DET KGL. DANSKE VIDENSKABERNES SELSKAB
MATEMATISK-FYSISKE MEDDELELSER, BIND XXV, NR. 8

ON THE PROOFS OF THE FUNDAMENTAL THEOREM ON ALMOST PERIODIC FUNCTIONS

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KØBENHAVN
I KOMMISSION HOS EJNAR MUNKSGAARD
1949

Printed in Denmark.
Bianco Lunos Bogtrykkeri

1. Introduction. The fundamental theorem of the theory of almost periodic functions states that any almost periodic function $f(x)$ with the Fourier series

$$f(x) \sim \sum a(\lambda)\, e^{i\lambda x}, \text{ where } a(\lambda) = M\{f(x)\, e^{-i\lambda x}\},$$

satisfies the Parseval equation

$$M\{|f(x)|^2\} = \Sigma\, |a(\lambda)|^2.$$

Many proofs of this theorem have been given. Among them the proof of Weyl [6] is, perhaps, the one which leads most directly to the goal. It depends on a systematic use of the process of convolution and on the methods of the theory of integral equations. Another proof, depending on the general theory of Fourier integrals, is due to Wiener [7]; it has been given a particularly simple form by Bochner [2, pp. 81—82].

Though these proofs give a clear insight in the whole theory, the more elementary proofs are not without interest. Among them the original proof of Bohr [3] is interesting by its crudeness. Its idea is to consider for every positive T the periodic function with the period T which coincides with $f(x)$ in the interval $(0, T)$, and to use Parseval's equation for this function. By making $T \to \infty$, one obtains the theorem. The passage to the limit is, however, of a complicated nature, and the whole proof is very long.

A considerable simplification was obtained by de la Vallée Poussin [5], who used the same idea together with the convolution process to prove the uniqueness theorem, which states that if $a(\lambda) = 0$ for all λ, then $f(x)$ vanishes identically. From this theorem Parseval's equation follows by a simple application of the convolution process. Since $f(x)$ vanishes identically if and only if $M\{|f(x)|^2\} = 0$, the proof of the uniqueness theorem amounts to a proof of Parseval's equation in the particular case where $a(\lambda) = 0$ for all λ. A simplification of de la Vallée Poussins proof has been given by M. Riesz [4].

It seems very natural to base a proof of the fundamental theorem on almost periodic functions on the corresponding theorem on periodic functions. It must, however, be mentioned, that the periodic function with the period T which coincides with $f(x)$ in the interval $(0, T)$ will generally be discontinuous in the points nT, so that it is not a special case of the theorem on almost periodic functions, which is used. Moreover, the periodic functions will generally not approximate $f(x)$ outside the interval $(0, T)$.

The truth is that actually it complicates matters to introduce this periodic function. As will be shown in the following pages, the proofs take a simpler form if, instead of the Fourier series of the periodic function with the period T, we consider the Fourier integral of the function $f_T(x)$ which coincides with $f(x)$ in the interval $(0, T)$ and is 0 elsewhere. Naturally, for large T this Fourier integral does not differ much from the Fourier series of the periodic function.

All we shall need on Fourier integrals is, that if $F(x)$ is a function, which is continuous in a certain closed interval and is 0 outside this interval, and if

$$\int_{-\infty}^{\infty} F(x)\, e^{-i\lambda x}\, dx = A(\lambda),$$

then in analogy to Parseval's equation

$$\int_{-\infty}^{\infty} |F(x)|^2\, dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} |A(\lambda)|^2\, d\lambda.$$

Thus our proofs are more elementary than the proof of Wiener referred to above, with which they have no connection.

Our proof of the Parseval equation follows step by step Bohr's proof. The main simplifications are in the beginning. In the later part a simplification in the exposition has been obtained by the use of a function introduced by Wiener [7, p. 495], connected with Bochner's translation function [1, p. 136].

In our proof of the uniqueness theorem we use de la Vallée Poussin's main lemma, which actually concerns the Fourier integral of the function $f_T(x)$. The simplification is in the remainder of the proof, where we avoid the artifice of choosing T as a fine translation number.

2. Proof of the Parseval equation. The inequality obtained from Parseval's equation by replacing $=$ by $\geq$ being an easy consequence of Bessel's formula, it is sufficient to prove the inequality obtained by replacing $=$ by $\leq$.

For an arbitrary $T > 0$ consider the function

$$\frac{1}{T}\int_0^T f(x)\, e^{-i\lambda x}\, dx = \frac{1}{T}\int_{-\infty}^{\infty} f_T(x)\, e^{-i\lambda x}\, dx = a_T(\lambda).$$

Then

$$\frac{1}{T}\int_0^T |f(x)|^2\, dx = \frac{1}{T}\int_{-\infty}^{\infty} |f_T(x)|^2\, dx = \frac{T}{2\pi}\int_{-\infty}^{\infty} |a_T(\lambda)|^2\, d\lambda.$$

It is therefore sufficient to prove:

To every $\delta > 0$ there exists a $T_0 > 0$ such that

$$\frac{T}{2\pi}\int_{-\infty}^{\infty} |a_T(\lambda)|^2\, d\lambda < \Sigma |a(\lambda)|^2 + \delta \quad \textit{for} \quad T > T_0.$$

3. We begin by proving

Lemma 1. *To every λ_0 and every $\delta > 0$ correspond an $\omega > 0$ and a $T_0 > 0$ such that*

$$\frac{T}{2\pi}\int_{\lambda_0-\omega}^{\lambda_0+\omega} |a_T(\lambda)|^2\, d\lambda < |a(\lambda_0)|^2 + \delta \quad \textit{for} \quad T > T_0.$$

Proof. If $f(x)$ is replaced by $f(x)\, e^{-i\lambda_0 x}$ the function $a_T(\lambda)$ is replaced by $a_T(\lambda + \lambda_0)$. It is therefore sufficient to consider the case $\lambda_0 = 0$.

(i) $a(0) = 0$, i. e. $M\{f(x)\} = 0$. — On placing for a given $c > 0$

$$\Phi(x) = \frac{1}{c}\int_x^{x+c} f_T(y)\,dy$$

we have by a simple computation

$$\frac{1}{T}\int_{-\infty}^{\infty}\Phi(x)\,e^{-i\lambda x}\,dx = a_T(\lambda)\,\frac{e^{i\lambda c}-1}{i\lambda c};$$

hence

$$\frac{1}{T}\int_{-\infty}^{\infty}|\Phi(x)|^2\,dx = \frac{T}{2\pi}\int_{-\infty}^{\infty}|a_T(\lambda)|^2\left|\frac{e^{i\lambda c}-1}{i\lambda c}\right|^2 d\lambda.$$

Since $\frac{1}{c}\int_x^{x+c} f(y)\,dy$ converges uniformly in x towards $M\{f(x)\}$ as $c\to\infty$ there exists to every $\varepsilon > 0$ a $c = c(\varepsilon)$, such that when $T > c$ then $|\Phi(x)| \leq \varepsilon$ in the interval $(0, T-c)$. In the intervals $(-c, 0)$ and $(T-c, T)$ we have $|\Phi(x)| \leq G = \sup|f(x)|$. Outside the interval $(-c, T)$ we have $\Phi(x) = 0$. Hence

$$\frac{1}{T}\int_{-\infty}^{\infty}|\Phi(x)|^2\,dx \leq \varepsilon^2 + \frac{2\,c\,G^2}{T}$$

and consequently

$$\frac{T}{2\pi}\int_{-\infty}^{\infty}|a_T(\lambda)|^2\left|\frac{e^{i\lambda c}-1}{i\lambda c}\right|^2 d\lambda \leq 2\,\varepsilon^2 \quad \text{for} \quad T > T_0 = \frac{2\,c\,G^2}{\varepsilon^2}.$$

For $|\lambda| <$ (some) $\omega = \omega(c)$ we have

$$\left|\frac{e^{i\lambda c}-1}{i\lambda c}\right| > \frac{1}{2}.$$

Hence

$$\frac{T}{2\pi}\int_{-\omega}^{\omega}|a_T(\lambda)|^2\,d\lambda \leq 4\cdot 2\,\varepsilon^2 = 8\,\varepsilon^2 \quad \text{for} \quad T > T_0.$$

(ii) $a(0) = a \neq 0$. — On placing $f(x) = a + h(x)$ we obtain a corresponding decomposition of $a_T(x)$ in two terms:

$$a_T(\lambda) = b_T(\lambda) + c_T(\lambda).$$

Hereby

$$\frac{1}{T}\int_0^T |a|^2\,dx = |a|^2 = \frac{T}{2\pi}\int_{-\infty}^{\infty} |b_T(\lambda)|^2\,d\lambda.$$

Hence by the triangle inequality we have for every $\omega > 0$

$$\left[\frac{T}{2\pi}\int_{-\omega}^{\omega} |a_T(\lambda)|^2\,d\lambda\right]^{\frac{1}{2}} \leq |a| + \left[\frac{T}{2\pi}\int_{-\omega}^{\omega} |c_T(\lambda)|^2\,d\lambda\right]^{\frac{1}{2}}.$$

Let $\varepsilon > 0$ be chosen such that $(|a| + 3\varepsilon)^2 < |a|^2 + \delta$, and next by (i), since $M\{h(x)\} = 0$, the numbers ω and T_0 such that

$$\frac{T}{2\pi}\int_{-\omega}^{\omega} |c_T(\lambda)|^2\,d\lambda < 9\,\varepsilon^2 \quad \text{for} \quad T > T_0.$$

Then

$$\frac{T}{2\pi}\int_{-\omega}^{\omega} |a_T(\lambda)|^2\,d\lambda \leq |a|^2 + \delta \quad \text{for} \quad T > T_0.$$

4. On account of Lemma 1 it is, in order to prove the Parseval equation, sufficient to prove the following

Lemma 2. *To every $\delta > 0$ there exists a finite set of numbers $\lambda_1, \cdots, \lambda_M$ such that for every $\omega > 0$*

$$\frac{T}{2\pi}\int_{\substack{|\lambda-\lambda_1| \geq \omega \\ \cdots \\ |\lambda-\lambda_M| \geq \omega}} |a_T(\lambda)|^2\,d\lambda < \delta \quad \textit{for} \quad T > (\textit{some})\; T_0(\omega).$$

We shall reduce this lemma to a lemma on the translation function

$$e(\tau) = \sup_x |f(x+\tau) - f(x)|.$$

On placing for a given $\tau > 0$

$$\Psi(x) = f_T(x+\tau) - f_T(x)$$

we have

$$\frac{1}{T}\int_{-\infty}^{\infty} \Psi(x)\, e^{-i\lambda x}\,dx = a_T(\lambda)\,(e^{i\lambda\tau} - 1);$$

hence
$$\frac{1}{T}\int_{-\infty}^{\infty}|\Psi(x)|^2\,dx = \frac{T}{2\pi}\int_{-\infty}^{\infty}|a_T(\lambda)|^2\,|e^{i\lambda\tau}-1|^2\,d\lambda.$$

Now $|\Psi(x)| \leq e(\tau)$ if the points x and $x+\tau$ both lie in the interval $(0, T)$, and $|\Psi(x)| \leq G$ if one of the points lie in $(0, T)$, whereas $\Psi(x) = 0$ if both points lie outside $(0, T)$. Hence
$$\frac{1}{T}\int_{-\infty}^{\infty}|\Psi(x)|^2\,dx \leq e(\tau)^2 + \frac{2\tau G^2}{T}$$
and consequently
$$\frac{T}{2\pi}\int_{-\infty}^{\infty}|a_T(\lambda)|^2\,|e^{i\lambda\tau}-1|^2\,d\lambda \leq 2\varepsilon^2 \quad \text{if} \quad e(\tau) \leq \varepsilon \quad \text{and} \quad T > \frac{2\tau G^2}{\varepsilon^2}.$$

On placing
$$\varphi(\tau) = \begin{cases} \varepsilon - e(\tau) & \text{when } e(\tau) \leq \varepsilon \\ 0 & \text{when } e(\tau) > \varepsilon \end{cases}$$
we therefore obtain for every $X > 0$
$$\frac{T}{2\pi}\int_{-\infty}^{\infty}|a_T(\lambda)|^2\left(\frac{1}{X}\int_0^X|e^{i\lambda\tau}-1|^2\,\varphi(\tau)\,d\tau\right)d\lambda \leq 2\varepsilon^2\,\frac{1}{X}\int_0^X\varphi(\tau)\,d\tau$$
$$\text{for} \quad T > \frac{2XG^2}{\varepsilon^2}.$$

In order to prove Lemma 2 it is therefore sufficient to prove

Lemma 3. *To every $\varepsilon > 0$ there exists a finite set of numbers $\lambda_1, \cdots, \lambda_M$ such that for every $\omega > 0$ there exists an $X > 0$ for which*
$$\frac{1}{X}\int_0^X|e^{i\lambda\tau}-1|^2\,\varphi(\tau)\,d\tau > \frac{1}{17}\,\frac{1}{X}\int_0^X\varphi(\tau)\,d\tau$$
when $|\lambda-\lambda_1| \geq \omega, \cdots, |\lambda-\lambda_M| \geq \omega$.

5. Translation numbers of $f(x)$ belonging to a given $\varrho > 0$, i. e. numbers τ for which $e(\tau) \leq \varrho$, will be denoted throughout by $\tau(\varrho)$. We shall first prove

Lemma 4. *For every* $\varrho > 0$ *the set* S *of numbers* λ *for which* $|e^{i\lambda t}-1| \leqq 1$ *for all* $t = \tau(\varrho) > 0$ *is finite.*

If S *consist of the numbers* $\lambda_1, \cdots, \lambda_M$, *there exists to every* $\omega > 0$ *an* $A > 0$, *such that if* $|\lambda-\lambda_1| \geqq \omega, \cdots, |\lambda-\lambda_M| > \omega$, *then* $|e^{i\lambda t}-1| > 1$ *for some positive* $t = \tau(\varrho) < A$.

Proof. By the uniform continuity of $f(x)$ there exists an $\eta > 0$ such that any positive $\tau < \eta$ is a $\tau(\varrho)$. Hence, if $|\lambda| > \pi/3\eta$ there exists a positive $t = \tau(\varrho) < \eta$ for which $|e^{i\lambda t}-1| > 1$. Thus S belongs to the interval $|\lambda| \leq \pi/3\eta$.

If λ' and λ'' both belong to S, i. e. if $|\lambda' t| \leqq \pi/3 \mod 2\pi$ and $|\lambda'' t| \leq \pi/3 \mod 2\pi$ for all $t = \tau(\varrho) > 0$, we have $|\lambda'-\lambda''| t \leq 2\pi/3 \mod 2\pi$ for all $t = \tau(\varrho) > 0$. In particular, the interval $2\pi/3|\lambda'-\lambda''| < t < 4\pi/3|\lambda'-\lambda''|$ of length $2\pi/3|\lambda'-\lambda''|$ will contain no $\tau(\varrho)$. Since every interval of a certain length $l = l(\varrho)$ contains a $\tau(\varrho)$, we obtain $|\lambda'-\lambda''| \geqq 2\pi/3l$. Hence S is finite.

Let now $\omega > 0$ be chosen, and consider the closed bounded set of numbers λ for which $|\lambda-\lambda_1| \geq \omega, \cdots, |\lambda-\lambda_M| \geqq \omega$, $|\lambda| \leq \pi/3\eta$. This set is covered by the open sets U_t defined by an inequality $|e^{i\lambda t}-1| > 1$ for a $t = \tau(\varrho) > 0$. Hence, by Borel's theorem, it is covered by a finite number of these sets, say by $U_{t_1}, \cdots, U_{t_n}$. As number A may then be used any number larger than the numbers $\eta, t_1, \cdots, t_n$.

6. We now turn to the proof of Lemma 3.

The translation function $e(\tau)$ being almost periodic, so is the function $\varphi(\tau)$. Since $\varphi(\tau)$ is non-negative and not identically zero, we have

$$M\{\varphi(\tau)\} = m > 0.$$

In Lemma 4 let $\varrho = \frac{1}{2}m$. Then the lemma gives numbers $\lambda_1, \cdots, \lambda_M$ and, when $\omega > 0$ is chosen, a number $A > 0$.

If $|\lambda-\lambda_1| \geqq \omega, \cdots, |\lambda-\lambda_M| \geqq \omega$, there exists a positive $t = \tau(\varrho) < A$ such that $|e^{i\lambda t}-1| > 1$. For $X > A$ we have

$$\frac{1}{X}\int_0^X |e^{i\lambda\tau}-1|^2 \varphi(\tau)\, d\tau \geqq \frac{1}{2X}\int_0^{X-A} [|e^{i\lambda\tau}-1|^2 \varphi(\tau) + |e^{i\lambda(\tau+t)}-1|^2 \varphi(\tau+t)]\, d\tau.$$

Now, since $|\lambda t| > \pi/3 \mod 2\pi$, the relations $|\lambda\tau| \leqq \pi/6 \mod 2\pi$ and $|\lambda(\tau+t)| \leqq \pi/6 \mod 2\pi$ cannot be valid together, i. e. we have for every τ

$$\max\{|e^{i\lambda\tau}-1|^2, |e^{i\lambda(\tau+t)}-1|^2\} > |e^{i\pi/6}-1|^2 > \tfrac{1}{4}.$$

Moreover, since t is a $\tau(\varrho)$, we have $e(\tau+t) \leqq e(\tau)+\varrho$, and consequently

$$\varphi(\tau+t) \geqq \varphi(\tau)-\varrho.$$

Hence we obtain

$$\frac{1}{X}\int_0^X |e^{i\lambda\tau}-1|^2\varphi(\tau)\,d\tau \geqq \frac{1}{2X}\int_0^{X-A}\tfrac{1}{4}[\varphi(\tau)-\varrho]\,d\tau.$$

Here the right hand side converges for $X\to\infty$ towards $\frac{1}{8}(m-\varrho)=\frac{1}{16}m$. The right hand side of the inequality in Lemma 3 converges for $X\to\infty$ towards $\frac{1}{17}m$. Hence the latter inequality is valid for some X and the proof is completed.

7. Proof of the uniqueness theorem. The main lemma in de la Vallée Poussin's proof states that when $a(\lambda)=0$ for all λ, then $a_T(\lambda)\to 0$ uniformly in λ as $T\to\infty$. Starting from this lemma the proof may by completed as follows.

For a given $\varepsilon>0$ let $T_0>0$ be chosen such that $|a_T(\lambda)|\leqq\varepsilon$ for all λ when $T>T_0$. For $U>T>T_0$ consider the function

$$g_{TU}(x) = \frac{1}{T}\int_{-\infty}^{\infty} f_U(x+t)\overline{f_T(t)}\,dt = \frac{1}{T}\int_0^T f_U(x+t)\overline{f(t)}\,dt.$$

Plainly, $g_{TU}(x)$ vanishes outside the interval $(-T, U)$ and coincides in the interval $(0, U-T)$ with the almost periodic function

$$g_T(x) = \frac{1}{T}\int_0^T f(x+t)\overline{f(t)}\,dt.$$

By a simple calculation

$$\int_{-\infty}^{\infty} g_{TU}(x)e^{-i\lambda x}\,dx = Ua_U(\lambda)\overline{a_T(\lambda)}.$$

Hence

$$\frac{1}{U}\int_0^{U-T}|g_T(x)|^2\,dx \leqq \frac{1}{U}\int_{-\infty}^{\infty}|g_{TU}(x)|^2\,dx = \frac{U}{2\pi}\int_{-\infty}^{\infty}|a_U(\lambda)|^2|a_T(\lambda)|^2\,d\lambda$$

$$\leqq \varepsilon^2\frac{U}{2\pi}\int_{-\infty}^{\infty}|a_U(\lambda)|^2\,d\lambda = \varepsilon^2\frac{1}{U}\int_0^U|f(x)|^2\,dx \leqq \varepsilon^2 G^2.$$

For $U \to \infty$ this gives

$$M\{|g_T(x)|^2\} \leq \varepsilon^2 G^2.$$

For $T \to \infty$ the function $g_T(x)$ converges uniformly in x towards the convolution

$$g(x) = \underset{t}{M}\{f(x+t)\overline{f(t)}\}.$$

Hence

$$M\{|g(x)|^2\} \leq \varepsilon^2 G^2.$$

Since this is true for all $\varepsilon > 0$, we have $M\{|g(x)|^2\} = 0$, which implies $g(x) \equiv 0$. In particular $g(0) = M\{|f(x)|^2\} = 0$, and hence $f(x) \equiv 0$.

8. Another variant of the proof of the uniqueness theorem. It may be remarked that a slight change in the above proof permits us to replace the use of Parseval's formula for Fourier integrals by Parseval's formula for periodic functions, which may be formulated as follows:

If $F(x)$ is continuous in a closed interval of length $\leq P$ and is 0 outside this interval, and if

$$\int_{-\infty}^{\infty} F(x)\, e^{-i\lambda x}\, dx = A(\lambda),$$

then

$$\int_{-\infty}^{\infty} |F(x)|^2\, dx = \frac{1}{P} \sum_{n=-\infty}^{\infty} \left| A\left(\frac{2\pi}{P} n\right)\right|^2.$$

Applying this formula to the function $g_{TU}(x)$, which vanishes outside the interval $(-T, U)$, and using that $f_U(x)$ also vanishes outside this interval we obtain

$$\int_0^{U-T} |g_T(x)|^2\, dx \leq \int_{-\infty}^{\infty} |g_{TU}(x)|^2\, dx = \frac{1}{T+U} \sum_{n=-\infty}^{\infty} U^2 \left| a_U\left(\frac{2\pi}{T+U} n\right)\right|^2 \left| a_T\left(\frac{2\pi}{T+U} n\right)\right|^2$$

$$\leq \varepsilon^2 \frac{1}{T+U} \sum_{n=-\infty}^{\infty} U^2 \left| a_U\left(\frac{2\pi}{T+U} n\right)\right|^2 = \varepsilon^2 \int_{-\infty}^{\infty} |f_U(x)|^2\, dx \leq \varepsilon^2 U G^2,$$

and the proof is completed as before.

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Indleveret til Selskabet den 29. november 1948.
Færdig fra trykkeriet den 30. Maj 1949.