DET KGL. DANSKE VIDENSKABERNES SELSKAB
MATEMATISK-FYSISKE MEDDELELSER, BIND XXV, NR. 19

# A PROOF OF THE SIMPLER PONTRJAGIN DUALITY THEOREMS BY HELP OF THE CONNECTION BETWEEN TWO INFINITE-DIMENSIONAL SPACES

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1950

Printed in Denmark
Bianco Lunos Bogtrykkeri

## 1. Two infinite-dimensional spaces, $\mathfrak{R}^\infty$ and $\mathfrak{R}_\infty$.

In a paper by H. Bohr and the author [1]—and more detailed in [2]—a connection between two infinite-dimensional spaces was established. We shall state explicitly those of the results which will be used in the sequel.

The space $\mathfrak{R}^\infty$ consists of all points $\boldsymbol{x} = (x_1, x_2, \cdots)$ with a countable number of coordinates which are arbitrary real numbers. The convergence notion in $\mathfrak{R}^\infty$ is defined by convergence in each of the coordinates, i. e. $(x_1^n, x_2^n, \cdots) \to (x_1, x_2, \cdots)$ if $x_1^n \to x_1, x_2^n \to x_2, \cdots$. This convergence notion arises from a topology defined by help of neighborhoods $U_{N,\varepsilon}$ of $(0, 0, \cdots)$ where $U_{N,\varepsilon}$ ($N$ positive integer, $\varepsilon > 0$) consists of all $\boldsymbol{x} = (x_1, x_2, \cdots)$ with $|x_i| < \varepsilon$ for $i = 1, 2, \cdots, N$.

The space $\mathfrak{R}_\infty$ consists of all points $\boldsymbol{a} = (a_1, a_2, \cdots)$ with a countable number of real coordinates, but so that they are all zero from a certain step (depending on the point), i. e. $a_n = 0$ for $n \geqq N = N(\boldsymbol{a})$. By the topology chosen in $\mathfrak{R}_\infty$—we need not state it here—the module of integral points in $\mathfrak{R}_\infty$, i. e. the points with mere integral coordinates, is discrete.

For an arbitrary closed module $M$ in $\mathfrak{R}^\infty$ we define its *dual module* $M'$ in $\mathfrak{R}_\infty$ as the set of points $\boldsymbol{a}$ in $\mathfrak{R}_\infty$ for which

$$\boldsymbol{a} \cdot \boldsymbol{x} = a_1 x_1 + a_2 x_2 + \cdots \equiv 0 \pmod 1 \text{ for every } \boldsymbol{x} \varepsilon M.$$

It is a closed module in $\mathfrak{R}_\infty$ (in the topology only referred to). We also introduce the analogous definition when $M$ is a closed module in $\mathfrak{R}_\infty$.

By a *substitution* $\boldsymbol{x} = T\boldsymbol{y}$ in $\mathfrak{R}^\infty$ we understand a linear transformation of the form

$$\begin{aligned} x_1 &= a_{11} y_1 + a_{12} y_2 + \cdots + a_{1p_1} y_{p_1} \\ x_2 &= a_{21} y_1 + a_{22} y_2 + \cdots + a_{2p_2} y_{p_2} \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned}$$

which establishes a one-to-one mapping of $\mathfrak{R}^\infty$ on (the whole) $\mathfrak{R}^\infty$. It turns out to be the same as a linear, one-to-one, bicontinuous transformation of $\mathfrak{R}^\infty$ onto itself.

The following theorems were proved.

**Theorem A.** *A closed module in the infinite-dimensional space $\mathfrak{R}^\infty$ is a point set $E$ which by a substitution can be transformed into a point set of a special form, in the following denoted by $S^\infty$, namely a point set $\{(y_1, y_2, \cdots)\}$ of the following structure: The indices $1, 2, \cdots, n, \cdots$ can be divided into three fixed classes $\{n_r\}, \{n_s\}, \{n_t\}$, such that the coordinates $y_{n_r}$ independently run through all numbers, and the coordinates $y_{n_s}$ independently run through all integers, while all the remaining coordinates $y_{n_t}$ are constantly zero. Conversely, each such point set $E$ is a closed module.*

**Theorem B.** *If $M$ is a closed module in $\mathfrak{R}^\infty$ or in $\mathfrak{R}_\infty$, then the dual module $M''$ of its dual module $M'$ is the module $M$ itself, i. e.*

$$M'' = M.$$

## 2. The Pontrjagin—van Kampen duality theorems.

Let $G$ be a locally compact abelian group satisfying the second axiom of countability. We use the additive notation for the group. By a continuous character on $G$ we understand (cp. [4], p. 127) a real multi-valued function $\alpha(x)$ uniquely defined modulo 1 on $G$ with the properties

1. $\alpha(x+y) \equiv \alpha(x) + \alpha(y) \pmod 1$.

2. To every $\varepsilon > 0$ can be found a neighborhood $U$ of $0$ such that $|\alpha(x)| < \varepsilon \pmod 1$ for $x \varepsilon U$.

We organize the set of continuous characters on $G$ so that it becomes a topological group. The sum $(\alpha_1 + \alpha_2)(x)$ of two characters $\alpha_1(x)$ and $\alpha_2(x)$ is defined by $(\alpha_1 + \alpha_2)(x) \equiv \equiv \alpha_1(x) + \alpha_2(x)$. With this addition the characters form a group. The zero-element is the character $\alpha(x) \equiv 0$. Corresponding to every $\varepsilon > 0$ and every compact set $F$ in $G$ we define a neighborhood of the zero-character as the set of characters $\alpha(x)$ satisfying

$$|\alpha(x)| < \varepsilon \pmod 1 \text{ for } x \varepsilon F.$$

In this way the group of characters becomes a topological group. We call it the character group of $G$ and denote it by $\widehat{G}$.

Pontrjagin ([4], p. 128) showed that $\widehat{G}$ is also a locally compact group satisfying the second axiom of countability, and furthermore he proved the following two fundamental theorems[1].

**Theorem 1.** *For a group $G$ of the type mentioned the character group $\widehat{\widehat{G}}$ of the character group $\widehat{G}$ is isomorphic with the group $G$ itself, i. e.*

$$\widehat{\widehat{G}} \simeq G.$$

*The isomorphism between $\widehat{\widehat{G}}$ and $G$ is realised in the natural way that the element $x \varepsilon G$ corresponds to the character $\chi(\alpha) = \alpha(x)$ on $\widehat{G}$.*

**Theorem 2.** *Let $H$ be a subgroup of a group $G$ of the type mentioned. If $H^*$ denotes the set of characters on $G$ which are $\equiv 0$ on $H$, and analogously $H^{**}$ denotes the set of characters on $\widehat{G}$ which are $\equiv 0$ on $H^*$ then the set $H^{**}$ by the identification of $\widehat{\widehat{G}}$ with $G$ is identical with the set $H$, i. e.*

$$H^{**} = H.$$

The purpose of this paper is to prove the following special case of these theorems by help of the connection between the spaces $\mathfrak{R}^{\infty}$ and $\mathfrak{R}_{\infty}$.

**Simpler Pontrjagin duality theorems.** *For compact and for discrete abelian groups satisfying the second axiom of countability the theorems 1 and 2 are valid. By the operation of passing to the character group, a group of one of the two types is transformed into a group of the other type.*

A group of the first type is in the sequel abbreviatively referred to as a compact group. A group of the second type, i. e. a countable discrete abelian group, is referred to as a discrete group.

By help of these simpler duality theorems and an investigation of the structure of locally compact groups, Pontrjagin and van Kampen obtained the theorems 1 and 2 in the general case.

[1] In this full generality first by van Kampen ([4], p. 126).

## 3. A realization of a compact group as a factor group inside $\mathfrak{R}^\infty$.

In this section we shall prove a theorem about a concrete way of realizing every compact group. For theorems used in the proof we shall, as before, refer the reader to [4].

**Theorem.** *Every compact group $G$ is isomorphic to a factor group $M/I$ where $I$ is the module of integral points in $\mathfrak{R}^\infty$ and $M$ is a closed module in $\mathfrak{R}^\infty$ containing $I$. The topology of $M/I$ is given in the natural way by help of the topology in $\mathfrak{R}^\infty$. Conversely, every factor group $M/I$ of the type mentioned, is a compact group.*

For the proof we take our starting point in the following theorem ([4], p. 46):

**Urysohn's lemma.** *Let $R$ be a compact regular topological space satisfying the second axiom of countability, and let $E$ and $F$ be two of its non-intersecting closed subsets. Then there exists a continuous function $f(x)$ defined on $R$ such that $0 \leq f(x) \leq 1$ for $x \varepsilon R$, $f(x) = 0$ for $x \varepsilon E$, and $f(x) = 1$ for $x \varepsilon F$.*

Now, let $E$ be a single point $a$ in $R$ and take a countable complete system of neighborhoods of $a$: $U_1, U_2, \cdots$. For $F$ successively equal to $R - U_1, R - U_2, \cdots$ we construct by Urysohn's lemma the functions $f_1(x), f_2(x), \cdots$. The function

$$g(x) = \sum_{n=1}^{\infty} \frac{f_n(x)}{n^2}$$

is then a continuous function on $R$ with $g(a) = 0$ and $g(x) > 0$ for $x \neq a$.

We may apply this to the compact group $G$ above since the underlying space of a topological group is always regular ([4], p. 56). Let $a$ be chosen as the zero of the group. In this way we get a continuous function $g(x)$ on $G$ with $g(0) = 0, g(x) > 0$ for $x \neq 0$.

As a *continuous function on a compact group*, $g(x)$ is uniformly continuous and hence also *almost periodic*. Thus $g(x)$ is a *continuous almost periodic function* on $G$. We shall use the *unicity theorem* for Fourier series of continuous almost periodic functions on a topological abelian group. Concerning the fact that we use such a deep-lying theorem we may remark that the main result of the Peter-Weyl theory on continuous functions on compact abelian groups, viz. the possibility of approximating every continuous function on the group by a linear combination of functions $e^{2\pi i\alpha(x)}$, is at the bottom of all proofs of the duality theorems. For a proof of the main results in the theory of almost periodic functions on an abelian group which utilizes the abelian type of the group, see my paper [3]. There no topology was considered, but it is a well-known and obvious fact that if such a topology exists and the almost periodic function $f(x)$ is continuous, then the characters in its Fourier series are all continuous since $C_n e^{2\pi i\alpha_n(x)} = \underset{t}{M}\left\{f(x-t)\, e^{2\pi i\alpha_n(t)}\right\}$ where $f(x)$ is uniformly continuous.

Let our function $g(x)$ above have the Fourier series

$$g(x) \sim \sum_{n=1}^{\infty}{}' C_n e^{2\pi i\alpha_n(x)}.$$

To the arbitrary element $h$ in $G$ we consider the translated function

$$g(x+h) \sim \sum_{n=1}^{\infty}{}' C_n e^{2\pi i\alpha_n(h)} e^{2\pi i\alpha_n(x)}.$$

If $\alpha_n(h) \equiv 0$ for $n = 1, 2, \cdots$, then $h$ must be equal to 0, for on account of the unicity theorem $g(x+h) = g(x)$, in particular $g(h) = g(0) = 0$.

We now map the arbitrary element $h \varepsilon G$ in the points $(\alpha_1(h), \alpha_2(h), \cdots)$ in $\Re^{\infty}$; these points form a coset in $\Re^{\infty}$ modulo the integral module $I$, i. e. an element in $\Re^{\infty}/I$. Let the image of $G$ in $\Re^{\infty}$ be (the module) $M$. Then, $G$ considered as an abstract group is mapped isomorphically on $M/I$ considered as an abstract group. Moreover, this mapping of the topological group $G$ is continuous when the topology in $\Re^{\infty}/I$ is given in the natural way by the topology in $\Re^{\infty}$. Since $G$ is compact and $M/I$ is a regular topological space satisfying the second axiom of countability, the mapping is bicontinuous ([4], p. 44). Hence we have an isomorphic mapping of the topological group $G$ on the togological group $M/I$,

$$\boxed{G \simeq M/I.}$$

As the image of a compact space by a continuous mapping, $M/I$ is closed in $\mathfrak{R}^\infty/I$. This implies that the image $M$ of $G$ in $\mathfrak{R}^\infty$ is closed in $\mathfrak{R}^\infty$ (since otherwise we could choose a sequence in $M$ converging to a point not in $M$, and the corresponding sequence in $M/I$ would then converge to the corresponding point in $\mathfrak{R}^\infty/I$, a point outside of $M/I$). Hence $M$, in the realization of $G$ above, is a closed module in $\mathfrak{R}^\infty$.

Conversely, every factor group $M/I$, where $M$ is a closed module in $\mathfrak{R}^\infty$ containing the integral module $I$, is a compact group since a sequence of points in $M$ can be reduced modulo 1 to lie in the compact set $0 \leqq x_1 \leqq 1, 0 \leqq x_2 \leqq 1, \cdots$ (the second axiom of countability being obviously fulfilled).

## 4. Proof of the simpler duality theorems.

Let $G$ be a compact group. We make use of the theorem of the preceding section which states that we can realize $G$ as a factor group $M/I$ inside $\mathfrak{R}^\infty$. By help of this we shall see that the character group $\widehat{G}$ can be realized as a factor group inside $\mathfrak{R}_\infty$.

Let $\alpha(X)$ be a continuous character on $M/I$ where $X$ is a variable coset in $M$ modulo $I$. We put $\alpha(\boldsymbol{x}) \equiv \alpha(X)$ for every $\boldsymbol{x} \varepsilon X$. In this way we get a continuous character $\alpha(\boldsymbol{x})$ on $M$. Our first task is to show that

$$\alpha(\boldsymbol{x}) \equiv \boldsymbol{a} \cdot \boldsymbol{x} \text{ where } \boldsymbol{a} \varepsilon \mathfrak{R}_\infty.$$

To see this we choose by theorem A a substitution $\boldsymbol{x} = T\boldsymbol{y}$ in $\mathfrak{R}^\infty$ which transforms $M$ into a module $\{(y_1, y_2, \cdots)\}$ of the simple form $S^\infty$. Since $M$ contains $I$, the class $\{n_i\}$ from theorem A must be empty. By this substitution the continuous character $\alpha(\boldsymbol{x})$ on $M$ is transformed into a continuous character $\beta(\boldsymbol{y}) = \alpha(T\boldsymbol{y})$ on the transformed module $\{(y_1, y_2, \cdots)\} = \{(\text{arbitrary, integral})\}$. Now, let

$$\beta(y_1, 0, 0, \cdots) \equiv b_1 y_1$$
$$\beta(0, y_2, 0, \cdots) \equiv b_2 y_2$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$

where in case $y_n$ is of "integral" type we may assume $b_n$ reduced modulo 1 to lie in the interval $0 \leq b < 1$. (It has been used here that a continuous character $\gamma(x)$ on the straight line, and on the integers, has the form $\gamma(x) = bx$.) Then

$$\beta(y_1, y_2, \cdots, y_n, 0, 0, \cdots) = b_1 y_1 + b_2 y_2 + \cdots + b_n y_n,$$

but for $n \to \infty$

$$(y_1, y_2, \cdots, y_n, 0, 0, \cdots) \to (y_1, y_2, \cdots)$$

and hence from the continuity of $\beta$ the sequence

$$(1) \qquad b_1 y_1 + b_2 y_2 + \cdots + b_n y_n$$

shall converge modulo 1 for every $(y_1, y_2, \cdots)$ from the transformed module.

Suppose now that $b_n$ was not $= 0$ for $n \geq$ a certain $N$. Then there would exist a sequence $n_1 < n_2 < \cdots$ such that $b_{n_p} \neq 0$ for $p = 1, 2, \cdots$. To obtain a contradiction we shall indicate a point from the transformed module such that the sequence (1) is not convergent modulo 1. We put $y_n = 0$ if $b_n = 0$. For the $n$ with $b_n \neq 0$, i. e. $n_1, n_2, \cdots$ we choose $y_n$ by induction. $y_{n_1}$ is chosen in accordance with its type (arbitrary or integral). Suppose $y_{n_p}$ chosen. Then we shall determine $y_{n_{p+1}}$ such that the numerical difference modulo 1 between

$$(2) \qquad b_{n_1} y_{n_1} + b_{n_2} y_{n_2} + \cdots + b_{n_p} y_{n_p}$$

and

$$b_{n_1} y_{n_1} + b_{n_2} y_{n_2} + \cdots + b_{n_p} y_{n_p} + b_{n_{p+1}} y_{n_{p+1}}$$

is $\geq \frac{1}{4}$, i. e. such that

$$(3) \qquad \left| b_{n_{p+1}} y_{n_{p+1}} \right| \geq \frac{1}{4} \pmod 1.$$

If $y_{n_{p+1}}$ is of the "arbitrary" type we only choose $y_{n_{p+1}}$ such that $b_{n_{p+1}} y_{n_{p+1}} = \frac{1}{2}$ which satisfies (3). If $y_{n_{p+1}}$ is of the "integral" type we write $b_{n_{p+1}}$, which is lying in the interval $0 < b < 1$, as a dyadic fraction. Since not all ciphers after the "point" in the fraction are zero or one we may choose $y_{n_{p+1}}$ as a power of 2 such that the first ciphers after the "point" in $b_{n_{p+1}}$ $y_{n_{p+1}}$ are 01 or 10. Then $b_{n_{p+1}} y_{n_{p+1}}$ reduced modulo 1 to the interval $0 \leq b < 1$ must in the first case lie in the interval $\frac{1}{4} \leq b \leq \frac{1}{2}$ and in the second case in the interval $\frac{1}{2} \leq b \leq \frac{3}{4}$. In both cases (3) is satisfied.

For this choice of the point $(y_1, y_2, \cdots)$ from the transformed module it is obvious that (1) cannot converge modulo 1 since the distance modulo 1 between consecutive elements in the subsequence (2) is always $\geq \frac{1}{4}$.

Thus we have seen that

$$\beta(\boldsymbol{y}) = \alpha(T\boldsymbol{y}) \equiv \boldsymbol{b} \cdot \boldsymbol{y} \text{ with } \boldsymbol{b} \varepsilon \mathfrak{R}_\infty,$$

and then

$$\alpha(\boldsymbol{x}) = \beta(T^{-1}\boldsymbol{x}) \equiv \boldsymbol{b} \cdot T^{-1}\boldsymbol{x} = \boldsymbol{a} \cdot \boldsymbol{x} \text{ with } \boldsymbol{a} \varepsilon \mathfrak{R}_\infty$$

where $\boldsymbol{a}$ is determined by $\boldsymbol{b} \cdot T^{-1}\boldsymbol{x} = \boldsymbol{a} \cdot \boldsymbol{x}$.

On the other hand every function $\alpha(\boldsymbol{x}) \equiv \boldsymbol{a} \cdot \boldsymbol{x}$ with $\boldsymbol{a} \varepsilon \mathfrak{R}_\infty$ obviously is a continuous character on $M$. But in order that it has arisen from a (continuous) character on $M/I$ a necessary and sufficient condition is that

$$\alpha(\boldsymbol{x}) \equiv \boldsymbol{a} \cdot \boldsymbol{x} \equiv 0 \text{ for } \boldsymbol{x} \varepsilon I$$

and this means $\boldsymbol{a} \varepsilon I'$ where $I'$ is the dual module in $\mathfrak{R}_\infty$ of $I$, i. e. the module of integral points in $\mathfrak{R}_\infty$ (see 1). Now, however, different $\boldsymbol{a}$'s in $I'$ may determine the same character on $M$, in fact

$$\boldsymbol{a}_1 \cdot \boldsymbol{x} \equiv \boldsymbol{a}_2 \cdot \boldsymbol{x} \text{ for } \boldsymbol{x} \varepsilon M$$

means $\boldsymbol{a}_1 - \boldsymbol{a}_2 \varepsilon M'$ where $M'$ is the dual module in $\mathfrak{R}_\infty$ of $M$ (see 1).

Hence, considered as abstract groups, the character group of $M/I$ and the group $I'/M'$ are isomorphic. Furthermore the arbitrary continuous character $\alpha(X)$ on $M/I$ is

$$\alpha(X) \equiv A \cdot X \text{ with } A \varepsilon I'/M' \quad (X \varepsilon M/I)$$

(the product $A \cdot X$ being defined by help of representatives $\boldsymbol{a}$ and $\boldsymbol{x}$ of $A$ and $X$).

The topology which is ascribed to the group $I'/M'$ in $\Re_\infty$ is the discrete one since already $I'$ is discrete (see 1). This, however, is also the topology ascribed to it as the character group of a compact group, for if in $\widehat{G}$ we consider the neighborhood of the zero-character determined by $F = G$ and $\varepsilon = \frac{1}{4}$ it consists of the characters $\alpha$ with

$$|\alpha(x)| < \frac{1}{4} \pmod 1 \text{ for } x \varepsilon G,$$

and the zero-character is the only such character. In fact, if $\alpha(x') \not\equiv 0$ for an element $x' \varepsilon G$ we could find a power $2^N$ of 2 such that $|\alpha(2^N x')| \geqq \frac{1}{4} \pmod 1$ (see top of p. 10).

Hence we have the result that the character group of $G \cong M/I$ is

$$\boxed{\widehat{G} \cong I'/M'.}$$

To prove theorem 1 for a compact group $G$ we have to prove that the character group of $I'/M'$ is isomorphic to $M/I$ by the correspondence mentioned in theorem 1. Let $\chi(A)$ be a (continuous) character[1] on $I'/M'$. For every $\boldsymbol{a} \varepsilon A$ we put $\chi(\boldsymbol{a}) \equiv \chi(A)$. Then $\chi(\boldsymbol{a})$ is a character on $I'$. Assume that

$$\begin{aligned} \chi(1, 0, 0, \cdots) &\equiv x_1 \\ \chi(0, 1, 0, \cdots) &\equiv x_2 \\ \cdots\cdots\cdots&\cdots\cdots\cdots. \end{aligned}$$

Then obviously

$$\chi(\boldsymbol{a}) \equiv \boldsymbol{x} \cdot \boldsymbol{a} \text{ with } \boldsymbol{x} = (x_1, x_2, \cdots) \varepsilon \Re^\infty.$$

On the other hand every function $\chi(\boldsymbol{a}) \equiv \boldsymbol{x} \cdot \boldsymbol{a}$ with $\boldsymbol{x} \varepsilon \Re^\infty$ is a character on $I'$. But in order that it arises from a character on $I'/M'$ a necessary and sufficient condition is that

$$\chi(\boldsymbol{a}) \equiv \boldsymbol{x} \cdot \boldsymbol{a} \equiv 0 \text{ for } \boldsymbol{a} \varepsilon M'$$

which by theorem B means that $\boldsymbol{x} \varepsilon M'' = M$. Now, however, different $\boldsymbol{x}$'s in $M$ may determine the same character on $I'$, in fact

$$\boldsymbol{x}_1 \cdot \boldsymbol{a} \equiv \boldsymbol{x}_2 \cdot \boldsymbol{a} \text{ for } \boldsymbol{a} \varepsilon I'$$

means $\boldsymbol{x}_1 - \boldsymbol{x}_2 \varepsilon I'' = I$.

[1] They are all continuous since the group is discrete.

Hence, considered as abstract groups, the character group of $I'/M'$ and the group $M/I$ are isomorphic. Furthermore an arbitrary character $\chi(A)$ on $I'/M'$ has the form

$$\chi(A) \equiv X \cdot A \text{ with } X \varepsilon M/I \ (A \varepsilon I'/M').$$

We shall now see that the topology of $M/I$ considered as a character group of $I'/M'$ coincides with the topology of $M/I$ induced by the topology in $\mathfrak{R}^{\infty}$.

In the first topology a neighborhood of zero is determined by an $\varepsilon > 0$ and a compact set $F$ from $I'/M'$, and since $I'/M'$ is discrete $F$ consists of a finite number of elements $A_1, A_2, \cdots, A_N$ from $I'/M'$. The neighborhood consists of all $X \varepsilon M/I$ with

$$(4) \qquad |X \cdot A_n| < \varepsilon \pmod 1, \; n = 1, 2, \cdots, N.$$

We now consider an arbitrary neighborhood of zero in the other topology. It consists of the $X \varepsilon M/I$ for which a representative $\boldsymbol{x} = (x_1, x_2, \cdots)$ satisfies

$$(5) \qquad \begin{array}{l} |x_1| < \varepsilon \pmod 1 \\ |x_2| < \varepsilon \pmod 1 \\ \cdots\cdots\cdots\cdots\cdots \\ |x_N| < \varepsilon \pmod 1 \end{array}$$

where $\varepsilon > 0$, and $N$ is a positive integer. In order to find a neighborhood (4) in the first topology contained in this neighborhood (5) we use the same $\varepsilon$ and $N$ in (4) as in (5) and choose for $A_1, A_2, \cdots, A_N$ the (not necessarily different) cosets with the respective representatives $(1, 0, 0, \cdots)$, $(0, 1, 0, \cdots)$, $\cdots$, $(0, 0, 0, \cdots, 0, 1, 0, 0, \cdots)$. In fact, for this choice the neighborhood (4) will coincide with (5).

Conversely, given an arbitrary neighborhood (4) it is possible to choose $\varepsilon$ and $N$ in (5) such that the neighborhood (5) is contained in the neighborhood (4). This is true since the $A_n$ have integral $\boldsymbol{a}_n$ as representatives in $\mathfrak{R}_{\infty}$.

Hence the two topologies are equivalent, and we have the result that the correspondence from theorem 1 is an isomorphism

$$\boxed{\hat{\hat{G}} \simeq G.}$$

This proves theorem 1 for a compact group $G$.

Theorem 1 for the case of a discrete group which is written in the form $\hat{G}$ where $G$ is compact, follows from the result above. In order to prove theorem 1 for an arbitrary discrete group it is therefore enough to prove that every such group is the character group of a compact group, a fact which is also stated in the "simpler theorems" on p. 5. This is easily done. Let $G$ be an arbitrary countable discrete group. We choose a system of generators $a_1, a_2, \cdots$ of $G$ (for instance all elements in $G$). An arbitrary element $a \varepsilon G$ may be written

$$(6) \qquad a = a_1^{n_1} a_2^{n_2} \cdots.$$

We map $a$ in the set of integral points $(n_1, n_2, \cdots)$ of $\mathfrak{R}_\infty$ for which (6) holds good. Let 0 by this procedure be mapped in the module $M_1$. Then obviously

$$G \simeq I'/M_1.$$

Hence, from the result on p. 11 and theorem B, the group $G$ is the character group of the compact group $M_1'/I$.

This proves theorem 1 for compact and discrete groups.

We now pass to the proof of theorem 2 for compact and discrete groups. Let $G$ be a compact group and $H$ a subgroup. By the isomorphism

$$G \simeq M/I$$

the set $H$ corresponds to the set $N/I$ where $N$ is a closed module in $\mathfrak{R}^\infty$, $I \subseteq N \subseteq M$. As found on pp. 10—11, the character group of $M/I$ is $I'/M'$ and an arbitrary continuous character $\alpha(X)$ on $M/I$ is of the form

$$\alpha(X) \equiv A \cdot X \quad (A \varepsilon I'/M', X \varepsilon M/I).$$

We shall now pick out the characters which are $\equiv 0$ on $N/I$, i. e. for which

$$A \cdot X \equiv 0 \text{ for } X \varepsilon N/I,$$

but this means (by the definition of dual module, p. 3) that the $A$'s from $I'/M'$ shall be taken from the subset $N'/M'$.

We repeat the procedure. As found on p. 12, an arbitrary character $\chi(A)$ on $I'/M'$ has the form

$$\chi(A) \equiv X \cdot A \quad (X \varepsilon M/I,\ A \varepsilon I'/M'),$$

and we have to pick out the characters which are $\equiv 0$ on $N'/M'$, i. e. for which

$$X \cdot A \equiv 0 \text{ for } A \varepsilon N'/M',$$

but this means (by the definition of dual module, p. 3) that the $X$'s from $M/I$ shall be taken from the subset $N''/I$ which by theorem B is equal to $N/I$, q. e. d.

Since $\widehat{G} \simeq I'/M'$ is an *arbitrary* discrete group and $H^* \simeq N'/M'$ is an *arbitrary* subgroup of $\widehat{G} \simeq I'/M'$, the theorem 2 is also proved for a discrete group.

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Indleveret til selskabet den 12. september 1949.
Færdig fra trykkeriet den 22. marts 1950.