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NON-CENTRAL COUPLING IN THE MIXED MESON THEORY OF NUCLEAR FORCES

BY

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The discovery by RABI and his collaborators [1] of the electric quadrupole moment of the deuteron has brought to light the existence in the law of interaction between a pair of nucleons of a term of non-central coupling, as a result of which the ground state of the deuteron is not a pure ^3S state, but contains also a D component. In fact, if the interaction is assumed to be independent of the charges (proton or neutron states) of the nucleons, it follows from general considerations of invariance of the interaction operator with respect to linear transformations of space [2] that for the two-body system the only type of non-central coupling to be taken into consideration involves a spin dependence of the "axial dipole" form*

$$D^{(12)} \equiv \left(\vec{\sigma}^{(1)} \vec{x}_0\right)\left(\vec{\sigma}^{(2)} \vec{x}_0\right) - \frac{1}{3}\left(\vec{\sigma}^{(1)} \vec{\sigma}^{(2)}\right) \tag{1}$$

* The following notations are used: To the i'th nucleon of the system considered correspond space coordinates $\vec{x}^{(i)}$ with conjugate momenta (multiplied by c) $\vec{p}^{(i)}$, spin coordinates $\vec{\sigma}^{(i)}$, isotopic coordinates $\boldsymbol{\tau}^{(i)}$, and coordinates discriminating the "large" and "small" components of the wave-functions $\varrho_l^{(i)}$ $(l = 1, 2, 3)$. The $\vec{\sigma}^{(i)}$, $\varrho_l^{(i)}$ are taken in the usual Dirac representation (except for the sign of the ϱ's). Relative space coordinates and momenta of the pair $(i\ k)$ will be defined as

$$\vec{x}^{(ik)} \equiv \vec{x}^{(i)} - \vec{x}^{(k)}, \qquad \vec{p}^{(ik)} \equiv \frac{1}{2}\left(\vec{p}^{(i)} - \vec{p}^{(k)}\right);$$

further

$$\left|\vec{x}^{(ik)}\right| = r^{(ik)}, \qquad \vec{x}_0^{(ik)} \equiv \vec{x}^{(ik)}/r^{(ik)};$$

if there is only one pair, the index $^{(12)}$ will be dropped. Summations over i, k must always be understood to exclude $i = k$.

The meson potential is represented by

$$\varphi(r) = e^{-\kappa r}/4\pi r;$$

the quantity κ, which is a measure for the inverse of the range of the potential, is connected with the meson mass M_m by the relation

$$\kappa = M_m\, c/\hbar.$$

The mass of the nucleon (neglecting the difference between proton and neutron mass) is denoted by M.

which just provides the desired mixing of ${}^3S_1$ with ${}^3D_1$ states. As it is well known, several forms of meson field theories of nuclear forces include such a term of non-central interaction even in the approximation in which only static effects are retained. However, this non-central interaction is introduced through the operator

$$S^{(12)} = -\frac{1}{\kappa^2}\left(\vec{\sigma}^{(1)}\operatorname{grad}^{(1)}\right)\left(\vec{\sigma}^{(2)}\operatorname{grad}^{(2)}\right)\varphi(r) \tag{2}$$

which can be decomposed into

$$\left.\begin{aligned}
S^{(12)} &= \frac{1}{3}V_\sigma^{(12)} + V_D^{(12)} - \frac{1}{3}C_{\bar{\sigma}}^{(12)} \\
V_\sigma^{(12)} &= \vec{\sigma}^{(1)}\vec{\sigma}^{(2)}\varphi(r) \\
V_D^{(12)} &= D^{(12)}F(r),\quad F(r) = \left(1+\frac{3}{\kappa r}+\frac{3}{\kappa^2 r^2}\right)\varphi(r) \\
C_{\bar{\sigma}}^{(12)} &= \frac{3}{4\pi\kappa^2}\lim_{\vec{\xi}\to 0}\int\delta\left(\vec{x}-\vec{\xi}\right)\left(\vec{\sigma}^{(1)}\vec{\xi}_0\right)\left(\vec{\sigma}^{(2)}\vec{\xi}_0\right)d\Omega,\quad \left(\vec{\xi}_0 = \vec{\xi}/|\vec{\xi}|\right),
\end{aligned}\right\} \tag{3}$$

i. e. it involves, besides a central spin-spin coupling $V_\sigma^{(12)}$ and an axial dipole coupling $V_D^{(12)}$, an additional term of "contact" interaction, the presence of which was first emphasized by BELINFANTE [3]. Even the distance dependence $F(r)$ of the axial dipole coupling, having a pole of the third order at the origin, would by itself cause the eigenvalue equation of the energy of the system to break down, and the same may be said of the still more singular contact interaction. A law of nuclear force of the form $S^{(12)}$ can therefore not lead, as it stands, to any well-defined result. It has been attempted [4] to avoid this inconvenience by arbitrarily "cutting off" the singularity, i. e. by replacing in $V_D^{(12)}$ the distance dependence $F(r)$ by some constant for distances smaller than a certain critical value r_0, and further ignoring the contact term $C_{\bar{\sigma}}^{(12)}$. But, apart from its arbitrariness, such a procedure would not seem [5] to be consistent, because the non-static interactions arising from the precession of the spin and isotopic variables under the influence of the static forces should, on this theory, be expected to become of an order of magnitude comparable to that of the static forces themselves. Another way out of the difficulty has therefore been put forward [5]. It consists in adopting a combination of two suitable types of meson fields, generated by the nucleons with such relative intensities that their respective contributions of the $S^{(12)}$ form to the static interaction just compensate each other. The mixing of states with different orbital momenta, revealed by the occurrence of the electric quadrupole moment of the deuteron, is then brought about by smaller coupling terms depending on the velocities of the nucleons.

The "mixed" meson theory contrasts with the "cut-off" theory also in another important respect, viz. as regards the assumed charges of the meson fields. As is well known, two possible choices of these charges are formally compatible with the charge-independence property of the nuclear forces: either are the meson fields neutral, or they enter into a definite, so-called symmetrical combination of both neutral and (positively and negatively) charged ones. Now, the cut-off theory yields the right sign and magnitude of the deuteron quadrupole moment only in its neutral form. This is certainly a very unsatisfactory feature of the cut-off theory, for it is hard to see by which physical arguments any neutral meson theory could be justified: since, namely, such a theory cannot have any point of contact with the evidence from cosmic ray mesons or β-radioactivity, the special distance dependence of the nuclear interaction to which it leads is nothing else than an arbitrarily assumed one, distinguished from any other only by its unhandiness. Worse still, a law of interaction independent of the isotopic variables of the nucleons, such as implied by a neutral meson theory, cannot be reconciled with the saturation properties of this interaction as exhibited in heavy nuclei. Indeed, if one assumes that the non-central forces give a large contribution to the total interaction (as is the case in the cut-off theory), it is doubtful [6] whether any saturation can at all be obtained. And, under the alternative assumption that the main part of the interaction can be represented by a central, charge-independent potential, one finds that the saturation requirements fix the dependence of this potential on the isotopic variables to a factor of the form $A + B\boldsymbol{\tau}^{(1)}\boldsymbol{\tau}^{(2)}$, with $A \ll B$: to a sufficient approximation, therefore, these requirements are fulfilled just by a symmetrical theory, characterized by a dependence on the $\boldsymbol{\tau}$'s of the type

$$T^{(12)} \equiv \mathbf{\tau}^{(1)}\mathbf{\tau}^{(2)}. \tag{4}$$

For these reasons, the symmetrical form of the mixed theory has been adopted and only this form, which involves a combination of pseudoscalar and vector meson fields, has so far been discussed, although a neutral type of mixed theory can formally be just as well set up by means of the dual combination of fields (scalar and pseudovector).

It could be shown, then, that the above-mentioned velocity-dependent coupling of the symmetrical mixed theory was able to yield the right sign and order of magnitude of the deuteron quadrupole moment, provided that the parameters expressing the intensities of the meson field sources were appropriately connected by simple relations [5, 7]. There subsisted, however, a flaw in this derivation, inasmuch as the same coupling apparently gave rise to a very large perturbation of the energy of the ground state. It was even contended by Ferretti [8] that we would actually here have to do with an essential divergence of the theory. The main purpose of the present note is to show how a closer analysis of the velocity-dependent coupling leads to a simple solution of the difficulty just mentioned, at the same time clearing up the physical meaning, hitherto rather obscure, of the coupling in question: in the two-nucleon case, this coupling effectively reduces to just an axial dipole interaction, in agreement with expectation on general invariance grounds.

The symmetrical mixed theory is, however, confronted with a much more serious difficulty in connexion with the recent Italian experiments [9] on the angular distribution of fast neutrons (of about 12–14 MeV energy) scattered by protons. The considerable asymmetry found in this angular distribution is namely in complete disagreement, as regards sign as well as absolute value, with the predictions of the symmetrical theory [10]. On the other hand, it has been pointed out that the asymmetry would agree in sign—and roughly, in order of magnitude—with the value deduced from the neutral cut-off theory [10]. In view of the serious objections against the latter theory, enumerated above, it would be rash to attribute much weight to this coincidence; it rather seems that, if the results of the Italian experiments are confirmed on further investigation, a much more radical departure from current theories of nuclear interaction would be required than a mere adoption of some form of neutral meson theory. Nevertheless, it might be of some interest, for the sake of information, to ascertain the kind of velocity-dependent coupling to which the mixed neutral meson theory gives rise: this discussion, carried out in the last section of the present paper, leads to the rather surprising result that in the two-nucleon case there is no velocity-dependent coupling between states of different orbital momenta. This form of meson theory is thus utterly unable to account for the deuteron quadrupole moment.

§ 1. Nuclear Forces on Symmetrical Mixed Theory.

The symmetrical mixed theory starts from the following expressions of the source densities of the meson fields in terms of the nucleon variables:

Vector field,

vector density
$$\begin{cases} \vec{\boldsymbol{M}} = g_1 \vec{\boldsymbol{m}}, & \vec{\boldsymbol{m}} = \sum_i \mathbf{T}^{(i)} \varrho_1^{(i)} \vec{\sigma}^{(i)} \delta\left(\vec{x} - \vec{x}^{(i)}\right) \\ \boldsymbol{N} = g_1 \boldsymbol{n}, & \boldsymbol{n} = \sum_i \mathbf{T}^{(i)} \delta\left(\vec{x} - \vec{x}^{(i)}\right) \end{cases}$$

tensor density*
$$\begin{cases} \vec{\boldsymbol{S}} \approx \frac{g_2}{\kappa} \vec{\boldsymbol{s}}, & \vec{\boldsymbol{s}} = \sum_i \mathbf{T}^{(i)} \vec{\sigma}^{(i)} \delta\left(\vec{x} - \vec{x}^{(i)}\right) \\ \vec{\boldsymbol{T}} = \frac{g_2}{\kappa} \vec{\boldsymbol{t}}, & \vec{\boldsymbol{t}} = -\sum_i \mathbf{T}^{(i)} \varrho_2^{(i)} \vec{\sigma}^{(i)} \delta\left(\vec{x} - \vec{x}^{(i)}\right). \end{cases}$$

Pseudoscalar field,

pseudoscalar density
$$\boldsymbol{R} = f_1 \boldsymbol{r}, \quad \boldsymbol{r} = \sum_i \mathbf{T}^{(i)} \varrho_2^{(i)} \delta\left(\vec{x} - \vec{x}^{(i)}\right)$$

pseudovector density
$$\begin{cases} \vec{\boldsymbol{P}} = \frac{f_2}{\kappa} \vec{\boldsymbol{s}}, & \\ \boldsymbol{Q} = \frac{f_2}{\kappa} \boldsymbol{q}, & \boldsymbol{q} = \sum_i \mathbf{T}^{(i)} \varrho_1^{(i)} \delta\left(\vec{x} - \vec{x}^{(i)}\right). \end{cases}$$

* The formula for $\vec{\boldsymbol{S}}$ is correct only to the first order in the nucleon velocities, a factor $\varrho_3^{(i)}$ having been replaced by 1 in $\vec{\boldsymbol{s}}$.

In terms of these density functions and intensity parameters g's and f's, the static interaction potential may be written in the form

$$\left.\begin{aligned} V &= g_1^2 V_{g_1} + g_2^2 V_{g_2} + f_2^2 V_{f_2} \\ V_{g_1} &= \frac{1}{2}\int \boldsymbol{n}(\vec{x})\,\boldsymbol{n}(\vec{x}')\,\varphi\left(\left|\vec{x}-\vec{x}'\right|\right) dv dv' \\ V_{g_2} &= -\frac{1}{2\kappa^2}\int \left[\vec{\boldsymbol{s}}(\vec{x}') \wedge \operatorname{grad}'\varphi\right] \operatorname{rot}\vec{\boldsymbol{s}}(\vec{x})\, dv dv' + \frac{1}{2\kappa^2}\int \vec{\boldsymbol{s}}^{\,2} dv \\ V_{f_2} &= \frac{1}{2\kappa^2}\int \left[\vec{\boldsymbol{s}}(\vec{x}') \cdot \operatorname{grad}'\varphi\right] \operatorname{div}\vec{\boldsymbol{s}}(\vec{x})\, dv dv'. \end{aligned}\right\} \quad (5)$$

After insertion of the above expressions of the source densities, the V's defined in (5) are easily reduced to

$$\left.\begin{aligned} V_{g_1} &= \frac{1}{2}\sum_{i,k} T^{(ik)}\,\varphi\left(r^{(ik)}\right) \\ V_{g_2} &= \frac{1}{2}\sum_{i,k} T^{(ik)}\left[V_\sigma^{(ik)} - S^{(ik)}\right] \\ V_{f_2} &= \frac{1}{2}\sum_{i,k} T^{(ik)} S^{(ik)}; \end{aligned}\right\} \quad (6)$$

in these formulae, use has been made of the notations (2), (3), and (4). By comparing the last expressions of V_{g_2} and V_{f_2}, it becomes obvious that the non-central interactions $S^{(ik)}$ can be suppressed by simply assuming

$$g_2^2 = f_2^2. \quad (7)$$

We are then left with a purely central potential; in particular, the spin-spin coupling $V_\sigma^{(12)}$ is entirely responsible for the separation of the ^3S and ^1S states of the deuteron.

Besides the static potential (5), the theory yields, to the first order of approximation in the nucleon velocities, a non-static coupling

$$\begin{aligned} W = \frac{g_1 g_2}{\kappa}\int \left\{\boldsymbol{n}(\vec{x}')\,\vec{\boldsymbol{t}}(\vec{x})\operatorname{grad}\varphi + \left[\vec{\boldsymbol{s}}(\vec{x}') \wedge \operatorname{grad}'\varphi\right] \vec{\boldsymbol{m}}(\vec{x})\right\} dv dv' \\ + \frac{f_1 f_2}{\kappa}\int \vec{\boldsymbol{s}}(\vec{x}')\,\boldsymbol{r}(\vec{x}) \operatorname{grad}\varphi\, dv dv'. \end{aligned} \quad (8)$$

We shall now analyze this formula by replacing the velocity-dependent factors $\vec{\boldsymbol{t}}, \vec{\boldsymbol{m}}, \vec{\boldsymbol{r}}$ by their approximate expressions resulting from a transformation analogous to the well-known Gordon decomposition of the current density in the Dirac electron theory. This transformation eliminates the variables $\varrho_1^{(i)}, \varrho_2^{(i)}$ and brings out explicitly the dependence on the momenta. It yields*

$$\left.\begin{aligned}
\vec{\boldsymbol{m}} &= \sum_i \overline{\boldsymbol{\tau}^{(i)} \frac{\vec{p}^{(i)}}{Mc^2}\,\delta\left(\vec{x}-\vec{x}^{(i)}\right)} + \frac{\hbar}{2\,Mc}\operatorname{rot}\vec{\boldsymbol{s}} - \frac{\hbar}{2\,Mc}\frac{d\vec{\boldsymbol{t}}}{cdt} \\
\vec{\boldsymbol{t}} &= \sum_i \overline{\boldsymbol{\tau}^{(i)}\vec{\sigma}^{(i)} \wedge \frac{\vec{p}^{(i)}}{Mc^2}\,\delta\left(\vec{x}-\vec{x}^{(i)}\right)} + \frac{\hbar}{2\,Mc}\operatorname{grad}\boldsymbol{n} + \frac{\hbar}{2\,Mc}\frac{d\vec{\boldsymbol{m}}}{cdt} \\
\boldsymbol{r} &= -\frac{\hbar}{2\,Mc}\operatorname{div}\vec{\boldsymbol{s}} - \frac{\hbar}{2\,Mc}\frac{d\boldsymbol{q}}{cdt} \\
\boldsymbol{q} &= \sum_i \overline{\boldsymbol{\tau}^{(i)}\vec{\sigma}^{(i)}\frac{\vec{p}^{(i)}}{Mc}\,\delta\left(\vec{x}-\vec{x}^{(i)}\right)} + \frac{\hbar}{2\,Mc}\frac{d\boldsymbol{r}}{cdt};
\end{aligned}\right\} \quad (9)$$

the terms containing time derivatives are of higher order in the velocities and must here consistently be neglected. The terms in $\vec{\boldsymbol{m}}$ and $\vec{\boldsymbol{t}}$ explicitly depending on the momenta give rise, when inserted in (8), to an interaction

$$\left.\begin{aligned}
W_{\text{sp.-orb.}} &= -\frac{2\,g_1 g_2}{\kappa}\cdot\frac{1}{Mc^2}\cdot\frac{1}{2}\sum_{i,k} T^{(ik)}\,\frac{\varphi'\left(r^{(ik)}\right)}{r^{(ik)}}\cdot \\
&\qquad\qquad \left(\vec{\sigma}^{(i)}+\vec{\sigma}^{(k)}\right)\left(\vec{x}^{(ik)}\wedge\vec{p}^{(ik)}\right) \\
&= \frac{2\,g_1 g_2}{\hbar c}\frac{M_m}{M}\cdot\frac{1}{2}\sum_{i,k} T^{(ik)}\left(\frac{1}{\kappa r^{(ik)}}+\frac{1}{\kappa^2 r^{(ik)2}}\right)\varphi\left(r^{(ik)}\right) \\
&\qquad\qquad \left(\vec{\sigma}^{(i)}+\vec{\sigma}^{(k)}\right)\left(\vec{x}^{(ik)}\wedge\vec{p}^{(ik)}\right)
\end{aligned}\right\} \quad (10)$$

of the usual "spin-orbit" type, involving, for each pair of nucleons, the scalar product of total spin and relative orbital momentum. The remaining terms of (9), though not explicitly containing the momenta, are nevertheless of the first order in

* In the derivation of these formulae, a convenient algebraic method, indicated in *l. c.* [11], § 5, has been followed. The bars mean that the underlying expressions have to be symmetrized. The time derivatives refer, strictly speaking, to a system of free nucleons.

the velocities on account of the factor $\hbar/2Mc$; they yield contributions to W of the same structure as the static potentials, apart from certain contact interactions of the form

$$C_0 = \frac{1}{2\kappa^2}\int \boldsymbol{n}^2\, dv = \frac{1}{2\kappa^2}\sum_{i,k} T^{(ik)}\,\delta\left(\vec{x}^{(ik)}\right) \tag{11}$$

and

$$\left.\begin{aligned} C_\sigma &= \frac{1}{2\kappa^2}\int \vec{\boldsymbol{s}}^2\, dv = \frac{1}{2}\sum_{i,k} T^{(ik)} C_\sigma^{(ik)} \\ C_\sigma^{(ik)} &= \frac{1}{\kappa^2}\vec{\sigma}^{(i)}\vec{\sigma}^{(k)}\,\delta\left(\vec{x}^{(ik)}\right). \end{aligned}\right\} \tag{12}$$

Indeed ,a comparison with (5) shows, after a partial integration and application of the equation $\Delta\varphi - \kappa^2\varphi = -\delta\left(\vec{x}-\vec{x}'\right)$, that these contributions reduce to

$$-g_1 g_2 \frac{M_m}{M}\left[V_{g_1} - C_0 + V_{g_2} - C_\sigma\right] + f_1 f_2 \frac{M_m}{M} V_{f_2}. \tag{13}$$

Summing up, and transforming (13) by means of (6) and (3), we finally get

$$\left.\begin{aligned} W = & -g_1 g_2 \frac{M_m}{M}\left(V_{g_1} - C_0\right) \\ & - \frac{2\,g_1 g_2 - f_1 f_2}{3}\frac{M_m}{M}\frac{1}{2}\sum_{i,k} T^{(ik)} V_\sigma^{(ik)} \\ & + \frac{M_m}{M}\frac{1}{2}\sum_{i,k} T^{(ik)}\left\{g_1 g_2\, C_\sigma^{(ik)} - \frac{g_1 g_2 + f_1 f_2}{3} C_\sigma^{(ik)}\right\} \\ & + (g_1 g_2 + f_1 f_2)\frac{M_m}{M}\frac{1}{2}\sum_{i,k} T^{(ik)} V_D^{(ik)} + W_{\text{sp.-orb.}}; \end{aligned}\right\} \tag{14}$$

the three first lines represent central and contact interactions, while the last one contains non-central couplings of axial dipole and spin-orbit types.

§ 2. The Deuteron Problem on Symmetrical Mixed Theory.

In the case of the two-nucleon system, we can think the wave-functions expanded in series of eigenfunctions of the energy corresponding to static central forces which are characterized by definite values of the orbital angular momentum. All matrixelements of the contact interaction operators between such stationary states vanish on account of the radial dependence of the eigenfunctions, except for the diagonal elements pertaining to S states. For the calculation of these matrix elements, the operator $C_{\bar{\sigma}}^{(12)}$ is clearly equivalent to $C_{\sigma}^{(12)}$, so that the non-static coupling (14) takes the somewhat simpler form

$$
\left.
\begin{aligned}
W = \frac{M_m}{M} T^{(12)} \cdot \Big\{ & -g_1 g_2 \varphi(r) - \frac{2\, g_1 g_2 - f_1 f_2}{3} V_{\sigma}^{(12)} \\
& + g_1 g_2 \frac{\delta(\vec{x})}{\kappa^2} + \frac{2\, g_1 g_2 - f_1 f_2}{3} C_{\sigma}^{(12)} \\
& + (g_1 g_2 + f_1 f_2)\, V_D^{(12)} \Big\} + W_{\text{sp.-orb.}}
\end{aligned}
\right\} \tag{15}
$$

In this operator, the only term bringing about a mixing of states with different orbital momenta is the last but one, the axial dipole coupling. In particular, the spin-orbit coupling in this case is diagonal with respect to orbital momentum.

By setting up the four-component wave-equations of the deuteron, including the operator W given by (8), Ferretti [8] came to the conclusion that there was no regular solution for S states. The same conclusion can be reached without this complicated procedure by considering the transformed operator (15): this presents in fact the same kind of singularities as the static operator $S^{(12)}$. However, the non-static operator is smaller than the static one by a factor of about M_m/M, representing the order of magnitude of the ratio of the velocities of the nucleons to the velocity of light: it is therefore in keeping with the general rules for the interpretation of the formalism of mixed theory [5] to treat it as a perturbation and to retain as significant only the effects which can be derived in an unambiguous way by application of the perturbation method. From this point of view, the contact terms, as already stated, do not offer any difficulty of convergence; and the same is true for the terms in r^{-3} contained in the axial dipole as well as the spin-orbit couplings, the situation being here entirely analogous to that in the atomic theory of fine and hyperfine structures. The apparent divergence of the expectation value of r^{-3} in S states is namely due solely to the neglect of a relativistic correction factor $\left(1+\frac{\varepsilon-V}{Mc^2}\right)^{-2}$ (ε being the *un*relativistic eigenvalue of the energy): the potential energy V becoming infinite at the origin, this factor is indeed sufficient to secure the convergence of the expectation value in question*. When account is taken of this circumstance, it is immediately clear that both the spin-orbit coupling and the diagonal elements of $V_D^{(12)}$ for S states vanish.

At this point, however, arises the difficulty mentioned in the introduction: if we determine the energy of the ground state, in the initial approximation, by the static central forces alone, and compute the correction due to the non-static coupling by the usual perturbation method, we find indeed a very large contribution** from the central and contact terms of W. Furthermore, as noted by FERRETTI [8], the corresponding perturbation of the eigenfunction is even represented by a diverging series. On closer examination, it is found that the interactions mainly responsible for this situation are the contact ones. But it now becomes clear how to remedy these defects. In the first place, contact interactions can always be eliminated by addition of suitably chosen invariants to the Hamiltonian of the system. So the term

$$\bar{g}_1 g_2 \frac{M_m}{M} C_0 = \frac{g_1 g_2}{2\,\kappa^2}\frac{M_m}{M}\int \boldsymbol{n}^2\, dv$$

disappears if the invariant

$$\frac{g_1 g_2}{2\,\kappa^2}\frac{M_m}{M}\int \left(\vec{\boldsymbol{m}}^2 - \boldsymbol{n}^2\right) dv$$

is added: it is then replaced by an interaction of the same type, but of higher order in the nucleon velocities, which must therefore be discarded according to the general rules for the interpretation of the formalism of mixed theory [5]. Similarly, the

* Cf. H. BETHE, Hdb. d. Phys. XXIV/1, p. 307, 385–386; W. PAULI, *ibid.* p. 237, equ. (89).

** I wish here to emphasize that a substantially correct statement of the reason for this abnormally large energy perturbation is already contained in SERPE's thesis [12], in which the decomposition (9) of the field sources was applied to the general interaction between nucleons and free mesons. However, SERPE did not follow up the matter.

term in C_σ can be reduced to higher order in the velocities by addition of the invariant $\int(\vec{\boldsymbol{s}}^2 - \vec{\boldsymbol{t}}^2)\,dv$ with a suitable coefficient. As regards the central interactions, on the other hand, they can simply be reckoned with those which determine the initial approximation; the central potential in that approximation thus becomes, on taking account of (7),

$$V_0 = \left(g_1^2 - \frac{M_m}{M} g_1 g_2\right) V_{g_1} + \left(g_2^2 - \frac{M_m}{M} \cdot \frac{2\, g_1 g_2 - f_1 f_2}{3}\right) \frac{1}{2} \sum_{i,k} T^{(ik)} V_\sigma^{(ik)}. \quad (16)$$

The remaining non-static interaction is then

$$W_{\text{eff}} = \frac{M_m}{M} (g_1 g_2 + f_1 f_2) \frac{1}{2} \sum_{i,k} T^{(ik)} V_D^{(ik)} + W_{\text{sp.-orb.}} \quad (17)$$

In the deuteron case, this interaction gives rise to a perturbation of the energy of the ground state which is of the second order only and may be neglected. There is, however, a first order perturbation of the eigenfunction of this state, resulting in an admixture of $^3D_1$-states and a corresponding quadrupole moment.

Although working with a non-central coupling of exactly the same type, viz. axial dipole coupling, the cut-off theory presents a quite different picture: the non-central coupling appears with a coefficient as large as that of the central interaction and can therefore not be treated as a perturbation. It gives rise to a large contribution to the energy of the ground state, sensitively depending on the cut-off radius r_0. All the same, the admixture of D state to the eigenfunction of the ground state is small (as required to explain the small quadrupole moment), the cut-off so to say compensating here the largeness of the coefficient of the axial dipole term. The mixed theory is unquestionably much simpler, and also more natural inasmuch as it deduces a small effect by means of a perturbation calculation and not by a device involving a considerable departure from the simple representation of nuclear forces by a central potential.

The intensity constants g_2, g_2 are primarily determined, in the mixed theory, by the binding energy of the ground state of the deuteron and the energy of the virtual ^1S state, which together fix the numerical coefficients of the central potential V_0; the inclusion in (16) of the non-static central interactions implies only relatively small modifications of the resulting values of these costants. In particular, even when account is taken of the relation (7), which further fixed $|f_2|$, the sign and magnitude of the calculated quadrupole moment, which is proportional to $g_1 g_2 + f_1 f_2$, can still be adjusted to fit observation. As a matter of fact, the experimental value of the quadrupole moment can be used together with the other evidence just mentioned to determine the four parameters g and f. One has the system of equations

$$\left.\begin{aligned} & g_1^2 - \frac{M_m}{M} g_1 g_2 = \alpha \\ & g_2^2 - \frac{1}{3}\frac{M_m}{M}(2\,g_1 g_2 - f_1 f_2) = \beta \\ & g_1 g_2 + f_1 f_2 = \gamma \\ & g_2^2 = f_2^2, \end{aligned}\right\} \tag{18}$$

α, β, γ, being given positive numerical quantities (of the dimensions of energy times length) of comparable orders of magnitude. It is easy to see that this system yields two essentially distinct solutions*, according to the sign assumed for the product $g_1 g_2$.

The mixed theory can in this respect admittedly be blamed for a wealth of adjustable parameters affording it an unfair amount of self-protection. A not unattractive possibility of restricting the number of independent source constants is suggested by the preceding discussion. If one would assume that

$$2\,g_1 g_2 - f_1 f_2 = 0, \tag{19}$$

the contact interaction C_σ would be completely eliminated as well as the additional spin-spin interaction, so that the value of g_2^2 would directly be fixed by the coefficient β of the zero-approximation central potential. Then, the absolute value, though not the sign, of the electric quadrupole moment would be predicted by the theory. It remains to be seen, however, whether this predicted value would not come out a little too small to fit observation: an estimate by Hulthén [7] would namely rather seem to suggest the relation $g_1 g_2 - f_1 f_2 = 0$.

* There subsists, of course, an arbitrariness in the signs corresponding to the invariance of the equations (18) for simultaneous change of sign of g_1 and g_2, or f_1 and f_2, or both pairs. But this arbitrariness is of no physical significance.

§ 3. Velocity-dependent Coupling on Neutral Mixed Theory.*

On the neutral mixed theory, the source densities of the meson fields are as follows:

Scalar field,

scalar density** $\approx f_1' n$

vector density $\frac{1}{\kappa} f_2' \vec{m}, \quad \frac{1}{\kappa} f_2' n;$

Pseudovector field,

pseudovector density $g_1' \vec{s}, \quad g_1' q$

pseudotensor density** $\frac{1}{\kappa} g_2' \vec{t}, \approx \frac{1}{\kappa} g_2' \vec{s};$

the functions $n, \vec{m}, \vec{s}, \vec{t}, q$ are the same as those denoted by the same letters in the symmetrical theory, except that the factors $\boldsymbol{\tau}^{(i)}$ are to be cancelled. The resulting static interaction may be written, in our notations,

$$V = -\frac{1}{2} \sum_{i,k} \left\{ f_1'^2 \varphi\left(r^{(ik)}\right) + g_1'^2 V_\sigma^{(ik)} - \left(g_1'^2 - g_2'^2\right) S^{(ik)} \right\}, \tag{20}$$

the last term disappearing if one puts

$$g_1'^2 = g_2'^2. \tag{21}$$

The non-static coupling in this case takes the form

$$\left. \begin{aligned} W = & -\frac{f_1' f_2'}{\kappa} \int n\left(\vec{x}'\right) \vec{m}\left(\vec{x}\right) \operatorname{grad} \varphi \, dv dv' \\ & + \frac{g_1' g_2'}{\kappa} \int \left\{ q\left(\vec{x}'\right) \vec{s}\left(\vec{x}'\right) \operatorname{grad} \varphi + \vec{t}\left(\vec{x}'\right) \left[\vec{s}\left(\vec{x}\right) \wedge \operatorname{grad} \varphi\right] \right\} dv dv'; \end{aligned} \right\} \tag{22}$$

After insertion of the expressions (9) (without the $\boldsymbol{\tau}$'s) for $\vec{m}$, $\vec{t}$ and q, it is immediately apparent that the terms not explicitly involving the momenta yield vanishing contributions. The scalar part thus reduces to

$$W_{\text{scal}} = -\frac{2 f_1' f_2'}{\kappa^2} \frac{M_m}{M} \frac{1}{2} \overline{\sum_{i,k} \frac{\vec{p}^{(i)}}{\hbar c} \operatorname{grad}^{(i)} \varphi\left(r^{(ik)}\right)}, \tag{23}$$

* *Note added in proof.* After completion of the present paper i have been informed that the results of this Section have been obtained indepedently by Bengt Holmberg in a paper published in Kungl. Fysiografiska Sällskapets Förhandlingar, Lund, 14, No. 22, 1944.

** The factors $\varrho_3^{(i)}$ have here been replaced by 1.

while the pseudovector part may successively be written:

$$
\left.\begin{aligned}
W_{\text{ps. vect.}} &= \frac{2\,g_1' g_2'}{\kappa^2}\frac{M_m}{M}\frac{1}{2}\sum_{i,k}\Bigg\{\overline{\left(\vec{\sigma}^{(i)}\frac{\vec{p}^{(i)}}{\hbar c}\right)\left(\vec{\sigma}^{(k)}\operatorname{grad}^{(k)}\varphi\left(r^{(ik)}\right)\right)} \\
&\quad + \overline{\left(\vec{\sigma}^{(i)}\wedge\frac{\vec{p}^{(i)}}{\hbar c}\right)\left(\vec{\sigma}^{(k)}\wedge\operatorname{grad}^{(k)}\varphi\left(r^{(ik)}\right)\right)}\Bigg\} \\
&= \frac{2\,g_1' g_2'}{\kappa^2}\frac{M_m}{M}\frac{1}{2}\sum_{i,k}\Bigg\{\left(\vec{\sigma}^{(i)}\wedge\vec{\sigma}^{(k)}\right)\overline{\left(\frac{\vec{p}^{(i)}}{\hbar c}\wedge\operatorname{grad}^{(k)}\varphi\right)} \\
&\quad - \left(\vec{\sigma}^{(i)}\vec{\sigma}^{(k)}\right)\overline{\frac{\vec{p}^{(i)}}{\hbar c}\operatorname{grad}^{(i)}\varphi}\Bigg\}.
\end{aligned}\right\} \tag{24}
$$

These expressions are not at all of a form which would be expected on general invariance considerations for first order velocity-dependent interactions, but, from this point of view, rather belong to the second order types [as may be seen by replacing grad φ by $\frac{i}{\hbar c}\left(\vec{p}\varphi - \varphi\vec{p}\right)$].

It is clear that a coupling of the form (23) will not give rise to any mixing of states with different orbital momenta. Indeed, the operator* W_{scal} commutes with that of the total orbital momentum $\sum_i \vec{x}^{(i)}\wedge\vec{p}^{(i)}$. The same applies to the last term of (24), while the first term, which has the general form

$$
\frac{2\,g_1' g_2'}{\kappa^2}\frac{M_m}{M}\frac{1}{2}\sum_{i,k}\frac{\varphi'\left(r^{(ik)}\right)}{r^{(ik)}}\left(\vec{\sigma}^{(i)}\wedge\vec{\sigma}^{(k)}\right)\left[\vec{x}^{(ik)}\wedge\frac{1}{2}\left(\vec{p}^{(i)}+\vec{p}^{(k)}\right)\right],
$$

vanishes for any two-nucleon system whose center of gravity is at rest. There is thus in the neutral theory no alternative to the cut-off procedure to explain the quadrupole moment of the deuteron.

* In the two-nucleon case, the operator W_{scal} may be written

$$
W_{\text{scal}} = -\frac{2\,f_1' f_2'}{\kappa}\frac{M_m}{M}\overline{\frac{\vec{p}}{\hbar c}\vec{x}\frac{\varphi'(r)}{r}};
$$

a more familiar form is obtained by going over to a representation in terms of quantized amplitudes ψ, $\psi^\dagger$: the energy density then becomes

$$
\psi^\dagger W_{\text{scal}}\psi = \frac{2\,f_1' f_2'}{\kappa^2}\frac{M_m}{M}\left\{\frac{1}{2i}\operatorname{div}\left[\psi^\dagger\operatorname{grad}\varphi\cdot\psi\right] - \frac{1}{2i}\left(\operatorname{grad}\psi^\dagger\cdot\psi - \psi^\dagger\operatorname{grad}\psi\right)\operatorname{grad}\varphi\right\}.
$$

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