

DET KGL. DANSKE VIDENSKABERNES SELSKAB
MATEMATISK-FYSISKE MEDDELELSER, BIND XXV, NR. 14

A THEOREM
ON ALMOST PERIODIC
FUNCTIONS OF INFINITELY
MANY VARIABLES

BY

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I KOMMISSION HOS EJNAR MUNKSGAARD
1950

Printed in Denmark
Bianco Lunos Bogtrykkeri

1. Introduction.

For an almost periodic function $f(x)$ of one variable the following theorem is true: *the function is periodic if and only if the set of all translated functions $f(x+h)$ is closed with respect to uniform convergence.* As remarked by B. JESSEN this can easily be proved directly from the structure definition of almost periodicity; also for the corresponding theorem on almost periodic functions $f(x_1, x_2, \dots, x_m)$ in the m -dimensional space he has shown me such a proof.

The way in which one generalizes the word "periodic" when passing from 1 to m dimensions with this theorem is to what might be called "fully periodic". Let $f(x_1, x_2, \dots, x_m)$ be a continuous function of (x_1, x_2, \dots, x_m) . A vector (h_1, h_2, \dots, h_m) is called a period vector of $f(x_1, x_2, \dots, x_m)$ if $f(x_1 + h_1, x_2 + h_2, \dots, x_m + h_m) = f(x_1, x_2, \dots, x_m)$ for all (x_1, x_2, \dots, x_m) . The set of all period vectors of $f(x_1, x_2, \dots, x_m)$ is obviously a closed module (a module being a set which with two points also contains their sum and difference). This module may consist of $(0, 0, \dots, 0)$ only, and the function is not periodic at all. If the dimension of the module is equal to the dimension m of the space we call the function *fully periodic*. Our theorem for almost periodic functions of m variables can then be stated, *such a function is fully periodic if and only if the set of all translated functions $f(x_1 + h_1, x_2 + h_2, \dots, x_m + h_m)$ is closed with respect to uniform convergence.*

JESSEN put the problem to decide whether this theorem also holds for almost periodic functions of an infinite number of variables. It will turn out—as a result of this paper—that it does hold in verbally the same formulation, the word "fully periodic" needing of course an appropriate definition. Incidentally, we

shall get another proof in the m -dimensional case than the one referred to above.

JESSEN also remarked that if the almost periodic function $f(x_1, x_2, \dots)$ in question is limit-periodic, the theorem is true as a simple consequence of a result of H. BOHR [2], [3] obtained in connection with a study of certain classes of almost periodic functions (see 4, and 6, p. 12 of the present paper). BOHR's result concerned an infinite system of linear congruences

$$r_{n1}x_1 + r_{n2}x_2 + \dots + r_{nq_n}x_{q_n} \equiv \theta_n \pmod{1}, n = 1, 2, \dots$$

with infinitely many real variables x_1, x_2, \dots and rational coefficients. It was later generalized by BOHR and FØLNER [4] to arbitrary coefficients, and this generalization will be a tool for the proof of the general case of our theorem.

2. Almost Periodic Functions of Infinitely Many Variables.

We start with recalling that an almost periodic function $f(x_1, x_2, \dots)$ of an infinite number of real variables x_1, x_2, \dots ¹ can be characterized as a (complex-valued) function, defined on the space \mathfrak{R}^∞ of points $\boldsymbol{x} = (x_1, x_2, \dots)$, which can be uniformly approximated by trigonometric polynomials

$$\sum c_n e^{2\pi i(a_1^n x_1 + a_2^n x_2 + \dots + a_{q_n}^n x_{q_n})}.$$

The space \mathfrak{R}^∞ is topologized in the following way. A sequence of points \boldsymbol{x}^n is said to converge towards \boldsymbol{x} , if $x_1^n \rightarrow x_1, x_2^n \rightarrow x_2, \dots$. Thus every trigonometric polynomial, and hence also every almost periodic function is continuous on \mathfrak{R}^∞ .

The exponent vectors $(a_1^n, a_2^n, \dots, a_{q_n}^n, 0, 0, \dots)$ in the trigonometric polynomial above have zeros on all coordinate places from a certain number. We define the space \mathfrak{R}_∞ as the set of all vectors $\boldsymbol{a} = (a_1, a_2, \dots)$ with zeros on all coordinate places from a certain number (depending on the point), and so the exponent vectors can be said to belong to \mathfrak{R}_∞ . The inner

¹ S. BOCHNER [1].

product $a_1x_1 + a_2x_2 + \dots$ between a vector $\mathbf{a} = (a_1, a_2, \dots)$ from \mathfrak{R}_∞ and a vector $\mathbf{x} = (x_1, x_2, \dots)$ from \mathfrak{R}^∞ is denoted by $\mathbf{a} \cdot \mathbf{x}$. The trigonometric polynomial above may then be written

$$\sum c_n e^{2\pi i \mathbf{a}^n \cdot \mathbf{x}} \quad (\mathbf{a}^n \in \mathfrak{R}_\infty, \mathbf{x} \in \mathfrak{R}^\infty).$$

To every almost periodic function $f(\mathbf{x})$ is associated a unique Fourier series

$$f(\mathbf{x}) \sim \sum A_n e^{2\pi i \mathbf{a}^n \cdot \mathbf{x}}.$$

To the translated function $f(\mathbf{x} + \mathbf{h})$ is associated the Fourier series

$$f(\mathbf{x} + \mathbf{h}) \sim \sum A_n e^{2\pi i \mathbf{a}^n \cdot \mathbf{h}} e^{2\pi i \mathbf{a}^n \cdot \mathbf{x}}.$$

A necessary and sufficient condition for a sequence of translated functions $f(\mathbf{x} + \mathbf{h}^1), f(\mathbf{x} + \mathbf{h}^2), \dots$ to converge uniformly to the (*eo ipso*) almost periodic function $g(\mathbf{x})$ is that the Fourier series of $f(\mathbf{x} + \mathbf{h}^n)$ converge formally to the Fourier series of $g(\mathbf{x})$. Therefore the necessary and sufficient condition for the set of all translated functions $f(\mathbf{x} + \mathbf{h})$ to be closed is that the set of points

$$\begin{pmatrix} e^{2\pi i \mathbf{a}^1 \cdot \mathbf{h}} \\ e^{2\pi i \mathbf{a}^2 \cdot \mathbf{h}} \\ \vdots \\ \vdots \end{pmatrix}$$

where \mathbf{h} runs through all vectors in \mathfrak{R}^∞ , be closed in \mathfrak{R}^∞ or—what comes to the same thing—that the module of points

$$\begin{pmatrix} \mathbf{a}^1 \cdot \mathbf{h} \\ \mathbf{a}^2 \cdot \mathbf{h} \\ \vdots \\ \vdots \end{pmatrix} + i_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ \vdots \end{pmatrix} + i_2 \begin{pmatrix} 0 \\ 1 \\ \vdots \\ \vdots \end{pmatrix} + \dots$$

where \mathbf{h} runs through all vectors in \mathfrak{R}^∞ and i_1, i_2, \dots through all integers, be closed in \mathfrak{R}^∞ .

3. Closed Modules and Substitutions in \mathfrak{R}^∞ .

Let $f(x_1, x_2, \dots) = f(\mathbf{x})$ be a continuous function on \mathfrak{R}^∞ . A vector $\mathbf{h} = (h_1, h_2, \dots)$ is called a period vector of $f(\mathbf{x})$ if $f(\mathbf{x} + \mathbf{h}) = f(\mathbf{x})$ for all points \mathbf{x} . The set of all period vectors of $f(\mathbf{x})$ is obviously a closed module in \mathfrak{R}^∞ . If it is not contained in any proper vector-subspace (i. e. *closed* linear subset¹) of \mathfrak{R}_∞ the function $f(\mathbf{x})$ is called fully periodic.

As a general example of a fully periodic function we consider an almost periodic function

$$f(\mathbf{x}) \sim \sum A_n e^{2\pi i \mathbf{a}^n \cdot \mathbf{x}}$$

which is periodic with the period 1 in all the coordinates x_1, x_2, \dots or—what comes to the same thing—a function which has all integral points as period vectors. Expressed as a property of the Fourier series this means that all the exponent vectors \mathbf{a}^n are integral vectors.

In the previously cited paper [4]—and in more detail in [5]—a structure theorem for closed modules in \mathfrak{R}^∞ was obtained. Incidentally, let me remark that this was done by introducing a suitable convergence notion in \mathfrak{R}_∞ to the effect that a duality took place between the closed modules in \mathfrak{R}^∞ and the closed modules in \mathfrak{R}_∞ . To formulate the structure theorem we have to introduce the notion of “substitution” in \mathfrak{R}^∞ . By this is understood a linear transformation $\mathbf{x} = T\mathbf{y}$ of the form

$$\begin{aligned} x_1 &= a_{11}y_1 + a_{12}y_2 + \dots + a_{1p_1}y_{p_1} \\ x_2 &= a_{21}y_1 + a_{22}y_2 + \dots + a_{2p_2}y_{p_2} \\ &\dots\dots\dots \end{aligned}$$

which establishes a one-to-one mapping $\mathbf{x} \leftrightarrow \mathbf{y}$ of the whole infinite-dimensional space on the whole infinite-dimensional space. It turns out to be the same as a linear, bicontinuous, one-to-one mapping of \mathfrak{R}^∞ onto itself. The structure theorem now runs as follows.

¹ A linear set in \mathfrak{R}^∞ need not be closed as is the case in the m -dimensional space. For instance the points in \mathfrak{R}_∞ form a linear subset of \mathfrak{R}^∞ and its closure is the whole \mathfrak{R}^∞ .

Structure theorem. A closed module in the infinite-dimensional space \mathfrak{R}^∞ is a point set E which by a substitution can be transformed into a point set of a special form, namely a point set $[(x_1, x_2, \dots)]$ of the following structure: The indices $1, 2, \dots, n, \dots$ can be divided into three fixed classes $\{n_r\}, \{n_s\}, \{n_t\}$, such that the coordinates x_{n_r} independently run through all numbers, and the coordinates x_{n_s} independently run through all integers, while all the remaining coordinates x_{n_t} are constantly zero. Conversely, each such point set E is a closed module.

The vector-subspaces (i. e. closed linear subsets) of \mathfrak{R}^∞ are of course characterized by an empty class $\{n_s\}$.

That a closed module is not contained in any proper vector-subspace of \mathfrak{R}^∞ —as is the demand to the period module of a fully periodic function—means that it has an empty class $\{n_t\}$ or, what is equivalent, that there exists a transformed set which contains all integral points.

If $f(\boldsymbol{x})$ is an almost periodic function on \mathfrak{R}^∞ and we subject \boldsymbol{x} to the substitution $\boldsymbol{x} = T\boldsymbol{y}$, then the transformed function $f(T\boldsymbol{y})$ is an almost periodic function of \boldsymbol{y} in \mathfrak{R}^∞ . In fact, a trigonometric polynomial in \boldsymbol{x} will be transformed into a trigonometric polynomial in \boldsymbol{y} by this process, and, to complete the reasoning, $|f(\boldsymbol{x}) - s(\boldsymbol{x})| \leq \varepsilon$ for all \boldsymbol{x} means the same as $|f(T\boldsymbol{y}) - s(T\boldsymbol{y})| \leq \varepsilon$ for all \boldsymbol{y} . Furthermore, if $f(\boldsymbol{x})$ has the Fourier series

$$(1) \quad \sum A_n e^{2\pi i \boldsymbol{a}^n \cdot \boldsymbol{x}},$$

then $f(T\boldsymbol{y})$ has the formally transformed Fourier series

$$(2) \quad \sum A_n e^{2\pi i \boldsymbol{b}^n \cdot \boldsymbol{y}},$$

where \boldsymbol{b}^n is determined by $\boldsymbol{a}^n \cdot T\boldsymbol{y} = \boldsymbol{b}^n \cdot \boldsymbol{y}$.¹ To see this, let $s_n(\boldsymbol{x})$ be a sequence of trigonometric polynomials which converges uniformly to $f(\boldsymbol{x})$. It will converge formally to the Fourier series (1), and therefore the sequence $s_n(T\boldsymbol{y})$ of trigonometric

¹ Incidentally we remark that the \boldsymbol{b}^n 's are obtained from the \boldsymbol{a}^n 's by a "substitution" T^* in \mathfrak{R}_∞ , T^* being the "adjoint substitution" of T (see [5]).

polynomials in \mathbf{y} converging uniformly to $f(T\mathbf{y})$ will converge formally to the series (2), which must therefore be the Fourier series of $f(T\mathbf{y})$.

Obviously, a period vector \mathbf{h} of $f(\mathbf{x})$ is transformed into a period vector \mathbf{k} ($\mathbf{h} = T\mathbf{k}$) of the function $f(T\mathbf{y})$.

It follows from what has been said that a necessary and sufficient condition for an almost periodic function $f(\mathbf{x})$ to be fully periodic is that there exists a substitution $\mathbf{x} = T\mathbf{y}$ such that $f(T\mathbf{y})$ is periodic with the period 1 in all the coordinates y_1, y_2, \dots , or—what comes to the same thing—that all the Fourier exponent vectors of $f(T\mathbf{y})$ are integral.

4. Infinitely Many Linear Congruences with Infinitely Many Variables.

We consider an arbitrary enumerable system of linear congruences with an enumerable number of real variables

$$(3) \quad \begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n_1}x_{n_1} &\equiv \theta_1 \pmod{1} \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n_2}x_{n_2} &\equiv \theta_2 \pmod{1} \\ \dots\dots\dots \end{aligned}$$

or, briefly written,

$$(3) \quad \begin{aligned} \mathbf{a}^1 \cdot \mathbf{x} &\equiv \theta_1 \\ \mathbf{a}^2 \cdot \mathbf{x} &\equiv \theta_2 \\ \dots\dots\dots \end{aligned}$$

where every congruence only contains a finite number of variables and the a 's and the θ 's are arbitrary (real) numbers. By π_1 we denote the module of points $(\theta_1, \theta_2, \dots)$ in \mathfrak{R}^∞ for which the corresponding infinite system (3) has a solution, and by π_2 the module of points $(\theta_1, \theta_2, \dots)$ for which any finite subsystem of (3) has a solution. The result of BOHR referred to in **1** was that in case of rational a 's a necessary and sufficient condition on the linear forms $\mathbf{a}^n \cdot \mathbf{x}$ in (3) in order that $\pi_1 = \pi_2$ is that there exists a substitution $\mathbf{x} = T\mathbf{y}$ in \mathfrak{R}^∞ which transforms them into linear forms with integral coefficients. The generalization in [4] of this result is that without the said restriction of rationality on

the a 's a necessary and sufficient condition on the linear forms in (3) in order that $\pi_1 = \pi_2$ is that there exists a substitution in \mathfrak{R}^∞ which transforms them into a system of linear forms of a certain simple type, denoted by S . By a system of linear forms of the type S we understand a system where certain of the variables (finite or infinite in number) have mere integral coefficients while each of the remaining variables (finite or infinite in number) necessarily becomes 0 if for a sufficiently large m one solves the m first "zero-congruences" corresponding to the linear forms, i. e. the congruences (3) with $\theta_1 = \theta_2 = \cdots = 0$.

By the proof of this theorem the structure theorem for closed modules in \mathfrak{R}^∞ played an important role.

5. Modules in the m -dimensional Space.

The duality between \mathfrak{R}^∞ and \mathfrak{R}_∞ loosely referred to above has its simpler origin in a duality considered by M. RIESZ [6] between two m -dimensional spaces $R_m = \{(x_1, x_2, \cdots, x_m)\}$ and $R_m = \{(a_1, a_2, \cdots, a_m)\}$. Since we shall use it in our proof we shall state this duality explicitly¹. To an arbitrary module M in R_m RIESZ considers the point set in (the other space) R_m consisting of all points $\mathbf{a} = (a_1, a_2, \cdots, a_m)$ from this latter R_m for which

$$\mathbf{a} \cdot \mathbf{x} = a_1 x_1 + a_2 x_2 + \cdots + a_m x_m \equiv 0 \pmod{1}$$

for every point $\mathbf{x} = (x_1, x_2, \cdots, x_m)$ from M . This point set is a closed module in R_m and is called the dual module of M . We denote it by M' . If we repeat the operation of passing to the dual module we get a closed module $M'' = (M')'$ in (the original space) R_m . The relation between M and M'' appears from the following important theorem.

Riesz's theorem. *If M is an arbitrary module in R_m , the dual M'' of its dual module M' is the closure \bar{M} of M , i. e.*

$$M'' = \bar{M}.$$

Before passing to the real proof of our theorem on almost periodic functions of infinitely many variables we shall make a

¹ For the proofs, see [5].

simple, and rather obvious, remark about the type of module M which is obtained by placing subspaces of the same dimension $\nu < m$ through all integral points in R_m and parallel to the same vector-subspace R of dimension ν , i. e. the module obtained by adding to the points in R all integral points in R_m . Our statement is that the largest vector-space contained in M^1 is R . This follows, by an indirect reasoning, from the fact that a usual space of dimension > 0 contains more than an enumerable number of points.

6. Proof of the Theorem.

We repeat the theorem to be proved.

Theorem. *An almost periodic function $f(\boldsymbol{x}) = f(x_1, x_2, \dots)$ of an infinite number of variables is fully periodic² if and only if the set of translated functions $f(\boldsymbol{x} + \boldsymbol{h}) = f(x_1 + h_1, x_2 + h_2, \dots)$ is closed with respect to uniform convergence.*

It is plain that this theorem contains the analogous theorem on almost periodic functions of a finite number of variables as a special case.

1. The one part of the theorem is easily dealt with. Let $f(\boldsymbol{x})$ be a fully periodic function. In order to show that the set of translated functions $f(\boldsymbol{x} + \boldsymbol{h})$ is closed we choose a substitution $\boldsymbol{x} = T\boldsymbol{y}$ such that the transformed function $f(T\boldsymbol{y}) = g(\boldsymbol{y}) = g(y_1, y_2, \dots)$ is periodic with the period 1 in all the coordinates y_1, y_2, \dots (see end of **3**). Let $\boldsymbol{h}^1, \boldsymbol{h}^2, \dots$ be a sequence of vectors such that $f(\boldsymbol{x} + \boldsymbol{h}^n)$ converges uniformly to a function $d(\boldsymbol{x})$. We have to show that $d(\boldsymbol{x})$ is equal to a translated function $f(\boldsymbol{x} + \boldsymbol{h})$. Let \boldsymbol{k}^n be the corresponding points of \boldsymbol{h}^n by the substitution above, i. e. $\boldsymbol{h}^n = T\boldsymbol{k}^n$. Then $g(\boldsymbol{y} + \boldsymbol{k}^n)$ will converge uniformly to $d(T\boldsymbol{y}) = j(\boldsymbol{y})$. We only have to prove that $j(\boldsymbol{y}) = g(\boldsymbol{y} + \boldsymbol{k})$ for some \boldsymbol{k} since then $d(\boldsymbol{x}) = j(T^{-1}\boldsymbol{x}) = g(T^{-1}\boldsymbol{x} + \boldsymbol{k}) = g(T^{-1}\boldsymbol{x} + T^{-1}\boldsymbol{h}) = f(\boldsymbol{x} + \boldsymbol{h})$, where \boldsymbol{h} is determined by $\boldsymbol{h} = T\boldsymbol{k}$.

The function $g(\boldsymbol{y}) = g(y_1, y_2, \dots)$ is periodic with the period 1 in all the coordinates, so we may assume that our

¹ The vector-space generated by all vector-spaces contained in M (which on account of the module property must be contained in M).

² See **3**.

\mathbf{k}^n -points are all lying in the "periodicity parallelotope" $0 \leq y_1 \leq 1, 0 \leq y_2 \leq 1, \dots$. This point set, however, is compact in the sense that every sequence $\mathbf{k}^1, \mathbf{k}^2, \dots$ of points from this set has a subsequence which converges to a point of the set. As our \mathbf{k} we may take any such limit-point of the sequence $\mathbf{k}^1, \mathbf{k}^2, \dots$, for on account of the continuity of $g(\mathbf{y}), j(\mathbf{y}) = \lim g(\mathbf{y} + \mathbf{k}^n) = g(\mathbf{y} + \mathbf{k})$, q. e. d.

2. We now pass to the more difficult part of the theorem. Let $f(\mathbf{x})$ be an almost periodic function for which the set of translated functions $f(\mathbf{x} + \mathbf{h})$ is closed. Our task is to prove that $f(\mathbf{x})$ is fully periodic or, what comes to the same thing, that there exists a substitution $\mathbf{x} = T\mathbf{y}$ such that the transformed function $f(T\mathbf{y})$ has mere integral Fourier exponent vectors (see end of 3). If

$$f(\mathbf{x}) \sim \sum A_n e^{2\pi i \mathbf{a}^n \cdot \mathbf{x}}, \text{ then } f(T\mathbf{y}) \sim \sum A_n e^{2\pi i \mathbf{b}^n \cdot \mathbf{y}},$$

where the linear forms $\mathbf{b}^n \cdot \mathbf{y}$ are obtained from the linear forms $\mathbf{a}^n \cdot \mathbf{x}$ by the substitution $\mathbf{x} = T\mathbf{y}$. So we have to find a substitution T which transforms the linear forms $\mathbf{a}^n \cdot \mathbf{x}$ into linear forms with integral coefficients. For this purpose we consider the corresponding system of congruences

$$(4) \quad \begin{aligned} \mathbf{a}^1 \cdot \mathbf{x} &\equiv \theta_1 \\ \mathbf{a}^2 \cdot \mathbf{x} &\equiv \theta_2 \\ &\dots \dots \dots \end{aligned}$$

Let π_1 and π_2 be the modules defined in 4 corresponding to these congruences. Our assumption that the set of translated functions $f(\mathbf{x} + \mathbf{h})$ is closed is equivalent to π_1 being closed (see 2).

It is plain that $\pi_1 \subseteq \pi_2$. Furthermore, the closure $Cl(\pi_1)$ of π_1 contains π_2 , $Cl(\pi_1) \supseteq \pi_2$. For the proof of this let $(\theta_1^0, \theta_2^0, \dots)$ be an arbitrary point from π_2 . In order to approximate $(\theta_1^0, \theta_2^0, \dots)$ by a point from π_1 we solve the m first congruences in (4) for this choice of $(\theta_1, \theta_2, \dots)$ and a "large" m . Let \mathbf{x}^0 be a solution. Then $(\theta_1^0, \theta_2^0, \dots, \theta_m^0, \mathbf{a}^{m+1} \cdot \mathbf{x}^0, \mathbf{a}^{m+2} \cdot \mathbf{x}^0, \dots)$ from π_1 will be a "good" approximation to $(\theta_1^0, \theta_2^0, \dots)$ in \mathfrak{R}^∞ . Since π_1 closed means $Cl(\pi_1) = \pi_1$, the relations

$$\pi_1 \subseteq \pi_2 \subseteq Cl(\pi_1)$$

imply $\pi_1 = \pi_2$. In case $f(\boldsymbol{x})$ is limit-periodic, i. e. all the \boldsymbol{x} -vectors are rational, the theorem of BOHR stated in 4 therefore immediately completes the proof.

In the general case we conclude by the theorem of BOHR and FÖLNER in 4 that there exists a substitution $\boldsymbol{x} = T\boldsymbol{y}$ which transforms the linear forms into a system of the type S . Our proof will be completed if we can show that the new system has mere integral coefficients. Let the new system of congruences be

$$(5) \quad \begin{aligned} \boldsymbol{b}^1 \cdot \boldsymbol{y} &\equiv \theta_1 \\ \boldsymbol{b}^2 \cdot \boldsymbol{y} &\equiv \theta_2 \\ &\dots\dots\dots \end{aligned}$$

For an arbitrary m we consider the module of points $(\theta_1, \theta_2, \dots, \theta_m)$ which is supplied by the m first congruences when \boldsymbol{y} runs through all points in \mathfrak{R}^∞ , and we wish to show that this module is closed in R_m . We assume to the contrary that it is not closed. Then there exists a point $(\theta'_1, \theta'_2, \dots, \theta'_m)$ which belongs to its closure but not to the module itself. We choose a sequence \boldsymbol{y}^r such that $(\boldsymbol{b}^1 \cdot \boldsymbol{y}^r, \boldsymbol{b}^2 \cdot \boldsymbol{y}^r, \dots, \boldsymbol{b}^m \cdot \boldsymbol{y}^r) \rightarrow (\theta'_1, \theta'_2, \dots, \theta'_m) \pmod{1}$ and next by the diagonal procedure a subsequence of \boldsymbol{y}^r such that $\boldsymbol{b}^{m+1} \cdot \boldsymbol{y}^r$ modulo 1 converges to a θ'_{m+1} , $\boldsymbol{b}^{m+2} \cdot \boldsymbol{y}^r$ modulo 1 converges to a θ'_{m+2} , etc., ad inf. The point $(\theta'_1, \theta'_2, \dots)$ will then obviously belong to $Cl(\pi_1)$ but not to π_1 (since not even the m first congruences can be fulfilled for this $(\theta'_1, \theta'_2, \dots)$). This is a contradiction since $Cl(\pi_1) = \pi_1$. Hence the module $\{(\theta_1, \theta_2, \dots, \theta_m)\}$ is closed. It will therefore be sufficient to prove that if the system of linear forms in (5) has not mere integral coefficients, then the module cannot be closed for all m .

To fix the matter assume for instance that the first column in our system of linear forms be not integral. The property of a system of linear forms of the type S (see 4) then implies that the variable y_1 necessarily becomes 0 if for a sufficiently large m one solves the m first "zero-congruences" corresponding to the linear forms, i. e. the congruences (5) with $\theta_1 = \theta_2 = \dots = 0$. Let m be chosen in this way. We shall then show that the module $M = \{(\theta_1, \theta_2, \dots, \theta_m)\}$ supplied by the m first congruences is not closed. To do this we write the m first congruences in (5) more fully as

$$\begin{pmatrix} i_1 \\ i_2 \\ \cdot \\ \cdot \\ i_m \end{pmatrix} = i_1 \begin{pmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ 0 \end{pmatrix} + i_2 \begin{pmatrix} 0 \\ 1 \\ \cdot \\ \cdot \\ 0 \end{pmatrix} + \cdots + i_m \begin{pmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ 1 \end{pmatrix}$$

by I we may write this

$$M = R + I.$$

In § we have seen that the largest vector-subspace in M is R . We now show that the closure \bar{M} of M contains a larger vector-subspace, i. e. a space of dimension $\geq \nu + 1$. Then \bar{M} must be different from M , and M cannot be closed as we had to prove.

We pass to the dual module M' of M (see §) which obviously consists of all integral points (k_1, k_2, \dots, k_m) in R_m that are orthogonal to R (for $(k_1, k_2, \dots, k_m) \cdot (0, 0, \dots, 0, 1, 0, \dots, 0) \equiv 0$ means k_r integral, and $(k_1, k_2, \dots, k_m) \cdot y_r (b_{1r}, b_{2r}, \dots, b_{mr}) \equiv 0$ for all y_r means $(k_1, k_2, \dots, k_m) \cdot (b_{1r}, b_{2r}, \dots, b_{mr}) = 0$).

The vector-space R^* of arbitrary points in R_m which are orthogonal to R has the dimension $m - \nu$. Suppose that the module $M' \subseteq R^*$ had the same dimension; then R could be characterized as the orthogonal vector-space to the module M' . But M' consists of integral points, so in particular a generating system $\mathbf{l}^1, \mathbf{l}^2, \dots, \mathbf{l}^{m-\nu}$ is integral. Hence R would be the set of points \mathbf{x} satisfying

$$\begin{aligned} \mathbf{l}^1 \cdot \mathbf{x} &= 0 \\ \mathbf{l}^2 \cdot \mathbf{x} &= 0 \\ \dots\dots\dots \\ \mathbf{l}^{m-\nu} \cdot \mathbf{x} &= 0. \end{aligned}$$

Solving algebraically we should get

$$\mathbf{x} = t_1 \mathbf{p}^1 + t_2 \mathbf{p}^2 + \cdots + t_\nu \mathbf{p}^\nu$$

with linearly independent *integral* vectors $\mathbf{p}^1, \mathbf{p}^2, \dots, \mathbf{p}^\nu$ and t_1, t_2, \dots, t_ν running through all numbers; and this contradicts the fact that the lattice of integral points in R has at most the

dimension $\nu - 1$. The argument therefore shows that the lattice M' has at most the dimension $m - \nu - 1$.

To complete the proof we use RIESZ's theorem (see 5) according to which $\bar{M} = (M')'$. Since M' is a lattice of dimension at most $m - \nu - 1$, the largest vector-subspace in \bar{M} has at least the dimension $m - (m - \nu - 1) = \nu + 1$ (for if M' is of dimension s , its orthogonal space is of dimension $m - s$ and this orthogonal space is obviously contained in the dual module $(M')' = \bar{M}$). This proves our theorem.

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