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INFINITE SYSTEMS OF LINEAR CONGRUENCES WITH INFINITELY MANY VARIABLES

BY

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§ 1. Introduction.

In the present paper we shall investigate a general problem concerning an arbitrary enumerable system of linear congruences with an enumerable number of variables

$$(1)\quad \begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n_1}x_{n_1} \equiv \theta_1 \pmod 1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n_2}x_{n_2} \equiv \theta_2 \pmod 1 \\ \cdots\cdots\cdots\cdots\cdots\cdots \end{array}$$

where every congruence only contains a finite number of variables and the a's and the θ's are arbitrary (real) numbers.

By the consideration of certain classifications of the almost periodic functions one of the authors[1)] met with a problem concerning a system of congruences of the above form but in the special case where all the a's were *rational* numbers. The problem was to give a convenient necessary and sufficient condition on the system of linear forms

$$(2)\quad \begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n_1}x_{n_1} \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n_2}x_{n_2} \\ \cdots\cdots\cdots\cdots\cdots\cdots \end{array}$$

in order that it possesses the following property: For every choice of the numbers $\theta_1, \theta_2, \cdots$ for which any finite subsystem of the system of congruences (1) has a solution[2)]—or, what amounts to the same, for which for any N the system of the N first of

1) H. Bohr: Unendlich viele lineare Kongruenzen mit unendlich vielen Unbekannten. Kgl. Danske Videnskabernes Selskab. Math.-fys. Meddelelser, Bind VII, 1925. In the following this paper is cited by (I). We do not, however, assume the reader to be acquainted with (I).

2) It will be convenient to interpret, not only a solution of the whole system (1), but also a solution of a finite subsystem of (1) as a point $(x_1, x_2, \cdots)$ in the infinite-dimensional space, although for a subsystem only a finite number of the variables really enters in the congruences in question (and the rest of the variables therefore can be chosen quite arbitrarily).

the congruences (1) has a solution—there shall exist a solution of the whole system (1).

If instead of the congruences (1) we consider the corresponding system of equations (now without limitation to rational coefficients) there exists no analogous problem. In fact, it follows from a general investigation of Toeplitz on such systems of equations that for an arbitrary given system the existence of a solution of any finite subsystem always will involve the existence of a solution of the whole system of equations. A direct proof of this special theorem can be found in the paper (I).

That the analogous theorem really is not true for congruences (not even if we restrict ourselves to rational coefficients) can be seen from the following simple example where, moreover, only a single variable x_1 explicitly enters (all the other variables $x_2, x_3, \cdots$ having the coefficients 0).

Example 1. We consider the system of congruences

$$\frac{1}{3}x_1 \equiv \theta_1 \pmod{1}$$
$$\frac{1}{9}x_1 \equiv \theta_2 \pmod{1}$$
$$\cdots\cdots\cdots\cdots$$
$$\frac{1}{3^n}x_1 \equiv \theta_n \pmod{1}$$
$$\cdots\cdots\cdots\cdots$$

for $\theta_1 = \theta_2 = \cdots = \frac{1}{2}$. The solutions of the n^{th} congruence are all points $(x_1, x_2, \cdots)$ where $x_2, x_3, \cdots$ are arbitrary numbers and x_1 is a number from the "lattice" $x_1 \equiv \frac{3^n}{2} \pmod{3^n}$. These solutions are also solutions of the $(n-1)^{\text{th}}$ congruence, for if $x_1 \equiv \frac{3^n}{2} \pmod{3^n}$ then also $x_1 \equiv \frac{3^n}{2} \pmod{3^{n-1}}$, i. e. $x_1 \equiv \frac{3^{n-1}}{2} \pmod{3^{n-1}}$, since $\frac{3^n}{2} = \frac{3^{n-1}}{2} + 3^{n-1}$. Hence for every N the N first congruences have solutions, viz. all the solutions $x_1 \equiv \frac{3^N}{2} \pmod{3^N}$ of the N^{th} congruence. But nevertheless there is no solution of the whole system of congruences, for if $(x_1, x_2, \cdots)$ is a solution of the N^{th} congruence then $|x_1| \geqq \frac{3^N}{2}$ which $\to \infty$ for $N \to \infty$.

For a given system of linear forms (2) we shall denote by π_1 the set of points $(\theta_1, \theta_2, \cdots)$ for which the corresponding infinite

system (1) has a solution, and by π_2 the set of points $(\theta_1, \theta_2, \cdots)$ for which any finite subsystem of (1) has a solution. It is plain that $\pi_1 \subseteq \pi_2$ and that both sets contain the point $(0, 0, \cdots)$.

The previous, in (I) treated, problem was to indicate a necessary and sufficient condition that a given system of linear forms (2) with rational coefficients have $\pi_1 = \pi_2$. Before stating the result we shall have to mention the notion of a *substitution* in an enumerable number of variables. A substitution is a linear transformation of the form

$$\begin{aligned} y_1 &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1p_1}x_{p_1} \\ y_2 &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2p_2}x_{p_2} \\ &\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned} \qquad (3)$$

which establishes a one-to-one mapping of the whole infinite-dimensional space on the *whole* infinite-dimensional space. As shown in (I) (cp. also § 4 of the present paper) a necessary and sufficient condition that the transformation (3) be a substitution is that no linear dependance exists amongst (any finite number of) the linear forms on the right-hand side of (3) and that each of the variables x_m can be "isolated", i. e. written as a linear combination of a finite number of the linear forms. In particular, any substitution has an "inverse substitution"

$$\begin{aligned} x_1 &= \beta_{11}y_1 + \beta_{12}y_2 + \cdots + \beta_{1q_1}y_{q_1} \\ x_2 &= \beta_{21}y_1 + \beta_{22}y_2 + \cdots + \beta_{2q_2}y_{q_2} \\ &\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned}$$

If a substitution is applied to a linear form we get a new linear form. The importance of substitutions in our problem is plain because a substitution applied to a system of linear forms will not change any of the sets π_1 and π_2 simply because two linear forms which correspond by the substitution will take the same value for corresponding values of the variables.

The solution of the former problem can now be stated as follows. *A necessary and sufficient condition that a system of linear forms with rational coefficients have $\pi_1 = \pi_2$ is that the system by a substitution can be transfered into an integral system, i. e. a system with mere integral coefficients.*

We remark, for orientation, that the *sufficiency* of the condition is easy to prove. In fact, on account of the invariance of the sets π_1 and π_2 by a substitution (applied to the linear forms) we need only show that every integral system (2) has $\pi_1 = \pi_2$. Denoting by $(\theta_1, \theta_2, \cdots)$ an arbitrary point from π_2 we shall show that it also lies in π_1. Let $P_N = (\xi_1^{(N)}, \xi_2^{(N)}, \cdots)$ be a solution of the N first congruences (1), $N = 1, 2, \cdots$. Since all a's are integral we can assume all ξ's reduced modulo 1 to lie in the interval $0 \leqq \xi < 1$. Hence we can choose a subsequence P_{N_p}, $p = 1, 2, \cdots$, of the sequence P_N, such that every coordinate sequence $\xi_i^{(N_p)}$ (i fixed) converges towards a number ξ_i for $p \to \infty$. The "limit-point" $(\xi_1, \xi_2, \cdots)$ will then be a solution of all the congruences (1), for if N_0 is an arbitrary positive integral number then $(\xi_1, \xi_2, \cdots)$ from continuity reasons will satisfy the N_0^{th} congruence because this congruence only contains a finite number of variables and the point $(\xi_1^{(N_p)}, \xi_2^{(N_p)}, \cdots)$ for every $p \geqq N_0$ is a solution of the congruence.—The real problem in (I) was to show the necessity of the condition, i. e. that amongst the rational systems there are no other systems than those mentioned above which have $\pi_1 = \pi_2$.

In the present paper we shall treat the corresponding problem for congruences with *arbitrary* coefficients. Also in this general case the systems with $\pi_1 = \pi_2$ can be characterized as systems which by substitutions can be transfered into systems of a certain simple type, denoted by S, which obviously has $\pi_1 = \pi_2$ and whose algebraic structure can be accounted for.

By *a system of linear forms of the type S* we shall understand a system where certain of the variables (finite or infinite in number) have mere integral coefficients while each of the other variables (finite or infinite in number) necessarily becomes 0 if for a sufficiently large N (i. e. for $N \geqq N_0$ where N_0 depends on the variable) one solves the N first "zero-congruences" corresponding to the linear forms, i. e. the congruences (1) with $\theta_1 = \theta_2 = \cdots = 0$.

Our purpose is to prove the following

Main Theorem[1]. *A necessary and sufficient condition that a system of linear forms have $\pi_1 = \pi_2$ is that the system by a substitution can be transfered into a system of the type S.*

1) Incidentally, our proof of the main theorem in reality yields a stronger form of this theorem than the one indicated here. For the formulation of the theorem in the stronger form we refer to § 5.

Also in this case it is easy to prove that the condition is *sufficient*. We only have to show that every system of the type S has $\pi_1 = \pi_2$. Denoting by $(\theta_1, \theta_2, \cdots)$ an arbitrary point from π_2 we shall show that it also belongs to π_1. Let $P_N = (\xi_1^{(N)}, \xi_2^{(N)}, \cdots)$ be a solution of the N first congruences (1). We may assume those coordinates which in all congruences have integral coefficients reduced modulo 1 to lie in the interval $0 \leqq \xi < 1$. Everyone of the remaining coordinates $\xi_i^{(N)}$ will possess a constant value ξ_i for $N \geqq N_0$ where $N_0 = N_0(i)$ is determined such that every solution $(x_1, x_2, \cdots)$ of the N_0 first zero-congruences will have $x_i = 0$; for as the two points $(\xi_1^{(N_0)}, \xi_2^{(N_0)}, \cdots)$ and $(\xi_1^{(N)}, \xi_2^{(N)}, \cdots)$ are both solutions of the N_0 first congruences (1) their difference $(\xi_1^{(N)} - \xi_1^{(N_0)}, \xi_2^{(N)} - \xi_2^{(N_0)}, \cdots)$ will be a solution of the N_0 first zero-congruences and hence $\xi_i^{(N)} - \xi_i^{(N_0)} = 0$, i. e. $\xi_i^{(N)} = \xi_i^{(N_0)} = \xi_i$ for $N \geqq N_0$. We now extract a subsequence from our sequence of points $P_N = (\xi_1^{(N)}, \xi_2^{(N)}, \cdots)$ such that any coordinate sequence $\xi_i^{(N)}$ (i fixed) which does not end in being a constant will converge towards a number ξ_i; this can be done since they are all lying in the interval $0 \leqq \xi < 1$. The limit point $(\xi_1, \xi_2, \cdots)$ will obviously (for continuity reasons) be a solution of all the congruences (1) and hence the point $(\theta_1, \theta_2, \cdots)$ will lie in π_1.

That the main theorem above contains the main theorem in (I) can be seen in the following way. Since every integral system is also a system of the type S the "trivial" part of the main theorem in (I) (concerning the sufficiency of the condition) is contained in the trivial part of the general main theorem. To show that the non-trivial part of the general main theorem involves the non-trivial part of the main theorem in (I) requires a little consideration. We are to show that any rational system (2) with $\pi_1 = \pi_2$ can be transfered into an integral system. The general main theorem only states that it can be transfered into a system of the type S. By using, however, that the system is rational we can easily prove that the resulting system of the type S always must be integral. Otherwise, in fact, there would exist in this system a variable y_m which for N sufficiently large necessarily becomes 0 by solution of the N first zero-congruences. The solutions of the N first zero-congruences in the original system would therefore satisfy an equation $a_{m1}x_1 + \cdots + a_{mp_m}x_{p_m} = 0$ whose left-hand side is that linear form which in the substitution used is put equal to y_m. Denoting, however, by G a common denominator of all the coefficients in the N first linear forms in the original system, obviously all points $(h_1 G, h_2 G, \cdots)$ where $h_1, h_2, \cdots$ are arbitrary integers will be solutions of the corresponding zero-congruences, and these points cannot possibly all satisfy the equation $a_{m1}x_1 + \cdots + a_{mp_m}x_{p_m} = 0$ (whose coefficients are not all 0). Hence our assumption has led to a contradiction.

That the proof of the general main theorem cannot follow quite the same line as the proof in the rational case given in (I) is due to the fact that certain finite-dimensional sets which enter in the investigation (see § 2), and which in (I) without real limitation could be supposed to be lattices, in the present case are modules of a more general kind. If, however, closures are taken of the sets in question these closures will get properties analogous to the sets in (I). But in order to obtain the substitution which transfers a given system of linear forms with $\pi_1 = \pi_2$ into a system of the special type S we should still as in (I) have to consider the mentioned sets themselves and not their closures. Now, however, from the properties of the closures it would be possible to get at analogous properties for the sets themselves which would allow the seeking out of the substitution wanted. This would be a similar, though more complicated line to that followed in (I) and until recently our intension had been to use this arrangement. Then, however, B. Jessen asked us whether in the infinite-dimensional space in question a structural theorem existed for closed modules analogous to that holding for such modules in a finite-dimensional space. That this is really the case we could answer affirmatively by help of our main theorem. Later on we found a more direct proof of this structural theorem for closed modules in the infinite-dimensional space by using the dual connection between our space and another infinite-dimensional space, a connection which in case of the finite-dimensional space was introduced by M. Riesz. Now, conversely, it turned out that a more perspicious proof of the main theorem could be obtained by first establishing the structural theorem for closed modules and then applying it to our problem. In fact, by applying this structural theorem to the closed module Γ formed by the set of all solutions of the zero-congruences corresponding to the given system of linear forms we could directly obtain the desired substitution, i. e. the substitution which takes our system (1) into a system of the type S and thus avoiding all difficulties arising from the consideration of the above mentioned non-closed modules.

In the present paper we have prefered to give the proof in this latter arrangement.

§ 2. Some important sets.

Already by the definition of a system of linear forms of the type S we had to consider the corresponding zero-congruences. In our treatment of the arbitrary system of congruences (1) the corresponding system of zero-congruences

$$(4)\quad \begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n_1}x_{n_1} &\equiv 0 \pmod 1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n_2}x_{n_2} &\equiv 0 \pmod 1 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots &\cdots\cdots \end{aligned}$$

will play an important role. In connection with the zero-congruences (4) we introduce the following notations.

Γ : The set of solutions of the zero-congruences (4).
Γ_m : The projection of Γ on the $x_1 \cdots x_m$-space.
H_m : The closure of Γ_m.
$\Lambda^{(N)}$: The set of solutions of the N first zero-congruences in (4).
$\Lambda_m^{(N)}$: The projection of $\Lambda^{(N)}$ on the $x_1 \cdots x_m$-space.
$H_m^{(N)}$: The closure of $\Lambda_m^{(N)}$.

Here Γ and $\Lambda^{(N)}$ are point sets in the infinite-dimensional space while the four other sets (with lower index m) are point sets in the m-dimensional $x_1 \cdots x_m$-space. Γ_m and $\Lambda_m^{(N)}$ are obviously (vector-) modules and hence H_m and $H_m^{(N)}$ are closed modules. Further, for $m_1 < m$, the module Γ_{m_1} is the projection of Γ_m on the $x_1 \cdots x_{m_1}$-space, and similarly $\Lambda_{m_1}^{(N)}$ is the projection of $\Lambda_m^{(N)}$.

As well-known the closed modules in the $x_1 \cdots x_m$-space have an especially simple structure. Let H be an arbitrary closed module in the m-dimensional space. Then it is possible to find a system of *linearly independent* vectors $F_1, \cdots, F_p, V_1, \cdots, V_q$ $(p + q \leqq m)$ such that H consists of all vectors (points) of the form

$$P = \xi_1 F_1 + \xi_2 F_2 + \cdots + \xi_p F_p + h_1 V_1 + \cdots + h_q V_q$$

where the ξ's are arbitrary numbers and the h's are arbitrary integers. Conversely, each such point set is a closed module. We shall say that the vectors $F_1, \cdots, F_p$ and $V_1, \cdots, V_q$ (together) generate H with respectively arbitrary and integral coefficients.

If H does not contain any vector space (with exception of the space 0 consisting only of the origin) there can be no F-vectors and H is a *lattice*. The parallelotope determined by the vectors $V_1, \cdots, V_q$ is then called a *fundamental parallelotope* of the lattice.

The general closed module H can be called a *lattice cylinder* erected on the lattice generated by the vectors $V_1, \cdots, V_q$ (integral coefficients) with the space determined by the vectors $F_1, \cdots, F_p$ as space of generatrix directions. Concerning the freedom by which one can choose a generating system of linearly independent vectors for a closed module in the m-dimensional space we state the following well-known

Theorem. *If H is a closed module and T an arbitrary (vector-) space both lying in the m-dimensional space we can determine a system of linearly independent vectors which generates H (with arbitrary, respectively integral coefficients) by determining first in an arbitrary manner such a generating system of the closed submodule $H \cap T$*[1]*, and next supplementing these vectors with certain other vectors (if necessary).*

Let us consider the sets (5) for a numerically given system of zero-congruences.

Example 2. Let the system of zero-congruences be

$$\begin{aligned} x_1 - x_2 &\equiv 0 \pmod 1 \\ \sqrt{2}\, x_2 &\equiv 0 \pmod 1 \\ \frac{1}{2}(x_1 - x_2) &\equiv 0 \pmod 1 \\ \frac{1}{2}\sqrt{2}\, x_2 &\equiv 0 \pmod 1 \end{aligned}$$

1) $H \cap T$ denotes the common part of H and T.

$$\begin{aligned} \frac{1}{4}(x_1 - x_2) &\equiv 0 \pmod 1 \\ \frac{1}{4}\sqrt{2}\, x_2 &\equiv 0 \pmod 1 \\ &\cdots\cdots\cdots \end{aligned}$$

Only the two variables x_1 and x_2 occur in these congruences. Hence for $m \geqq 2$ the set $\Lambda_m^{(N)}$ consists of all points $(x_1, \cdots, x_m)$ whose projections on the x_1x_2-plane lie in $\Lambda_2^{(N)}$, just as $\Lambda^{(N)}$ consists of all points $(x_1, x_2, \cdots)$ whose projections on the x_1x_2-plane lie in $\Lambda_2^{(N)}$. The set $\Lambda_2^{(1)}$ is the closed module in the x_1x_2-plane determined by $x_1 - x_2 \equiv 0 \pmod 1$ (it may for instance be generated by $F_1 = (1, 1)$ and $V_1 = (1, 0)$). The sets $\Lambda_2^{(2)} \supset \Lambda_2^{(3)} \supset \cdots$ form a strictly decreasing sequence of lattices in the x_1x_2-plane; for instance $\Lambda_2^{(2)}$ is the lattice generated by the vectors $V_1 = (1, 0)$ and $V_2 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, and more generally $\Lambda_2^{(2n)}$ is the lattice generated by the vectors $V_1 = (2^{n-1}, 0)$ and $V_2 = \left(\frac{2^{n-1}}{\sqrt{2}}, \frac{2^{n-1}}{\sqrt{2}}\right)$. As to the projections on the x_1-axis we see that $\Lambda_1^{(1)}$ is the whole x_1-axis while $\Lambda_1^{(2)} \supset \Lambda_1^{(3)} \supset \cdots$ is a strictly decreasing sequence of non-closed modules which are all lying everywhere dense on the x_1-axis. All these modules can be generated by a finite number of vectors, though of course not by linearly independent vectors; for instance $\Lambda_1^{(2)}$ is generated by the vectors $V_1 = 1$ and $V_2 = \frac{1}{\sqrt{2}}$ and more generally $\Lambda_1^{(2n)}$ is generated by the vectors $V_1 = 2^{n-1}$ and $V_2 = \frac{2^{n-1}}{\sqrt{2}}$.[1] Since the sets $\Lambda_1^{(n)}$ are everywhere dense on the x_1-axis it follows that their closures $H_1^{(n)}$ are all equal to the whole x_1-axis. Finally we see that $\Gamma = \{(0, 0, x_3, x_4, \cdots)\}$ where $x_3, x_4, \cdots$ are arbitrary numbers so that the sets Γ_1 and Γ_2 consist only of the origin.

In the rational case the knowledge of Γ is sufficient to decide whether $\pi_1 = \pi_2$ or not. In fact, by help of the main theorem in the rational case we can easily show that a necessary and sufficient condition that $\pi_1 = \pi_2$ is that Γ by a substitution can be transfered into a set which contains the "unit lattice" in the infinite-dimensional space, i. e. the set $\{(h_1, h_2, \cdots)\}$ where the h's are arbitrary integers. This can be seen in the following way.

1) It can easily be seen that for any m and N the set $\Lambda_m^{(N)}$ also in the case of an *arbitrary* system of linear forms may be generated by a *finite* number of (generally non-independent) vectors with arbitrary, respectively integral coefficients. In fact if $M > m$ denotes a positive integer so large that no variable with larger index than M really occurs (i. e. has a coefficient different from 0) in any of the N first linear forms we see that $\Lambda_M^{(N)}$ is a closed module in the $x_1 \cdots x_M$-space and that $\Lambda_m^{(N)}$ is its projection on the $x_1 \cdots x_m$-space. The projection of a system of (linearly independent) generators of the closed module $\Lambda_M^{(N)}$ will therefore be a system of (in general linearly dependent) generators of $\Lambda_m^{(N)}$.

(i). If Γ by a substitution can be transfered into a set which contains the unit lattice, then the linear forms by the substitution must be transfered into linear forms whose corresponding zero-congruences amongst their solutions have all points $(h_1, h_2, \cdots)$. If this is used for the points $(1, 0, 0, \cdots)$, $(0, 1, 0, 0, \cdots)$, $\cdots$ it follows that the coefficients of x_1, the coefficients of x_2, $\cdots$ are all integral. Hence, on account of the main theorem, $\pi_1 = \pi_2$.

(ii). If $\pi_1 = \pi_2$, the linear forms can, on account of the main theorem, be transfered into an integral system. The corresponding system of zero-congruences of this integral system is obviously satisfied by all points from the unit lattice. Hence, by the substitution, Γ is transfered into a set which contains the unit lattice.

In the general case where the coefficients are arbitrary numbers the knowledge of Γ is not sufficient to decide whether $\pi_1 = \pi_2$. In fact we can easily indicate two systems of linear forms which have the same Γ but such that $\pi_1 \neq \pi_2$ for the one system and $\pi_1 = \pi_2$ for the other. This we do in the following example.

Example 3. We consider the two systems of linear forms

$$\begin{array}{ll} \frac{1}{3}x_1 & x_1 \\ \frac{1}{9}x_1 & \sqrt{2}\,x_1 \\ \frac{1}{27}x_1 & 0x_1 \\ \vdots & \vdots \\ \frac{1}{3^n}x_1 & 0x_1 \\ \vdots & \vdots \end{array}$$

where the first system is the same as that used in example 1, § 1. In both systems only the one variable x_1 really occurs. It is clear that the two systems have the same Γ, namely the set $\{(0, x_2, x_3, \cdots)\}$ where $x_2, x_3, \ldots$ are arbitrary numbers. The first system, however, has $\pi_1 \neq \pi_2$—in fact we proved in example 1 that the point $\left(\frac{1}{2}, \frac{1}{2}, \cdots\right)$ was lying in π_2 but not in π_1—while the second system obviously has $\pi_1 = \pi_2$ since in reality it only contains a finite number (namely 2) of linear forms.

While, thus, a consideration of Γ alone cannot decide whether $\pi_1 = \pi_2$ we shall see in the following paragraph that the knowledge of the sets $H_m^{(N)}$ is sufficient for that purpose.

§ 3. The sets $H_m^{(N)}$, H_m and the condition $\pi_1 = \pi_2$.

In this paragraph we shall indicate as a statement on the sets $H_m^{(N)}$ a necessary and sufficient condition for the validity of $\pi_1 = \pi_2$. Moreover, in the case $\pi_1 = \pi_2$ we shall find a connection between the sets $H_m^{(N)}$ and H_m.

Theorem. *A necessary and sufficient condition that $\pi_1 = \pi_2$ is that for every $m = 1, 2, \cdots$ the sequence of m-dimensional sets*

$$H_m^{(1)} \supseteq H_m^{(2)} \supseteq H_m^{(3)} \supseteq \cdots$$

is constant from a certain step (depending on m).

Additional Theorem. *If $H_m^{(1)} \supseteq H_m^{(2)} \supseteq H_m^{(3)} \supseteq \cdots$ for every m is constant from a certain step (and hence $\pi_1 = \pi_2$) this constant set is just the set H_m.*

We remark that if for a given m the sequence

$$H_m^{(1)} \supseteq H_m^{(2)} \supseteq H_m^{(3)} \supseteq \cdots$$

is constant ($= \Phi_m$) from a certain step N_0 then for every $m_1 < m$ the sequence

$$H_{m_1}^{(1)} \supseteq H_{m_1}^{(2)} \supseteq H_{m_1}^{(3)} \supseteq \cdots$$

will also—at the latest from the same step—be constant ($=$ the closure of the projection of Φ_m on the $x_1 \cdots x_{m_1}$-space); for two sets (viz. Φ_m and $\Lambda_m^{(N)}$ for $N \geq N_0$) in the $x_1 \cdots x_m$-space with identical closures (viz. Φ_m) are projected into two sets in the $x_1 \cdots x_{m_1}$-space with identical closures, because the condition that two sets have identical closures is that every point in each of the sets can be approximated by points in the other and this property obviously is preserved by projection.

We divide the theorem above, together with its addition, in a theorem A for the sufficiency and the addition and a theorem B for the necessity.

Theorem A. *If for every m the sequence*

$$H_m^{(1)} \supseteq H_m^{(2)} \supseteq H_m^{(3)} \supseteq \cdots$$

is constant from a certain step, then $\pi_1 = \pi_2$ and the constant set is equal to H_m.

Proof. We first show that $\pi_1 = \pi_2$. Denoting by $(\theta_1, \theta_2, \cdots)$ an arbitrary point from π_2 we are to show that it also lies in π_1, i. e. that there exists a solution $Y = (y_1, y_2, \cdots)$ of all the congruences (1). Let

$$H_m^{(1)} \supseteqq H_m^{(2)} \supseteqq H_m^{(3)} \supseteqq \cdots \supseteqq H_m^{(N)} \supseteqq \cdots$$

be constant for $N \geqq N_m$ where the integral sequence N_m moreover is chosen to be strictly increasing (and hence $\to \infty$).

We take our starting-point in an arbitrary positive integer M[1] and in an arbitrary chosen solution $Y^{(M)} = (y_1^{(M)}, y_2^{(M)}, \cdots)$ of the N_M first congruences (1). Next we choose a solution $X^{(M+1)} = (x_1^{(M+1)}, x_2^{(M+1)}, \cdots)$ of the N_{M+1} first congruences. This solution can be altered by an arbitrary point $Z^{(M+1)}$ from $\Lambda^{(N_{M+1})}$, i. e. for any point $Z^{(M+1)}$ from $\Lambda^{(N_{M+1})}$ (and no other points) the point $X^{(M+1)} + Z^{(M+1)}$ is again a solution of the N_{M+1} first congruences; this is true since $\Lambda^{(N_{M+1})}$ is the set of solutions of the N_{M+1} first zero-congruences (4). *Hence we can alter the solution* $X^{(M+1)} = (x_1^{(M+1)}, x_2^{(M+1)}, \cdots)$ *such that the projected point* $(x_1^{(M+1)}, \cdots, x_M^{(M+1)})$ *is altered by an arbitrary point from* $\Lambda_M^{(N_{M+1})}$ when only the other coordinates of $X^{(M+1)}$ are altered in a suitable manner. Our wish is now that the altered point $X^{(M+1)} + Z^{(M+1)}$ shall lie "near to" $Y^{(M)}$. Since $N_{M+1} > N_M$ the point $X^{(M+1)}$ is as $Y^{(M)}$ a solution of the N_M first congruences and hence their difference $Y^{(M)} - X^{(M+1)}$ is lying in $\Lambda^{(N_M)}$. The difference of the projected points $(y_1^{(M)}, \cdots, y_M^{(M)}) - (x_1^{(M+1)}, \cdots, x_M^{(M+1)})$ will therefore lie in $\Lambda_M^{(N_M)}$ and hence a fortiori in $H_M^{(N_M)}$ *and hence also in* $H_M^{(N_{M+1})}$. Since, as mentioned above, the solution $X^{(M+1)}$ of the N_{M+1} first congruences can be altered to another solution $Y^{(M+1)} = (y_1^{(M+1)}, y_2^{(M+1)}, \cdots)$ of these congruences such that the difference $(y_1^{(M+1)}, \cdots, y_M^{(M+1)}) - (x_1^{(M+1)}, \cdots, x_M^{(M+1)})$ becomes an arbitrarily chosen point of $\Lambda_M^{(N_{M+1})}$ and since the previous difference $(y_1^{(M)}, \cdots, y_M^{(M)}) - (x_1^{(M+1)}, \cdots, x_M^{(M+1)})$ is lying in the closure $H_M^{(N_{M+1})}$ of the set $\Lambda_M^{(N_{M+1})}$ it is clear that to every $\varepsilon_M > 0$ we can choose our solution $Y^{(M+1)}$ such that the first of the two M-dimensional point-differences ε_M-approximates the latter, i. e. such that

1) For the proof of $\pi_1 = \pi_2$ we could choose $M = 1$. When M is chosen arbitrarily it is in view of the proof of the additional theorem.

$$\left|(y_1^{(M+1)} - x_1^{(M+1)}) - (y_1^{(M)} - x_1^{(M+1)})\right| = \left|y_1^{(M+1)} - y_1^{(M)}\right| < \varepsilon_M$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\left|(y_M^{(M+1)} - x_M^{(M+1)}) - (y_M^{(M)} - x_M^{(M+1)})\right| = \left|y_M^{(M+1)} - y_M^{(M)}\right| < \varepsilon_M.$$

Next, let $X^{(M+2)} = (x_1^{(M+2)}, x_2^{(M+2)}, \cdots)$ be a solution of the N_{M+2} first congruences (1). This solution can be altered by an arbitrary point from $\Lambda^{(N_{M+2})}$ and hence $X^{(M+2)}$ can be altered such that the projected point $(x_1^{(M+2)}, \cdots, x_{M+1}^{(M+2)})$ is altered by an arbitrary point from $\Lambda_{M+1}^{(N_{M+2})}$ when only the other coordinates of $X^{(M+2)}$ are altered in a suitable manner. Our wish is that the altered point shall lie "near to" $Y^{(M+1)}$. Since $N_{M+2} > N_{M+1}$ the point $X^{(M+2)}$ is as $Y^{(M+1)}$ a solution of the N_{M+1} first congruences. The difference $(y_1^{(M+1)}, \cdots, y_{M+1}^{(M+1)}) - (x_1^{(M+2)}, \cdots, x_{M+1}^{(M+2)})$ is therefore lying in $\Lambda_{M+1}^{(N_{M+1})}$ and hence a fortiori in $H_{M+1}^{(N_{M+1})}$ *and hence also in* $H_{M+1}^{(N_{M+2})}$. Since, as mentioned above, the solution $X^{(M+2)}$ of the N_{M+2} first congruences can be altered to another solution $Y^{(M+2)} = (y_1^{(M+2)}, y_2^{(M+2)}, \cdots)$ of these congruences such that the difference $(y_1^{(M+2)}, \cdots, y_{M+1}^{(M+2)}) - (x_1^{(M+2)}, \cdots, x_{M+1}^{(M+2)})$ becomes an arbitrarily chosen point of $\Lambda_{M+1}^{(N_{M+2})}$ and since the previous difference $(y_1^{(M+1)}, \cdots, y_{M+1}^{(M+1)}) - (x_1^{(M+2)}, \cdots, x_{M+1}^{(M+2)})$ is lying in the closure $H_{M+1}^{(N_{M+2})}$ of the set $\Lambda_{M+1}^{(N_{M+2})}$ it is clear that to every $\varepsilon_{M+1} > 0$ we can choose the solution $Y^{(M+2)}$ such that the first of the two $(M+1)$-dimensional point-differences ε_{M+1}-approximates the latter, i. e. such that

$$\left|(y_1^{(M+2)} - x_1^{(M+2)}) - (y_1^{(M+1)} - x_1^{(M+2)})\right| = \left|y_1^{(M+2)} - y_1^{(M+1)}\right| < \varepsilon_{M+1}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\left|(y_{M+1}^{(M+2)} - x_{M+1}^{(M+2)}) - (y_{M+1}^{(M+1)} - x_{M+1}^{(M+2)})\right| = \left|y_{M+1}^{(M+2)} - y_{M+1}^{(M+1)}\right| < \varepsilon_{M+1}.$$

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In general, i. e. for an arbitrary $n \geqq M + 1$, let the point $X^{(n)} = (x_1^{(n)}, x_2^{(n)}, \cdots)$ be a solution of the N_n first congruences (1). This solution can be altered by an arbitrary point from $\Lambda^{(N_n)}$ and hence $X^{(n)}$ can be altered such that the projected point $(x_1^{(n)}, \cdots, x_{n-1}^{(n)})$ is altered by an arbitrary point from $\Lambda_{n-1}^{(N_n)}$ when only the other coordinates of $X^{(N)}$ are altered in a suitable manner. Our wish is that the altered point shall lie "near to" $Y^{(n-1)}$. Since $N_n > N_{n-1}$ the point $X^{(n)}$ is as $Y^{(n-1)}$ a solution of the N_{n-1} first congruences. The difference $(y_1^{(n-1)}, \cdots, y_{n-1}^{(n-1)}) - (x_1^{(n)}, \cdots, x_{n-1}^{(n)})$ is therefore lying in $\Lambda_{n-1}^{(N_{n-1})}$ and hence a fortiori in $H_{n-1}^{(N_{n-1})}$ *and hence also in* $H_{n-1}^{(N_n)}$. Since, as mentioned above, the solution $X^{(n)}$ of the N_n first congruences can be altered to another solution $Y^{(n)} = (y_1^{(n)}, y_2^{(n)}, \cdots)$ of these congruences such that the difference $(y_1^{(n)}, \cdots, y_{n-1}^{(n)}) - (x_1^{(n)}, \cdots, x_{n-1}^{(n)})$ becomes an arbitrarily chosen point of $\Lambda_{n-1}^{(N_n)}$ and since the previous difference $(y_1^{(n-1)}, \cdots, y_{n-1}^{(n-1)}) - (x_1^{(n)}, \cdots, x_{n-1}^{(n)})$ is lying in the closure $H_{n-1}^{(N_n)}$ of the set $\Lambda_{n-1}^{(N_n)}$ it is clear that to every $\varepsilon_{n-1} > 0$ we can choose the point $Y^{(n)}$ such that the first of the two $(n-1)$-dimensional point-differences ε_{n-1}-approximates the other, i.e. such that

$$\left| (y_1^{(n)} - x_1^{(n)}) - (y_1^{(n-1)} - x_1^{(n)}) \right| = \left| y_1^{(n)} - y_1^{(n-1)} \right| < \varepsilon_{n-1}$$

. . . .

$$\left| (y_{n-1}^{(n)} - x_{n-1}^{(n)}) - (y_{n-1}^{(n-1)} - x_{n-1}^{(n)}) \right| = \left| y_{n-1}^{(n)} - y_{n-1}^{(n-1)} \right| < \varepsilon_{n-1}.$$

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Choosing our ε's such that $\sum_M^\infty \varepsilon_r$ is convergent we consider the sequence

$$Y^{(M)} = (y_1^{(M)}, y_2^{(M)}, \cdots)$$
$$Y^{(M+1)} = (y_1^{(M+1)}, y_2^{(M+1)}, \cdots)$$
$$Y^{(M+2)} = (y_1^{(M+2)}, y_2^{(M+2)}, \cdots)$$

. . . .

The M first coordinate sequences $y_\nu^{(M)}, y_\nu^{(M+1)}, y_\nu^{(M+2)}, \cdots$ $(\nu = 1, 2, \cdots, M)$ satisfy

$$\left| y_\nu^{(p)} - y_\nu^{(q)} \right| \leqq \sum_q^\infty \varepsilon_r \quad \text{for} \quad p > q \geqq M$$

while each of the following coordinate sequences $y_n^{(M)}, y_n^{(M+1)}, y_n^{(M+2)}, \cdots$ $(n \geqq M+1)$ satisfy

$$\left| y_n^{(p)} - y_n^{(q)} \right| \leqq \sum_q^\infty \varepsilon_r \quad \text{for} \quad p > q \geqq n.$$

Hence, in particular, all the coordinate sequences converge towards respective numbers $y_1, y_2, \cdots$. The limit point

$$Y = (y_1, y_2, \cdots)$$

will then be a solution of all the congruences (1). In fact, to see that Y is a solution of the N^{th} congruence we observe that $Y^{(n)} = (y_1^{(n)}, y_2^{(n)}, \cdots)$ from a certain step is a solution of this congruence. Since only a finite number of variables really occurs in the congruence the statement follows from continuity reasons. Thus $(\theta_1, \theta_2, \cdots)$ is lying in π_1 and hence $\pi_1 = \pi_2$. Out of regard to the following we observe that the M first coordinates $y_1, \cdots, y_M$ of Y satisfy the inequalities

$$\left| y_1 - y_1^{(M)} \right| \leqq \sum_M^\infty \varepsilon_r$$

(6) . . . .

$$\left| y_M - y_M^{(M)} \right| \leqq \sum_M^\infty \varepsilon_r.$$

Now, to conclude the proof of theorem A, we have to show that the constant final set $H_M^{(N_M)}$ in the sequence

$$H_M^{(1)} \supseteqq H_M^{(2)} \supseteqq H_M^{(3)} \supseteqq \cdots$$

for every $M = 1, 2, \cdots$ is equal to H_M. Since $\Gamma_M \subseteqq \Lambda_M^{(N_M)}$, it is plain that $H_M \subseteqq H_M^{(N_M)}$. In order to show that, conversely, $H_M^{(N_M)} \subseteqq H_M$ for an arbitrarily given M we use the proof above in the case $\theta_1 = \theta_2 = \cdots = 0$ with our present M as the M in the proof. The previous point $Y^{(M)} = (y_1^{(M)}, y_2^{(M)}, \cdots)$ is then an arbitrary point from $\Lambda^{(N_M)}$ and the projected point $(y_1^{(M)}, \cdots, y_M^{(M)})$ is therefore *an arbitrary point from* $\Lambda_M^{(N_M)}$. We are to show that $(y_1^{(M)}, \cdots, y_M^{(M)})$ can be approximated by points from Γ_M. But this is an immediate consequence of the fact that (in the present case $\theta_1 = \theta_2 = \cdots = 0$) the point Y constructed in the proof above is lying in Γ and that its M first coordinates satisfy the inequalities (6) where $\sum_M^\infty \varepsilon_r$ can be chosen arbitrarily small.

Theorem B. *If* $\pi_1 = \pi_2$ *the sequence*

$$H_m^{(1)} \supseteqq H_m^{(2)} \supseteqq H_m^{(3)} \supseteqq \cdots$$

will for every m be constant from a certain step.

Proof. Indirectly, we assume that there exists an m_0 for which $H_{m_0}^{(1)} \supseteqq H_{m_0}^{(2)} \supseteqq H_{m_0}^{(3)} \supseteqq \cdots$ is not constant from a certain step and are to show that $\pi_1 \neq \pi_2$, i. e. that there exists a $(\theta_1, \theta_2, \cdots)$ which belongs to π_2 but not to π_1. We first consider the geometric appearance of the sequence of modules $H_{m_0}^{(n)}$ $(n = 1, 2, \cdots)$. This sequence is an essentially decreasing[1)] sequence of lattice cylinders (see § 2). It is therefore plain that from a certain step $n \geqq N_0$ the least space (vector space) which contains $H_{m_0}^{(n)}$, and the space of generatrix directions of the cylinder $H_{m_0}^{(n)}$, will be constant spaces R_p and R_{p_1} of dimensions (say) p and p_1. Furthermore from this step the lattice base G_n of $H_{m_0}^{(n)}$ can be chosen in such a way that the least space which contains G_n is *a fixed space* R_q (dimension q with $p = p_1 + q$). The lattices G_n form from this step an essentially decreasing sequence in their common least space R_q. Therefore the q-dimensional content of the fundamental parallelotope of G_n ("fundamental content G_n") is an essentially increasing sequence which $\to \infty$ (since the fundamental content is at least doubled by the transition from one lattice to the next every time the lattices are different).

By $K(\varrho)$ we denote the open sphere in R_q with radius ϱ and center O as also the q-dimensional content of this sphere. By $C(\varrho)$ we denote the corresponding sphere cylinder in R_p with the sphere $K(\varrho)$ as base and the space of generatrix directions R_{p_1}. We also consider spheres in R_q whose centers are not lying in O and the corresponding sphere cylinders in R_q. In the following we denote for abbreviation sphere cylinders with base-sphere in R_q and space of generatrix directions R_{p_1} as "sphere cylinders" without further specification. By the sphere cylinder around the point P in R_p with radius ϱ we understand the sphere cylinder corresponding to the sphere with radius ϱ and center in the projection of P on R_q in the direction of R_{p_1}.

We first determine a sequence of strictly increasing positive numbers $N_1, N_2, \cdots, N_\nu, \cdots$ and the corresponding positive numbers $\varrho_1, \varrho_2, \cdots, \varrho_\nu, \cdots$ by the following procedure.

1°. Let $N_1 \geqq N_0$ be chosen such that the fundamental content G_{N_1} is larger than the sphere content $K(1)$. Then the sphere $K(1)$ cannot contain a complete system of representatives in R_q modulo G_{N_1} and hence the sphere cylinder $C(1)$ cannot contain a complete system of representatives in R_p modulo $H_{m_0}^{(N_1)}$. To this N_1 we choose the positive number ϱ_1 so large that every sphere in R_q with radius ϱ_1 contains a complete system of representatives in R_q modulo G_{N_1} and hence also a complete system of representatives in R_p modulo $H_{m_0}^{(N_1)}$. In particular, everyone of our sphere cylinders in R_p with radius ϱ_1 will contain a complete system of representatives in R_p modulo $H_{m_0}^{(N_1)}$.

1) An essentially decreasing sequence of sets is here and in the following sequence where every element is contained in the preceding and which is not constant from a certain step. The expression, an essentially increasing sequence of numbers, used below, has an analogous meaning.

2°. Next we determine $N_2 > N_1$ such that the fundamental content G_{N_2} is larger than $K(\varrho_1 + 2)$. Then the sphere cylinder $C(\varrho_1 + 2)$ cannot contain a complete system of representatives in R_p modulo $H_{m_0}^{(N_2)}$. To this N_2 we determine the positive number ϱ_2 so large that everyone of our sphere cylinders in R_p with radius ϱ_2 contains a complete system of representatives in R_p modulo $H_{m_0}^{(N_2)}$.

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ν°. After having determined $N_{\nu-1}$ and $\varrho_{\nu-1}$ we determine $N_\nu > N_{\nu-1}$ such that the fundamental content G_{N_ν} is larger than $K(\varrho_{\nu-1} + \nu)$. Then the sphere cylinder $C(\varrho_{\nu-1} + \nu)$ cannot contain a complete system of representatives in R_p modulo $H_{m_0}^{(N_\nu)}$. To this N_ν we determine the positive number ϱ_ν so large that everyone of our sphere cylinders in R_p with radius ϱ_ν contains a complete system of representatives in R_p modulo $H_{m_0}^{(N_\nu)}$.

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After having determined N_ν and ϱ_ν $(\nu = 1, 2, \cdots)$ we now pass to the direct searching of a point $(\theta_1, \theta_2, \cdots)$ which belongs to the set π_2 but not to the set π_1. The idea in this (successive) determination modulo 1 of the numbers $\theta_1, \theta_2, \cdots$, the kernel of which can be found in example 1, § 1, is that we try to see that the set of projections $(x_1, \cdots, x_{m_0})$ on the $x_1 \cdots x_{m_0}$-space of all solutions $(x_1, x_2, \cdots)$ of the N first congruences (1) will lie farther and farther away from O for increasing values of N. More precisely, we will see that the set of projections for $N = N_\nu$ will lie in R_p and outside $C(\nu)$.

1st step. We first choose an arbitrary point $P^{(1)} = (x_1^{(1)}, x_2^{(1)}, \cdots)$ in the infinite-dimensional space which only satisfies the condition that the projected point $P_{m_0}^{(1)} = (x_1^{(1)}, \cdots, x_{m_0}^{(1)})$ is lying in R_p and has no equivalent point modulo $H_{m_0}^{(N_1)}$ lying in $C(1)$.

Such a point exists on account of 1° since $C(1)$ does not contain a complete system of representatives in R_p modulo $H_{m_0}^{(N_1)}$. We substitute $(x_1, x_2, \cdots) = (x_1^{(1)}, x_2^{(1)}, \cdots)$ in the N_1 first linear forms (2). The numbers thus determined (but only considered modulo 1) shall be our numbers $\theta_1, \theta_2, \cdots, \theta_{N_1}$. We observe that the total set of solutions of the N_1 first congruences (1) (with the θ's just chosen) is the set $P^{(1)} + \Lambda^{(N_1)}$ because $\Lambda^{(N_1)}$ is the set of solutions of the N_1 first zero-congruences. From the choice of $P^{(1)}$ it follows that the projection of this set $P^{(1)} + \Lambda^{(N_1)}$ on the $x_1 \cdots x_{m_0}$-space—i. e. the set $P_{m_0}^{(1)} + \Lambda_{m_0}^{(N_1)}$ which consists of all points equivalent to $P_{m_0}^{(1)}$ modulo $\Lambda_{m_0}^{(N_1)}$—is lying in R_p and outside $C(1)$.

2^{nd} *step.* Next we choose (which is possible from the choice of N_2) a point $D_{m_0}^{(2)} = (d_1^{(2)}, \cdots, d_{m_0}^{(2)})$ in R_p which has no equivalent point modulo $H_{m_0}^{(N_2)}$ in $C(\varrho_1 + 2)$. The sphere cylinder in R_p around $D_{m_0}^{(2)}$ with radius ϱ_1 contains (on account of the choice of ϱ_1) a point equivalent to $P_{m_0}^{(1)}$ modulo $H_{m_0}^{(N_1)}$. Since $\Lambda_{m_0}^{(N_1)}$ is lying everywhere dense in $H_{m_0}^{(N_1)}$ this cylinder also contains a point $P_{m_0}^{(2)} = (x_1^{(2)}, \cdots, x_{m_0}^{(2)})$ equivalent to $P_{m_0}^{(1)}$ modulo $\Lambda_{m_0}^{(N_1)}$. We can therefore choose a point $P^{(2)} = (x_1^{(2)}, x_2^{(2)}, \cdots)$ whose projection on the $x_1 \cdots x_{m_0}$-space is $P_{m_0}^{(2)}$ and which is equivalent to $P^{(1)}$ modulo $\Lambda^{(N_1)}$. In particular $P^{(2)}$ is a solution of the N_1 first congruences (1). We now substitute $(x_1, x_2, \cdots) = (x_1^{(2)}, x_2^{(2)}, \cdots)$ in the N_2 first linear forms (2) and denote the numbers thus determined (modulo 1) by $\theta_1, \cdots, \theta_{N_2}$. The N_1 first of these numbers coincide with the numbers $\theta_1, \cdots, \theta_{N_1}$ determined by the first step, since $P^{(2)}$ satisfies the N_1 first congruences (formed with these θ's). We now consider the set of solutions $(x_1, x_2, \cdots)$ of the N_2 first (with the above θ's formed) congruences (1), i. e. the set $P^{(2)} + \Lambda^{(N_2)}$. Then the projection of this set on the $x_1 \cdots x_{m_0}$-space—i. e. the set $P_{m_0}^{(2)} + \Lambda_{m_0}^{(N_2)}$ which consists of all points equivalent to $P_{m_0}^{(2)}$ modulo $\Lambda_{m_0}^{(N_2)}$—is lying in R_p and outside $C(2)$; that the set is lying in R_p is plain, and the second statement follows from the fact that $P_{m_0}^{(2)}$ is lying in a sphere cylinder around $D_{m_0}^{(2)}$ with radius ϱ_1 where $D_{m_0}^{(2)}$ has no equivalent point in $C(\varrho_1 + 2)$ modulo $H_{m_0}^{(N_2)}$ and hence a fortiori no equivalent point modulo $\Lambda_{m_0}^{(N_2)}$.

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ν^{th} *step.* We choose (which is possible from the choice of N_ν) a point $D_{m_0}^{(\nu)} = (d_1^{(\nu)}, \cdots, d_{m_0}^{(\nu)})$ in R_p which has no equivalent point modulo $H_{m_0}^{(N_\nu)}$ in $C(\varrho_{\nu-1} + \nu)$. The sphere cylinder in R_p around $D_{m_0}^{(\nu)}$ with radius $\varrho_{\nu-1}$ contains (on account of the choice of $\varrho_{\nu-1}$) a point equivalent to $P_{m_0}^{(\nu-1)}$ modulo $H_{m_0}^{(N_{\nu-1})}$ and hence also a point $P_{m_0}^{(\nu)} = (x_1^{(\nu)}, \cdots, x_{m_0}^{(\nu)})$ equivalent to $P_{m_0}^{(\nu-1)}$ modulo $\Lambda_{m_0}^{(N_{\nu-1})}$. We can therefore choose a point $P^{(\nu)} = (x_1^{(\nu)}, x_2^{(\nu)}, \cdots)$ whose projection on the $x_1 \cdots x_{m_0}$-space is $P_{m_0}^{(\nu)}$ and which is equivalent to $P^{(\nu-1)}$ modulo $\Lambda^{(N_{\nu-1})}$. In particular $P^{(\nu)}$ is a solution of the $N_{\nu-1}$ first congruences (1). We now substitute $(x_1, x_2, \cdots) = (x_1^{(\nu)}, x_2^{(\nu)}, \cdots)$ in the N_ν first linear forms (2) and denote the numbers thus determined (modulo 1) by $\theta_1, \cdots, \theta_{N_\nu}$. The $N_{\nu-1}$ first of these numbers coincide with the numbers $\theta_1, \cdots, \theta_{N_{\nu-1}}$ determined by the $(\nu - 1)^{th}$ step. We consider the set of solutions $(x_1, x_2, \cdots)$ of the N_ν first (with the above θ's formed) congruences (1). Then the projection of this set on the $x_1 \cdots x_{m_0}$-space—i. e. the set $P_{m_0}^{(\nu)} + \Lambda_{m_0}^{(N_\nu)}$ which consists of all points equivalent to $P_{m_0}^{(\nu)}$ modulo $\Lambda_{m_0}^{(N_\nu)}$—lies in R_p and outside $C(\nu)$; that the set is lying in R_p is plain, and the second statement follows from the fact that $P_{m_0}^{(\nu)}$ is lying in a sphere cylinder around $D_{m_0}^{(\nu)}$ with radius $\varrho_{\nu-1}$ where $D_{m_0}^{(\nu)}$ has no equivalent point in $C(\varrho_{\nu-1} + \nu)$ modulo $H_{m_0}^{(N_\nu)}$ and hence a fortiori no equivalent point modulo $\Lambda_{m_0}^{(N_\nu)}$.

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In this manner we have got a point $(\theta_1, \theta_2, \cdots)$ with the desired properties. In fact, the point is belonging to π_2 since for every ν the N_ν first (with these θ's formed) congruences (1) have a solution $P^{(\nu)} = (x_1^{(\nu)}, x_2^{(\nu)}, \cdots)$, and here $N_\nu \to \infty$ for $\nu \to \infty$. On the other hand the point $(\theta_1, \theta_2, \cdots)$ does not belong to π_1, i. e. there is no solution of the whole system of congruences (1); for every solution of the N_ν first congruences has a projection on the $x_1 \cdots x_{m_0}$-space which lies in R_p and outside $C(\nu)$.

Remark. The theorems of this paragraph connect the condition $\pi_1 = \pi_2$ with the closures $H_m^{(N)}$ and H_m of the modules $\Lambda_m^{(N)}$ and Γ_m. We shall mention that analogous theorems hold for the sets $\Lambda_m^{(N)}$ and Γ_m themselves, viz.

Theorem. *A necessary and sufficient condition that $\pi_1 = \pi_2$ is that for every $m = 1, 2, \cdots$ the sequence*

$$\Lambda_m^{(1)} \supseteqq \Lambda_m^{(2)} \supseteqq \Lambda_m^{(3)} \supseteqq \cdots$$

is constant from a certain step (depending on m).

Additional Theorem. *If $\Lambda_m^{(1)} \supseteqq \Lambda_m^{(2)} \supseteqq \Lambda_m^{(3)} \supseteqq \cdots$ for every m is constant from a certain step this constant set is just the set Γ_m.*

If these theorems, as their analogues for the closures, are divided in a theorem A for the sufficiency and the addition and a theorem B for the necessity, the theorem A is even simpler to prove than the previous theorem A. Theorem B, however, lies deeper than its analogue. We can obtain the new theorem B from the old one by the following

Theorem. *For an arbitrary system of linear forms* (1) *(with $\pi_1 = \pi_2$ or $\pi_1 \neq \pi_2$) there exists to every positive integer m an integer $M \geqq m$ and a positive integer N such that the sequence $\Lambda_m^{(N)} \supseteqq \Lambda_m^{(N+1)} \supseteqq \Lambda_m^{(N+2)} \supseteqq \cdots$ is the projection on the $x_1 \cdots x_m$-space of the sequence $H_M^{(N)} \supseteqq H_M^{(N+1)} \supseteqq H_M^{(N+2)} \supseteqq \cdots$.*

We omit, however, the proofs of these theorems which are unnecessary for the proof of our main theorem in its present framing (cp. p. 8-9).

§ 4. The structure of closed modules in the infinite-dimensional space.

In this paragraph we shall study the closed modules in our infinite-dimensional space—which from now on is denoted by R^∞—where the underlying convergence notion, occasionally used in the previous paragraphs, is that of convergence in each of the coordinates. As we shall see the closed modules in the space R^∞ possess quite a similar structure as that of the closed modules in the usual m-dimensional space R_m (see § 2).

In order to prove the structure theorem in R^∞ we shall use the analogous structure theorem in R_m, $m = 1, 2, \cdots$. The transition from the finite-dimensional case is, however, not a trivial one. We shal have to put in an intermediate space R_∞ between the finite-dimensional spaces R_m and the space R^∞. The space R_∞ is as R^∞ an infinite-dimensional space, but while a point $X = (x_1, x_2, \cdots)$ in R^∞ may have quite arbitrary coor-

dinates, a point $A = (a_1, a_2, \circ\circ\circ)$[1] in R_∞ always has coordinates which from a certain step (depending on the point) are 0, i. e. $a_n = 0$ for $n \geqq N = N(A)$.

Between the spaces R_∞ and R^∞ there exists, when a convergence notion in R_∞ is suitably chosen, a *duality*. Once established this duality permits us to get at the structure theorem for closed modules in R^∞ from an analogous structure theorem for closed modules in R_∞. Now, as mentioned, the space R_∞ is lying nearer to the finite-dimensional spaces R_m than does R^∞, in fact it can be exhausted by the $a_1 a_2 \cdots a_m$-space for $m \to \infty$. This is the reason why, as we shall see, the structure theorem in R_∞ can easily be obtained from the finite-dimensional case.

The duality, mentioned above, between R_∞ and R^∞ is analogous to a duality considered by M. Riesz between two m-dimensional spaces $R_m = \{(a_1, \cdots, a_m)\}$ and $R_m = \{(x_1, \cdots, x_m)\}$. If M is an arbitrary module in R_m Riesz considers the point set in (the other space) R_m consisting of all points $A = (a_1, \cdots, a_m)$ from this latter R_m for which

$$A \cdot X = a_1 x_1 + a_2 x_2 + \cdots + a_m x_m \equiv 0 \pmod 1$$

for every point $X = (x_1, x_2, \cdots, x_m)$ from M. This point set is a closed module in R_m and is called the *dual* module of M. We denote it by M'. If we repeat the operation of passing to the dual module we get a closed module $M'' = (M')'$ in (the original space) R_m. The relation between M and M'' appears from the following important theorem.

Riesz's Theorem. *If M is an arbitrary module in R_m the dual module M'' of its dual module M' is the closure $\overline{M}$ of M, i. e.*

$$M'' = \overline{M}.$$

For a closed module H in R_m we get in particular $H'' = H$.

We now pass to the establishment of the duality between R_∞ and R^∞, or rather that side of the duality which will be needed in the following. A full account of the duality can be found in another paper[2] where the topic of this paragraph is discussed in more detail.

1) For points in R_∞ we use the notation $(a_1, a_2, \circ\circ\circ)$ in order to make apparent that their coordinates are all zero from a certain step.

2) H. Bohr and E. Følner: On a structure theorem for closed modules in an infinite-dimensional space, to appear elsewhere.

Let T be an arbitrary linear transformation in R_∞ and let the fundamental points $(1, 0, 0, \circ\circ\circ)$, $(0, 1, 0, \circ\circ\circ)$, $\cdots$ by the transformation be taken into the points

$$T\{(1, 0, 0, \circ\circ\circ)\} = S_1 = (t_{11}, t_{21}, \circ\circ\circ)$$
$$T\{(0, 1, 0, \circ\circ\circ)\} = S_2 = (t_{12}, t_{22}, \circ\circ\circ)$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

from R_∞. The arbitrary point $A = (a_1, a_2, \circ\circ\circ)$ from R_∞ will then be carried into the point

$$B = T(A) = a_1 S_1 + a_2 S_2 + \circ\circ\circ.$$

Introducing the matrix $T = \{t_{rs}\}$ the linear transformation may be written $B = TA$. In the following we denote a linear transformation in R_∞ and the corresponding (uniquely determined) matrix by the same letter T.

Conversely, each such matrix equation

$$\begin{Bmatrix} b_1 \\ b_2 \\ \circ \\ \circ \\ \circ \end{Bmatrix} = \begin{Bmatrix} t_{11} t_{12} & \cdot & \cdot & \cdot \\ t_{21} t_{22} & \cdot & \cdot & \cdot \\ \circ \ \circ & & & \\ \circ \ \circ & & & \\ \circ \ \circ & & & \end{Bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ \circ \\ \circ \\ \circ \end{Bmatrix}$$

where the column vectors are arbitrary points from R_∞ is a linear transformation in R_∞.

We now define the scalar product between two points $A = (a_1, a_2, \circ\circ\circ)$ and $X = (x_1, x_2, \cdots)$ from R_∞ and R^∞ respectively. We put

$$A \cdot X = X \cdot A = a_1 x_1 + a_2 x_2 + \circ\circ\circ.$$

In matrix notation the scalar product is expressed by $A^* X$ or $X^* A$[1] when we agree on considering the points as column vectors (for convenience we usually write them horizontally).

For a given linear transformation T in R_∞ and two variable points X and Y from R^∞ we now set up the condition

(7) $$A \cdot X = T(A) \cdot Y \text{ for every } A \text{ from } R_\infty.$$

1) The star denotes the operation of transposing a matrix.

We shall show that this condition on X and Y is equivalent to a linear transformation in R^∞ (expressed by linear expressions as (3), § 1) of Y into X (and thus, in particular, that to any given Y there exists one and only one X satisfying (7)).

In matrix notation the condition runs as follows

$$A^* X = (TA)^* Y \quad \text{or} \quad A^* X = A^* T^* Y.$$

Putting successively $A^* = (1, 0, 0, \circ\circ\circ)$, $(0, 1, 0, \circ\circ\circ)$, $\cdots$ in this relation we get

(8) $$X = T^* Y$$

and conversely the former condition follows from (8) by left-multiplying it with A^*.

Putting (8) into (7) and changing Y to X we get the relation

(9) $$A \cdot T^*(X) = T(A) \cdot X \text{ for every } A \text{ from } R_\infty \text{ and every } X \text{ from } R^\infty.$$

We now define a *substitution in* R_∞ as a linear, one-to-one transformation of R_∞ onto R_∞.

If T is a substitution the condition (7) is equivalent to the condition

(10) $$A \cdot Y = T^{-1}(A) \cdot X \text{ for every } A \text{ from } R_\infty,$$

in fact we have only substituted $T^{-1}(A)$ for A and interchanged the two sides of the equation (7). Here T^{-1} denotes the inverse substitution of T. Since (7) is equivalent to (8) we see that (10) is equivalent to

(11) $$Y = (T^{-1})^* X.$$

Hence also the relations (8) and (11) are equivalent which shows that T^* is a one-to-one transformation of R^∞ onto R^∞ and therefore what we have called a substitution in R^∞ (see § 1). Putting $T' = (T^*)^{-1}$ and replacing X by $T'(X)$ in (9) we obtain the following

Theorem 1. *If T is a substitution in R_∞ then T^* is a substitution in R^∞ and there exists a uniquely determined substitution T' in R^∞ such that*

(12) $A \cdot X = T(A) \cdot T'(X)$ *for every* A *from* R_∞ *and every* X *from* R^∞,

viz. the substitution $T' = (T^*)^{-1} = (T^{-1})^*$.

We call T' the *dual substitution* of T.

In order to speak of *closed* modules in R_∞ and R^∞ we must know the underlying convergence notion of the two spaces. We have already mentioned that in R^∞ our convergence notion is that of convergence in every coordinate. In order to define a suitable convergence notion in R_∞ we first observe that our convergence notion in R^∞ may also be stated as follows:

A sequence $X^{(n)}$ *converges towards* X *if and only if*

$$A \cdot X^{(n)} \rightarrow A \cdot X \text{ for every } A \text{ from } R_\infty.$$

In fact, since a point A from R_∞ only contains a finite number of non-zero coordinates the former condition involves the latter, and conversely, the former condition is obtained from the latter by putting successively $A = (1, 0, 0, \circ\circ\circ), (0, 1, 0, \circ\circ\circ), \cdots$.

In the new form the notion of convergence in R^∞ has a dual notion of convergence in R_∞:

A sequence $A^{(n)}$ *of points from* R_∞ *is said to converge towards a point* A *from* R_∞ *if and only if*

$$X \cdot A^{(n)} \rightarrow X \cdot A \text{ for every } X \text{ from } R^\infty.$$

This is going to be our convergence notion in R_∞.[1)]

Remark. Our substitutions in R^∞ are obviously *bicontinuous*. In order to show that our substitutions in R_∞ are also bicontinuous we remark that on account of (9) every linear transformation T in R_∞ is continuous; in fact, when $A^{(n)} \rightarrow A$ we get from (9)

$$X \cdot T(A^{(n)}) = T^*(X) \cdot A^{(n)} \rightarrow T^*(X) \cdot A = X \cdot T(A)$$

for every X from R^∞ which shows that $T(A^{(n)}) \rightarrow T(A)$. It can easily be shown that our substitutions in R^∞ or R_∞ are just the linear, one-to-one, bicontinuous transformations of the space onto itself (in the case of R_∞ nothing is left to prove).

1) In the following we shall only use the definition of convergence in R_∞ in the above form; we may, however, mention that this definition, as easily seen, is equivalent to the following (more direct) one: Convergence of a sequence in R_∞ means convergence in every coordinate and moreover the existence of a p only depending on the sequence, such that all points of the sequence have 0 on the coordinate places with higher number than p

For an arbitrary closed module H in R_∞ we consider the point set H' in R^∞ which consists of all points X for which

$$A \cdot X \equiv 0 \pmod 1 \text{ for every } A \text{ from } H.$$

Obviously the set H' is a module. Furthermore H' is closed, for if $X^{(n)} \rightarrow X$ in R^∞ and all $X^{(n)}$ are lying in H', then for every A from H we have $0 \equiv A \cdot X^{(n)} \rightarrow A \cdot X$ so that $A \cdot X \equiv 0$. We call the closed module H' the *dual module* of the closed module H. The following simple theorem indicates the connection between the two notions, dual module and dual substitution.

Theorem 2. *If we subject a closed module* H *in* R_∞ *to a substitution* T *and subject the dual module* H' *in* R^∞ *to the dual substitution* T' *then the resulting module* $T'(H')$ *in the latter case is the dual module of the resulting module* $T(H)$ *in the former case, i. e.*

$$T'(H') = (T(H))'.$$

This is an immediate consequence of the relation (12) when we only observe that $T(A)$ runs through $T(H)$ and $T'(X)$ runs through $T'(H')$ when A runs through H and X through H'.

We have defined above the dual module of a closed module from R_∞. Analogously, we define the dual module H' of a closed module H from R^∞ as the point set (eo ipso closed module) consisting of the points A from R_∞ for which

$$X \cdot A \equiv 0 \pmod 1 \text{ for every } X \text{ from } H.$$

Then we have the following important

Theorem 3. *For an arbitrary closed module* H *in* R^∞ *the dual module* H'' *of its dual module* H' *is the module itself, i. e.*

$$H'' = H.$$

Obviously $H'' \supseteq H$. Thus we only have to prove that $H'' \subseteq H$. Let then $Y = (y_1, y_2, \cdots)$ be an arbitrary point from H''. In order to show that Y is lying in H, let m be an arbitrary positive integer. We consider the points $(a_1, a_2, \cdots, a_m, 0, 0, \circ\circ\circ) = (a_1, a_2, \cdots, a_m)$ from the common part L of H' and the $a_1 a_2 \cdots a_m$-space. Then for every point in L we have

$$(13) \qquad (y_1, y_2, \cdots, y_m) \cdot (a_1, a_2, \cdots, a_m) \equiv 0 \pmod 1.$$

Next, let M denote the projection of H on the $x_1 x_2 \cdots x_m$-space (i. e. the set of points $(x_1, x_2, \cdots, x_m)$ arising from the points $(x_1, x_2, \cdots)$ of H by cancelling all coordinates with indices $> m$). M is again a module, but not necessarily a closed module. Plainly, $L = M'$ and thus on account of (13) the point $(y_1, y_2, \cdots, y_m)$ belongs to M''. Now, according to Riesz's theorem

$$M'' = \bar{M}$$

and hence $(y_1, y_2, \cdots, y_m)$ can be approximated by points $(x_1, x_2, \cdots, x_m)$ from M. Since m is arbitrary it follows that $Y = (y_1, y_2, \cdots)$ can be approximated by points $(x_1, x_2, \cdots)$ from H, i. e. Y must lie in $\bar{H} = H$, q. e. d.

We shall now prove the following structure theorem for closed modules in R_∞.

Structure Theorem R_∞. *A closed module H in the infinite-dimensional space R_∞ is a point set E which by a substitution can be transfered into a point set of a special form, in the following denoted by S_∞, namely a point set $\{(a_1, a_2, \circ\circ\circ)\}$ of the following structure: The indices $1, 2, \cdots, n, \cdots$ can be divided into three fixed classes $\{n_r\}$, $\{n_s\}$, $\{n_t\}$ depending only on the point set, such that the coordinates a_{n_r} independently run through all numbers, and the coordinates a_{n_s} independently run through all integers, while all the remaining coordinates a_{n_t} are constantly zero. Only, of course, the simultaneous variation of the a_{n_r} and the a_{n_s} in the set is limited by the obvious demand that $(a_1, a_2, \circ\circ\circ)$ always shall lie in R_∞, i. e. have 0 from a certain coordinate place (depending on the point). Conversely, each such point set E is a closed module.*

The latter part of the theorem follows immediately from the remark on p. 26.

In order to prove the first (and real) part of the theorem let H_m denote the common part of H and the $x_1 \cdots x_m$-space. Then, obviously, H_m is in the usual sense a closed module in the $a_1 \cdots a_m$-space. Furthermore, H_m is the common part of H_{m+1} and the $a_1 \cdots a_m$-space. Hence it follows from the theorem on p. 10, for $m = 1, 2, \cdots$, that we can generate successively the closed modules $H_1, H_2, \cdots$ by linearly independent vectors with arbitrary and integral coefficients in such a way that the generating vectors of H_{m+1} are the generating vectors of H_m with the same types of coefficients, in connection with other vectors (if necessary). In this way we get a sequence of linearly independent vectors $G_1, G_2, \cdots$ which provided with suitable types of coefficients (integral or arbitrary) will generate H (generation of course in the sense that for each vector of H only a finite number of generators is used). With arbitrary coefficients the vectors spann a subspace $R(H)$ of R_∞. Let R_1 denote the common part of $R(H)$ and the x_1-axis. If the space R_1 is not the whole x_1-axis, but only the 0-vector we place a non-zero vector on the x_1-axis. Then this vector together with $R(H)$ will spann a space $R^{(1)}$ which contains the x_1-axis. If $R(H)$ itself contains the x_1-axis we put $R^{(1)} = R(H)$. Next, let R_2 denote the common part of $R^{(1)}$ and the $x_1 x_2$-plane. If the space R_2 is not the whole $x_1 x_2$-plane, but only the x_1-axis we place a vector in the $x_1 x_2$-plane outside the x_1-axis. Then this vector together with $R^{(1)}$ will spann a space $R^{(2)}$ which contains the $x_1 x_2$-plane. If $R^{(1)}$ itself contains the $x_1 x_2$-plane we put $R^{(2)} = R^{(1)}$. In this way we continue. If the vectors thus found in some way or other are put into a sequence with the vectors $G_1, G_2, \cdots$ we get a sequence of linearly independent vectors $U_1, U_2, \cdots$ which provided with suitable types of coefficients (zero, integral or arbitrary) will generate H and with mere arbitrary coefficients the whole space R_∞. The linear independence of $U_1, U_2, \cdots$ secures that each point in R_∞ has only one representation by this generation. Hence

$$B = a_1 U_1 + a_2 U_2 + \circ\circ\circ$$

is a substitution in R_∞ of $A = (a_1, a_2, \circ\circ\circ)$ into B. It takes the fundamental vectors $(1, 0, 0, \circ\circ\circ)$, $(0, 1, 0, \circ\circ\circ)$, $\cdots$ into the vectors $U_1, U_2, \cdots$. Therefore the inverse substitution, which takes $U_1, U_2, \cdots$ into the fundamental vectors, will take the closed module H into a set $\{(a_1, a_2, \circ\circ\circ)\}$ determined by $a_i = 0$ for certain i, a_i arbitrary integral for certain i, and

a_i arbitrary for the remaining i. This proves structure theorem R_∞.

By help of structure theorem R_∞ and the duality between R_∞ and R^∞ we shall now obtain the main result of this paragraph, viz.

Structure Theorem R^∞. *A closed module in the infinite-dimensional space R^∞ is a point set E which by a substitution can be transfered into a point set of a special form, denoted by S^∞, namely a point set $\{(x_1, x_2, \cdots)\}$ of the following structure: The indices $1, 2, \cdots, n, \cdots$ can be divided into three fixed classes $\{n_r\}, \{n_s\}, \{n_t\}$ which depend only on the point set, such that the coordinates x_{n_r} independently run through all numbers, and the coordinates x_{n_s} independently run through all integers, while all the remaining coordinates x_{n_t} are constantly zero. Conversely, each such point set E is a closed module.*

Again, the latter part of the theorem follows immediately from the remark on p. 26.

In order to obtain a proof of the first (and real) part of the theorem by help of the corresponding theorem in R_∞ let us first show that the dual module of a closed module of the special form S_∞ is a closed module of the special form S^∞. More precisely we shall prove

Theorem 4. *For a closed module H in R_∞ of the special form S_∞, explicitly $\{(a_1, a_2, \circ\circ\circ)\}$ with the coordinates a_{n_r} arbitrary, the coordinates a_{n_s} integral, and the coordinates a_{n_t} zero, the dual module H' in R^∞ is of the special form S^∞, and more precisely the dual module is $\{(x_1, x_2, \cdots)\}$ where the x_{n_r} are zero, the x_{n_s} integral, and the x_{n_t} arbitrary.*

We first observe that obviously all points X of the form mentioned are lying in H'. Conversely, we have to show that all points in H' have the form mentioned. Since the points X in H' have to fulfill

$$(\circ\circ\circ\ \xi_{n_r}\ \circ\circ\circ) \cdot X \equiv 0 \pmod 1 \text{ for all values } \xi_{n_r}$$

it follows that the n_r^{th} coordinate of X must be zero, and since

$$(\underset{1\,2\cdots}{\circ\circ\circ}\ \underset{n_s}{1}\ \underset{\cdots}{\circ\circ\circ}) \cdot X \equiv 0 \pmod 1$$

it follows that the n_s^{th} coordinate of X must be integral. This proves theorem 4.

We have now got all means necessary to prove structure theorem R^∞. Let first H be an arbitrary closed module in R_∞. Then on account of structure theorem R_∞ there exists a substitution T in R_∞ such that $T(H)$ has the special form S_∞. The dual module H' of H is a closed module in R^∞. We shall show that H' by a substitution can be taken into a closed module of the special form S^∞. In fact, the dual substitution T' of T has this property, for it follows from theorem 2 that $T'(H') = (T(H))'$ and from theorem 4 that $(T(H))'$, as the dual module of a closed module of the special form S_∞, is itself a closed module of the special form S^∞. Hence we see that every closed module in R^∞ which is the dual module of a closed module in R_∞ by a substitution can be taken into a set of the form S^∞. In order to complete the proof of structure theorem R^∞ we therefore only have to show that *every* closed module H in R^∞ can be written in the form K' where K is a closed module in R_∞. This, however, is a consequence of theorem 3 which tells that $H = H''$ so that for K we may use H'.

§ 5. Proof of the main theorem.

Already in § 1 we have formulated the main theorem and proved the simple "half" of it, namely that a sufficient condition that a system of linear forms (2) have $\pi_1 = \pi_2$ is that the system by a substitution can be taken into a system of the type S. We shall now show that this condition is also necessary, i. e. that every system of linear forms which has $\pi_1 = \pi_2$ by a substitution can be taken into a system of the type S.

For a system of congruences (1) the set Γ of solutions of the corresponding zero congruences is obviously always (i. e. whether $\pi_1 = \pi_2$ or not) a closed module in R^∞. Hence the structure theorem R^∞ from § 4 states that there exists a substitution T which takes Γ into a point set of the form S^∞, corresponding (say) to the classes $\{n_r\}, \{n_s\}, \{n_t\}$. By this substitution T the system of linear forms will be taken into a system where the coefficient columns corresponding to the variables x_{n_r} are zero columns while the coefficient columns corresponding to the variables x_{n_s} are integral columns. This is seen by putting $(\circ\circ\circ\ \xi_{n_r}\ \circ\circ\circ)$ with arbitrary ξ_{n_r}, respectively $\underset{1\,2\,\ldots\,n_s\,\ldots}{(\circ\circ\circ\ 1\ \circ\circ\circ)}$ into the zero-congruences. Conversely, a coefficient column of zero's corresponds to a variable x_{n_r} and an integral coefficient column which is not a zero column to a variable x_{n_s}.

Now, we shall show that if $\pi_1 = \pi_2$ for the original system, and hence also for the transformed system, the latter of these systems will be of the type S.

Obviously it makes no real difference if all the coefficient columns corresponding to the variables x_{n_r} are removed together with their respective variables. For since all these columns consist of zero's this removal will neither change the property of having or not having $\pi_1 = \pi_2$, nor the property of being or not being a system of the type S.

We shall use theorem A and B from § 3 on the system after the removal. Since $\pi_1 = \pi_2$ the modules $H_m^{(N)}$ of this system will for each m be constant from a certain step $N \geqq N_0 = N_0(m)$ and equal to the modul H_m. Since Γ_m is a module of the form $\{(x_1, x_2, \cdots, x_m)\}$ where the indices $1, 2, \cdots, m$ can be divided into two classes $\{n_s\}$ and $\{n_t\}$ such that the coordinates x_{n_s} are integral and the coordinates x_{n_t} are zero, Γ_m is in particular a closed modul so that $H_m = \Gamma_m = \{(x_1, x_2, \cdots, x_m)\} = \{(\text{integral, zero})\}$. Hence from the step N_0 also $H_m^{(N)} = \{(\text{integral, zero})\}$. Finally, using that $\Lambda_m^{(N)} \subseteqq H_m^{(N)}$ we find the following property of our new system: Each of the variables x_{n_t} becomes zero if one solve the N first zero-congruences for sufficiently large N (depending on the variable). Hence the system is of the type S. *This proves the main theorem.* Furthermore we see that each of the variables x_{n_s} becomes integral if one solve the N first zero-congruences for sufficiently large N (depending on the variable). The same of course is also true for the system before the removal of the variables x_{n_r} with mere zero coefficients. This proves the following

Stronger form of the main theorem. *A necessary (and sufficient) condition that a system of linear forms have $\pi_1 = \pi_2$ is that the linear forms by a substitution can be transfered into a system which is of the type S and moreover possesses the property*

that each of the variables belonging to the integral columns necessarily becomes integral if one solve the N first zero-congruences, corresponding to the linear forms, for sufficiently large N (depending on the variable).

Remark. A (necessary and) sufficient condition that a system of linear forms of the type S have the additional property mentioned in the theorem above is that the variables mentioned necessarily become integral if one solve the system of *all* the zero-congruences corresponding to the linear forms.

In fact, to prove this, we may use theorem A and B from § 3 in a similar way as above.

§ 6. A remark on the algebraic structure of a system of the special type *S*.

The notion of a system of linear forms of the type S was defined in § 1 as a system of linear forms where certain variables had mere integral coefficients while each of the other variables necessarily became 0 by solution of a suitable finite selection of the zero-congruences corresponding to the linear forms.

The question, therefore, naturally arises how a finite system of zero-congruences (in a finite number of variables) can force one of the variables to be zero. In this final paragraph we treat this problem by giving *a necessary and sufficient condition that a system of linear zero-congruences in* $x_1, \cdots, x_n$

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\equiv 0 \pmod 1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &\equiv 0 \pmod 1 \\ \cdots\cdots\cdots\cdots\cdots\cdots&\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &\equiv 0 \pmod 1 \end{aligned}$$

will involve $x_1 = 0$.

Let in the corresponding matrix

$$\left\{\begin{matrix} a_{11}a_{12}\cdots a_{1n} \\ a_{21}a_{22}\cdots a_{2n} \\ \cdots\cdots\cdots\cdots \\ a_{m1}a_{m2}\cdots a_{mn} \end{matrix}\right\}$$

the system of row vectors $R_1, R_2, \cdots, R_m$ have the maximal rank ϱ. Then we can find ϱ linearly independent vectors amongst these row vectors. Let it be, for instance, $R_1, R_2, \cdots, R_\varrho$. Then numbers α exist such that

$$\begin{aligned} R_{\varrho+1} &= \alpha_{11}R_1 + \alpha_{12}R_2 + \cdots + \alpha_{1\varrho}R_\varrho \\ R_{\varrho+2} &= \alpha_{21}R_1 + \alpha_{22}R_2 + \cdots + \alpha_{2\varrho}R_\varrho \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ R_m &= \alpha_{m-\varrho,1}R_1 + \alpha_{m-\varrho,2}R_2 + \cdots + \alpha_{m-\varrho,\varrho}R_\varrho. \end{aligned}$$

The column vectors in the abridged matrix

$$\begin{Bmatrix} a_{11} \cdots a_{1n} \\ \cdots\cdots\cdots \\ a_{\varrho 1} \cdots a_{\varrho n} \end{Bmatrix}$$

are denoted by $S_1, \cdots, S_n$. They have the maximal rank ϱ. The column vectors in the matrix

$$\begin{Bmatrix} \alpha_{11} \cdots\cdots\cdots \alpha_{1\varrho} \\ \cdots\cdots\cdots\cdots\cdots \\ \alpha_{m-\varrho,1} \cdots \alpha_{m-\varrho,\varrho} \end{Bmatrix}$$

are denoted by $\mathfrak{S}_1, \cdots, \mathfrak{S}_\varrho$.

Instead of the congruences we can equally well consider the equations

$$(14) \qquad \begin{aligned} a_{11}x_1 &+ \cdots + a_{1n}x_n = h_1 \\ a_{21}x_1 &+ \cdots + a_{2n}x_n = h_2 \\ &\cdots\cdots\cdots\cdots\cdots \\ a_{\varrho 1}x_1 &+ \cdots + a_{\varrho n}x_n = h_\varrho \\ a_{\varrho+1,1}x_1 &+ \cdots + a_{\varrho+1,n}x_n = h_{\varrho+1} \\ &\cdots\cdots\cdots\cdots\cdots \\ a_{m1}x_1 &+ \cdots + a_{mn}x_n = h_m \end{aligned}$$

where the h's are new integral variables. This system of equations can be solved for a given choice of $h_1, \cdots, h_\varrho$ if and only if

$$h_1\mathfrak{S}_1 + h_2\mathfrak{S}_2 + \cdots + h_\varrho\mathfrak{S}_\varrho \equiv 0 \pmod 1)^{1)};$$

1) Here, by $A \equiv 0 \pmod 1$ we mean that A is an integral vector.

for the ϱ first equations can always be solved and they involve the validity of the others if the condition above is satisfied, while otherwise at least one equation is not satisfied. In particular, the condition is satisfied if $h_1 = h_2 = \cdots = h_\varrho = 0$.

If the vectors $S_2, \cdots, S_n$ have the maximal rank ϱ we can choose x_1 arbitrarily by the solution of the ϱ first equations with $h_1 = \cdots = h_\varrho = 0$. If our congruences have no solutions with $x_1 \neq 0$ it follows that $S_2, \cdots, S_n$ must have the maximal rank $\varrho - 1$. Let this necessary condition be satisfied. The integral solutions $(h_1, h_2, \cdots, h_\varrho)$ of

$$h_1\mathfrak{S}_1 + h_2\mathfrak{S}_2 + \cdots + h_\varrho\mathfrak{S}_\varrho \equiv 0 \pmod 1$$

form a lattice. Then obviously a necessary and sufficient condition that every solution of the equations (14) has $x_1 = 0$ is that the lattice $\{(h_1, h_2, \cdots, h_\varrho)\}$ is contained in the space spanned by $S_2, \cdots, S_n$. Hence we have the result:

A necessary and sufficient condition that the congruences involve $x_1 = 0$ is that $S_2, \cdots, S_n$ have the maximal rank $\varrho - 1$ and that the lattice $\{(h_1, h_2, \cdots, h_\varrho)\}$ of integral solutions $(h_1, h_2, \cdots, h_\varrho)$ of

$$h_1\mathfrak{S}_1 + h_2\mathfrak{S}_2 + \cdots + h_\varrho\mathfrak{S}_\varrho \equiv 0 \pmod 1$$

is contained in the space spanned by $S_2, \cdots, S_n$.

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