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# A COMPOSITE GIANT STAR MODEL WITH ISOTHERMAL CORE

BY

ANDERS REIZ

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1. During the last few years a number of papers [1] have appeared dealing with the problems of energy production and evolution of main-sequence and giant stars, which start from discussions of composite stellar models consisting of isothermal cores and point-source envelopes.

Gamow [2] in particular has considered the evolution of a star with energy production in a convective core according to the carbon cycle mechanism, and suggested that such a star, as a consequence of exhaustion of the hydrogen in the convective core, would evolve into a giant star, of very large radius for its mass, and built on a model in which the energy is produced by the carbon cycle in a sufficiently hot shell surrounding the inert core.

Although the investigations quoted above [1] would appear to throw some doubts on Gamow's suggestion, it nevertheless seemed worth while to examine whether a model for the giant stars, characterized by a core devoid of hydrogen, might lead to sufficiently high temperatures and densities in the region immediately surrounding the core to explain the energy production in the giant stars according to the carbon cycle.

The aim of the present paper is to investigate, by numerical methods, the possibilities of a model with non-productive core to explain the energy production of giant stars. The investigation was restricted to a single giant star (*Capella A*) of specified mass, radius, and luminosity.

The problem has been simplified through the assumption that the entire energy production takes place in an infinitely narrow shell surrounding the core. This means that the total outward net-flux of energy $L(r)$ is assumed to be constant, and equal to the luminosity $L$, right up to the core, and zero inside the core.

The temperature of the shell in which the energy production takes place is assumed to be 20 million degrees. The choice of the value of the temperature is governed by the following considerations. The simplified model should approximate as closely as possible a model which has these properties: 1. The change of $L(r)$ with distance $r$ from the centre throughout the envelope is given by the energy production according to the carbon cycle mechanism. 2. The particular distance $r$ from the centre at which $L(r)$ becomes zero, coincides with the radius of the non-productive core.

It may be remarked here already that the results of the present investigations tend to show that the energy production takes place in a very narrow zone, and indicate that a repetition of the investigations with a considerably higher value of the representative temperature would be desirable[1].

**2.** As already mentioned the model considered is specified by given values of the mass, $M$, the radius $R$, and the luminosity $L$. The hydrogen content $X$ is considered as a variable parameter. We assume that the helium content of the envelope is negligible, and that the non-hydrogen part contains the heavier elements in the relative proportions of the RUSSELL-mixture. The mean molecular weight of the envelope can then be calculated from $X$. With regard to the core we assume that each gram consists of $X$ gram helium, and $1-X$ gram of the RUSSELL-mixture (no hydrogen). The mean molecular weight of the core can then also be calculated as a function of $X$.

We thus assume that the model is composed by an envelope with constant $L(r)$ surrounding an isothermal core, the mean molecular weights being $\mu_1$ and $\mu_2$, respectively. The transition from envelope to core is assumed to take place when the temperature, going inward, reaches 20 million degrees. The following question is now formulated: given mass, radius, and luminosity

[1] Note added in proof: After this paper was sent to the press calculations (to appear in Arkiv för Astronomi, Band 1:1, 1948) have been performed with a temperature of 33 million degrees as representative for the carbon cycle. The energy-output computed by means of BETHE's law from the new temperatures and densities is for $X = 0.34$, $L = 4.0 \cdot 10^{35}$ erg/sec closely agreeing with $L_{\text{obs}} = 4.5 \cdot 10^{35}$ The central conditon $M(r) = 0$ for $r = 0$ for the composite configuration is satisfied for the above hydrogen content. The representative temperature almost coincides with the theoretical value 32 million degrees derived by BETHE for Capella A (Phys. Rev., **55**, 434, 1939).

of the star, does there exist a certain value of the hydrogen content $X$, which is varied as a parameter, for which such a composite model can be constructed, satisfying the condition $M(r) = 0$ for $r = 0$? In fact, from our investigations it will appear that for $X$ situated between 0.31 and 0.32, there does exist such a configuration, built up by an envelope with zones in radiative and convective equilibrium surrounding an isothermal core.

The differential equations which form the starting point are the classical ones given by EDDINGTON (cf. [5])

$$\left.\begin{aligned} \frac{dP}{dr} &= -\frac{GM(r)}{r^2}\varrho \\ P &= p_g + p_r = \frac{k}{\mu H}\varrho T + \frac{1}{3}aT^4 \\ \frac{dp_r}{dr} &= -\frac{\varkappa L(r)}{4\pi c r^2}\varrho \\ \frac{dM(r)}{dr} &= 4\pi r^2 \varrho \\ \frac{dL(r)}{dr} &= 4\pi r^2 \varrho \varepsilon(r), \end{aligned}\right\} \quad (1)$$

where the symbols used mean: $r$ distance from the centre, $P$ total pressure, composed by $p_g$, gas-pressure, and $p_r$, radiation-pressure, $T$ temperature, $\varrho$ density, $M(r)$ mass inside sphere of radius $r$, $L(r)$ net-flux through sphere of radius $r$, $\varkappa$ coefficient of opacity, $\mu$ molecular weight, $G$ constant of gravitation, $a$ STEFAN's constant, $k$ BOLTZMANN's constant, $H$ mass of the proton, $c$ velocity of light.

Starting from the boundary values of the variables $T = T_0$, $\varrho \simeq 0$, $M(R) = M$, $L(R) = L$, the differential equations can be integrated inwards, thus giving temperature, density, and mass at any point in the configuration. As to the function $\varepsilon(r)$ expressing the energy generation per unit mass at the distance $r$ from the centre we can make use of the formula derived by BETHE [3]; in the following calculations, however, as has already been mentioned, we make use of the approximation $L(r) = constant = L$ in the envelope. In order to simplify the integrations the following variables are conveniently introduced [4]

$$\left.\begin{aligned} x = 1/r, \quad x_0 = 1/R, \quad t = 12 + \log(x - x_0) \\ y = \log T, \quad z = \log \varrho, \quad u = M(r)/M. \end{aligned}\right\} \quad (2)$$

Substituting at the same time the expression, cf. [5],

$$\varkappa = \varkappa_0 \frac{1}{\tau} \frac{\varrho}{T^{3.5}},$$

where $$\varkappa_0 = 3.89 \cdot 10^{25} (1 - X^2).$$

$X$ being the hydrogen abundance, the helium content being zero, the equations (1) are transformed as follows [4]:

$$\left.\begin{aligned} \frac{dy}{dt} &= \gamma \frac{x - x_0}{T} \frac{\varrho^2}{T^{6.5}} \frac{1}{\tau} \\ \frac{dz}{dt} &= \alpha \frac{x - x_0}{T} u - \frac{dy}{dt} \left\{ 1 + \delta \frac{T^3}{\varrho} \right\} \\ \frac{du}{dt} &= -\zeta \frac{x - x_0}{x^4} \varrho, \end{aligned}\right\} \quad (3)$$

which govern the variations in the variables $y$, $z$, and $u$ in the case of radiative equilibrium. The constants $\alpha$, $\delta$, $\gamma$, and $\zeta$ are to be found in [4].[1] As to the guillotine factor $\tau$ the values given by B. Strömgren [6] have been used.

3. The integrations of the equations (3) were started using an analytical development [7] from the surface to a certain point well below the surface; from this the integrations were carried out inwards by means of standard methods until a temperature of 20 million degrees was reached. The observational data for *Capella A* on which the calculations are based are due to Kuiper [8]

$$L = 120\, L_\odot, \quad M = 4.18\, M_\odot, \quad R = 15.8\, R_\odot$$

[1] In the expression for $\zeta$ given there the factor $M$ has dropped out in the denominator; further the constant here denoted by $\delta$ is there called $\beta$.

Table 1

| | $X = 0.28$ | | | $X = 0.30$ | | |
|---|---|---|---|---|---|---|
| $t$ | $y = \log T$ | $z = \log \varrho$ | $u = \frac{M(r)}{M}$ | $y = \log T$ | $z = \log \varrho$ | $u = \frac{M(r)}{M}$ |
| 0.0—1 | 5.191 | 4.150—10 | 1.000 | 5.179 | 4.107—10 | 1.000 |
| .1 | .290 | .473 | 1.000 | .278 | .430 | 1.000 |
| .2 | .390 | .798 | 1.000 | .378 | .754 | 1.000 |
| .3 | .490 | 5.124 | 1.000 | .478 | 5.080 | 1.000 |
| .4 | .591 | .451 | 0.999 | .578 | .407 | 0.999 |
| .5 | .691 | .779 | .996 | .679 | .734 | .997 |
| .6 | .792 | 6.106 | .992 | .779 | 6.061 | .993 |
| .7 | .892 | .431 | .984 | .880 | .387 | .985 |
| .8 | 5.990 | .755 | .967 | 5.978 | .711 | .970 |
| .9—1 | 6.087 | 7.076 | .937 | 6.074 | 7.032 | .943 |
| 0.0 | .182 | .384 | .886 | .171 | .343 | .897 |
| .1 | .274 | .673 | .808 | .263 | .637 | .826 |
| .2 | .362 | .932 | .701 | .352 | .902 | .726 |
| .3 | .443 | 8.153 | .570 | .435 | 8.132 | .601 |
| .4 | .514 | .329 | .430 | .509 | .320 | .467 |
| .5 | .575 | .461 | .303 | .572 | .466 | .340 |
| .6 | .625 | .555 | .201 | .625 | .575 | .236 |
| .7 | .666 | .618 | .128 | .670 | .653 | .158 |
| .8 | .700 | .657 | .081 | .706 | .707 | .106 |
| .9 | .728 | .682 | .053 | .738 | .747 | .074 |
| 1.0 | .752 | .697 | .036 | .764 | .779 | .053 |
| .1 | .772 | .709 | .027 | .787 | .807 | .042 |
| .2 | .790 | .720 | .022 | .808 | .834 | .036 |
| .3 | .808 | .732 | .019 | .828 | .866 | .032 |
| .4 | .824 | .749 | .017 | .848 | .904 | .030 |
| .5 | .840 | .771 | .016 | .868 | .951 | .028 |
| .6 | .856 | .800 | .016 | .890 | 9.009 | .028 |
| .7 | .873 | .838 | .016 | .912 | .079 | .027 |
| .8 | .892 | .888 | .016 | .938 | .164 | .027 |
| .9 | .912 | .950 | .016 | 6.967 | .266 | .027 |
| 2.0 | .935 | 9.026 | .016 | 7.001 | .387 | .027 |
| .1 | .961 | .118 | .015 | .040 | .526 | . |
| .2 | 6.992 | .228 | . | .085 | .685 | . |
| .3 | 7.028 | .357 | . | .137 | 9.863—10 | . |
| .4 | .070 | .507 | . | .195 | 0.061 | . |
| .5 | .118 | .676 | . | .259 | .276 | . |
| .6 | .173 | 9.866—10 | . | .329 | .510 | . |
| .7 | .235 | 0.074 | . | .401 | .765 | . |
| .8 | .303 | .300 | .015 | .475 | 1.044 | .027 |

(to be continued)

Table 1 (continued).

| | $X = 0.32$ | | | $X = 0.34$ | | |
|---|---|---|---|---|---|---|
| $t$ | $y = \log T$ | $z = \log \varrho$ | $u = \frac{M(r)}{M}$ | $y = \log T$ | $z = \log \varrho$ | $u = \frac{M(r)}{M}$ |
| 0.0—1 | 5.168 | 4.065—10 | 1.000 | | | |
| .1 | .266 | .388 | 1.000 | 5.255 | 4.348—10 | 1.000 |
| .2 | .366 | .712 | 1.000 | .354 | .672 | 1.000 |
| .3 | .466 | 5.038 | 1.000 | .454 | .998 | 1.000 |
| .4 | .566 | .364 | 0.999 | .555 | 5.324 | 0.999 |
| .5 | .667 | .691 | .997 | .655 | .651 | .997 |
| .6 | .767 | 6.018 | .994 | .756 | .977 | .994 |
| .7 | .868 | .344 | .986 | .856 | 6.303 | .988 |
| .8 | 5.966 | .668 | .973 | 5.955 | .628 | .975 |
| .9—1 | 6.063 | .992 | .948 | 6.052 | .950 | .952 |
| 0.0 | .159 | 7.305 | .906 | .148 | 7.265 | .914 |
| .1 | .253 | .601 | .840 | .242 | .565 | .854 |
| .2 | .343 | .873 | .748 | .333 | .843 | .768 |
| .3 | .427 | 8.111 | .630 | .418 | 8.089 | .658 |
| .4 | .503 | .309 | .501 | .497 | .297 | .533 |
| .5 | .569 | .468 | .376 | .565 | .467 | .410 |
| .6 | .625 | .590 | .269 | .624 | .602 | .302 |
| .7 | .672 | .682 | .188 | .674 | .706 | .217 |
| .8 | .712 | .750 | .131 | .717 | .788 | .156 |
| .9 | .746 | .804 | .095 | .753 | .854 | .116 |
| 1.0 | .775 | .850 | .072 | .784 | .913 | .090 |
| .1 | .801 | .893 | .058 | .812 | .970 | .074 |
| .2 | .825 | .938 | .050 | .839 | 9.028 | .065 |
| .3 | .848 | .987 | .046 | .865 | .093 | .059 |
| .4 | .871 | 9.045 | .043 | .891 | .166 | .056 |
| .5 | .895 | .114 | .041 | .919 | .252 | .053 |
| .6 | .921 | .195 | .040 | .949 | .351 | .052 |
| .7 | .950 | .292 | .040 | 6.983 | .466 | .051 |
| .8 | 6.982 | .405 | .039 | 7.021 | .598 | .050 |
| .9 | 7.020 | .537 | .039 | .064 | .748 | .050 |
| 2.0 | .063 | .687 | .039 | .113 | 9.918—1 | .050 |
| .1 | .112 | 9.858—10 | .039 | .169 | 0.106 | .049 |
| .2 | .167 | 0.047 | .039 | .230 | .312 | . |
| .3 | .229 | .254 | .038 | .298 | .534 | . |
| .4 | .296 | .478 | . | .368 | .776 | . |
| .5 | .368 | .722 | . | .440 | 1.039 | .049 |
| .6 | .440 | .987 | .038 | | | |

(to be continued)

Table 1 (continued).

| | $X = 0.38$ | | | $X = 0.40$ | | |
|---|---|---|---|---|---|---|
| $t$ | $y = \log T$ | $z = \log \varrho$ | $u = \frac{M(r)}{M}$ | $y = \log T$ | $z = \log \varrho$ | $u = \frac{M(r)}{M}$ |
| 0.1—1 | 5.233 | 4.272—10 | 1.000 | 5.223 | 4.237—10 | 1.000 |
| .2 | .332 | .596 | 1.000 | .322 | .560 | 1.000 |
| .3 | .432 | .921 | 1.000 | .422 | .885 | 1.000 |
| .4 | .532 | 5.246 | 0.999 | .522 | 5.210 | 0.999 |
| .5 | .632 | .573 | .998 | .622 | .537 | .998 |
| .6 | .733 | .900 | .995 | .722 | .864 | .996 |
| .7 | .833 | 6.225 | .990 | .822 | 6.190 | .991 |
| .8 | 5.932 | .550 | .979 | 5.922 | .514 | .981 |
| .9—1 | 6.030 | .874 | .960 | 6.020 | .839 | .965 |
| 0.0 | .126 | 7.192 | .928 | .116 | 7.158 | .935 |
| .1 | .221 | .497 | .877 | .212 | .466 | .888 |
| .2 | .314 | .784 | .803 | .304 | .756 | .818 |
| .3 | .402 | 8.043 | .706 | .394 | 8.020 | .726 |
| .4 | .484 | .268 | .591 | .477 | .253 | .616 |
| .5 | .557 | .459 | .473 | .552 | .452 | .502 |
| .6 | .621 | .616 | .364 | .618 | .619 | .394 |
| .7 | .676 | .744 | .275 | .676 | .757 | .302 |
| .8 | .724 | .849 | .207 | .726 | .873 | .231 |
| .9 | .765 | .940 | .159 | .769 | .974 | .180 |
| 1.0 | .801 | 9.022 | .127 | .807 | 9.067 | .145 |
| .1 | .833 | .102 | .106 | .842 | .156 | .121 |
| .2 | .864 | .184 | .093 | .874 | .247 | .106 |
| .3 | .895 | .271 | .084 | .907 | .342 | .096 |
| .4 | .926 | .367 | .079 | .941 | .447 | .089 |
| .5 | .960 | .475 | .075 | 6.977 | .562 | .085 |
| .6 | 6.997 | .596 | .073 | 7.016 | .691 | .082 |
| .7 | 7.037 | .733 | .071 | .059 | .834 | .079 |
| .8 | .083 | 9.887—10 | .070 | .107 | 9.994—10 | .078 |
| .9 | .133 | 0.058 | .069 | .159 | 0.170 | .077 |
| 2.0 | .190 | .246 | .068 | .218 | .363 | .076 |
| .1 | .252 | .451 | .068 | .281 | .571 | .075 |
| .2 | .319 | .672 | .067 | .348 | .796 | .075 |
| .3 | .388 | .912 | .067 | | | |
| .4 | .458 | 1.174 | .066 | | | |

(to be continued)

Table 1 (continued).

| | $X = 0.42$ | | | $X = 0.50$ | | |
|---|---|---|---|---|---|---|
| $t$ | $y = \log T$ | $z = \log \varrho$ | $u = \frac{M(r)}{M}$ | $y = \log T$ | $z = \log \varrho$ | $u = \frac{M(r)}{M}$ |
| 0.1—1 | 5.213 | 4.203—10 | 1.000 | 5.175 | 4.080—10 | 1.000 |
| .2 | .312 | .526 | 1.000 | .273 | .402 | 1.000 |
| .3 | .411 | .850 | 1.000 | .372 | .725 | 1.000 |
| .4 | .511 | 5.176 | 0.999 | .472 | 5.049 | 0.999 |
| .5 | .611 | .502 | .998 | .572 | .374 | .998 |
| .6 | .712 | .828 | .996 | .672 | .700 | .997 |
| .7 | .812 | 6.154 | .991 | .772 | 6.026 | .994 |
| .8 | 5.911 | .479 | .982 | .872 | .350 | .987 |
| .9—1 | 6.009 | .802 | .966 | 5.971 | .674 | .975 |
| 0.0 | .106 | 7.122 | .939 | 6.067 | .997 | .955 |
| .1 | .201 | .433 | .895 | .164 | 7.312 | .922 |
| .2 | .295 | .726 | .831 | .258 | .617 | .872 |
| .3 | .385 | .996 | .744 | .352 | .902 | .804 |
| .4 | .469 | 8.235 | .640 | .440 | 8.163 | .718 |
| .5 | .546 | .442 | .529 | .524 | .396 | .620 |
| .6 | .615 | .618 | .422 | .599 | .603 | .521 |
| .7 | .675 | .767 | .330 | .667 | .784 | .429 |
| .8 | .727 | .894 | .257 | .727 | .945 | .350 |
| .9 | .773 | 9.005 | .203 | .780 | 9.091 | .286 |
| 1.0 | .812 | .109 | .164 | .827 | .230 | .238 |
| .1 | .849 | .209 | .138 | .871 | .362 | .202 |
| .2 | .884 | .309 | .121 | .913 | .493 | .176 |
| .3 | .919 | .414 | .109 | .955 | .626 | .158 |
| .4 | .956 | .528 | .101 | 6.999 | .763 | .144 |
| .5 | 6.994 | .651 | .096 | 7.044 | 9.908—10 | .135 |
| .6 | 7.036 | .788 | .092 | .092 | 0.063 | .128 |
| .7 | .081 | 9.939—10 | .090 | .143 | .229 | .123 |
| .8 | .132 | 0.106 | .088 | .200 | .407 | .119 |
| .9 | .187 | .288 | .086 | .259 | .599 | .116 |
| 2.0 | .247 | .486 | .085 | .323 | .803 | .114 |
| .1 | .312 | .700 | .084 | | | |

(to be continued)

Table 1 (continued).

| | $X = 0.70$ | | | $X = 0.80$ | | |
|---|---|---|---|---|---|---|
| $t$ | $y = \log T$ | $z = \log \varrho$ | $u = \frac{M(r)}{M}$ | $y = \log T$ | $z = \log \varrho$ | $u = \frac{M(r)}{M}$ |
| 0.2—1 | 5.193 | 4.183—10 | 1.000 | 5.160 | 4.134—10 | 1.000 |
| .3 | .292 | .503 | 1.000 | .258 | .452 | 1.000 |
| .4 | .390 | .825 | 1.000 | .357 | .773 | 1.000 |
| .5 | .490 | 5.148 | 0.999 | .456 | 5.095 | 0.999 |
| .6 | .589 | .472 | .998 | .555 | .418 | .998 |
| .7 | .689 | .796 | .996 | .654 | .741 | .997 |
| .8 | .789 | 6.121 | .992 | .754 | 6.065 | .993 |
| .9—1 | .888 | .444 | .985 | .853 | .387 | .987 |
| 0.0 | 5.986 | .767 | .973 | 5.952 | .709 | .977 |
| .1 | 6.083 | 7.088 | .954 | 6.048 | 7.031 | .960 |
| .2 | .179 | .404 | .924 | .144 | .347 | .933 |
| .3 | .274 | .709 | .881 | .239 | .655 | .895 |
| .4 | .367 | .998 | .824 | .333 | .950 | .845 |
| .5 | .458 | 8.268 | .755 | .425 | 8.227 | .782 |
| .6 | .544 | .517 | .677 | .513 | .485 | .711 |
| .7 | .624 | .747 | .597 | .596 | .725 | .636 |
| .8 | .698 | .959 | .520 | .673 | .948 | .562 |
| .9 | .764 | 9.159 | .451 | .743 | 9.158 | .494 |
| 1.0 | .824 | .351 | .391 | .806 | .361 | .433 |
| .1 | .879 | .537 | .340 | .865 | .559 | .381 |
| .2 | .933 | .718 | .299 | .921 | .752 | .337 |
| .3 | 6.987 | 9.894—10 | .267 | 6.978 | 9.941—10 | .301 |
| .4 | 7.042 | 0.070 | .241 | 7.035 | 0.128 | .272 |
| .5 | .098 | .246 | .221 | .094 | .314 | .250 |
| .6 | .156 | .426 | .206 | .154 | .502 | .232 |
| .7 | .215 | .611 | .194 | .215 | .694 | .217 |
| .8 | .277 | .803 | .184 | .279 | .890 | .206 |
| .9 | .340 | 1.004 | .177 | .343 | 1.095 | .196 |

The hydrogen abundance has been varied as follows: $X = 0.25$, 0.28, 0.30, 0.32, 0.34, 0.38, 0.40, 0.42, 0.50, 0.70, 0.80. The solutions are listed in Table 1 except for $X = 0.25$, in which case the mass was used up before arriving at the above temperature. The computations were carried out with one more figure than given in the tables. The quantities listed are supposed to be correct to the last figure.

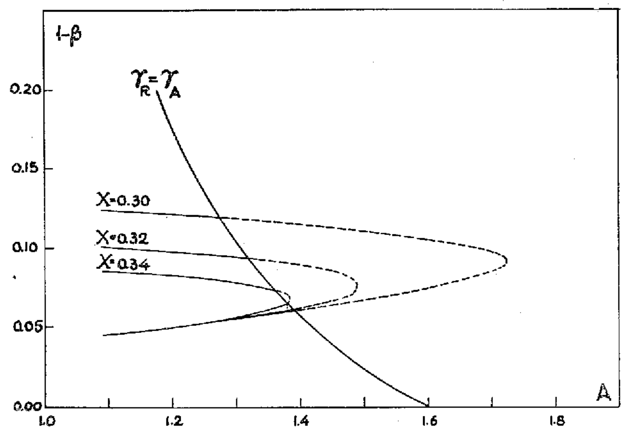


Fig. 1

4. The above-mentioned integrations have been performed on the tacit assumption that radiative equilibrium is stable. This is the case if the radiative gradient $\gamma_R$ is less than the corresponding adiabatic gradient $\gamma_A$. The gradients can be written as follows [9]

$$\gamma_R = \frac{\beta}{1-\left(1-\frac{3}{4}\beta\right)A} \tag{4}$$

$$\gamma_A = \beta + \frac{2}{3}\frac{(4-3\beta)^2}{(8-7\beta)}, \tag{5}$$

where

$$A = \frac{\varkappa \frac{L(r)}{M(r)}}{4\pi c G}\cdot\frac{1}{(1-\beta)} \tag{6}$$

and

$$p_g = \beta P. \tag{7}$$

An instructive picture of the conditions is obtained by plotting the quantity $A$ against the corresponding value $(1 - \beta)$ [10]. In the same diagram has been drawn the curve for which $\gamma_A = \gamma_R$, that is the limiting curve of the domain of convection. It appears that in the actual case convection is possible only for $X < 0.34$, and that the convection zones are increasing through the star for decreasing hydrogen content. It is thus concluded that it is necessary to revise the calculations, taking into account the modifications due to the convective equilibrium. In the convection zone the following differential equations are valid [9]:

$$\left.\begin{aligned} \frac{1}{P}\frac{dP}{dr} &= \gamma_A \frac{1}{\varrho}\frac{d\varrho}{dr} \\ \frac{dP}{dr} &= -\frac{GM(r)}{r^2}\varrho \\ \frac{dM(r)}{dr} &= 4\pi r^2 \varrho. \end{aligned}\right\} \quad (8)$$

Transforming these by means of the variables (2) we are left with

$$\left.\begin{aligned} \frac{dy}{dt} &= \frac{2(4-3\beta)}{3(8-7\beta)}\frac{dz}{dt} \\ \frac{dz}{dt} &= \frac{GMH}{k}\mu\frac{(x-x_0)}{T}\frac{u\beta}{\gamma_A}. \end{aligned}\right\} \quad (9)$$

The third equation remains unchanged. The integrations are thus performed by means of these equations as long as the condition $\gamma_R > \gamma_A$ is fulfilled. From the point where the convection zone is changed back into the radiative zone the corresponding differential equations are to be used until a temperature of 20 million degrees has been reached.

5. At this interface has been fitted an isothermal core devoid of hydrogen. The fitting conditions imply continuity in pressure, temperature, and mass; the density, however, has a discontinuity at the interface. The ratio of the densities on both sides of the interface is equal to the ratio of the respective molecular weights. The differential equations of the isothermal core are immediately

derived from (3) by putting $\frac{dy}{dt}$ equal to zero and changing the molecular weight $\mu_1$ entering in the constant $\alpha$ to $\mu_2$.

The integrations are now continued until the stage of non-relativistic degeneracy defined through the equality in pressure at the interface

$$\frac{k}{\mu_2 H}\varrho T = \frac{K_1}{\mu_3^{5/3}}\varrho^{5/3} \tag{10}$$

or

$$\varrho = \left(\frac{kT}{HK_1}\frac{\mu_3^{5/3}}{\mu_2}\right)^{3/2} \tag{11}$$

is reached [9, ciph. 56c]. $\mu_3$ denotes the mean molecular weight of the degenerate core, the constant $K_1$ is equal to $9.91 \cdot 10^{12}$. The mean molecular weights $\mu_1$, $\mu_2$, and $\mu_3$ are derived from the formulae

$$\left.\begin{aligned} \mu_1^{-1} &= 2X + n_R(1-X) \\ \mu_2^{-1} &= \frac{3}{4}X + n_R(1-X) \\ \mu_3^{-1} &= \frac{1}{2}X + n_R(1-X), \end{aligned}\right\} \tag{12}$$

where the quantity $n_R$ has been put equal to 0.54, or the value given for a completely ionized Russell-mixture [11].

In the case of non-relativistic degeneracy the equation of state is given by the right member of (10), which means that the configuration is a Lane-Emden polytrope of index $n = 3/2$. It appears from the integrations that the density in the relevant parts of our models stays below the limit where relativistic degeneracy sets in. Before performing the actual integrations it is advisable to investigate if a fitting of a polytrope of that type is really possible. The fitting problem can be solved in a convenient way by a method due to Russell [12], cf. also [5]. Defining the variables[1]

$$A = \frac{4\pi\varrho r^3}{3M(r)}, \quad B = 4\pi G\frac{\varrho^2 r^2}{P}, \tag{13}$$

[1] This variable $A$ must not, of course, be confused with the variable $A$ defined by means of (6).

RUSSELL in a diagram with the ordinate $A$ and the abscissa $B$ plots the function $A(B)$ corresponding to the $E$-solutions of different polytropes. In order to determine the type of solution in a certain point of a model we have only to calculate the polytropic index $n$ and the quantities $A$ and $B$. The solution will be an $E$-solution, an $M$-solution, or an $F$-solution according as the point $(A, B)$ lies upon, inside, or outside the $E$-curve for the given polytropic index. The quantities $A$, $B$ for $n = 3/2$ calculated for the hydrogen contents 0.30, 0.31, 0.32 are listed in Table 2.

Table 2.

| $X$ | $A$ | $B$ |
|---|---|---|
| 0.30 | 0.13 | 4.72 |
| 0.31 | 0.23 | 7.79 |
| 0.32 | 0.34 | 11.33 |

It is remarked in passing that the integrations for $X = 0.31$ are based upon interpolated $y$, $z$, and $u$ values; by means of these the numerical integrations are started at the point where the convection zone begins. The points $(A, B)$ are plotted in the diagram (Figure 2), from which is clearly demonstrated that an $E$-solution does exist for an $X$-value between 0.31 and 0.32.

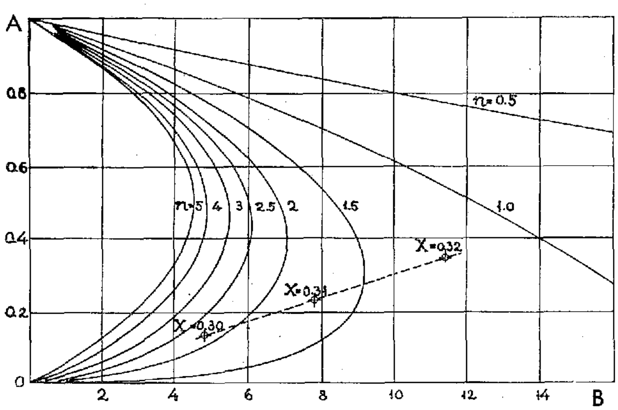


Fig. 2.

The differential equations governing the variation in $z$ in the degenerate zone can be written as follows

Table 3.

## $X = 0.30$

Zone of convective equilibrium.

| $t$ | $\log T$ | $\log \varrho$ | $u = \frac{M(r)}{M}$ |
|---|---|---|---|
| 0.70 | 6.668 | 8.654—10 | 0.1586 |
| .80 | .702 | .715 | .1063 |
| .90 | .728 | .763 | .0726 |
| 1.00 | .749 | .803 | .0522 |
| .10 | .768 | .839 | .0400 |
| .20 | .786 | .873 | .0331 |
| .30 | .804 | .908 | .0292 |
| .40 | .824 | .946 | .0271 |
| .50 | .846 | 8.990 | .0258 |
| .60 | .871 | 9.039 | .0250 |
| .70 | .899 | .096 | .0246 |

Second zone of radiative equilibrium.

| $t$ | $\log T$ | $\log \varrho$ | $u$ |
|---|---|---|---|
| 1.80 | 6.931 | 9.162—10 | 0.0243 |
| .90 | .962 | .248 | .0242 |
| 2.00 | 6.995 | .352 | .0241 |
| .10 | 7.032 | .475 | .0240 |
| .20 | .073 | .619 | .0240 |
| .30 | .120 | .782 | .0239 |
| .40 | .174 | 9.966—10 | .0239 |
| .50 | .234 | 0.168 | .0239 |
| .60 | .301 | .388 | .0239 |

Isothermal zone.

| $t$ | $\log \varrho$ | $u = \frac{M(r)}{M}$ |
|---|---|---|
| 2.60 | 0.598 | 0.0239 |
| .64 | .820 | .0239 |

## $X = 0.31$

Zone of convective equilibrium.

| $t$ | $\log T$ | $\log \varrho$ | $u = \frac{M(r)}{M}$ |
|---|---|---|---|
| 0.70 | 6.671 | 8.668—10 | 0.1731 |
| .80 | .707 | .734 | .1190 |
| .90 | .735 | .787 | .0838 |
| 1.00 | .759 | .832 | .0620 |
| .10 | .781 | .873 | .0490 |
| .20 | .802 | .914 | .0414 |
| .30 | .824 | 8.956 | .0370 |
| .40 | .847 | 9.001 | .0345 |
| .50 | .874 | .052 | .0330 |

Second zone of radiative equilibrium.

| $t$ | $\log T$ | $\log \varrho$ | $u$ |
|---|---|---|---|
| 1.60 | 6.901 | 9.112—10 | 0.0321 |
| .70 | .929 | .188 | .0316 |
| .80 | .959 | .280 | .0313 |
| .90 | 6.992 | .390 | .0311 |
| 2.00 | 7.029 | .518 | .0310 |
| .10 | .071 | 9.667 | .0309 |
| .20 | .120 | 9.834—10 | .0308 |
| .30 | .174 | 0.022 | .0308 |
| .40 | .236 | .227 | .0307 |
| .50 | .303 | .450 | .0307 |

Isothermal zone.

| $t$ | $\log \varrho$ | $u = \frac{M(r)}{M}$ |
|---|---|---|
| 2.50 | 0.674 | 0.0307 |
| .54 | 0.899 | .0307 |

## $X = 0.32$

Zone of convective equilibrium.

| $t$ | $\log T$ | $\log \varrho$ | $u = \frac{M(r)}{M}$ |
|---|---|---|---|
| 0.70 | 6.672 | 8.682—10 | 0.1879 |
| .80 | .711 | .752 | .1313 |
| .90 | .742 | .810 | .0946 |
| 1.00 | .769 | .860 | .0716 |
| .10 | .793 | .906 | .0577 |
| .20 | .817 | .951 | .0494 |
| .30 | .842 | 8.999 | .0445 |

Second zone of radiative equilibrium.

| $t$ | $\log T$ | $\log \varrho$ | $u$ |
|---|---|---|---|
| 1.40 | 6.868 | 9.052—10 | 0.0417 |
| .50 | .893 | .116 | .0401 |
| .60 | .920 | .193 | .0390 |
| .70 | .949 | .286 | .0384 |
| .80 | 6.981 | .395 | .0380 |
| .90 | 7.018 | .522 | .0378 |
| 2.00 | .059 | .668 | .0376 |
| .10 | .107 | 9.834—10 | .0374 |
| .20 | .161 | 0.020 | .0374 |
| .30 | .222 | .223 | .0373 |
| .40 | .288 | .444 | .0372 |
| .50 | .359 | .683 | .0372 |

Isothermal zone.

| $t$ | $\log \varrho$ | $u = \frac{M(r)}{M}$ |
|---|---|---|
| 2.42 | 0.715 | 0.0372 |
| .46 | 0.941 | .0372 |

| | | |
|---|---|---|
| .76 | 1.621 | .0238 |
| .80 | 1.940 | .0238 |
| .84 | 2.289 | .0237 |
| .88 | 2.671 | .0236 |
| .92 | 3.087 | .0234 |
| .96 | 3.537 | .0229 |
| 3.00 | 4.016 | .0220 |

Degeneracy at $t = 2.974$.

| | | |
|---|---|---|
| 3.00 | 3.866 | 0.0221 |
| .04 | 4.064 | .0210 |
| .08 | .221 | .0197 |
| .12 | .350 | .0183 |
| .16 | .459 | .0170 |
| .20 | .554 | .0157 |
| .24 | .638 | .0145 |
| .28 | .713 | .0134 |
| .32 | .782 | .0124 |
| .36 | .844 | .0115 |
| .40 | .903 | .0108 |
| .44 | 4.958 | .0101 |
| .48 | 5.010 | .0096 |
| .52 | .061 | .0091 |
| .56 | .110 | .0087 |
| 3.60 | .157 | .0083 |
| 3.70 | 5.273 | .0077 |
| .80 | .387 | .0073 |
| .90 | .502 | .0070 |
| 4.00 | .619 | .0068 |
| .10 | .740 | .0067 |
| .20 | .864 | .0066 |
| .30 | 5.991 | .0066 |
| .40 | 6.122 | .0065 |

| | | |
|---|---|---|
| .66 | 1.713 | .0305 |
| .70 | 2.037 | .0304 |
| .74 | 2.390 | .0303 |
| .78 | 2.775 | .0300 |
| .82 | 3.192 | .0294 |
| .86 | 3.637 | .0283 |
| .90 | 4.096 | .0259 |

Degeneracy at $t = 2.865$.

| | | |
|---|---|---|
| 2.88 | 3.793 | 0.0274 |
| .92 | 3.998 | .0252 |
| .96 | 4.154 | .0227 |
| 3.00 | .277 | .0201 |
| .04 | .376 | .0175 |
| .08 | .459 | .0151 |
| .12 | .528 | .0129 |
| .16 | .586 | .0110 |
| .20 | .637 | .0094 |
| .24 | .680 | .0080 |
| .28 | .719 | .0068 |
| .32 | .753 | .0058 |
| .36 | .784 | .0051 |
| .40 | .812 | .0044 |
| .44 | .838 | .0039 |
| .48 | .862 | .0035 |
| .52 | .885 | .0032 |
| .56 | .908 | .0029 |
| .60 | .930 | .0027 |
| 3.70 | 4.984 | 0.0024 |
| .80 | 5.040 | .0022 |
| .90 | .100 | .0020 |
| 4.00 | .166 | .0020 |
| .10 | .238 | .0019 |
| .20 | .318 | .0019 |

| | | |
|---|---|---|
| .58 | 1.755 | .0369 |
| .62 | 2.078 | .0367 |
| .66 | 2.431 | .0364 |
| .70 | 2.813 | .0359 |
| .74 | 3.223 | .0349 |
| .78 | 3.654 | .0328 |
| .82 | 4.085 | .0286 |

Degeneracy at $t = 2.784$.

| | | |
|---|---|---|
| 2.80 | 3.798 | 0.0312 |
| .88 | 4.129 | .0232 |
| 2.96 | .316 | .0150 |
| 3.04 | .424 | .0084 |
| .12 | .483 | .0037 |
| .20 | .507 | .0008 |

$$\frac{dz}{dt} = \frac{3}{5}\frac{GM}{K_1}\mu_3^{5/3}\frac{x - x_0}{\varrho^{2/3}}u, \tag{14}$$

the second equation being identical with the last one in (3). They are integrated numerically in the above three cases and the corresponding $y$, $z$, and $u$ values are collected in Table 3. The existence of an $E$-solution for $X$ between 0.31 and 0.32 is clearly exhibited. Finally it should be noted that the higher the $X$-value the smaller the remaining relative mass, contrary to the case of the point-source models without an isothermal core (cf. Table 1).

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