mu'}\, d\omega \tag{32}$$

or, by means of (4),

$$\vec{\mathcal{M}} = iB \sum_{\mu,\mu'} \alpha_\mu^* \dot{\alpha}_{\mu'} \vec{M}_{\mu'\mu}, \tag{33}$$

where the components of $\vec{M}$ along the x, y, and z-axes are given by the M_1, M_2, and M_3 of the previous paragraph.

In introducing the transformation (11), we make use of the relation

$$\sum_{\mu,\mu'} D_{\mu\nu}^* \vec{M}_{\mu\mu'} D_{\mu'\nu'} = \vec{M}'_{\nu\nu'}, \tag{34}$$

where the vector on the right hand side has components M_1, M_2, and M_3 along the axes of the ellipsoid. According to (18), the angular momentum becomes a sum of two terms. The first term contains $\dot{a}_\nu$ and can be shown to vanish; the second gives, by application of (21),

$$\mathcal{M}_\varkappa = B \sum_{\nu,\nu'} a_\nu a_{\nu'} \sum_{\varkappa'} q_{\varkappa'} (M_{\varkappa'} M_\varkappa)_{\nu'\nu} \tag{35}$$

or, by means of (26),

$$\mathcal{M}_\varkappa = q_\varkappa \mathcal{J}_\varkappa, \tag{36}$$

corresponding to (25). We may thus write

$$T_{\mathrm{rot}} = \frac{1}{2} \sum_\varkappa \frac{\mathcal{M}_\varkappa^2}{\mathcal{J}_\varkappa} \tag{37}$$

as an alternative form for the rotational energy.

III. Quantum Theory of Nuclear Surface Oscillations.

1) Stationary states of oscillating nucleus.

We assume that the quantum mechanical description of nuclear hydrodynamics can be derived from the classical equations of motion by the usual procedure of quantization.

In order to classify the stationary states of the nucleus it is

convenient to take the Hamiltonian in the form (8), a form well known from the quantum theory of fields (cf. WENTZEL[21], p. 33). The excitation of the nucleus may then be described in terms of the five variables n_μ, the occupation numbers, with eigenvalues 0, 1, 2, ...

In the usual way, we put

$$\left.\begin{aligned} \alpha_\mu &= \sqrt{\frac{\hbar}{2B\omega}}\,(b_\mu + (-)^\mu b^*_{-\mu}) \\ \pi_\mu &= i\sqrt{\frac{\hbar B\omega}{2}}\,(b^*_\mu - (-)^\mu b_{-\mu}), \end{aligned}\right\} \tag{38}$$

where ω is given by (9). We are restricting ourselves to $\lambda = 2$ and have dropped the index λ. The matrices b_μ and their Hermitian conjugates b^*_μ obey the relations

$$\left.\begin{aligned} b_\mu b^*_\mu &= n_\mu + 1 \\ b^*_\mu b_\mu &= n_\mu \end{aligned}\right\} \tag{39}$$

and are the destruction and creation operators.

In the n_μ-representation, the Hamiltonian is diagonal and has the eigenvalues

$$E_N = \hbar\omega \sum_\mu \left(\frac{1}{2} + n_\mu\right) = \hbar\omega\left(\frac{5}{2} + N\right), \tag{40}$$

where $N = \sum_\mu n_\mu = 0, 1, 2, \ldots\ldots$

Each energy level is as many times degenerate as the N excitation quanta can be distributed on the five individual modes of oscillation. In order to determine the angular momentum quantum numbers of the excited states, we consider the operator $\mathfrak{M}_z$ given by (33). By means of (7), (38), and (39) one finds that $\mathfrak{M}_z$ is diagonal in the n_μ-representation and has the eigenvalues

$$\mathfrak{M}_z = \hbar \sum_{\mu=-2}^{2} \mu n_\mu. \tag{41}$$

It follows that the excitation quanta are equivalent to Bose-Einstein particles, phonons, of spin 2. The number n_μ represents

the number of phonons having a component $\mu\hbar$ of angular momentum along the z-axis.

From this equivalence one can determine the angular momenta I of the excitation states having a given N. For the first few values of N one finds

$$\left.\begin{array}{ll} N = 0 & I = 0 \\ N = 1 & I = 2 \\ N = 2 & I = 0, 2, 4 \\ N = 3 & I = 0, 2, 3, 4, 6. \end{array}\right\} \quad (42)$$

In the same manner, the states corresponding to the oscillations with $\lambda > 2$ can be characterized as excitation of Bose-Einstein quanta of spin λ.

As regards the order of magnitude of the level spacing corresponding to $\lambda = 2$, it may be noted that, for a medium heavy nucleus with $A = 100$, the approximate expressions (4) and (5) give a value for $\hbar\omega$ of 2.3 MeV. For a heavy nucleus ($A = 200$), one finds $\hbar\omega = 1.3$ MeV.

Whereas these values indicate the general order of magnitude of the excitation energies for nuclear surface oscillations, simple level systems of this type are not to be expected in actual nuclei. Apart from the influence of shell structure on the nuclear deformability, the coupling of the surface modes to the motion of individual nucleons will in general give rise to a more composite level structure. For the treatment of this coupling effect, it is often convenient to consider the surface oscillations in the coordinates β_μ.

2) Transformation to coordinates β_μ.

In Section II, the Hamiltonian has been given in terms of the coordinates β_μ and their time derivatives. In constructing the wave equation in this set of coordinates we form the expression

$$ds^2 = 2\,T\,dt^2 = \sum_{\mu,\nu} G_{\mu\nu}\,d\beta_\mu\,d\beta_\nu. \quad (43)$$

The matrix $G_{\mu\nu}$ may be found from equations (20), (23), (25), and (27).

The Hamiltonian now takes the form (cf. PAULI[22], p. 120)

$$H_S = T + V = -\frac{\hbar^2}{2}\sum_{\mu\nu} G^{-\frac{1}{2}}\frac{\partial}{\partial\beta_\mu} G^{\frac{1}{2}} G^{\mu\nu}\frac{\partial}{\partial\beta_\nu} + V, \tag{44}$$

where $G^{\mu\nu}$ is the reciprocal of $G_{\mu\nu}$. The determinant of $G_{\mu\nu}$ is denoted by G and is found to be

$$G = 4B^5\beta^8 \sin^2 3\gamma \sin^2\theta \tag{45}$$

by means of the relation

$$\sin\gamma \sin\left(\gamma - \frac{2\pi}{3}\right)\sin\left(\gamma + \frac{2\pi}{3}\right) = -\frac{1}{4}\sin 3\gamma. \tag{46}$$

The volume element is proportional to $\left|G^{\frac{1}{2}}\right|$ and may be taken as

$$d\tau = \beta^4 \left|\sin 3\gamma\right| \sin\theta\, d\beta\, d\gamma\, d\theta\, d\varphi\, d\psi. \tag{47}$$

The potential energy is given by (14), while the kinetic energy may be written in the form (19) with

$$T_{\text{vib}} = -\frac{\hbar^2}{2B}\left\{\frac{1}{\beta^4}\frac{\partial}{\partial\beta}\beta^4\frac{\partial}{\partial\beta} + \frac{1}{\beta^2}\frac{1}{\sin 3\gamma}\frac{\partial}{\partial\gamma}\sin 3\gamma\frac{\partial}{\partial\gamma}\right\}. \tag{48}$$

The rotational energy T_{rot} has the form of the kinetic energy of a top (in general asymmetric) with moments of inertia $\mathcal{J}_\varkappa$.

It is convenient to express T_{rot} in terms of the angular momentum components along the axes of the ellipsoid. Putting

$$\mathcal{M}_\varkappa = \hbar\, Q_\varkappa, \tag{49}$$

we have from (37)

$$T_{\text{rot}} = \sum_\varkappa \frac{\hbar^2}{2\mathcal{J}_\varkappa} Q_\varkappa^2. \tag{50}$$

The operators $Q_\varkappa$ may be expressed as differential operators in the θ_i (cf. CASIMIR[29], p. 57) or they may be considered as matrices, obeying the commutation relations

$$Q_\varkappa Q_\lambda - Q_\lambda Q_\varkappa = -i Q_{\varkappa\times\lambda}. \tag{51}$$

We use the representation in which Q_3 is diagonal. The $Q_\varkappa$ commute with the components $\hbar Q_x$, $\hbar Q_y$, $\hbar Q_z$ of the angular momentum along the fixed coordinate axes. The latter operators obey relations similar to (51), but with a change of sign of i.

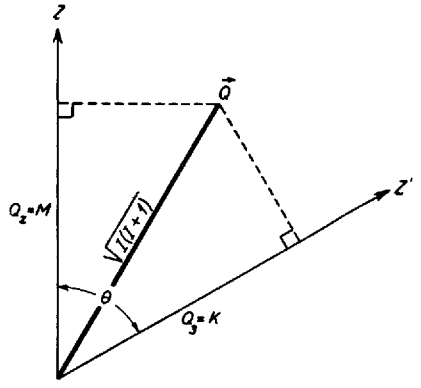

Fig. 2. Angular momentum diagram for the oscillating nucleus. The angular momentum vector $\vec{Q}$, of length $\sqrt{I(I+1)}$, has components Q_z, with eigenvalues M, about the fixed z-axis, and Q_3, with eigenvalues K, about the z'-axis of the nucleus.

Since T_{rot} involves β only as a factor β^{-2}, the stationary state wave functions separate in the following way:

$$\Psi(\beta_\mu) = f(\beta)\Phi(\gamma, \theta_i), \tag{52}$$

with Φ and f obeying the equations

$$\left\{-\frac{1}{\sin 3\gamma}\frac{\partial}{\partial\gamma}\sin 3\gamma\frac{\partial}{\partial\gamma}+\frac{1}{4}\sum_{\varkappa}\frac{Q_\varkappa^2}{\sin^2\left(\gamma-\varkappa\frac{2\pi}{3}\right)}\right\}\Phi = \Lambda\Phi \tag{53}$$

and

$$\left\{-\frac{\hbar^2}{2B}\frac{1}{\beta^4}\frac{\partial}{\partial\beta}\beta^4\frac{\partial}{\partial\beta}+\frac{1}{2}C\beta^2+\frac{\Lambda\hbar^2}{2B\beta^2}\right\}f = E_S f, \tag{54}$$

where E_S is the total oscillation energy. For a particular value of $Q^2 = I(I+1)$ and $Q_z = M$, we may write Φ in the form

$$\Phi_M^{I,\tau}(\gamma, \theta_i) = \sum_{K=-I}^{I} g_K^{I,\tau}(\gamma)\mathcal{D}_{M,K}^I(\theta_i), \tag{55}$$

where the functions $\mathfrak{D}^I_{MK}(\theta_i)$ give a $(2I+1)$-dimensional representation of the rotation group. These functions are eigenfunctions for the operators Q_z and Q_3 belonging to the eigenvalues M and K (see Fig. 2). The symbol τ stands for the two quantum numbers which, in addition to I and M, characterize the eigenstates Φ.

3) Symmetry properties of the wave function.

A configuration of the nucleus determines the surface coordinates α_μ uniquely, but, as already mentioned, there is no one to one correspondence between the coordinates α_μ and β_μ. The α_μ determine the principal axes of the ellipsoid, but the β_μ depend also on the designation of these axes as the 1,2,3 axis, respectively, and on the choice of positive direction for the three axes.

Restricting ourselves to right hand coordinate systems, there are 24 sets of β_μ values which correspond to the same set of α_μ. In the special case of symmetry with respect to one or more of the axes, the arbitrariness in the choice of β_μ is even continuously manifold.

Since the wave function is one-valued in the α_μ, it must be invariant with respect to those transformations of the β_μ which leave the α_μ unaltered. These may be expressed in terms of three basic transformation operators R_1, R_2, and R_3.

R_1 corresponds to a reversal of the 2 and 3 axes and may be defined by the scheme

$$R_1(\gamma, \theta, \varphi, \psi) = (\gamma, \pi - \theta, \varphi + \pi, -\psi) \tag{56}$$

which implies

$$R_1 \mathfrak{D}^I_{MK}(\theta_i) = \exp\{i\pi(I+K)\}\mathfrak{D}^I_{M,-K}(\theta_i). \tag{57}$$

The total deformation parameter β is uniquely determined by the α_μ and is left unaltered by all the transformations in question. We have $R_1^2 = 1$.

R_2 corresponds to a rotation through 90° of the 1 and 2 axes with respect to the 3 axis and is given by

$$R_2(\gamma, \theta, \varphi, \psi) = \left(-\gamma, \theta, \varphi, \psi + \frac{\pi}{2}\right) \tag{58}$$

and

$$R_2 \mathcal{D}^I_{MK}(\theta_i) = \exp\left\{i\frac{\pi}{2}K\right\} \mathcal{D}^I_{M,K}(\theta_i). \tag{59}$$

It is seen that $R_4^2 = 1$.

Finally, R_3 corresponds to a cyclic permutation of the three axes. The transformation of the θ_i is expressed most simply in terms of the rotation with Eulerian angles $\varphi_i = \left(\frac{\pi}{2}, 0, \frac{\pi}{2}\right)$, which performs the permutation of the axes. We have

$$\left.\begin{aligned} R_3(\gamma) &= \left(\gamma - \frac{2\pi}{3}\right) \\ R_3 \mathcal{D}^I_{MK}(\theta_i) &= \sum_{K'} \mathcal{D}^I_{MK'}(\theta_i) \mathcal{D}^I_{K',K}(\varphi_i) \end{aligned}\right\} \tag{60}$$

and $R_3^3 = 1$.

The 24 transformations of the β_μ can all be written in the form

$$R = R_1^{s_1} R_2^{s_2} R_3^{s_3}, \tag{61}$$

where s_1, s_2, and s_3 are integers. Therefore, the solutions to (53), which represent nuclear states, are invariant with respect to the transformations R_1, R_2, and R_3.

If we use the form (55) for the wave function, the symmetry requirements impose certain restrictions on the functions $g_K(\gamma)$. From (57) and (59) follows

$$g_K(\gamma) = \exp\{i\pi(I+K)\} g_{-K}(\gamma) \tag{62}$$

$$g_K(\gamma) = \exp\left\{i\frac{\pi}{2}K\right\} g_K(-\gamma). \tag{63}$$

Similarly, the relation

$$g_K(\gamma) = \sum_{K'} \mathcal{D}^I_{KK'}(\varphi_i) g_{K'}\left(\gamma - \frac{2\pi}{3}\right) \tag{64}$$

is demanded to ensure invariance with respect to R_3.

Applying (63) twice, one sees that g_K vanishes if K is odd. In the terminology used to classify the stationary states of an asymmetric top (cf. CASIMIR[20], p. 61) we may say that only the even ($\mathcal{E}$) classes of states exist. From (62) follows

$$g_K(\gamma) = (-)^I g_{-K}(\gamma), \tag{65}$$

showing that, for I even, only the symmetrical class $\mathcal{E}^+$ is allowed, for I odd only the antisymmetrical class $\mathcal{E}^-$. It is a particular consequence that no state of $I = 1$ exists. The conditions (63) and (64) effectively limit the variation interval for γ to $0 \leq \gamma < \frac{\pi}{3}$.

The symmetry requirements for the states of the vibrating and rotating nucleus are analogous to those relating to certain types of molecules containing identical nuclei of spin zero.

4) States with $I = 0$.

In general, the wave equation (53) consists of a number of coupled differential equations in γ of rather complicated character, and for free oscillations of the nuclear surface it is often easier to consider the equations of motion in the coordinates α_μ. As an example of the solution of the wave equation in the β_μ-coordinates we take the simple case of $I = 0$, in which the wave function is independent of θ_i. Moreover, (63) and (64) demand that Φ be a one-valued function of $\cos 3\gamma$. The equation (53) thus is reduced to the familiar equation for Legendre polynomials $P_\lambda(\cos 3\gamma)$ with

$$\Lambda = 9\lambda(\lambda + 1) \qquad \lambda = 0, 1, 2 \ldots . \tag{66}$$

as eigenvalues.

Equation (54) for $f(\beta)$ may now be solved by putting

$$f(\beta) = h(\beta) \exp\left\{-\frac{1}{2}\alpha B \beta^2\right\} \tag{67}$$

with

$$\alpha = \frac{1}{\hbar}\sqrt{\frac{C}{B}} = \frac{\omega}{\hbar} \tag{68}$$

and $h(\beta)$ a polynomial. For small β, the leading term in $h(\beta)$ is $\beta^{3\lambda}$. If the highest power occurring in $h(\beta)$ is denoted by ν, the eigenvalues for the oscillation energy are given by

$$E_\nu = \hbar\omega\left(\frac{5}{2}+\nu\right) \quad \nu = 3\lambda,\ 3\lambda+2, \cdots. \tag{69}$$

The ground state of the nucleus corresponds to $\nu = \lambda = 0$ and has the wave function

$$\Psi(\beta_\mu) = f(\beta) = \frac{8}{3}\pi^{-\frac{1}{2}}(\alpha B)^{5/2}\exp\left\{-\frac{1}{2}\alpha B\beta^2\right\}, \tag{70}$$

normalized for a volume element $d\tau = \beta^4 d\beta$ (cf. (47)). From (70) one finds

$$\beta_0^2 = \overline{\beta^2} = \frac{5}{2}\frac{1}{\alpha B} = \frac{5}{2}\frac{\hbar}{\sqrt{BC}} \tag{71}$$

as a measure of the zero point amplitude.

IV. Motion of Nucleons in a Deformed Nucleus.

The motion of a particle in a deformed nucleus has been considered by RAINWATER[8] who treated the nuclear field as a potential well with infinitely high walls. FEENBERG and HAMMACK[23] have discussed the problem of a finite well. These authors restricted themselves to nuclear deformations possessing cylindrical symmetry, in which case the particle wave equation can be studied in spheroidal coordinates. The shift of the energy levels, with respect to those of a spherical potential, were calculated in the first approximation containing linear terms in the deformation parameter.

In this approximation, the energy levels may also, for arbitrary deformations, be calculated by ordinary perturbation theory. For an infinite well one may use the method of the perturbed boundary conditions[24), 25]. We shall write the energy of the particle

$$H = H_p + H_{\text{int}}, \tag{72}$$

where

$$H_p = T_p + V_p(r) + D(\vec{l}\cdot\vec{s}) \tag{73}$$

is the particle energy for a spherical nucleus. We have included a spin orbit coupling of strength D.

H_{int} represents the interaction of the particle with the nuclear deformation and may, to first order in α_μ, be written in the form

$$H_{\text{int}} = -k(r)\sum_\mu \alpha_\mu Y_\mu(\vartheta, \varphi) \tag{74}$$

if we restrict ourselves to the harmonics of second order. The polar coordinates of the particle are denoted by r, ϑ, φ. If we can assume that the change in potential accompanying the nuclear deformation takes place only at the surface, k will contain the factor $\delta(r - R_0)$. For a potential well with infinite walls, the expectation value of k is given by

$$\overline{k} = 2\,\overline{T_p}, \tag{75}$$

where $\overline{T_p}$ is the average kinetic energy of the particle in the nucleus. For more general models, the expectation value of k will depend somewhat on the state of the particle, but it appears[23), 26)] that it will not deviate essentially from the simple expression (75); for a finite rectangular well it has been estimated to be some 25 % smaller[26)] for most particle states.

If the magnitude l of the orbital angular momentum of the particle remains a good quantum number for a deformed nucleus, the significant matrix elements of H_p take a simple form. This assumption implies that no near-lying single particle states with different l exist, which can perturb the state in question.

If, moreover, the spin orbit coupling is large compared with H_{int}, also j, the magnitude of the total angular momentum of the particle, will be an approximate constant of the motion. The component j_z will only be a constant in the special case of deformations which are symmetrical with respect to the z-axis ($\alpha_\mu = 0$ for $\mu \neq 0$).

For fixed j, the interaction energy may be written

$$H_{\text{int}} = kc_j \sum_{\mu} \alpha_\mu \pi_\mu, \tag{76}$$

where

$$c_j = \sqrt{\frac{5}{4\pi}} \frac{1}{2j(j+1)}, \tag{77}$$

and where the operators π_μ are given by

$$\left.\begin{aligned} \pi_0 &= \left(\frac{3}{2} j_z^2 - \frac{1}{2} j(j+1)\right) \\ \pi_{\pm 1} &= \mp \sqrt{\frac{3}{8}} \left\{ j_z (j_x \pm i j_y) + (j_x \pm i j_y) j_z \right\} \\ \pi_{\pm 2} &= \sqrt{\frac{3}{8}} (j_x \pm i j_y)^2. \end{aligned}\right\} \tag{78}$$

From this form of H_{int} the matrix elements can be immediately evaluated, for instance in a representation in which j_z is diagonal.

If the spin orbit coupling is not strong compared with the coupling of the particle to the nuclear deformation, j is no longer a good quantum number and the form (76) for H_{int} is not valid. Provided l remains approximately constant, we can still use a similar expression for H_{int}, obtained by replacing c_j by

$$c_l = \sqrt{\frac{5}{4\pi}} \frac{2}{(2l-1)(2l+3)} \tag{79}$$

and by replacing, in π_μ, the components of $\vec{j}$ with the components of $\vec{l}$.

H_{int} may also be expressed in relative coordinates which describe the motion of the particle with respect to the principal axes of the deformed nucleus. If we denote by j_1, j_2, and j_3 the components of $\vec{j}$ along these axes, we find from (76) and (78)

$$H_{\text{int}} = \frac{1}{2} kc_j \beta \left\{ \cos\gamma \left(3j_3^2 - j(j+1)\right) + \sqrt{3} \sin\gamma \, (j_1^2 - j_2^2) \right\}, \tag{80}$$

using the notation of (12).

For fixed β and γ, this interaction energy has the same form as the Hamiltonian for an asymmetric top, except for the fact that the effective moments of inertia, corresponding to (80), may be negative. The eigenvalues for H_{int} can be derived in complete analogy to the procedure used for the top.

The formulae given in this section apply most directly to nuclei which possess a single particle in addition to closed shells. However, for nuclei having closed shells minus one particle, the only necessary modification is a change of sign of the quantities k and D.

In the case of more complicated nuclei, which have to be described in terms of several particles in addition to, or lacking in, closed shells, the interaction with the nuclear surface takes a simple form, analogous to (76) or (80), only if the magnitude J of the total angular momentum of the particles is a constant of the motion. If the angular momenta of the particles are decoupled under the influence of the surface deformation, the interaction acquires a more complex character.

V. Coupling of Single Particle Motion to Nuclear Surface Oscillations.

1) Equations of motion.

The total system, nuclear surface + single particle, is described by the Hamiltonian

$$H = H_S + H_p + H_{\text{int}}. \tag{81}$$

Expressing the energy H_S of the surface oscillations in terms of the coordinates β, we have

$$H_S = T_{\text{vib}} + T_{\text{rot}} + \text{V}, \tag{82}$$

where the three terms are given by (48), (50), and (14), respectively. The particle energy H_p is given by (73) and, for the interaction energy, we shall use (76) or (80), assuming j to be a constant of the motion.

Even with this simplification, the equations of motion, describing the coupled system, are somewhat complex, but, in limiting cases, simple approximative solutions may be obtained.

If the interaction is weak, or more precisely, if H_{int} is small compared with the level spacing of the uncoupled system, the motion of the surface and of the particle may, in first approximation, be considered as independent. The effect of H_{int} may then be treated as a small perturbation, and it will be convenient to use the form (76) for H_{int}.

If the interaction is strong, we may use a procedure analogous to the treatment of molecular structures. In this approximation, one may consider the motion of the particle relative to the axes of the nucleus, corresponding to the form (80) for H_{int}. The vibration and rotation of the nucleus, to a first approximation, have only an adiabatic influence on the particle motion. Non-adiabatic effects can be calculated as small perturbations.

The value of H_{int} depends on the single particle state in question and on its orientation with respect to the deformed nucleus. For the various nuclear species, we may expect levels of both types, corresponding to weak and strong interaction, respectively. We shall, in particular, consider the strong coupling case which appears, in many instances, to give the best representation of nuclear ground states, as is indicated by the large empirical quadrupole moments.

The total angular momentum $\vec{I}$ of the nucleus is the sum of two parts, $\vec{Q}$ and $\vec{j}$, referring to the surface oscillations and the particle motion, respectively. Eliminating $\vec{Q}$ from (50), we may write

$$T_{\text{rot}} = \sum_{\varkappa} \frac{\hbar^2}{2\mathcal{J}_\varkappa} (I_\varkappa - j_\varkappa)^2 \tag{83}$$

for the rotational energy of the nucleus.

The commutation rules for the components of $\vec{I}$ and $\vec{j}$ along the axes of the nucleus are

$$\left.\begin{aligned} I_\varkappa I_\lambda - I_\lambda I_\varkappa &= -iI_{\varkappa\times\lambda} \\ j_\varkappa j_\lambda - j_\lambda j_\varkappa &= ij_{\varkappa\times\lambda} \\ I_\varkappa j_\lambda - j_\lambda I_\varkappa &= 0. \end{aligned}\right\} \tag{84}$$

From these expressions the commutators involving the Q_k may be derived.

In the strong coupling approximation, it is convenient to use a representation in which I_3 and j_3 are diagonal. Their eigenvalues will be denoted by K and Ω, respectively (cf. Fig. 3). The wave function for a stationary state of the system thus takes the form

$$\Psi_M^{I,\tau} = \sum_{\Omega,K} \varphi_{\Omega,K}^{I,\tau}(\beta,\gamma)\,\chi_\Omega \mathfrak{D}_{MK}^I(\theta_i), \tag{85}$$

which is a generalization of (55). The function χ_Ω represents a single particle state with $j_3 = \Omega$. The states of the nucleus are

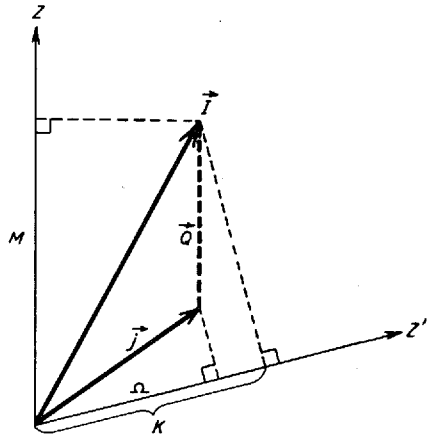

Fig. 3. Angular momentum diagram for the quasi-molecular nuclear model. The total angular momentum $\vec{I}$ of the nucleus is the resultant of the single particle angular momentum $\vec{j}$ and the rotation vector $\vec{Q}$ of the oscillating nucleus. The projections of $\vec{j}$ and $\vec{I}$ along the z'-axis of the nucleus, which become approximate constants of the motion in the strong coupling approximation, have eigenvalues Ω and K, respectively.

characterized by the quantum numbers I and M, referring to the absolute value of the nuclear angular momentum and its component along a fixed z-axis, respectively. Additional quantum numbers are specified by the index τ.

2) Symmetry properties of the wave function.

If the single particle motion is described relatively to the axes of the deformed nucleus, the transformation operators R_1, R_2, and R_3, introduced in (III, 3), also act on the particle wave functions. In the representation corresponding to (85) one finds

$$R_1 \chi_\Omega = \exp\{-i\pi(j+\Omega)\}\,\chi_{-\Omega} \tag{86}$$

$$R_2 \chi_\Omega = \exp\left\{-i\frac{\pi}{2}\Omega\right\}\chi_\Omega \tag{87}$$

$$R_3 \chi_\Omega = \sum_{\Omega'} \mathfrak{D}^{j*}_{\Omega',\Omega}(\varphi_i)\,\chi_{\Omega'} \tag{88}$$

in analogy to (57), (59), and (60).

The invariance of Ψ with respect to R_2^2 thus implies that $\varphi_{\Omega,K}$ vanishes except for

$$K - \Omega = 2\nu \qquad \nu = 0, \pm 1, \pm 2, \ldots . \tag{89}$$

corresponding to the even class of states. The condition $R_1\,\Psi = \Psi$ next demands

$$\varphi_{\Omega,K}(\beta,\gamma) = (-)^{I-j}\varphi_{-\Omega,-K}(\beta,\gamma), \tag{90}$$

and Ψ may, therefore, be expressed in the form

$$\Psi^{I,\tau}_M = \sum_{\substack{\Omega \geq 0 \\ K}} \varphi^{I,\tau}_{\Omega,K}\left\{\chi_\Omega \mathfrak{D}^I_{MK} + (-)^{I-j}\chi_{-\Omega}\mathfrak{D}^I_{M,-K}\right\}. \tag{91}$$

For $I-j$ even, only the class $\mathfrak{E}^+$ exists, for $I-j$ odd, only the class $\mathfrak{E}^-$.

The invariance with respect to R_2 and R_3 imposes the further restrictions

$$\varphi_{\Omega,K}(\beta,\gamma) = (-)^\nu \varphi_{\Omega,K}(\beta,-\gamma) \tag{92}$$

and

$$\varphi_{\Omega,K}(\beta,\gamma) = \sum_{\Omega',K'} D^{j*}_{\Omega,\Omega'}(\varphi_i)\mathfrak{D}^I_{K,K'}(\varphi_i)\varphi_{\Omega',K'}\left(\beta,\gamma-\frac{2\pi}{3}\right) \tag{93}$$

on the functions $\varphi_{\Omega,K}$ (cf. (63) and (64)). The variation interval for γ is thereby effectively limited to $0 \leq \gamma < \frac{\pi}{3}$.

3) Strong coupling approximation.

In the case of strong interaction between the single particle and the nuclear surface oscillations, the wave equation possesses solutions for which the angular momentum components j_3 and I_3 are approximately constants of the motion.

In considering nuclear states of this type, it is convenient to divide the Hamiltonian into two parts of which the first, H_0, commutes with j_3 and I_3, while the second part, U, contains no diagonal elements in the j_3, I_3 representation.

We thus have

$$H = H_0 + U, \tag{94}$$

where

$$\left.\begin{aligned} H_0 = {} & T_{\text{vib}} + V + \left(\frac{\hbar^2}{4\mathscr{J}_1} + \frac{\hbar^2}{4\mathscr{J}_2}\right)\left(I(I+1) - I_3^2 + j(j+1) - j_3^2\right) \\ & + \frac{\hbar^2}{2\mathscr{J}_3}(I_3 - j_3)^2 + H_p + \frac{1}{2}kc_j\beta\cos\gamma\left(3j_3^2 - j(j+1)\right), \end{aligned}\right\} \tag{95}$$

and where

$$\left.\begin{aligned} U = {} & U_1 + U_2 + U_3 = -\left(\frac{\hbar^2}{\mathscr{J}_2} I_1 j_1 + \frac{\hbar^2}{\mathscr{J}_2} I_2 j_2\right) \\ & + \left(\frac{\hbar^2}{4\mathscr{J}_1} - \frac{\hbar^2}{4\mathscr{J}_2} + \frac{\sqrt{3}}{2}kc_j\beta\sin\gamma\right)(j_1^2 - j_2^2) + \left(\frac{\hbar^2}{4\mathscr{J}_1} - \frac{\hbar^2}{4\mathscr{J}_2}\right)(I_1^2 - I_2^2) \end{aligned}\right\} \tag{96}$$

according to (81), (82), (83), and (80). We first consider the solutions corresponding to the unperturbed Hamiltonian H_0; the influence of U will be discussed in the following section.

If U can be neglected, the nuclear state may be characterized by the quantum numbers I, M, K, and Ω, and the wave function (91) reduces to a single term if we look apart from the symmetry requirement (93).

The function $\varphi_{\Omega,K}(\beta,\gamma)$ is given by the equation

$$H_0(I_3 = K,\ j_3 = \Omega)\ \varphi(\beta,\gamma) = E\varphi(\beta,\gamma), \tag{97}$$

where E is the energy of the nucleus. This Hamiltonian may be considered as a sum of a kinetic energy T_{vib} and a potential energy $W(\beta,\gamma)$ which does not contain derivatives with respect to β or γ. From (14) and (27) one finds

$$\left.\begin{aligned} V(\beta,\gamma) = {} & H_p + \frac{1}{2}C\beta^2 + \frac{1}{2}kc_j\beta\cos\gamma\left(3\Omega^2 - j(j+1)\right) + \frac{\hbar^2}{8B\beta^2\sin^2\gamma}(K-\Omega)^2 \\ & + \frac{\hbar^2}{16B\beta^2}\left(\frac{1}{\sin^2\left(\gamma - \frac{2\pi}{3}\right)} + \frac{1}{\sin^2\left(\gamma + \frac{2\pi}{3}\right)}\right)\left(I(I+1) - K^2 + j(j+1) - \Omega^2\right). \end{aligned}\right\} \tag{98}$$

The particle energy H_p is a constant with respect to β and γ.

We shall in particular consider states with $K = \Omega$ which are found to have the smallest energy. The ground state of the nucleus is expected to be of this type. For $K = \Omega$ the potential (98) has a minimum for $\beta = \beta_1$ and $\gamma = \gamma_1$, where

$$\gamma_1 = \begin{cases} 0 \\ \pi \end{cases} \text{ for } 3\Omega^2 - j(j+1) \begin{cases} < 0 \\ > 0 \end{cases}, \tag{99}$$

while β_1 is given by the equation

$$C\beta_1 - \frac{1}{2} kc_j \left| 3\Omega^2 - j(j+1) \right| - \frac{\hbar^2}{3B\beta_1^3} \left(I(I+1) + j(j+1) - 2\Omega^2 \right) = 0, \tag{100}$$

which has a single positive root. The potential (98) also possesses other minima, but these lie at higher energies. The potential function is illustrated in Figs. 4 and 5.

The β_1, γ_1 configuration represents the equilibrium shape of the nucleus deformed under the influence of the single particle.

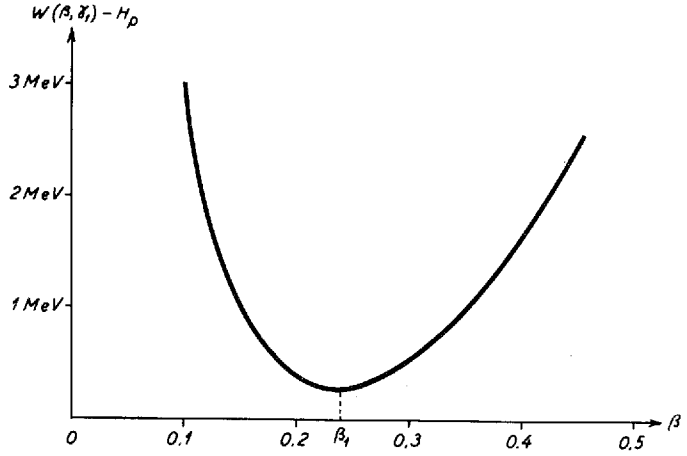

Fig. 4. Potential energy for nuclear vibrations, as a function of β for $\gamma = \gamma_1$. The potential has a minimum for $\beta = \beta_1$. For larger β, the increase of W is due to the surface tension, while the increase of W for small β is due partly to the coupling to the single particle motion and partly to the rotation vibration interaction. The numerical values given on the figure correspond to the example discussed on page 34.

This equilibrium shape possesses rotational symmetry with respect to the 3 axis of the nucleus, and we have therefore a wide-going analogy to the structure of linear molecules.

If we neglect the third term on the left hand side of (100), the equation for β_1 is equivalent to that obtained by RAINWATER[8] for the equilibrium deformation of the nucleus. The extra term represents the rotation-vibration interaction. The effect of this term becomes small in the limit of very strong coupling between

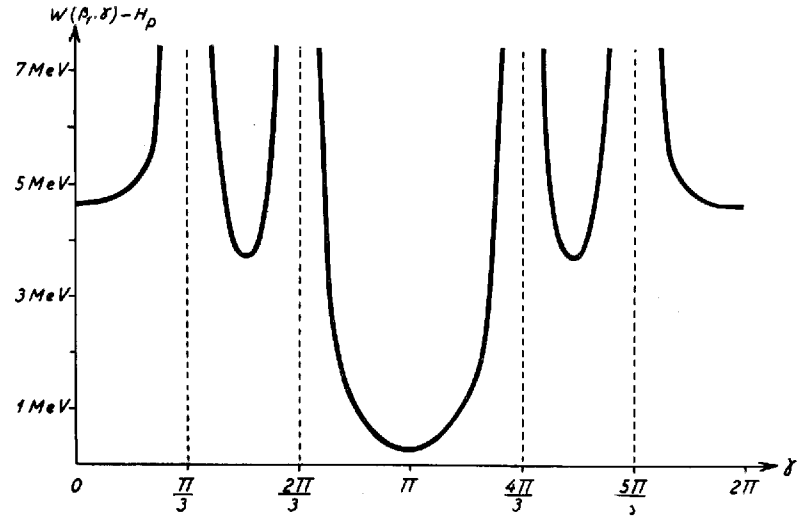

Fig. 5. Potential energy for nuclear vibrations as a function of γ, for $\beta = \beta_1$. The singularities in W for $\gamma = \pi \pm \frac{\pi}{3}$ and $\pi \pm \frac{2\pi}{3}$ are due to the rotation vibration interaction. The validity of the strong coupling approximation implies that the amplitude of the wave function is small for $|\pi - \gamma| \gtrsim \frac{\pi}{3}$. The curve illustrates the numerical example considered on page 34.

single particle and nuclear surface. In actual cases, however, the effect may be quite appreciable.

The nucleus will vibrate around the equilibrium shape and, for small oscillations, the potential may be approximated by the expression

$$W(\beta, \gamma) = W(\beta_1, \gamma_1) + \frac{1}{2} p (\beta - \beta_1)^2 + \frac{1}{2} q (\gamma - \gamma_1)^2 \quad (101)$$

with

$$p = C + \frac{\hbar^2}{B \beta_1^4} \left(I(I+1) + j(j+1) - 2\Omega^2 \right) \quad (102)$$

and

$$q = \frac{1}{2} k c_j \beta_1 |3\Omega^2 - j(j+1)| + \frac{2\hbar^2}{3B\beta_1^2}\left(I(I+1) + j(j+1) - 2\Omega^2\right). \quad (103)$$

With this form for the potential, the wave equation (97) may be solved approximately by separation of the coordinates β and γ. Putting

$$\varphi(\beta, \gamma) = \xi(x)\,\eta(y), \quad (104)$$

where $\beta_1 x = \beta - \beta_1$ and $y = \gamma - \gamma_1$, the wave equations for ξ and η become, by means of (48),

$$\left\{-\frac{\hbar^2}{2B\beta_1^2}\left(\frac{\partial^2}{\partial x^2} + 4\frac{\partial}{\partial x}\right) + \frac{1}{2} p \beta_1^2 x^2\right\} \xi(x) = E_\beta \xi(x) \quad (105)$$

and

$$\left\{-\frac{\hbar^2}{2B\beta_1^2}\frac{1}{y}\frac{\partial}{\partial y} y \frac{\partial}{\partial y} + \frac{1}{2} q y^2\right\} \eta(y) = E_\gamma \eta(y), \quad (106)$$

assuming throughout that $x \ll 1$ and $y \ll 1$. We have

$$E = W(\beta_1, \gamma_1) + E_\beta + E_\gamma \quad (107)$$

for the total energy of the nucleus.

Equation (105) may be solved by noting that the function $e^{2x}\xi(x)$ satisfies the wave equation for the linear harmonic oscillator. The eigenvalues of E_β are

$$E_\beta = \frac{2\hbar^2}{B\beta_1^2} + \hbar\sqrt{\frac{p}{B}}\left(n_\beta + \frac{1}{2}\right) \qquad n_\beta = 0, 1, 2 \cdots \quad (108)$$

and the wave function representing the state $n_\beta = 0$ is given by

$$\xi_0(x) = \left(\frac{2s}{\pi}\right)^{\frac{1}{4}} \exp\left\{-s\left(x + \frac{1}{s}\right)^2\right\}, \quad (109)$$

where

$$s = \frac{\beta_1^2}{2\hbar}\sqrt{pB}. \quad (110)$$

As a consequence of the zero point vibration, the mean value of x is shifted from zero to

$$\bar{x} = -\frac{1}{s}, \tag{111}$$

while

$$\overline{(x-\bar{x})^2} = \frac{1}{4s} \tag{112}$$

gives the mean square deviation.

Equation (106) is equivalent to the radial equation of a two-dimensional harmonic oscillator with zero angular momentum. The eigenvalues of E_γ are

$$E_\gamma = \hbar \sqrt{\frac{q}{B\beta_1^2}}(n_\gamma + 1) \qquad n_\gamma = 0, 1, 2 \cdots \tag{113}$$

and the ground state wave function is given by

$$\eta_0(y) = \sqrt{2t}\exp\{-ty^2\} \tag{114}$$

with

$$t = \frac{\beta_1}{2\hbar}\sqrt{qB}. \tag{115}$$

In accordance with (47), the volume element has been taken to be $|y|\,dy$ For the state (114),

$$y^2 = \frac{1}{2t} \tag{116}$$

gives the mean square deviation from the equilibrium value $y = 0$.

In the limit of very strong coupling, the last term in (100) may be neglected, as already mentioned, and, according to (102) and (103), the values of p and q will be approximately equal to C and $C\beta_1^2$, respectively. From (110) and (115) it then follows that, as regards order of magnitude,

$$s \sim t \sim \left(\frac{\beta_1}{\beta_0}\right)^2, \tag{117}$$

where β_0 is the zero point amplitude for free nuclear surface oscillations, given by (71). The above approximate solution to the wave equation (97) thus becomes valid for $\beta_1 \gg \beta_0$.

If one uses the tentative values for B, C, and k given by (4), (5), (6), and (75), one finds that, for most nuclear states, β_1 is of the same order of magnitude as β_0. In such cases, the above solution to (97) cannot be expected to be very accurate.

As a numerical example, we may consider a heavy nucleus with $Z = 80$, $A = 200$. We assume a single particle state with $j = {}^9/_2$ and take the angular momentum quantum numbers of the nucleus to be $I = \Omega = K = j$. This choice is expected to represent the nuclear ground state[10].

From (4), (5), and (6) we find $C = 63$ MeV and $\hbar^2/B = 25$ keV. Taking k to be 40 MeV, we find from (99) and (100), by means of (77), the values $\beta_1 = 0.24$, $\gamma_1 = \pi$. This deformation represents an oblate spheroid with $\delta R_3 = -0.15\, R_0$ (cf. (15)).

From (102) and (103) one obtains $p = 130$ MeV and $q = 4.86$ MeV, and the spacings between the eigenvalues E_β and E_γ are, from (108) and (113), found to be 1.8 MeV and 1.4 MeV, respectively. For s and t one gets the values 2.1 and 1.7 from (110) and (115), respectively. Thus, according to (112) and (116), we have $\overline{(x-\bar{x})^2} = 0.12$ and $\overline{y^2} = 0.30$.

Due to the magnitude of the deviations from equilibrium, these estimates are, of course, quite crude and can only indicate orders of magnitude. A more accurate solution to (97) would fall off more rapidly than (109) and (114) as we move away from equilibrium. Actually, the potential (98) is singular for $\beta = 0$ as well as for $\gamma = \pm\frac{\pi}{3}$ or $\pm\frac{2\pi}{3}$ (cf. Figs. 4 and 5).

For many applications, such as in the estimate of the nuclear quadrupole moment, a more accurate solution to (97) may be required. In the present connection, where we are primarily concerned with the classification of the quantum states of the nucleus, we shall not consider the problem in more detail.

In order to obtain nuclear wave functions, valid in the strong coupling limit, we still have to take into account the symmetry requirements discussed in the preceding paragraph. The two degenerate eigenfunctions to H_0 with j_3, I_3 equal to Ω, K and $-\Omega$, $-K$, respectively, have to be combined according to (91). For the particular case of $I = j$, the symmetrical combination must be taken.

The condition (92) requires φ to be an even function of γ

since we have assumed $\nu = 0$. Thus, only the eigenvalues (113), corresponding to even n_γ, are acceptable. Finally, the condition (93) may be satisfied by taking for the wave function

$$\left.\begin{aligned} &\Psi(I, M, K, \Omega, n_\beta, n_\gamma) = \\ &(1 + R_3 + R_3^2)\, \varphi_{\Omega,K}^{I, n_\beta, n_\gamma} \left(\chi_\Omega \mathfrak{D}_{MK}^I + (-)^{I-j} \chi_{-\Omega} \mathfrak{D}_{M,-K}^I\right), \end{aligned}\right\} \quad (118)$$

where R_3 acts on χ_Ω, $\mathfrak{D}_{MK}^I$, and $\varphi_{\Omega,K}$ according to the equations (88) and (60). In the wave function (118) the three axes of the nucleus enter in a symmetrical manner.

4) Effect of perturbation terms.

In order to estimate the influence of the perturbation terms (96) we consider their matrix elements in the j_3, I_3 representation. We first note that the non-vanishing elements of j_1, j_2, I_1 and I_2 are given by

$$(\Omega | j_1 \pm i j_2 | \Omega \mp 1) = \sqrt{(j \pm \Omega)(j \mp \Omega + 1)} \quad (119)$$

and

$$(K | I_1 \mp i I_2 | K \mp 1) = \sqrt{(I \pm K)(I \mp K + 1)}. \quad (120)$$

The operators occurring in U have therefore the following matrix elements

$$(K, \Omega | j_1^2 - j_2^2 | \Omega \mp 2, K) = \frac{1}{2} \sqrt{(j \pm \Omega)(j \pm \Omega - 1)(j \mp \Omega + 1)(j \mp \Omega + 2)} \quad (121)$$

$$(K, \Omega | I_1^2 - I_2^2 | \Omega, K \mp 2) = \frac{1}{2} \sqrt{(I \pm K)(I \pm K - 1)(I \mp K + 1)(I \mp K + 2)} \quad (122)$$

$$\left.\begin{aligned} &(K, \Omega | j_1 I_1 | \Omega - 1, K \mp 1) = \pm (K, \Omega | j_2 I_2 | \Omega - 1, K \mp 1) = \\ &\frac{1}{4} \sqrt{(j + \Omega)(j - \Omega + 1)(I \pm K)(I \mp K + 1)} \end{aligned}\right\} \quad (123)$$

$$\left.\begin{aligned} &(K, \Omega | j_1 I_1 | \Omega + 1, K \mp 1) = \mp (K, \Omega | j_2 I_2 | \Omega + 1, K \mp 1) = \\ &\frac{1}{4} \sqrt{(j - \Omega)(j + \Omega + 1)(I \pm K)(I \mp K + 1)}. \end{aligned}\right\} \quad (124)$$

From these expressions all non-vanishing matrix elements of U may be derived.

The effect of the perturbation depends on the ratio of the matrix elements to the energy difference between the corresponding unperturbed energy levels. If this ratio is small, U will have only a small influence on the stationary state of the nucleus.

The perturbation term U_1 in (96) connects the state (Ω, K) with states $(\Omega \pm 1, K \pm 1)$ and $(\Omega \pm 1, K \mp 1)$. Perturbations of this type are well known from the theory of diatomic molecules, where they tend to decouple the electronic angular momenta from the molecular axis. If we disregard numerical factors depending on the angular momentum quantum numbers, the order of magnitude of the matrix elements of U_1 is given by

$$U_1 \sim \frac{\hbar^2}{B\beta^2} \tag{125}$$

according to (27). Since the operator U_1 changes the value of Ω, the β_1-values of the combining states are essentially different and, according to (98) and (100), their energy difference will therefore be of the order of

$$\Delta_1 E \sim C\beta_1^2, \tag{126}$$

again neglecting numerical factors depending on the Ω-values. From (71) we thus find

$$\frac{U_1}{\Delta_1 E} \sim \left(\frac{\beta_0}{\beta_1}\right)^4, \tag{127}$$

showing that the perturbations are small, provided $\beta_1 \gg \beta_0$. The same condition was found in the preceding paragraph for the validity of the approximate solution of the wave equation (97).

Even if U_1 is small, it does remove the degeneracy between the states (Ω, K) and $(-\Omega, -K)$. In molecular spectra, this effect gives rise to the so-called Ω-doubling (or Λ-doubling) of the energy levels. In nuclei, however, no doubling phenomenon of this type will occur since, according to (91), only a particular combination of the (Ω, K) and $(-\Omega, -K)$ wave functions, either the symmetrical or the antisymmetrical, is allowed. The situation is similar to the case of diatomic molecules with identical nuclei of spin zero.

The perturbation U_2 represents the effect on the particle motion of deviations from rotational symmetry in the shape of the

nucleus. The operator U_2 in (96) connects (Ω, K) with $(\Omega \pm 2, K)$ according to (121). For strong interaction, the largest term of U_2 is that involving $kc_j\beta \sin \gamma$. From (100) we have, as regards order of magnitude,

$$U_2 \sim C\beta_1^2\gamma, \tag{128}$$

while $\Delta_2 E \sim \Delta_1 E$ is given by (126). We thus find

$$\frac{U_2}{\Delta_2 E} \sim \gamma \sim \frac{\beta_0}{\beta_1} \tag{129}$$

according to (116) and (117). Again U_2 is small, provided $\beta_1 \gg \beta_0$.

One might expect to obtain an improved treatment by including the large part of U_2 in the unperturbed Hamiltonian. In analogy to the case of polyatomic molecules, one would then first solve the particle motion for fixed, but arbitrary values of β and γ and, subsequently, consider the vibration and rotation of the nucleus. In contrast to the molecular case, however, the deformation of the nucleus depends essentially on the motion of the single particle. For this reason, the matrix element of U_2 will in general be smaller, and the value of $\Delta_2 E$ larger, than the value corresponding to the motion of the particle in a fixed field. The effect of U_2 is, therefore, smaller in the nuclear case and cannot be simply accounted for in analogy to the molecular case.

Finally, the perturbation U_3, like U_2, arises from the asymmetry in the nuclear shape, and represents the anomaly in the nuclear motion as compared with the motion of a symmetrical top. The operator U_3 connects (Ω, K) with $(\Omega, K \pm 2)$. The order of magnitude of U_3 is found to be

$$U_3 \sim \frac{\hbar^2}{B\beta_1^2}\gamma, \tag{130}$$

while

$$\Delta_3 E \sim \frac{\hbar^2}{B\beta_1^2}\frac{1}{\gamma^2} \tag{131}$$

gives an estimate of $\Delta_3 E$, arising primarily from the term in (98) involving $(K - \Omega)^2$. Thus,

$$\frac{U_3}{\Delta_3 E} \sim \gamma^3 \sim \left(\frac{\beta_0}{\beta_1}\right)^3 \tag{132}$$

according to (116) and (117).

The above estimates of the perturbation effects show that, in the strong coupling limit ($\beta_1 \gg \beta_0$), the treatment outlined in the preceding paragraph becomes valid for a large class of nuclear states. For these we may consider the particle as moving in a field possessing cylindrical symmetry, while the nuclear surface vibrates around its symmetrical equilibrium shape. Of course, such a treatment will break down if the nucleus vibrates or rotates too strongly and in addition the states may become strongly perturbed in cases where, by coincidence, the energy denominators ΔE can become small.

As already mentioned in the preceding paragraph, the tentative estimates (4, 5, and 6) and (75) of the parameters B, C, and k entering into the description of the nucleus, lead in general to comparable values of β_1 and β_0. In order to estimate the magnitude of the perturbation terms, we consider, as an example, a heavy nucleus with $A = 200$, $Z = 80$, $j = {}^9/_2$, the case already discussed in the preceding paragraph. It may be noted that the situation is not greatly changed if we consider lighter nuclei or smaller values of j, provided $j \geq {}^3/_2$ (or $l \geq 1$). In the case of $j = {}^1/_2$, the value of H_{int} used above vanishes.

We take the ground state to have the quantum numbers $I = K = \Omega = j = {}^9/_2$ and $n\beta = n_\gamma = 0$. The perturbation U_1 connects this state with states having $K = \Omega = {}^7/_2$. By means of the numerical values quoted in the preceding paragraph, one finds from (123) a value for U_1 of about ${}^1/_2$ MeV to 1 MeV, depending somewhat on the effective value chosen for β and γ. The most important regions for β and γ are $0.15 < \beta < 0.30$ and $|\pi - \gamma| < \frac{\pi}{6}$. We may here disregard the symmetrization rule leading to wave functions of the type (118). The energy difference between the ground state and the state $K = \Omega = {}^7/_2$, $n\beta = n_\gamma = 0$ is estimated to be some 3 MeV, but, since the wave functions $\varphi(\beta, \gamma)$ for the two states are far from overlapping completely, the effective value for $\Delta_1 E$ will include some excitation of the β, γ vibrations. If $\Delta_1 E$ is taken to be about 5 MeV, the ratio $\frac{U_1}{\Delta_1 E}$ may be expected to be of the order of 10—20 %.

The term U_2 in (96) connects the ground state with states having $(\Omega, K) = ({}^5/_2, {}^9/_2)$. From (96) the matrix elements have

been found to be about 0.5 MeV. The value of $\Delta_2 E$ appears to be somewhat larger than $\Delta_1 E$ and is estimated to be some 5 to 10 MeV. Finally, U_3 introduces admixtures of states of the type ($^9/_2$, $^5/_2$), and from (122) values for U_3 of the order of 0.3 MeV have been obtained, while $\Delta_3 E$ is estimated to be about 5 MeV.

It appears that the amplitudes of the connecting states may not exceed some 10—20 %. In some applications, these amplitudes only enter squared and, in such cases, the strong coupling approximation may be rather accurate.

In conclusion, it must be stressed that an analysis like the above rests on a highly simplified nuclear model. Anomalies in the nuclear deformability and the influence of additional particles may greatly change the picture. In general, a more detailed analysis will be needed to decide which coupling model gives the best representation of a particular nuclear state under consideration*.

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* A discussion of various nuclear properties implied by the model considered is being prepared by B. MOTTELSON and the writer.

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