DET KGL. DANSKE VIDENSKABERNES SELSKAB
MATEMATISK-FYSISKE MEDDELELSER, BIND XXIII, NR. 1

DEDICATED TO PROFESSOR NIELS BOHR ON THE OCCASION OF HIS 60TH BIRTHDAY

GENERAL PROPERTIES OF THE CHARACTERISTIC MATRIX IN THE THEORY OF ELEMENTARY PARTICLES I

BY

C. MØLLER

KØBENHAVN
I KOMMISSION HOS EJNAR MUNKSGAARD
1945

CONTENTS

Printed in Denmark
Bianco Lunos Bogtrykkeri A/S

INTRODUCTION

According to HEISENBERG[1], the well-known convergence difficulties, inherent in all quantum field theories so far developed, are due to the existence of a new universal constant of the dimension of a length. This constant plays the role of a minimal length representing a limit to the application of the ordinary concepts of quantum field theory in a similar way as the existence of Planck's constant limits the unambiguous application of classical mechanical concepts to atomic systems. The correct incorporation of this universal length into the theory is still unknown.

Recently, however, HEISENBERG[2] has taken an important step towards the future theory. In ordinary quantum mechanics an atomic system is completely defined by the Hamiltonian function of the system. Now, the assumption of a Hamiltonian which by means of the Schrödinger differential equation defines a continuous time-displacement of the wave function seems to be in contradiction with the existence of a universal minimal length. HEISENBERG therefore concludes that the Hamiltonian function will lose its predominant importance in the future theory, and that the atomic systems in this theory must be defined by other fundamental functions.

A primary problem will be to determine these functions. This problem is intimately connected with the question which quantities of the current theory will keep their meaning or, in other words, which quantities will still be regarded as 'observable' in the future theory. Although it is difficult to give an exhaustive answer to this question at present, it is natural to assume that any quantity, whose determination is unaffected by the existence of the minimal length, may be considered as 'observable.'

Such quantities are the energy and momentum of a free particle, the cross section of any collision process with or without creation and annihilation of particles, and the discrete energy values of atomic systems in closed stationary states. In quantum mechanics these quantities may be calculated by means of the Schrödinger equation when the Hamiltonian of the system is known. The collision cross sections, however, are given more directly by the matrix elements of a certain unitary matrix S, which in a rather complicated way depends on the Hamiltonian. Therefore HEISENBERG assumes that in the future theory this characteristic matrix S or an Hermitian matrix η connected with S by the relation

$$S = e^{i\eta}$$

will take over the role played by the Hamiltonian in quantum theory, i. e. in future the atomic systems should be defined by giving the matrices S or η.

In quantum mechanics the Hamiltonian of a special system could be obtained by a simple procedure from the classical Hamiltonian of the system. The most urgent, and until now entirely unsolved, problem will now be that of finding the procedure by which the characteristic matrix may be derived in each special case. As a first step towards a solution of this problem we may try to find the general conditions satisfied by the matrix S in any case, and it seems natural to assume that all conditions satisfied by S in quantum mechanics independently of the special form of the Hamiltonian, will hold also in the future theory.

A condition of this kind, stated by HEISENBERG[2)] in his first paper, is the equation

$$S^\dagger S = S S^\dagger = 1,$$

expressing that S is a unitary matrix. In section 1 of the present paper, we shall give an alternative, and perhaps a somewhat more rigorous, proof of this equation.

Another important condition stated by HEISENBERG is the invariance of the matrix S under Lorentz transformations. In section 2 we shall prove this property by using the transformation properties of cross sections and the connection between these quantities and the elements of the matrix S. The invariance of the fundamental matrix S brings about a considerable simplification in the description of atomic systems, as compared with the quantum mechanical description in which the fundamental matrix—the Hamiltonian—has rather complicated transformation properties. In section 3 it is shown how the invariance property of S may be used to find a number of general 'constants of collision', i. e. variables which commute with S and with the total kinetic energy. Such quantities as have the same values or mean values before and after the collision will probably in the future theory play a similar important role as the constants of motion in ordinary quantum mechanics.

When the matrix elements of S are given as functions of the (real) momentum variables of the system, we are thus able to calculate the cross sections for all collision processes, but, as shown in section 1, the discrete energy values in closed stationary states of the system are so far completely undetermined. However, if S is assumed to be an analytic function of the momentum variables now regarded as complex variables, these energy values may, as remarked by KRAMERS and HEISENBERG[8],* be obtained as the energies corresponding to those purely imaginary values of the momentum variables which make S equal to zero. In a sequel to the present paper these problems will be considered in more detail and new general conditions for the matrix elements of S will be derived.

Thus, all the quantities which according to HEISENBERG are to be considered 'observable,' are in this way derivable from the matrix S, which therefore in fact plays a similar role as the Hamiltonian in the old theory. It remains to be seen if the atomic systems are completely defined by a given S, or if the future theory will contain observable quantities which cannot be calculated from the matrix S.

* I am greatly indebted to Professor HEISENBERG for an opportunity of seeing his manuscript before publication.

1. The characteristic matrix S. The unitarity condition. Independence of S of the energy values in closed stationary states.

In this section we shall consider a collision between a certain number of like particles from the point of view of ordinary quantum mechanics. Let $\boldsymbol{k}_i$ be the momentum operator of the i'th particle. If $\varkappa$ is the rest mass of the particle, the corresponding kinetic energy is $W_i = \sqrt{\varkappa^2 + k_i^2}$, and the total kinetic energy and the total momentum of the particles are given by

$$\left.\begin{aligned} W &= \sum_i W_i = \sum_i \sqrt{\varkappa^2 + k_i^2} \\ \boldsymbol{K} &= \sum_i \boldsymbol{k}_i , \end{aligned}\right\} \qquad (1)$$

respectively.*

If the eigenvalues of $\boldsymbol{k}_i$ are denoted by $\boldsymbol{k}_i'$, the wave function in momentum space will be a function of the vectors $\boldsymbol{k}_i'$. Since we want to treat the general case of a collision in which annihilation and creation processes may take place, the total number n of particles is not a constant of the motion, and we shall have to consider a succession of wave functions[5)]

$$\text{const.},\ \psi\left(\boldsymbol{k}_1'\right),\ \psi\left(\boldsymbol{k}_1', \boldsymbol{k}_2'\right), \cdots \psi\left(\boldsymbol{k}_1', \cdots \boldsymbol{k}_i', \cdots \boldsymbol{k}_{n'}'\right)$$

corresponding to the different eigenvalues n' of n. For simplicity we shall in what follows treat the different particles as distinguishable, which means neglecting all exchange effects. However, if the particles for instance have Bose statistics, all the wave functions are of course symmetrical in the variables $\boldsymbol{k}_i'$.

In a representation where the $\boldsymbol{k}_i$ are diagonal matrices (a '$\boldsymbol{k}$-representation'), any quantity like the potential energy V will be represented by a matrix $\left(\boldsymbol{k}_1' \cdots \boldsymbol{k}_{n'}' \middle| V \middle| \boldsymbol{k}_1'' \cdots \boldsymbol{k}_{n''}''\right)$, where n' may be different from n'' corresponding to a transition in which the number of particles is changed. We shall often simply write $(\boldsymbol{k}' | V | \boldsymbol{k}'')$ for these general matrix elements.

The representation is not uniquely determined by the condition that the $\boldsymbol{k}_i$ are diagonal, since the phase factors in the

* Throughout this paper are used the same units as in HEISENBERG's paper, where $\boldsymbol{k}$, W, and $\varkappa$ all have the dimensions of a reciprocal length.

representation may be chosen arbitrarily. If $(\boldsymbol{k}' \,|\, V \,|\, \boldsymbol{k}'')^{\times}$ denotes the representative of the operator V in an other $\boldsymbol{k}$-representation we have

$$(\boldsymbol{k}' \,|\, V \,|\, \boldsymbol{k}'')^{\times} = e^{i\,\alpha(\boldsymbol{k}')}\,(\boldsymbol{k}' \,|\, V \,|\, \boldsymbol{k}'')\,e^{-i\,\alpha(\boldsymbol{k}'')}, \tag{2}$$

where $\alpha(\boldsymbol{k}') = \alpha(\boldsymbol{k}'_1, \cdots \boldsymbol{k}'_{n'})$ may be any real function of the variables $(\boldsymbol{k}') = (\boldsymbol{k}'_1, \cdots \boldsymbol{k}'_{n'})$. Similarly, if $(\boldsymbol{k}' \,|)^{\times}$ and $(\boldsymbol{k}' \,|)$ are the representatives of the same state in the two representations, we have

$$(\boldsymbol{k}' \,|)^{\times} = e^{i\,\alpha(\boldsymbol{k}')}\,(\boldsymbol{k}' \,|). \tag{2'}$$

Let us consider a stationary state of the system with the energy E. The corresponding time independent Schrödinger equations in the different momentum spaces may then be written

$$\left.\begin{array}{c} (E - W')\,\Psi(\boldsymbol{k}'_1 \cdots \boldsymbol{k}'_{n'}) = \\ \sum\limits_{n''} \int (\boldsymbol{k}'_1 \cdots \boldsymbol{k}'_{n'} \,|\, V \,|\, \boldsymbol{k}''_1 \cdots \boldsymbol{k}''_{n''})\, \boldsymbol{dk}''_1\, \boldsymbol{dk}''_2 \cdots \boldsymbol{dk}''_{n''}\, \Psi(\boldsymbol{k}''_1 \cdots \boldsymbol{k}''_{n''}) \\ n' = 1,\, 2,\, 3 \cdots \end{array}\right\} \tag{3}$$

Here W' is the eigenvalue of W corresponding to the values $(\boldsymbol{k}'_1, \cdots \boldsymbol{k}'_{n'})$ of the momentum vectors, and $\boldsymbol{dk}''_i$ is a volume-element in the momentum space of the i'th particle. The integration and summation on the right hand side of (3) is to be extended over all the different momentum spaces. In what follows we shall simply write $\int \boldsymbol{dk}''$ instead of $\sum\limits_{n''} \int \boldsymbol{dk}''_1 \cdots \boldsymbol{dk}''_{n''}$ and (3) will be written

$$(E - W')\,\Psi(\boldsymbol{k}') = \int (\boldsymbol{k}' \,|\, V \,|\, \boldsymbol{k}'')\, \boldsymbol{dk}''\, \Psi(\boldsymbol{k}''), \tag{4}$$

where $(\boldsymbol{k}')$ is short for $(\boldsymbol{k}'_1, \cdots \boldsymbol{k}'_{n'})$.

We shall now in particular consider a stationary collision process in which the primary particles have the momentum values $(\boldsymbol{k}^0_1, \cdots \boldsymbol{k}^0_{n^0}) = (\boldsymbol{k}^0)$. Thus, the corresponding wave function being a function of both $(\boldsymbol{k}^0)$ and $(\boldsymbol{k}')$, it will be denoted by

$$(\boldsymbol{k}'_1 \cdots \boldsymbol{k}'_{n'} \,|\, \Psi \,|\, \boldsymbol{k}^0_1 \cdots \boldsymbol{k}^0_{n^0}) = (\boldsymbol{k}' \,|\, \Psi \,|\, \boldsymbol{k}^0). \tag{5}$$

If we consider all possible collisions with varying initial momentum vectors $(\boldsymbol{k}^0)$, the functions (5) define a matrix Ψ, which may be called the wave matrix.

According to (4) the components of the wave matrix satisfy the equations

$$(W^0 - W')\,(\boldsymbol{k}' \,|\, \Psi \,|\, \boldsymbol{k}^0) = \int (\boldsymbol{k}' \,|\, V \,|\, \boldsymbol{k}'')\, d\boldsymbol{k}''\, (\boldsymbol{k}'' \,|\, \Psi \,|\, \boldsymbol{k}^0), \tag{6}$$

where $E = W^0 = \sum_i \sqrt{\varkappa^2 + k_i^{02}}$ is the total energy of the system for the given initial conditions. Using the ordinary rule for matrix multiplication, (6) may be written

$$\Psi W - W \Psi = V \Psi. \tag{7}$$

The wave matrix Ψ is a sum of two parts:

$$\Psi = \Psi^0 + T, \tag{8}$$

where Ψ^0 and T represent the incident and the scattered waves, respectively. In configuration space the function Ψ^0 represents a set of plane waves. In the $\boldsymbol{k}$-representation Ψ^0 is simply

$$\left.\begin{array}{l} (\boldsymbol{k}_1' \cdots \boldsymbol{k}_{n'}' \,|\, \Psi^0 \,|\, \boldsymbol{k}_1^0 \cdots \boldsymbol{k}_{n^0}^0) = \delta(\boldsymbol{k}' - \boldsymbol{k}^0) = \\ \quad \begin{cases} 0 \text{ for } n' \neq n^0 \\ \delta(\boldsymbol{k}_1' - \boldsymbol{k}_1^0) \cdots \delta(\boldsymbol{k}_{n'}' - \boldsymbol{k}_{n^0}^0) \text{ for } n' = n^0. \end{cases} \end{array}\right\} \tag{9}$$

Thus we get for Ψ

$$\Psi = 1 + T, \tag{10}$$

where 1 denotes the unit matrix.

We shall from now on treat the quantity Ψ as an operator whose representatives in different representations are connected by the ordinary rules of quantum mechanical transformation theory. In particular a change in phase leads to a change in the representatives of Ψ given by (2). Thus Ψ will in any representation have the form (10) with the first term equal to the unit matrix.

If we put

$$U = -2\pi i V \Psi = -2\pi i V - 2\pi i V T, \tag{11}$$

we get, from (6), (10), and (11),

$$(W^0 - W')\,(\boldsymbol{k}' \,|\, T \,|\, \boldsymbol{k}^0) = -\frac{1}{2\pi i}(\boldsymbol{k}' \,|\, U \,|\, \boldsymbol{k}^0). \tag{12}$$

Solving with respect to T, we obtain the general solution

$$(\boldsymbol{k}'|T|\boldsymbol{k}^0) = -\frac{1}{2\pi i}(\boldsymbol{k}'|U|\boldsymbol{k}^0)\left[\frac{1}{W^0-W'}+\lambda\delta(W^0-W')\right], \quad (13)$$

where λ is an arbitrary constant. If T is to represent outgoing waves only, λ must, according to DIRAC[6] and HEISENBERG[2], be chosen equal to $-i\pi$. Here it is understood that an integral over W' containing the singular factor $\frac{1}{W^0-W'}$ must be taken as the Cauchy principle value, defined as the limit for $\varepsilon \to 0$ of the integral when the small domain $W^0-\varepsilon$ to $W^0+\varepsilon$ is excluded from the range of integration.

If we introduce the improper functions

$$\delta_{\pm}(W'-W^0) = \frac{\pm 1}{2\pi i(W'-W^0)}+\frac{1}{2}\delta(W'-W^0) \quad (14)$$

(13) may be written

$$(\boldsymbol{k}'|T|\boldsymbol{k}^0) = \delta_{+}(W'-W^0)(\boldsymbol{k}'|U|\boldsymbol{k}^0). \quad (15)$$

If T is eliminated from (11) by means of (15), we get an integral equation, which completely determines U when the potential V is given.

Since the total momentum is conserved, the elements of the wave matrices Ψ, T, and U must have the form

$$(\boldsymbol{k}'|U|\boldsymbol{k}^0) = \delta(\boldsymbol{K}'-\boldsymbol{K}^0)(\boldsymbol{k}'|U_{\boldsymbol{K}^0}|\boldsymbol{k}^0), \quad (16)$$

where $(\boldsymbol{k}'|U_{\boldsymbol{K}^0}|\boldsymbol{k}^0)$ is a submatrix of U corresponding to a fixed value $\boldsymbol{K}'=\boldsymbol{K}^0$ of the total momentum.

If $A^{\dagger}$ is the Hermitian conjugate of a matrix A defined by

$$(\boldsymbol{k}'|A^{\dagger}|\boldsymbol{k}'') = (\boldsymbol{k}''|A|\boldsymbol{k}')^*, \quad (17)$$

we get from (11)

$$U^{\dagger} = 2\pi i \Psi^{\dagger} V, \quad (18)$$

since $V^{\dagger}=V$, on account of V being Hermitian. Multiplying (11) by $\Psi^{\dagger}$ on the left, and (18) by Ψ on the right, and adding, we get

$$\Psi^{\dagger} U + U^{\dagger}\Psi = 0, \quad (19)$$

or, using (10),

$$U + U^{\dagger} + T^{\dagger}U + U^{\dagger}T = 0. \tag{20}$$

This equation, in which the potential V has disappeared, represents a general condition for the wave matrices. From (15), (14), and (17) we get

$$(\boldsymbol{k}' \,|\, T^{\dagger} \,|\, \boldsymbol{k}^0) = \delta_+ (W' - W^0)\; (\boldsymbol{k}' \,|\, U^{\dagger} \,|\, \boldsymbol{k}^0), \tag{21}$$

and the matrix equation (20) becomes

$$\left.\begin{aligned} &(\boldsymbol{k}' \,|\, U + U^{\dagger} \,|\, \boldsymbol{k}^0) + \\ &\int (\boldsymbol{k}' \,|\, U^{\dagger} \,|\, \boldsymbol{k}'')\, d\boldsymbol{k}''\, (\boldsymbol{k}'' \,|\, U \,|\, \boldsymbol{k}^0)\, \left[\delta_+ (W' - W'') + \delta_+ (W'' - W^0)\right] = 0. \end{aligned}\right\}$$

We now define a new matrix R by

$$\left.\begin{aligned} (\boldsymbol{k}' \,|\, R \,|\, \boldsymbol{k}^0) &= \delta (W' - W^0)\; (\boldsymbol{k}' \,|\, U \,|\, \boldsymbol{k}^0) = \\ \delta (\boldsymbol{K}' - \boldsymbol{K}^0)\, &\delta (W' - W^0)\; (\boldsymbol{k}' \,|\, U_{\boldsymbol{K}^0 W^0} \,|\, \boldsymbol{k}^0), \end{aligned}\right\} \tag{23}$$

where $(\boldsymbol{k}' \,|\, U_{\boldsymbol{K}^0 W^0} \,|\, \boldsymbol{k}^0)$ is a submatrix corresponding to fixed values $\boldsymbol{K}^0$ and W^0 for the total momentum and energy. When (22) is multiplied by $\delta(W' - W^0)$, the integral in (22) will contain a factor

$$\left.\begin{aligned} \delta (W' - W^0)\, &\left[\delta_+ (W^0 - W'') + \delta_+ (W'' - W^0)\right] = \\ \delta (W' - W^0)\, \delta (W'' - W^0) &= \delta (W' - W'')\, \delta (W'' - W^0) \end{aligned}\right\}$$

on account of (14), and by (23) we thus get the simple matrix equation

$$R + R^{\dagger} + R^{\dagger}R = 0. \tag{25}$$

Now, if we define HEISENBERG's characteristic matrix S by the equation

$$S = 1 + R, \tag{26}$$

(25) becomes identical with the equation

$$S^{\dagger}S = 1. \tag{27}$$

This condition alone is not, however, sufficient to make S a unitary matrix. We must also have

$$SS^{\dagger} = 1. \tag{28}$$

In order to prove this last equation we consider that solution Ψ_- of the Schrödinger equation (6) in which the outgoing waves T in (10) are replaced by ingoing waves. We then have

$$\Psi_- = 1 + T_- \tag{29}$$

with

$$(\boldsymbol{k}' \,|\, T_- \,|\, \boldsymbol{k}^0) = \delta_-(W' - W^0)\,(\boldsymbol{k}' \,|\, U_- \,|\, \boldsymbol{k}^0), \tag{30}$$

where the function δ_+ in (15) has been replaced by δ_-. Since Ψ_- is a solution of (6) and, by (14),

$$(W^0 - W')\,\delta_-(W' - W^0) = \frac{1}{2\pi i},$$

we have

$$U_- = 2\pi i V \Psi_- \tag{31}$$

on the analogy of (11). From (31) and the Hermitian conjugate equation

$$U_-^\dagger = -2\pi i \Psi_-^\dagger V \tag{32}$$

we get as before

$$\Psi_-^\dagger U_- + U_-^\dagger \Psi_- = 0, \tag{33}$$

or, by (29),

$$U_- + U_-^\dagger + T_-^\dagger U_- + U_-^\dagger T_- = 0 \tag{34}$$

analogously with (20). If this matrix equation is written out, we obviously get an equation obtained from (22) by replacing U and δ_+ by U_- and δ_-, respectively. Therefore, if we define a matrix R_- by

$$(\boldsymbol{k}' \,|\, R_- \,|\, \boldsymbol{k}^0) = \delta(W' - W^0)\,(\boldsymbol{k}' \,|\, U_- \,|\, \boldsymbol{k}^0), \tag{35}$$

we get from (34) analogously with (25)

$$R_- + R_-^\dagger + R_-^\dagger R_- = 0. \tag{36}$$

Further, multiplying (18) by Ψ_- on the right, and (31) by $\Psi^\dagger$ on the left, and subtracting, we get

$$U^\dagger \Psi_- - \Psi^\dagger U_- = 0, \tag{37}$$

or

$$U^\dagger - U_- + U^\dagger T_- - T^\dagger U_- = 0 \tag{38}$$

by (10) and (29).

Written in terms of representatives this equation reads

$$\left.\begin{array}{c}(\boldsymbol{k}' \,|\, U^{\dagger} - U_{-} \,|\, \boldsymbol{k}^{0}) + \\ \int (\boldsymbol{k}' \,|\, U^{\dagger} \,|\, \boldsymbol{k}'')\, \boldsymbol{dk}''\, (\boldsymbol{k}'' \,|\, U \;\, |\, \boldsymbol{k}^{0}) \left[\delta_{-}(W'' - W^{0}) - \delta_{+}(W' - W'')\right] = 0\end{array}\right\} \; ($$

on account of (21) and (30).

Since

$$\delta_{+}(W^{0} - W'') = \delta_{-}(W'' - W^{0}),$$

we get by multiplication of (39) with $\delta(W' - W^{0})$ and using (23) and (35)

$$R^{\dagger} = R_{-}. \tag{40}$$

(36) may thus be written

$$R + R^{\dagger} + R R^{\dagger} = 0. \tag{41}$$

This equation together with (25) shows that R and $R^{\dagger}$ are commuting:

$$R R^{\dagger} = R^{\dagger} R. \tag{42}$$

Therefore also the matrices S and $S^{\dagger}$ commute, and the equation (28) holds as a consequence of (27). On account of the unitarity conditions (27) and (28) S may now be written in the form

$$S = e^{i\eta}, \tag{43}$$

where η is an Hermitian matrix—HEISENBERG's η-matrix.

In a perturbation theory, where V is considered as small, U, T, R, and η are also small, as is seen from (11), (15), (23), and (43). In the first approximation we get from (11)

$$U = -2\pi i V, \tag{44}$$

and from (26), (43), and (23)

$$\eta = \frac{1}{i} R, \tag{45}$$

and

$$(\boldsymbol{k}' \,|\, \eta \,|\, \boldsymbol{k}^{0}) = -2\pi\delta(W' - W^{0})\, (\boldsymbol{k}' \,|\, V \,|\, \boldsymbol{k}^{0}), \tag{46}$$

showing that the η-matrix is essentially equal to the potential energy in this approximation.

Instead of the $3n'$ variables $(\boldsymbol{k}'_1 \cdots \boldsymbol{k}'_{n'})$ we now introduce the total momentum $\boldsymbol{K}'$, the total energy W', and $3n'-4$ other variables $(x') = (x'_1 \cdots x'_{3n'-4})$ as independent variables. When $\Delta' = \dfrac{\partial(\boldsymbol{K}' W' x')}{\partial(\boldsymbol{k}'_1 \cdots \boldsymbol{k}'_{n'})}$ denotes the functional determinant corresponding to this transformation, the connection between the matrix elements of any matrix A in the two representations is given by

$$(\boldsymbol{k}' \,|\, A \,|\, \boldsymbol{k}^0) = \sqrt{\Delta'}\, (\boldsymbol{K}' W' x' \,|\, A \,|\, \boldsymbol{K}^0 W^0 x^0) \sqrt{\Delta^0}, \tag{47}$$

provided the phase factors are unaffected by the transformation. For the representatives of any state in the two representations we have similarly

$$(\boldsymbol{k}' \,|) = \sqrt{\Delta'}\, (\boldsymbol{K}' W' x' \,|). \tag{47'}$$

In the new representation any of the three matrices R, S, and η have the form

$$(\boldsymbol{K}' W' x' \,|\, R \,|\, \boldsymbol{K}^0 W^0 x^0) = \delta(\boldsymbol{K}' - \boldsymbol{K}^0)\, \delta(W' - W^0)\, (x' \,|\, R \,|\, x^0). \tag{48}$$

Here

$$(x' \,|\, R \,|\, x^0) = (x' \,|\, U_{\boldsymbol{K}^0 W^0} \,|\, x^0) \tag{49}$$

is a submatrix corresponding to fixed values $\boldsymbol{K}' = \boldsymbol{K}^0$ and $W' = W^0$ of the total momentum and energy. For the submatrices an equation like (27) takes the form

$$\ldots' \,|\, S^{\dagger} \,|\, x'')\, dx''\, (x'' \,|\, S \,|\, x^0) = \int (x'' \,|\, S \,|\, x')^*\, dx''\, (x'' \,|\, S \,|\, x^0) = \delta(x' - x^0). \tag{50}$$

It should be remembered, however, that $\int dx''$ is an abbreviation for $\sum_{n''} \int dx''_1 \cdots dx''_{3n''-4}$, just like

$$\int \boldsymbol{dk}'' = \sum_{n''} \int \boldsymbol{dk}''_1 \cdots \boldsymbol{dk}''_{n''}. \tag{51}$$

Returning now to the general equation (22) we get in the new representation with $y = (\boldsymbol{K}, x)$

$$\left.\begin{array}{c} (W' y' \,|\, U + U^{\dagger} \,|\, W^0 y^0) + \\ \int (W' y' \,|\, U^{\dagger} \,|\, W'' y'')\, dW''\, dy''\, (W'' y'' \,|\, U \,|\, W^0 y^0)\, \left[\delta_+ (W' - W'') + \right. \\ \left. \delta_+ (W'' - W^0)\right] = 0. \end{array}\right\} \tag{52}$$

If we multiply this equation by $\frac{1}{W'-W^0}$, or by $\delta_+(W'-W^0)$, a special investigation of the singularity at $W'=W^0$ will be necessary. By (15) and (21) the result of a multiplication with $\delta_+(W'-W^0)$ can be written

$$\left.\begin{aligned}&(W'y'\,|\,T+T^\dagger\,|\,W^0y^0)+\\ &\int (W'y'\,|\,U^\dagger\,|\,W''y'')\,dW''\,dy''\,(W''y''\,|\,U\,|\,W^0y^0)\\ &\qquad \delta_+(W'-W^0)\left[\delta_+(W'-W'')+\delta_+(W''-W^0)\right]=\\ &\qquad A(W^0,y',y^0)\,\delta(W'-W^0),\end{aligned}\right\}\ \text{(5.}$$

where A so far is an arbitrary function of W^0, y', and y^0. The right hand side is zero except for $W'=W^0$ and vanishes entirely if multiplied by $W'-W^0$, thus in fact (53) reduces to (52) by multiplication with $W'-W^0$. The function A must be determined in such a way that an integration of the equation (53) with respect to W' over a range containing the value W^0 leads to a correct result. We need only integrate (53) from $W^0-\varepsilon$ to $W^0+\varepsilon$ and afterwards we can let ε go to zero.

Now we get from (14) by a simple calculation

$$\left.\begin{aligned}&\delta_+(W'-W^0)\left[\delta_+(W'-W'')+\delta_+(W''-W^0)\right]=\\ &\frac{1}{(2\pi i)^2(W'-W'')(W''-W^0)}+\frac{1}{4\pi i}\,\frac{\delta(W''-W')}{W''-W^0}+\frac{1}{4\pi i}\,\frac{\delta(W''-W^0)}{W'-W''}+\\ &\frac{1}{2}\,\delta(W'-W^0)\,\delta(W''-W^0)=\delta_+(W'-W'')\,\delta_+(W''-W^0)+\\ &\qquad\frac{1}{4}\,\delta(W'-W^0)\,\delta(W''-W^0).\end{aligned}\right\}\ \text{(5}$$

Further we have, for any function $f(W')$ which is continuous at $W'=W^0$,

$$\lim_{\varepsilon\to 0}\int_{W^0-\varepsilon}^{W^0+\varepsilon} f(W')\,\delta_+(W'-W^0)\,dW'=f(W^0)\lim_{\varepsilon\to 0}\int_{W^0-\varepsilon}^{W^0+\varepsilon}\delta_+(W'-W^0)\,dW'=\frac{f(W^0)}{2},\ \text{(5}$$

and

$$\lim_{\varepsilon\to 0}\int_{W^0-\varepsilon}^{W^0+\varepsilon}\frac{f(W')\,dW'}{(2\pi i)^2\,(W'-W'')\,(W''-W^0)}=\frac{f(W^0)\,\delta(W''-W^0)}{4}. \tag{56}$$

In fact, we have, with $\xi = W'' - W^0$

$$\left.\begin{aligned}&\lim_{\varepsilon\to 0}\int_{W^0-\varepsilon}^{W^0+\varepsilon}\frac{dW'}{(2\pi i)^2\,(W'-W'')\,(W''-W^0)} = \\ &\lim_{\varepsilon\to 0}\begin{cases}\dfrac{1}{4\pi^2\xi}\,ln\,\dfrac{-\varepsilon-\xi}{\varepsilon-\xi} & \text{for } \xi<-\varepsilon\\[2ex] \dfrac{1}{4\pi^2\xi}\,ln\,\dfrac{\varepsilon+\xi}{\varepsilon-\xi} & \text{for } -\varepsilon<\xi<\varepsilon\\[2ex] \dfrac{1}{4\pi^2\xi}\,ln\,\dfrac{\varepsilon+\xi}{\xi-\varepsilon} & \text{for } \xi>\varepsilon,\end{cases}\end{aligned}\right\}\quad(57)$$

and this function is easily shown to be equal to $\frac{1}{4}\,\delta(\xi) = \frac{1}{4}\,\delta(W''-W^0)$ in the limit $\varepsilon\to 0$.

When the operation $\lim_{\varepsilon\to 0}\int_{W^0-\varepsilon}^{W^0+\varepsilon} dW'$ is applied to the equation (53), we thus, using (15), (21), the first equation (54), (55), and (56), get

$$\left.\begin{aligned}&A\,(W^0, y', y^0) = \frac{1}{2}(W^0 y'\,|\,U+U^\dagger\,|\,W^0 y^0) + \\ &\int (W^0 y'\,|\,U^\dagger\,|\,W'' y'')\,dW''\,dy''\,(W'' y''\,|\,U\,|\,W^0 y^0)\\ &\qquad\left[\frac{1}{4}\,\delta\,(W''-W^0) + \frac{1}{2}\,\delta\,(W''-W^0)\right] = \\ &\frac{1}{4}\int (W^0 y'\,|\,U^\dagger\,|\,W^0 y'')\,dy''\,(W^0 y''\,|\,U\,|\,W^0 y^0).\end{aligned}\right\}\quad(58)$$

Here we have also used the equation

$$\left.\begin{aligned}&(W^0 y'\,|\,U+U^\dagger\,|\,W^0 y^0) + \\ &\int (W^0 y'\,|\,U^\dagger\,|\,W^0 y'')\,dy''\,(W^0 y''\,|\,U\,|\,W^0 y^0) = 0,\end{aligned}\right\}\quad(59)$$

which follows from (23) and (25) by integration of (25) with respect to W'.

Introducing the expression (58) for A into (53) we get, using the last equation (54),

$$(W'y' \,|\, T + T^\dagger \,|\, W^0 y^0) +$$
$$\int (W'y' \,|\, U^\dagger \,|\, W''y'')\, dW''dy''\, (W''y'' \,|\, U \,|\, W^0 y^0)\, .$$
$$\delta_+ (W' - W'')\, \delta_+ (W'' - W^0) = 0,$$

or

$$T + T^\dagger + T^\dagger T = 0. \qquad (61)$$

For the wave matrix Ψ in (10) we have therefore

$$\Psi^\dagger \Psi = 1, \qquad (62)$$

which shows that the matrix elements $(\boldsymbol{k}' \,|\, \Psi \,|\, \boldsymbol{k}^0)$, considered as functions of $(\boldsymbol{k}')$ with fixed $(\boldsymbol{k}^0)$, are normalized eigenfunctions of the Hamiltonian $H = W + V$ belonging to the continuous eigenvalues $E = W^0 = \sum_i \sqrt{\varkappa^2 + k_i^{0\,2}}$.

The matrix Ψ, however, will not in general be a unitary matrix, since the equation

$$\Psi \Psi^\dagger = 1 \qquad (63)$$

will hold in special cases only. When (63) holds, the reciprocal of Ψ exists, and $\Psi^{-1} = \Psi^\dagger$. From the Schrödinger equation (7), which may be written

$$\Psi W = H \Psi, \qquad (64)$$

we then, by multiplication with Ψ^{-1} on the right, get

$$H = \Psi W \Psi^{-1}, \qquad (65)$$

showing that the matrices H and W have the same eigenvalues. (63) can thus only be true for systems which do not have any closed states.

In the general case we may define an Hermitian operator

$$\mathscr{E} = 1 - \Psi \Psi^\dagger \qquad (66)$$

which, on account of (62), satisfies the equation

$$\mathscr{E}^2 = 1 - 2\,\Psi\Psi^\dagger + \Psi\Psi^\dagger\Psi\Psi^\dagger = 1 - \Psi\Psi^\dagger = \mathscr{E}. \qquad (67)$$

Hence $\mathscr{E}$ has the eigenvalues 0 and 1.

Further we have

$$H\Psi\Psi^\dagger - \Psi\Psi^\dagger H = 0. \qquad (68)$$

To prove this equation, multiply (64) by $\Psi^\dagger$ on the right and the conjugate equation

$$W\Psi^\dagger = \Psi^\dagger H \tag{69}$$

by Ψ on the left, and subtract.

(68) shows that $\Psi\Psi^\dagger$ and therefore also $\mathscr{E}$ commute with H. Thus $\mathscr{E}$ and H have a common system of eigenfunctions.

Since

$$\mathscr{E}\,\Psi = 0 \tag{70}$$

according to (66) and (62), the functions $(\boldsymbol{k}'\,|\,\Psi\,|\,\boldsymbol{k}^0)$ belonging to the continuous eigenvalues of H, are also eigenfunctions of $\mathscr{E}$ belonging to the eigenvalue 0. Moreover these functions are the only eigenfunctions of that kind. To prove this, let us assume that $\Phi(\boldsymbol{k}')$ is another independent eigenfunction of $\mathscr{E}$ with the eigenvalue 0, then Φ may be taken orthogonal to all the functions $(\boldsymbol{k}'\,|\,\Psi\,|\,\boldsymbol{k}^0)$, i. e.

$$\left.\begin{aligned} \Psi^\dagger\,\Phi &= 0 \\ \text{and} \qquad \mathscr{E}\,\Phi &= 0 \end{aligned}\right\} \tag{71}$$

which, by (66), leads to the equation

$$\Phi = 0 .$$

The eigenfunctions $\Psi_r(\boldsymbol{k}')$ of H belonging to the discrete* eigenvalues are therefore also eigenfunctions of $\mathscr{E}$ with the eigenvalue 1. If $\Psi_r(\boldsymbol{k}')$ is considered as a matrix with one column only, while $\Psi_r^\dagger(\boldsymbol{k}') = \Psi_r(\boldsymbol{k}')^*$ is considered as a matrix with one row only, the orthogonality relations for the discrete eigenstates may be written

$$\left.\begin{aligned} \Psi_r^\dagger\,\Psi_s &= \delta_{rs} \\ \Psi_r^\dagger\,\Psi &= 0 . \end{aligned}\right\} \tag{72}$$

Since the functions $\Psi_r(\boldsymbol{k}')$ are the only eigenfunctions of $\mathscr{E}$ belonging to the eigenvalue 1, all other eigenfunctions belonging to the eigenvalue 0, we obviously have

$$(\boldsymbol{k}'\,|\,\mathscr{E}\,|\,\boldsymbol{k}'') = \sum_r \Psi_r(\boldsymbol{k}')\,\Psi_r^\dagger(\boldsymbol{k}''), \tag{73}$$

* The index r numerating the 'discrete' energy values will in general also include continuous variables such as the total momentum $\boldsymbol{K}^0$ of the system.

and the equation (66) may be written

$$(\boldsymbol{k}' \,|\, \Psi\Psi^{\dagger} \,|\, \boldsymbol{k}'') + \sum_{r} \Psi_r(\boldsymbol{k}')\, \Psi_r^{\dagger}(\boldsymbol{k}'') = (\boldsymbol{k}' \,|\, 1 \,|\, \boldsymbol{k}''), \tag{74}$$

which in the general case replaces (63).

We shall now discuss the question whether the discrete energy values in the closed stationary states are at least partly determined by the S-matrix, as originally indicated by HEISENBERG[7]. *A priori*, this seems possible, since the asymptotic form of the wave function in great distances which determines the collision cross sections, depends chiefly on the form of the potential function in small distances, which again is essential for the position of the discrete energy levels. From the preceding developments it follows, however, that the discrete energy values are completely independent of the form of the S-matrix. To see this, let us assume that we do not know only the submatrices S and R, but the whole matrices U and Ψ. With given Ψ the operator $\mathfrak{S}$ is given by (66). Determining the eigenfunctions of $\mathfrak{S}$ belonging to the eigenvalue 1, we get simultaneously the eigenfunctions $\Psi_r(\boldsymbol{k}')$ of H belonging to the discrete energy values or at least linear combinations of these functions. The values of the energy in these states, however, are completely undetermined. The energy levels in the closed states may be changed in an arbitrary way without any change in Ψ, i. e. without any effect on the results of collision processes. New fundamental assumptions about the S-matrix will thus be necessary in order to determine the energy values in the closed states. To this question we shall return in the sequel to this paper.

2. General definition of cross sections. Connection between the cross sections and the characteristic matrix S. Proof of the invariance of S.

In order to investigate the transformation properties of the matrix S under Lorentz transformations it seems natural to use the connection between the matrix elements of S and the cross sections. For this purpose we need a general definition of cross sections in an arbitrary Lorentz frame of reference. The current textbooks give definitions of cross sections only for the special frames of reference, where either the center of gravity or one of the particles are initially at rest. In the general case it will now be most convenient to use a definition which makes the cross sections entirely independent of the frame of reference. This condition, together with the well-known expression for the cross section in the system where the center of gravity is at rest, uniquely determines the definition to be adopted in the general case.

We shall first consider a collision between particles 1 and 2 without creation of new particles. Let ϱ_1^0 and ϱ_2^0 be the densities of particles 1 and 2 in the incident beams and let $\boldsymbol{u}_1^0$ and $\boldsymbol{u}_2^0$ be the corresponding particle velocities. If $dQ = \sigma d\Omega$ denotes the differential cross section for a scattering of particles 2 say into a solid angle $d\Omega$, we have in the center of gravity system

$$dQ = \frac{dN}{\varrho_1^0 \varrho_2^0 \, |\boldsymbol{u}_1^0 - \boldsymbol{u}_2^0|},$$

where dN is the number of particles 2, which, pr. unit volume and unit time, i. e. pr. unit four-dimensional volume, is scattered into the solid angle $d\Omega$.

Since the four dimensional volume is invariant, the number dN must be independent of the frame of reference. Thus the cross section must in general be given by

$$dQ = \frac{dN}{F^0}, \tag{75}$$

where the factor F^0 is an invariant, which in the center of gravity system reduces to $\varrho_1^0 \varrho_2^0 |\boldsymbol{u}_1^0 - \boldsymbol{u}_2^0|$. As is easily seen, F^0 must then be given by

$$F^0 = \varrho_1^0 \varrho_2^0 \sqrt{|\boldsymbol{u}_1^0 - \boldsymbol{u}_2^0|^2 - |\boldsymbol{u}_1^0 \times \boldsymbol{u}_2^0|^2}\,. \tag{76}$$

Here $\boldsymbol{u}_1^0 \times \boldsymbol{u}_2^0$ is the vector product of the vectors $\boldsymbol{u}_1^0$ and $\boldsymbol{u}_2^0$, which is zero if $\boldsymbol{u}_1^0$ and $\boldsymbol{u}_2^0$ are parallel, as is the case in the center of gravity system.

To prove the invariance of F^0 we use the fact that the quantities $k_i^\mu = (\boldsymbol{k}_i, W_i)$ and $k_{i\mu} = (\boldsymbol{k}_i, -W_i)$ by any Lorentz transformation transform like the components of a four-vector. The quantities $k_1^\mu k_2^\nu - k_1^\nu k_2^\mu$ are thus the components of an antisymmetrical tensor and the quantity

$$\left.\begin{aligned} B &= \sqrt{|\boldsymbol{k}_1 W_2 - \boldsymbol{k}_2 W_1|^2 - |\boldsymbol{k}_1 \times \boldsymbol{k}_2|^2} = \\ &= \sqrt{\frac{-1}{2}\left(k_1^\mu k_2^\nu - k_1^\nu k_2^\mu\right)\left(k_{1\mu} k_{2\nu} - k_{1\nu} k_{2\mu}\right)} \end{aligned}\right\} \quad (77)$$

is an invariant.* Further we have for any Lorentz transformation connecting the variables of two frames of reference K and $\overline{K}$.

$$\left.\begin{aligned} \bar{\varrho}_i &= \varrho_i \, \frac{1 - \boldsymbol{v}\boldsymbol{u}_i}{\sqrt{1-\boldsymbol{v}^2}} \\ \overline{W}_i &= W_i \, \frac{1 - \boldsymbol{v}\boldsymbol{u}_i}{\sqrt{1-\boldsymbol{v}^2}}, \quad \boldsymbol{u}_i = \frac{\boldsymbol{k}_i}{W_i}, \end{aligned}\right\} \quad (78)$$

where $\boldsymbol{v}$ represents the velocity of $\overline{K}$ relative to K. Hence $\frac{\varrho_1 \varrho_2}{W_1 W_2} = \frac{\bar{\varrho}_1 \bar{\varrho}_2}{\overline{W}_1 \overline{W}_2}$ and the quantity F^0 in (76) may be written as a product of two invariant quantities:

$$F^0 = \frac{\varrho_1^0 \varrho_2^0}{W_1^0 W_2^0} \cdot B^0$$

with B^0 given by (77).

Thus the cross section defined by (75) and (76) is invariant. Further it is symmetrical in the two particles and reduces to the usual definition of cross sections in the center of gravity system. The same holds for that system in which one of the particles, say particle 1, is initially at rest, since in this case F^0 reduces to $F^0 = \varrho_1^0 \varrho_2^0 u_2^0$.

We shall now trace the connection between the scattering cross section dQ and the matrix elements of S. For this purpose we need the Schrödinger function $\Psi(\boldsymbol{x}_1 \boldsymbol{x}_2)$ in configuration space. This function is obtained from the wave matrix $(\boldsymbol{k}_1' \boldsymbol{k}_2' \,|\, \Psi \,|\, \boldsymbol{k}_1^0 \boldsymbol{k}_2^0)$ by means of the equation

$$\Psi(\boldsymbol{x}_1 \boldsymbol{x}_2) = \int (\boldsymbol{x}_1 \boldsymbol{x}_2 \,|\, \boldsymbol{k}_1' \boldsymbol{k}_2')\, d\boldsymbol{k}_1' \, d\boldsymbol{k}_2' \, (\boldsymbol{k}_1' \boldsymbol{k}_2' \,|\, \Psi \,|\, \boldsymbol{k}_1^0 \boldsymbol{k}_2^0) \quad (79)$$

with the transformation function

* Here the usual convention is made regarding the summation from 1 to 4 over dummy indices like μ and ν in (77).

$$(\boldsymbol{x}_1 \boldsymbol{x}_2 \,|\, \boldsymbol{k}'_1 \boldsymbol{k}'_2) = (2\pi)^{-3}\, e^{i(\boldsymbol{k}'_1 \boldsymbol{x}_1 + \boldsymbol{k}'_2 \boldsymbol{x}_2)}\, e^{i\gamma(\boldsymbol{k}'_1 \boldsymbol{k}'_2)}, \tag{80}$$

where $\gamma' = \gamma(\boldsymbol{k}'_1 \boldsymbol{k}'_2)$ is a real function of $\boldsymbol{k}'_1$ and $\boldsymbol{k}'_2$ depending on the phases in the chosen $\boldsymbol{k}$-representation.[8] γ' may also be a function of the time. For the incident waves we get, by (8) and (9),

$$(\boldsymbol{x}_1 \boldsymbol{x}_2) = (2\pi)^{-3}\, e^{i(\boldsymbol{k}^0_1 \boldsymbol{x}_1 + \boldsymbol{k}^0_2 \boldsymbol{x}_2)}\, e^{i\gamma^0} = (2\pi)^{-3} e^{i\boldsymbol{K}^0 \boldsymbol{x}_1}\, e^{i\boldsymbol{k}^0_2(\boldsymbol{x}_2 - \boldsymbol{x}_1)}\, e^{i\gamma^0}, \tag{81}$$

while the scattered waves on account of (8), (15), and (16), are given by

$$\left.\begin{aligned} \boldsymbol{x}_1 \boldsymbol{x}_2) = {} & (2\pi)^{-3} \int e^{i(\boldsymbol{k}'_1 \boldsymbol{x}_1 + \boldsymbol{k}'_2 \boldsymbol{x}_2)}\, e^{i\gamma'}\, \delta(\boldsymbol{K}' - \boldsymbol{K}^0) \\ & \delta_+ (W'_1 + W'_2 - W^0)\, (\boldsymbol{k}'_1 \boldsymbol{k}'_2 \,|\, U_{\boldsymbol{K}^0} \,|\, \boldsymbol{k}^0_1 \boldsymbol{k}^0_2)\, d\boldsymbol{k}'_1\, d\boldsymbol{k}'_2 = \\ 2\pi)^{-3} & e^{i\boldsymbol{K}^0 \boldsymbol{x}_1} \int e^{i\boldsymbol{k}'_2(\boldsymbol{x}_2 - \boldsymbol{x}_1)}\, e^{i\gamma(\boldsymbol{K}^0 - \boldsymbol{k}'_2, \boldsymbol{k}'_2)} \\ \sqrt{\kappa^2 + |\boldsymbol{K}^0 - \boldsymbol{k}'_2|^2} & + \sqrt{\kappa^2 + k'^2_2} - W^0\big)\, \big(\boldsymbol{K}^0 - \boldsymbol{k}'_2, \boldsymbol{k}'_2 \,|\, U_{\boldsymbol{K}^0} \,|\, \boldsymbol{k}^0_1 \boldsymbol{k}^0_2\big)\, d\boldsymbol{k}'_2 . \end{aligned}\right\} \tag{82}$$

Now we are only interested in the asymptotic values of T for $r = |\boldsymbol{x}_2 - \boldsymbol{x}_1| \rightarrow \infty$. Introducing polar coordinates in the $\boldsymbol{k}'_2$-space with the direction $\boldsymbol{e}'_2 = \dfrac{\boldsymbol{x}_2 - \boldsymbol{x}_1}{|\boldsymbol{x}_2 - \boldsymbol{x}_1|}$ as polar axis, the integration with respect to the angles in the integral in (82) leads to the expression[9]

$$\left.\begin{aligned} \boldsymbol{x}_1 \boldsymbol{x}_2)_{r\rightarrow\infty} = {} & (2\pi)^{-3} e^{i\boldsymbol{K}^0 \boldsymbol{x}_1} \frac{2\pi}{ir} \left\{ \int_0^\infty e^{ik'_2 r} (\boldsymbol{K}^0 - k'_2 \boldsymbol{e}'_2, k'_2 \boldsymbol{e}'_2 \,|\, U_{\boldsymbol{K}^0} \,|\, \boldsymbol{k}^0_1, \boldsymbol{k})^0_2 \right. \\ & e^{i\gamma(\boldsymbol{K}^0 - k'_2 \boldsymbol{e}'_2, k'_2 \boldsymbol{e}'_2)}\, \delta_+ \left(\sqrt{\kappa^2 + |\boldsymbol{K}^0 - k'_2 \boldsymbol{e}'_2|^2} + \sqrt{\kappa^2 + k'^2_2} - W^0\right) k'_2\, dk'_2 - \\ {}^{ik'_2 r} & (\boldsymbol{K}^0 + k'_2 \boldsymbol{e}'_2, -k'_2 \boldsymbol{e}'_2 \,|\, U_{\boldsymbol{K}^0} \,|\, \boldsymbol{k}^0_1, \boldsymbol{k}^0_2)\, e^{i\gamma(\boldsymbol{K}^0 + k'_2 \boldsymbol{e}'_2, -k'_2 \boldsymbol{e}'_2)} \\ & \left. \delta_+ \left(\sqrt{\kappa^2 + |\boldsymbol{K}^0 + k'_2 \boldsymbol{e}'_2|^2} + \sqrt{\kappa^2 + k'^2_2} - W^0\right) k'_2\, dk'_2, \right\} \end{aligned}\right\} \tag{83}$$

where only terms of the order $\dfrac{1}{r}$ have been retained. When the argument of the δ_+-function is denoted by $A(k'_2)$, and when $f(k'_2)$ is any function of k'_2, we have[9], neglecting terms of the order $\dfrac{1}{r^2}$,

$$\int_0^\infty e^{ik_2' r} f(k_2')\, \delta_+(A)\, k_2'\, dk_2' = \frac{e^{i\bar{k}_2' r}\, \bar{k}_2'}{\left.\dfrac{\partial A(\bar{k}_2')}{\partial \bar{k}_2'}\right|_{\text{const. } \boldsymbol{e}_2'}} f(\bar{k}_2'), \tag{84}$$

where $\bar{k}_2'$ is the value of k_2' determined by

$$A(k_2') = 0, \tag{85}$$

and the differentiation of $A(k_2')$ with respect to k_2' is to be performed by constant $\boldsymbol{e}_2'$.

In the same approximation we have

$$\int_0^\infty e^{-ik_2' r} f(k_2')\, \delta_+(A)\, k_2'\, dk_2' = 0, \tag{86}$$

so that the second integral in (83) may be neglected.

In the first integral we put

$$\boldsymbol{k}_2' = \bar{k}_2 \boldsymbol{e}_2' \quad \text{and} \quad \boldsymbol{k}_1' = \boldsymbol{K}^0 - \bar{k}_2' \boldsymbol{e}_2'. \tag{87}$$

Thus $\boldsymbol{k}_1'$ and $\boldsymbol{k}_2'$ are now the values of the momentum variables following from the theorems of conservation of momentum and energy. When W_1', W_2', $\boldsymbol{u}_1'$ and $\boldsymbol{u}_2'$ are the corresponding energies and velocities, we get, since the differentiation in $\dfrac{\partial A}{\partial k_2'}$ is to be performed by constant $\boldsymbol{e}_2'$,

$$\left.\frac{\partial A}{\partial k_2'}\right|_{\text{const. } \boldsymbol{e}_2'} = \left(\frac{\boldsymbol{k}_2'}{W_2'} - \frac{\boldsymbol{k}_1'}{W_1'}\right) \boldsymbol{e}_2' = (\boldsymbol{u}_2' - \boldsymbol{u}_1')\, \boldsymbol{e}_2'. \tag{88}$$

Using (84) and (88) in the calculation of the first integral in (83) we finally get for the total wave function in configuration space the asymptotic expression*

$$\left.\begin{aligned} \Psi(\boldsymbol{x}_1 \boldsymbol{x}_2)_{r\to\infty} &= \Psi^0(\boldsymbol{x}_1 \boldsymbol{x}_2) + T(\boldsymbol{x}_1 \boldsymbol{x}_2)_{r\to\infty} = \\ = (2\pi)^{-3} e^{i\boldsymbol{K}^0 \boldsymbol{x}_1} &\left\{ e^{i\boldsymbol{k}_2^0(\boldsymbol{x}_2 - \boldsymbol{x}_1)} e^{i\gamma^0} - 2\pi i \frac{e^{ik_2' r} e^{i\gamma'}}{r} \frac{k_2' (\boldsymbol{k}_1' \boldsymbol{k}_2' \,|\, U_{\boldsymbol{K}^0 W^0} \,|\, \boldsymbol{k}_1^0 \boldsymbol{k}_2^0)}{(\boldsymbol{u}_2' - \boldsymbol{u}_1')\, \boldsymbol{e}_2'} \right. . \end{aligned}\right\} ($$

The probability of finding the particle 2 in a distance $r = |\boldsymbol{x}_2 - \boldsymbol{x}_1|$ from the particle 1 after the collision is thus proportional to

* This asymptotic expression for Ψ is seen to be correct only if the potential function goes to zero faster than $\frac{1}{r}$ as r tends to infinity. Thus, a slight modification of the theory is necessary in the limiting case of a Coulomb field.

$$\frac{(2\pi)^2}{r^2}\frac{k_2'^2\,|(\boldsymbol{k}_1'\boldsymbol{k}_2'\,|\,U_{\boldsymbol{K}^0W^0}|\,\boldsymbol{k}_1^0\boldsymbol{k}_2^0)|^2}{|(\boldsymbol{u}_2'-\boldsymbol{u}_1')\,\boldsymbol{e}_2'|^2}. \tag{90}$$

To get the number of particles 2 which pr. unit time are scattered into the solid angle $d\Omega$ in the direction $\boldsymbol{e}_2'$, we have to multiply (90) by the factor $|(\boldsymbol{u}_2'-\boldsymbol{u}_1')\,\boldsymbol{e}_2'|\cdot r^2 d\Omega$, which determines the current through a surface element $r^2 d\Omega$ placed at right angles to the direction $\boldsymbol{e}_2'$ in a constant distance r from the particle 1. Taking account of the particle densities in the incident waves we finally for the scattering cross section get

$$dQ_{\text{scat.}} = 4\pi^2\frac{k_2'^2\,|(\boldsymbol{k}_1'\boldsymbol{k}_2'\,|\,U_{\boldsymbol{K}^0W^0}\,|\,\boldsymbol{k}_1^0\boldsymbol{k}_2^0)|^2}{\sqrt{|\boldsymbol{u}_1^0-\boldsymbol{u}_2^0|^2-|\boldsymbol{u}_1^0\times\boldsymbol{u}_2^0|^2}\cdot|(\boldsymbol{u}_2'-\boldsymbol{u}_1')\,\boldsymbol{e}_2'|}\,d\Omega. \tag{91}$$

The matrix $U_{\boldsymbol{K}^0W^0}$ occurring in this expression is, apart from the δ-functions, identical with the matrix R in (23).

If we put

$$I = \sqrt{W_1'W_2'}\,(\boldsymbol{k}_1'\boldsymbol{k}_2'\,|\,U_{\boldsymbol{K}^0W^0}|\,\boldsymbol{k}_1^0\boldsymbol{k}_2^0)\sqrt{W_1^0W_2^0} \tag{92}$$

(91) may be written

$$dQ_{\text{scat.}} = 4\pi^2\frac{|I|^2}{B^0}\frac{k_2'^2\,d\Omega}{|(\boldsymbol{u}_2'-\boldsymbol{u}_1')\,\boldsymbol{e}_2'|\cdot W_1'W_2'}, \tag{93}$$

where B^0 is the invariant quantity defined by (77).

Since the vectors $\boldsymbol{k}_1'$ and $\boldsymbol{k}_2'$ are determined by the conservation theorems, we have

$$(\boldsymbol{u}_2'-\boldsymbol{u}_1')\,\boldsymbol{e}_2' = \frac{\partial(W_1'+W_2')}{\partial k_2'} = \frac{\partial W'}{\partial k_2'}, \tag{94}$$

where $\boldsymbol{k}_1'$ in W_1' is to be put equal to $\boldsymbol{K}^0-k_2'\boldsymbol{e}_2'$ before the differentiation, which then is performed by constant $\boldsymbol{e}_2'$. (93) may therefore also be written

$$dQ_{\text{scat.}} = 4\pi^2\frac{|I|^2}{B^0}\iint\delta(\boldsymbol{K}'-\boldsymbol{K}^0)\,\delta(W'-W^0)\frac{\boldsymbol{dk}_1'}{W_1'}\frac{k_2'^2\,dk_2'\,d\Omega}{W_2'}, \tag{95}$$

where the integration in $\int\boldsymbol{dk}_1'$ is to be extended over the whole $\boldsymbol{k}_1'$-space, while the integration in $\int dk_2'$ ranges from 0 to ∞. For small U and V, where (44) holds, the expression (95) is easily seen to be identical with the usual quantum mechanical expression for the cross section, derived by a perturbation method in which the potential V is treated as small. The integral in (95) is obviously invariant, $\delta(\boldsymbol{K}'-\boldsymbol{K}^0)\,\delta(W'-W^0)$ and $\frac{d\boldsymbol{k}'_i}{W'_i}$ being invariant quantities (cf. equation (129) section 3). Since the same holds for dQ and B^0, (95) shows that the modulus of the quantity I, defined by (92), must be relativistically invariant.

Let us now consider the case where a new particle 3 is created during the collision. The cross section for a process in which the new particle obtains a momentum between $\boldsymbol{k}'_3$ and $\boldsymbol{k}'_3+d\boldsymbol{k}'_3$, while particle 2 is scattered into a solid angle $d\Omega$ in the direction $\boldsymbol{e}'_2$, may be derived as before. We simply get

$$_{\text{scat.,emis.}} = 4\pi^2 \frac{k'^2_2 \,|(\boldsymbol{k}'_1\boldsymbol{k}'_2\boldsymbol{k}'_3\,|\,\boldsymbol{U}_{\boldsymbol{K}^0W^0}\,|\,\boldsymbol{k}^0_1\boldsymbol{k}^0_2)|^2}{\sqrt{|\boldsymbol{u}^0_1-\boldsymbol{u}^0_2|^2-|\boldsymbol{u}^0_1\times\boldsymbol{u}^0_2|^2}\cdot|(\boldsymbol{u}'_2-\boldsymbol{u}'_1)\,\boldsymbol{e}'_2|}\,d\Omega\, d\boldsymbol{k}'_3 \qquad (\text{}$$

on a close analogy of (91). By the same arguments as before we find that the modulus of the quantity

$$\sqrt{W'_1W'_2W'_3}\,(\boldsymbol{k}'_1\boldsymbol{k}'_2\boldsymbol{k}'_3\,|\,U_{\boldsymbol{K}^0W^0}\,|\,\boldsymbol{k}^0_1\boldsymbol{k}^0_2)\sqrt{W^0_1W^0_2}$$

is a relativistic invariant. The same holds for a general matrix element of $\boldsymbol{U}_{\boldsymbol{K}^0W^0}$ and therefore also for the elements of the matrix R defined by (23).

Let us now consider an arbitrary Lorentz transformation. In the momentum space of the i'th particle, say, this will in general be a non-linear transformation

$$\bar{\boldsymbol{k}}_i = \mathscr{L}(\boldsymbol{k}_i). \qquad (97)$$

If all transformed quantities are distinguished by a bar, the representative of the transformed matrix $\bar{R}$ in a $\bar{\boldsymbol{k}}$-representation will be denoted by $(\bar{\boldsymbol{k}}'\,|\,\bar{R}\,|\,\bar{\boldsymbol{k}}^0)$. Further let $\bar{\boldsymbol{k}}'_i$ and $\bar{\boldsymbol{k}}^0_i$ denote those eigenvalues of $\bar{\boldsymbol{k}}_i$ which are connected with the eigenvalues $\boldsymbol{k}'_i$ and $\boldsymbol{k}^0_i$ by (97), i. e.

$$\bar{\boldsymbol{k}}'_i = \mathscr{L}(\boldsymbol{k}'_i); \quad \bar{\boldsymbol{k}}^0_i = \mathscr{L}(\boldsymbol{k}^0_i). \qquad (98)$$

The result of the preceding investigation may then be written

$$\sqrt{[W'_i]}\,\left|(\boldsymbol{k}'\,|\,R\,|\,\boldsymbol{k}^0)\right|\sqrt{[W^0_i]} = \sqrt{[\overline{W}'_i]}\,\left|(\bar{\boldsymbol{k}}'\,|\,\bar{R}\,|\,\bar{\boldsymbol{k}}^0)\right|\sqrt{[\overline{W}^0_i]}\,, \qquad (99)$$

where the symbol $[W_i']$ is used for the product of the energies corresponding to the values $(\boldsymbol{k}_1', \cdots \boldsymbol{k}_{n'}')$ of the momentum variables.

In his papers quoted above HEISENBERG[2] has stated a more general equation viz.

$$\sqrt{[W_i']}\,(\boldsymbol{k}'|R|\boldsymbol{k}^0)\sqrt{[W_i^0]} = \sqrt{[\overline{W}_i']}\,(\overline{\boldsymbol{k}}'|\overline{R}|\overline{\boldsymbol{k}}^0)\sqrt{[\overline{W}_i^0]} \tag{100}$$

where the modulus $|(\boldsymbol{k}'|R|\boldsymbol{k}^0)|$ in (99) is replaced by $(\boldsymbol{k}'|R|\boldsymbol{k}^0)$. In contrast to the equation (99) which holds for any choice of the phases in the $\overline{\boldsymbol{k}}$-representation, the equation (100) cannot be true for all $\overline{\boldsymbol{k}}$-representations, since a change in phase in the $\overline{\boldsymbol{k}}$-representation will change the right hand side of (100) in accordance with (2) while the left hand side remains unchanged. By means of the equation (25) and the corresponding equation

$$\overline{R} + \overline{R}^\dagger + \overline{R}^\dagger \overline{R} = 0 \tag{101}$$

for the transformed matrix $\overline{R}$, it may be shown, however, that it is possible, for given phases of the $\boldsymbol{k}$-representation, to choose the phases in the $\overline{\boldsymbol{k}}$-representation in such a way that (100) is true.

According to the special theory of relativity the momentum and energy variables of a particle transform like the components of a four vector. Hence we find by a simple calculation that the functional determinant Δ', corresponding to the transformation $(\boldsymbol{k}_1', \cdots \boldsymbol{k}_{n'}') \to (\overline{\boldsymbol{k}}_1', \cdots \overline{\boldsymbol{k}}_{n'}')$ given by (98), is

$$\Delta' = \frac{\partial(\overline{\boldsymbol{k}}_1' \cdots \overline{\boldsymbol{k}}_{n'}')}{\partial(\boldsymbol{k}_1' \cdots \boldsymbol{k}_{n'}')} = \frac{[\overline{W}_i']}{[W_i']}. \tag{102}$$

We may now put

$$(\boldsymbol{k}'|R|\boldsymbol{k}^0) = |(\boldsymbol{k}'|R|\boldsymbol{k}^0)|\,e^{i\,r(\boldsymbol{k}';\boldsymbol{k}^0)} \tag{103}$$

and

$$(\overline{\boldsymbol{k}}'|\overline{R}|\overline{\boldsymbol{k}}^0) = |(\overline{\boldsymbol{k}}'|\overline{R}|\overline{\boldsymbol{k}}^0)|\,e^{i\,\overline{r}(\overline{\boldsymbol{k}}';\overline{\boldsymbol{k}}^0)}, \tag{104}$$

where the arguments $r(\boldsymbol{k}';\boldsymbol{k}^0)$ and $\overline{r}(\overline{\boldsymbol{k}}';\overline{\boldsymbol{k}}^0)$ are real functions of the variables $(\boldsymbol{k}', \boldsymbol{k}^0)$ and $(\overline{\boldsymbol{k}}', \overline{\boldsymbol{k}}^0)$, respectively.

If we change the phases in the $\boldsymbol{k}$-representation in accordance with (2), the argument $r(\boldsymbol{k}';\boldsymbol{k}^0)^\times$ of the new representative of R, defined by

$$(\boldsymbol{k}' \,|\, R \,|\, \boldsymbol{k}^0)^{\times} = |(\boldsymbol{k}' \,|\, R \,|\, \boldsymbol{k}^0)|\, e^{i\, r(\boldsymbol{k}';\boldsymbol{k}^0)^{\times}} \tag{105}$$

is obviously given by

$$r(\boldsymbol{k}';\boldsymbol{k}^0)^{\times} = r(\boldsymbol{k}';\boldsymbol{k}^0) + \alpha(\boldsymbol{k}') - \alpha(\boldsymbol{k}^0). \tag{106}$$

When we use (104) and (17), the equation (101), written in terms of representatives, becomes

$$\left.\begin{aligned} &|(\bar{\boldsymbol{k}}' \,|\, \bar{R} \,|\, \bar{\boldsymbol{k}}^0)|\, e^{i\bar{r}(\bar{\boldsymbol{k}}';\bar{\boldsymbol{k}}^0)} + |(\bar{\boldsymbol{k}}^0 \,|\, \bar{R} \,|\, \bar{\boldsymbol{k}}')|\, e^{-i\bar{r}(\bar{\boldsymbol{k}}^0;\bar{\boldsymbol{k}}')} + \\ &+ \int |(\bar{\boldsymbol{k}}'' \,|\, \bar{R} \,|\, \bar{\boldsymbol{k}}')|\, \boldsymbol{d}\bar{\boldsymbol{k}}''\, |(\bar{\boldsymbol{k}}'' \,|\, \bar{R} \,|\, \bar{\boldsymbol{k}}^0)|\, e^{i[\bar{r}(\bar{\boldsymbol{k}}'';\bar{\boldsymbol{k}}^0) - \bar{r}(\bar{\boldsymbol{k}}'';\bar{\boldsymbol{k}}')]} = 0, \end{aligned}\right\} \tag{107}$$

where $(\bar{\boldsymbol{k}}'', \bar{\boldsymbol{k}}', \bar{\boldsymbol{k}}^0)$ are connected with $(\boldsymbol{k}'', \boldsymbol{k}', \boldsymbol{k}^0)$ by equations of the form (98). On account of (99) and (102), (107) may be written

$$\begin{aligned} &|(\boldsymbol{k}' \,|\, R \,|\, \boldsymbol{k}^0)|\, e^{i\bar{r}(\bar{\boldsymbol{k}}';\bar{\boldsymbol{k}}^0)} + |(\boldsymbol{k}^0 \,|\, R \,|\, \boldsymbol{k}')|\, e^{-i\bar{r}(\bar{\boldsymbol{k}}^0;\bar{\boldsymbol{k}}')} + \\ &+ \int |(\boldsymbol{k}'' \,|\, R \,|\, \boldsymbol{k}')|\, \boldsymbol{d}\boldsymbol{k}''\, |(\boldsymbol{k}'' \,|\, R \,|\, \boldsymbol{k}^0)|\, e^{i[\bar{r}(\bar{\boldsymbol{k}}'';\bar{\boldsymbol{k}}^0) - \bar{r}(\bar{\boldsymbol{k}}'';\bar{\boldsymbol{k}}')]} = 0 \end{aligned}$$

or, by means of (103)

$$\left.\begin{aligned} &(\boldsymbol{k}' \,|\, R \,|\, \boldsymbol{k}^0)\, e^{i[\bar{r}(\bar{\boldsymbol{k}}';\bar{\boldsymbol{k}}^0) - r(\boldsymbol{k}';\boldsymbol{k}^0)]} + (\boldsymbol{k}' \,|\, R^{\dagger} \,|\, \boldsymbol{k}^0)\, e^{-i[\bar{r}(\bar{\boldsymbol{k}}^0;\bar{\boldsymbol{k}}') - r(\boldsymbol{k}^0;\boldsymbol{k}')]} + \\ &\int (\boldsymbol{k}' \,|\, R^{\dagger} \,|\, \boldsymbol{k}'')\, \boldsymbol{d}\boldsymbol{k}''\, (\boldsymbol{k}'' \,|\, R \,|\, \boldsymbol{k}^0)\, e^{i[\bar{r}(\bar{\boldsymbol{k}}'';\bar{\boldsymbol{k}}^0) - \bar{r}(\bar{\boldsymbol{k}}'';\bar{\boldsymbol{k}}') - r(\boldsymbol{k}'';\boldsymbol{k}^0) + r(\boldsymbol{k}'';\boldsymbol{k}')]} = 0. \end{aligned}\right\} \tag{108}$$

The equation (108) must, for arbitrary $\mathscr{L}$ in (98), be identical with the equation (25), which, in terms of representatives, reads

$$(\boldsymbol{k}' \,|\, R \,|\, \boldsymbol{k}^0) + (\boldsymbol{k}' \,|\, R^{\dagger} \,|\, \boldsymbol{k}^0) + \int (\boldsymbol{k}' \,|\, R^{\dagger} \,|\, \boldsymbol{k}'')\, \boldsymbol{d}\boldsymbol{k}''\, (\boldsymbol{k}'' \,|\, R \,|\, \boldsymbol{k}^0) = 0. \tag{109}$$

This is possible only if the three exponential functions in (108) are equal for all values of the momentum variables. Thus we get the following two equations:

$$\bar{r}(\bar{\boldsymbol{k}}';\bar{\boldsymbol{k}}^0) + \bar{r}(\bar{\boldsymbol{k}}^0;\bar{\boldsymbol{k}}') = r(\boldsymbol{k}';\boldsymbol{k}^0) + r(\boldsymbol{k}^0;\boldsymbol{k}') \tag{110}$$

$$\bar{r}(\bar{\boldsymbol{k}}'';\bar{\boldsymbol{k}}') - \bar{r}(\bar{\boldsymbol{k}}'';\bar{\boldsymbol{k}}^0) + \bar{r}(\bar{\boldsymbol{k}}';\bar{\boldsymbol{k}}^0) = r(\boldsymbol{k}'';\boldsymbol{k}') - r(\boldsymbol{k}'';\boldsymbol{k}^0) + r(\boldsymbol{k}';\boldsymbol{k}^0). \tag{111}$$

These equations are not independent, however, since (110) may be derived from (111) by adding the equation obtained from (111) by interchanging $(\boldsymbol{k}')$ and $(\boldsymbol{k}^0)$.

We can now put

$$r(\boldsymbol{k}';\boldsymbol{k}^0) = \bar{r}(\bar{\boldsymbol{k}}';\bar{\boldsymbol{k}}^0) + \bar{f}(\bar{\boldsymbol{k}}';\bar{\boldsymbol{k}}^0), \tag{112}$$

where the function $\bar{f}$, by means of (98), may be regarded as a function of the transformed variables. If we introduce (112) into (110) we get

$$\bar{f}(\bar{\boldsymbol{k}}^0;\bar{\boldsymbol{k}}') = -\bar{f}(\bar{\boldsymbol{k}}';\bar{\boldsymbol{k}}^0). \tag{113}$$

Similarly we get from (111), (112), and (113)

$$\bar{f}(\bar{\boldsymbol{k}}';\bar{\boldsymbol{k}}'') + \bar{f}(\bar{\boldsymbol{k}}'';\bar{\boldsymbol{k}}^0) = \bar{f}(\bar{\boldsymbol{k}}';\bar{\boldsymbol{k}}^0). \tag{114}$$

This equation must hold for all values of the independent variables $(\bar{\boldsymbol{k}}', \bar{\boldsymbol{k}}'', \bar{\boldsymbol{k}}^0)$ and represents a functional equation for $\bar{f}$. The general solution of (114) is

$$\bar{f}(\bar{\boldsymbol{k}}';\bar{\boldsymbol{k}}^0) = \bar{\alpha}(\bar{\boldsymbol{k}}') - \bar{\alpha}(\boldsymbol{k}^0), \tag{115}$$

where $\bar{\alpha}(\bar{\boldsymbol{k}}')$ is an arbitrary function of the variables $(\bar{\boldsymbol{k}}')$ only.

Thus, (112) takes the form

$$r(\boldsymbol{k}';\boldsymbol{k}^0) = \bar{r}(\bar{\boldsymbol{k}}';\bar{\boldsymbol{k}}^0) + \bar{\alpha}(\bar{\boldsymbol{k}}') - \bar{\alpha}(\bar{\boldsymbol{k}}^0). \tag{116}$$

By a suitable change of the phases in the $\bar{\boldsymbol{k}}$-representation, in accordance with (106), the right hand side of (116) may be made equal to the argument function $\bar{r}(\bar{\boldsymbol{k}}';\bar{\boldsymbol{k}}^0)^{\times}$ in the new $\bar{\boldsymbol{k}}$-representation. If we afterwards omit the cross by which all functions in the new $\bar{\boldsymbol{k}}$-representation are distinguished, we thus have

$$r(\boldsymbol{k}';\boldsymbol{k}^0) = \bar{r}(\bar{\boldsymbol{k}}';\bar{\boldsymbol{k}}^0), \tag{117}$$

showing that the quantity $r(\boldsymbol{k}';\boldsymbol{k}^0)$ is invariant provided the phases of the $\bar{\boldsymbol{k}}$-representation are suitably chosen. With this choice (100) is seen to be a consequence of (99), (103), (104), and (117). The equation (100) will then hold also for the matrices S and η defined by (26) and (43).

Even on condition that (117) and (100) are to hold, the phases of the $\bar{\boldsymbol{k}}$-representation are not at all uniquely determined by the phases adopted in the $\boldsymbol{k}$-representation, since the representatives of R, on account of the δ-functions in (23) will be invariant under all such transformations of the type (2), where α is an arbitrary function of the total momentum and energy only. In particular we may choose the same phases in the $\bar{\boldsymbol{k}}$-representation as in the $\boldsymbol{k}$-representation, in which case the representatives of a matrix A in the two representations, according to (47) and (102), are connected by the equation

$$(\boldsymbol{k}' \,|\, A \,|\, \boldsymbol{k}^0) = \sqrt{\frac{[\overline{W}_i']}{[W_i']}}\,(\bar{\boldsymbol{k}}' \,|\, A \,|\, \bar{\boldsymbol{k}}^0)\sqrt{\frac{[\overline{W}_i^0]}{[W_i^0]}}. \tag{118}$$

Now the representative of $\bar{S}$ in the $\boldsymbol{k}$-representation is, by (118) and (100) applied to S instead of R, given by

$$(\boldsymbol{k}' \,|\, \bar{S} \,|\, \boldsymbol{k}^0) = \sqrt{\frac{[\overline{W}_i']}{[W_i']}}\,(\bar{\boldsymbol{k}}' \,|\, \bar{S} \,|\, \bar{\boldsymbol{k}}^0)\sqrt{\frac{[\overline{W}_i^0]}{[W_i^0]}} = (\boldsymbol{k}' \,|\, S \,|\, \boldsymbol{k}^0), \tag{119}$$

or

$$\bar{S} = S. \tag{120}$$

Thus Heisenberg's characteristic matrix is an invariant matrix, i. e. S and $\bar{S}$ have the same eigenvalues and the same eigenstates.

The matrix S, however, is not only invariant in the sense of equation (100). Since all frames of reference moving with constant velocity, are entirely equivalent as regards the development of the collision process, all functions like $(\boldsymbol{k}' \,|\, \Psi \,|\, \boldsymbol{k}^0)$, $(\boldsymbol{k}' \,|\, U \,|\, \boldsymbol{k}^0)$, and $(\boldsymbol{k}' \,|\, S \,|\, \boldsymbol{k}^0)$ must be what may be called invariant in form, i. e. $(\bar{\boldsymbol{k}}' \,|\, \bar{S} \,|\, \bar{\boldsymbol{k}}^0)$ is the same function of the transformed variables $(\bar{\boldsymbol{k}}')$ and $(\bar{\boldsymbol{k}}^0)$ as the function $(\boldsymbol{k}' \,|\, S \,|\, \boldsymbol{k}^0)$ of the variables $(\boldsymbol{k}')$ and $(\boldsymbol{k}^0)$. This means that the invariant quantities in (100) must be functions of the four-dimensional scalar products $\boldsymbol{k}_i'\boldsymbol{k}_l' - W_i' W_l'$, $\boldsymbol{k}_i'\boldsymbol{k}_l^0 - W_i' W_l'$, or $\boldsymbol{k}_i^0\boldsymbol{k}_l^0 - W_i^0 W_l^0$, only.

In what follows we shall exclusively have to do with form-invariant matrices. Any such matrix A will obviously have the same eigenvalues as the corresponding transformed matrix $\bar{A}$, but A and $\bar{A}$ will not in general have the same eigenstates. As an example we may take the magnitude M^2 of the total angular momentum, the total kinetic energy, or the matrix U in (11). Only if A is invariant in the sense of (100) or (120), the eigenvalues and eigenstates of A and $\bar{A}$ will be identical.

3. The eigenvalue problem in the new theory. Constants of collision. Consequences of the invariance of S.

The main problem in the usual quantum mechanics is that of finding the canonical transformation which brings the Hamiltonian on diagonal form, or in other words of determining the eigenvalues and eigenfunctions of the Hamiltonian. This problem is equivalent to the problem of finding a complete set of commuting constants of the motion, i. e. variables which commute with the Hamiltonian and with each other. In the representation where these constants of the motion are represented by diagonal matrices, the Hamiltonian will also be on diagonal form. In some cases it is possible at once to write down a number of constants of the motion. For instance, if the Hamiltonian is a form-invariant under all rotations in ordinary space, the components of the total angular momentum commute with the Hamiltonian, and the magnitude of the angular momentum together with the component in a fixed direction then will form an (incomplete) set of commuting constants of the motion.

In the new theory we are faced with the analogous problem of transforming the characteristic matrix on diagonal form. Since, however, the matrix S is invariant under a larger group of transformations than the Hamiltonian, we are able at once to write down a larger number of quantities, which commute with S. The most important among these quantities are those which also commute with the total kinetic energy W. Such quantities commuting with S and W we may call constants of collision or collision constants, since they have the same values (or mean values) in the initial and final states.[10)]

On account of the factor $\delta(\boldsymbol{K}'-\boldsymbol{K}^0)\,\delta(W'-W^0)$ in (23), R and therefore also $S = 1+R$ commute with the components of the total momentum $\boldsymbol{K}$ and with the total kinetic energy W. The four quantities

$$(K^\mu) = (K_x, K_y, K_z, W) \tag{121}$$

thus represent a set of commuting collision constants. This result is connected with the invariance of S under space-time translations. Similarly the invariance of S under all rotations in four-dimensional space gives us at once six other quantities commuting with S.

Let us for a moment consider a space of m dimensions with coordinates $(x^r) = (x^1, x^2, \cdots, x^m)$. A general infinitesimal transformation of coordinates in this space is then given by

$$\overline{x}^r = x^r + \varepsilon f^r(x^1, \cdots, x^m), \tag{122}$$

where ε is an infinitesimal parameter, while f^r is an arbitrary function of the coordinates. Now, consider a quantity $a = a(x^1, \cdots, x^m)$ depending on the coordinates (x^r). If $\overline{a}$ is the corresponding transformed quantity, the 'substantial' variation of a is defined by

$$\delta a = \overline{a}(\overline{x}) - a(x), \tag{123}$$

while the 'local' variation is given by

$$\delta^* a = \overline{a}(x) - a(x). \tag{124}$$

Neglecting terms of the second order in ε, we obviously have

$$\delta a = \delta^* a + \varepsilon \sum_r f^r \frac{\partial}{\partial x^r} a. \tag{125}$$

If a is a form-invariant quantity, i. e. if $\overline{a}(\overline{x})$ is the same function of the transformed variables $(\overline{x})$ as a of the variables (x), we have $\overline{a}(x) = a(x)$, or

$$\delta^* a = 0. \tag{126}$$

Hence

$$\delta a = \varepsilon \sum_r f^r \frac{\partial}{\partial x^r} a. \tag{127}$$

Now, let the variables (x^r) be identified with the components of the momentum vectors $\{(\boldsymbol{k}'), (\boldsymbol{k}^0)\} = \{\boldsymbol{k}'_1 \cdots \boldsymbol{k}'_{n'}, \boldsymbol{k}^0_1 \cdots \boldsymbol{k}^0_{n^0}\}$ and let a be the function $a(\boldsymbol{k}', \boldsymbol{k}^0) = \delta(\boldsymbol{K}' - \boldsymbol{K}^0)\, \delta(W' - W^0)$. For an infinitesimal Lorentz transformation in the direction of the x-axis we have for all particles, if ε is the infinitesimal relative velocity,

$$\overline{k}'_{ix} = k'_{ix} - \varepsilon W'_i, \ \overline{k}'_{iy} = k'_{iy}, \ \overline{k}'_{iz} = k'_{iz} \tag{128}$$

and similar transformation equations for the variables $\boldsymbol{k}_i^0$. By means of (127) we get in this case, since $x\,\delta(x) = 0$,

$$\begin{aligned}\delta a &= \delta(\overline{\boldsymbol{K}}' - \overline{\boldsymbol{K}}^0)\,\delta(\overline{W}' - \overline{W}^0) - \delta(\boldsymbol{K}' - \boldsymbol{K}^0)\,\delta(W' - W^0)\\ &= -\varepsilon \sum_i \left(W_i' \frac{\partial}{\partial k_{ix}'} + W_i^0 \frac{\partial}{\partial k_{ix}^0}\right) \delta(\boldsymbol{K}' - \boldsymbol{K}^0)\,\delta(W' - W^0)\\ &= -\varepsilon\,\big[\delta'(K_x' - K_x^0)\,(W' - W^0)\,\delta(W' - W^0)\\ &\qquad + (K_x' - K_x^0)\,\delta(K_x' - K_x^0)\,\delta'(W' - W^0)\big]\,\delta(K_y' - K_y^0)\,\delta(K_z' - K_z^0)\\ &= 0.\end{aligned}$$

In the same way it is shown that the function

$$\delta(\boldsymbol{K}' - \boldsymbol{K}^0)\,\delta(W' - W^0) = \delta(\overline{\boldsymbol{K}}' - \overline{\boldsymbol{K}}^0)\,\delta(\overline{W}' - \overline{W}^0) \quad (129)$$

is invariant for a general Lorentz transformation, in contrast to the function $\delta(\boldsymbol{K}' - \boldsymbol{K}^0)\,\delta_+(W' - W^0)$, which has more complicated transformation properties.

Now, let a be given by

$$a(\boldsymbol{k}', \boldsymbol{k}^0) = \sqrt{[W_i']}\,(\boldsymbol{k}' \,|\, A \,|\, \boldsymbol{k}^0)\,\sqrt{[W_i^0]},$$

where A is any form-invariant matrix. A need not, however, be really invariant in the sense of (100) and (120). For the Lorentz transformation (128) we then by means of (127) get

$$\left.\begin{aligned}\delta a &= \sqrt{[\overline{W}_i']}\,(\overline{\boldsymbol{k}}' \,|\, \bar{A} \,|\, \overline{\boldsymbol{k}}^0)\,\sqrt{[\overline{W}_i^0]} - \sqrt{[W_i']}\,(\boldsymbol{k}' \,|\, A \,|\, \boldsymbol{k}^0)\,\sqrt{[W_i^0]}\\ &= -\varepsilon \sum_i \left(W_i' \frac{\partial}{\partial k_{ix}'} + W_i^0 \frac{\partial}{\partial k_{ix}^0}\right) \sqrt{[W_i']}\,(\boldsymbol{k}' \,|\, A \,|\, \boldsymbol{k}^0)\,\sqrt{[W_i^0]}\\ &= -\varepsilon \sqrt{[W_i']\,[W_i^0]} \sum_i \left\{\frac{1}{2}\left(W_i' \frac{\partial}{\partial k_{ix}'} + \frac{\partial}{\partial k_{ix}'} W_i'\right)\right.\\ &\qquad\qquad \left. + \frac{1}{2}\left(W_i^0 \frac{\partial}{\partial k_{ix}^0} + \frac{\partial}{\partial k_{ix}^0} W_i^0\right)\right\} (\boldsymbol{k}' \,|\, A \,|\, \boldsymbol{k}^0).\end{aligned}\right\} \quad (130)$$

Here we have used equations of the type

$$W_i' \frac{\partial}{\partial k_{ix}'} \sqrt{[W_i']} = \sqrt{[W_i']}\,\frac{1}{2}\left(W_i' \frac{\partial}{\partial k_{ix}'} + \frac{\partial}{\partial k_{ix}'} W_i'\right), \quad (131)$$

which follow immediately from the definition of $[W_i']$ in (99).

Now, let $\boldsymbol{\xi}_i = (\xi_i, \eta_i, \zeta_i)$ be the vector operator, which, in the $\boldsymbol{k}$-representation, is identical with the operator of differentiation

$$i\frac{\partial}{\partial \boldsymbol{k}_i'} = i\left(\frac{\partial}{\partial k_{ix}'}, \frac{\partial}{\partial k_{iy}'}, \frac{\partial}{\partial k_{iz}'}\right)$$

when operating to the right, and with

$$-i\frac{\partial}{\partial \boldsymbol{k}_i^0} = -i\left(\frac{\partial}{\partial k_{ix}^0}, \frac{\partial}{\partial k_{iy}^0}, \frac{\partial}{\partial k_{iz}^0}\right).$$

when operating to the left. When we define a new vector operator $\boldsymbol{N} = (N_x, N_y, N_z)$ by

$$\boldsymbol{N} = \sum_i \overline{W_i \boldsymbol{\xi}_i} = \frac{1}{2}\sum_i (W_i \boldsymbol{\xi}_i + \boldsymbol{\xi}_i W_i), \tag{132}$$

the equation (130) may, after division with $\sqrt{[W_i']}\sqrt{[W_i^0]}$, be written

$$(\boldsymbol{k}' | \bar{A} | \boldsymbol{k}^0) - (\boldsymbol{k}' | A | \boldsymbol{k}^0) = -\frac{\varepsilon}{i}(\boldsymbol{k}' | N_x A - A N_x | \boldsymbol{k}^0), \tag{133}$$

where we have made use of (118).

Introducing the quantum Poisson Bracket (P. B.) of two variables A and B by

$$[A, B] = \frac{1}{i}(AB - BA), \tag{134}$$

we may write (133) as

$$\delta A = \bar{A} - A = \varepsilon [A, N_x]. \tag{135}$$

For a finite Lorentz transformation with the relative velocity v we then have

$$\bar{A} = e^{i\varkappa N_x} A e^{-i\varkappa N_x} \tag{136}$$

with $\varkappa = \mathrm{tgh}^{-1} v$. If instead we had considered Lorentz transformations in the directions of the other coordinate axes, we had got similar formulae with N_x replaced by N_y or by N_z.

We shall now consider a rotation through an infinitesimal angle ε in ordinary space around the z-axis. The transformation equations are for each particle

$$\left.\begin{aligned} \bar{k}'_{ix} &= k'_{ix} + \varepsilon k'_{iy} \\ \bar{k}'_{iy} &= k'_{iy} - \varepsilon k'_{ix} \\ \bar{k}'_{iz} &= k'_{iz} \end{aligned}\right\} \quad (137)$$

and similar equations for $\boldsymbol{k}_i^0$.

By means of (127) we get in this case in the same way as before

$$\delta A = \bar{A} - A = \varepsilon [A, M_z], \quad (138)$$

where M_z is the z-component of the vector operator $\boldsymbol{M}$ defined by

$$\boldsymbol{M} = \sum_i (\boldsymbol{\xi}_i \times \boldsymbol{k}_i). \quad (139)$$

Hence for a finite rotation through an angle θ

$$\bar{A} = e^{i\theta M_z} A e^{-i\theta M_z}. \quad (140)$$

For rotations around the x- and y-axes we have simply to replace M_z in these formulae by M_x and M_y, respectively. The transformations (136) and (140) are contact transformations. Thus any functional relation between (form-invariant) matrices, as for instance commutation relations, will be covariant under Lorentz transformations.

According to its definition the operator $\boldsymbol{\xi}_i$ depends on the phases in the $\boldsymbol{k}$-representation and, by a suitable choice of the phases, $\boldsymbol{\xi}_i$ may be made equal to the coordinate vector $\boldsymbol{x}_i$ of the i'th particle. In this latter case the phase constant γ' in (80) is zero and the vector $\boldsymbol{M}$ is identical with the total angular momentum. In another Lorentz frame of reference we have a similar operator $\bar{\boldsymbol{\xi}}_i$ which in the $\bar{\boldsymbol{k}}$-representation is represented by differentiation operators. Provided the phases in the $\bar{\boldsymbol{k}}$-representation are chosen in accordance with the transformation formula (118), the connection between $\bar{\boldsymbol{\xi}}_i$ and $\boldsymbol{\xi}_i$ is given by (136) and (140).

The variables $\bar{\boldsymbol{\xi}}_i$ defined in this way, however, in general will not be equal to the coordinate vectors $\bar{\boldsymbol{x}}_i$ of the i'th particle in the new Lorentz frame, even if the phases in the $\boldsymbol{k}$-representation are chosen in such a way that $\boldsymbol{\xi}_i$ is equal to $\boldsymbol{x}_i$. This may be simply illustrated by considering the non-relativistic approximation, where the Lorentz transformation takes the simple form of a Galilei transformation

$$\left.\begin{aligned} &\bar{x}_i = x_i - vt, \quad \bar{y}_i = y_i, \quad \bar{z}_i = z_i \\ &\bar{k}_{ix} = k_{ix} - v\kappa, \quad \bar{k}_{iy} = k_{iy}, \quad \bar{k}_{iz} = k_{iz}. \end{aligned}\right\} \quad (141)$$

In this approximation we have

$$\boldsymbol{N} = \sum_i \kappa \boldsymbol{\xi}_i. \tag{142}$$

Thus $\boldsymbol{N}$ commutes with ξ_i, η_i, and ζ_i, and we get from (136)

$$\bar{\xi}_i = \xi_i, \tag{143}$$

so that $\bar{\xi}_i$ will be different from $\bar{x}_i$ defined by (141), even if $\boldsymbol{\xi}_i$ is equal to $\boldsymbol{x}_i$. In this latter case we have $\gamma' = 0$ in (80), but the corresponding phase constant $\bar{\gamma}'$ in the new Lorentz frame will be given by

$$\bar{\gamma}' = vt\bar{K}_x'. \tag{144}$$

When A in (135) and (138) is the matrix S, we have $\delta S = \bar{S} - S = 0$ on account of the invariance of S under all rotations in space-time. Thus S must commute with the components

$$M_x,\ M_y,\ M_z,\ N_x,\ N_y,\ N_z \tag{145}$$

of the vectors $\boldsymbol{M}$ and $\boldsymbol{N}$. The components of the former vector are even constants of collision, since $\boldsymbol{M}$ commutes with W. This follows from (138), remembering that W is invariant under all rotations in ordinary space.

The concept of a constant of collision is obviously more comprehensive than the concept of a constant of motion. The latter quantity can only be defined in cases where a Hamiltonian exists, while a constant of collision for its definition only requires the existence of the characteristic matrix. If a Hamiltonian of the system exists, six of the seven collision constants in (121) and (145), viz. the components of $\boldsymbol{K}$ and $\boldsymbol{M}$, are constants of the motion, while W is a collision constant only. The total kinetic energy of the particles has the same value in the initial and final states and is thus a constant of collision, but it is not, of course, a constant of the motion, since the kinetic energy is partly transformed into potential energy during the collision. In the new theory, which renounces a detailed description of the collision process and considers the results of the collision as observable only, the constants of collision will probably take over the role played by the constants of motion in quantum mechanics.

The quantities (145) are not commuting. Our next problem will be to construct a maximal number of functions of the variables (121) and (145), which commute with each other and with the variables (121). These functions will then also commute with S and they will together with the variables (121) form a set of commuting constants of collision. Now let (α) denote a complete set of commuting collision constants. In the α-representation the matrix S (or η) will then be diagonal and the eigenvalues of S will be functions of the eigenvalues of (α). Thus S may in any representation be regarded as a function of the α's. Since S is an invariant matrix, it will, however, be a function of invariant combinations of the α's only. Such invariant combinations are also constants of collision and it is convenient from the beginning to choose these invariant collision constants as members of the complete set. We shall therefore begin with an investigation of the transformation properties of the variables (121) and (145). For this purpose we must establish the commutability relations for these variables.

For the variables $\boldsymbol{\xi}_i$ and $\boldsymbol{k}_i$ we have the canonical relations

$$[k_{ix}, k_{jy}] = 0, \; [\xi_i, \eta_j] = 0, \; [\xi_i, k_{jx}] = \delta_{ij}, [\xi_i, k_{jy}] = 0, \qquad (146)$$

etc. Hence for the components of the vector $\boldsymbol{M}$ in (139) we get the well-known commutation relations for the components of the angular momentum in quantum mechanics

$$[M_x, M_y] = M_z, \cdots \qquad (147)$$

Here we have explicitly written down one only of the three equations, following from each other by cyclic permutation of the letters x, y, z. In the same way we get, by (132), (139), and (146),

$$\left.\begin{array}{l} [N_x, M_x] = 0, \; [N_x, M_y] = N_z, \; [N_x M_z] = -N_y, \cdots \\ [N_x, N_y] = -M_z, \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{array}\right\} (148)$$

where again, as everywhere in what follows, the dots indicate equations, which may be obtained from those explicitly written down by cyclic permutation of the letters x, y, z.

Finally for the P.B.'s of a variable (121) with a variable (145) we get

$$\left.\begin{array}{l} [M_x, K_x] = 0, \; [M_x, K_y] = K_z, [M_x, K_z] = -K_y, [M_x, W] = 0 \cdots \\ [\boldsymbol{N}, W] = \boldsymbol{K}, [N_x, K_x] = W, \; [N_x, K_y] = 0 \cdots\cdots\cdots\cdots \end{array}\right\} (149)$$

The commutation relations (149) simply express the four-vector character of $(\boldsymbol{K}^{\mu}) = (\boldsymbol{K}, W)$. For an infinitesimal Lorentz transformation in the direction of the x-axis, we have, for instance, by (135) and (149)

$$\left.\begin{aligned} \delta K_x &= -\varepsilon W,\ \delta K_y = \delta K_z = 0 \\ \delta W &= -\varepsilon K_x, \end{aligned}\right\} \tag{150}$$

which are the transformation equations of a four-vector.

Thus the P.B.-relations (149) must remain true, if $(\boldsymbol{K}, W)$ is replaced by any other matrix four-vector.

Similarly it follows from (147) and (148) that the quantities

$$(M^{\mu\nu}) = \begin{pmatrix} 0 & M_z & -M_y & N_x \\ -M_z & 0 & M_x & N_y \\ M_y & -M_x & 0 & N_z \\ -N_x & -N_y & -N_z & 0 \end{pmatrix} \tag{151}$$

are the components of an antisymmetrical tensor; in fact we get as before, by (135) and (148), the equations

$$\left.\begin{aligned} \delta M_x &= 0, & \delta M_y &= -\varepsilon N_z, & \delta M_z &= \varepsilon N_y, \\ \delta N_x &= 0, & \delta N_y &= \varepsilon M_z, & \delta N_z &= -\varepsilon M_y, \end{aligned}\right\} \tag{152}$$

which are just the transformation equations for the components of an antisymmetrical tensor. The square of this tensor

$$\frac{1}{4} M^{\mu\nu} M_{\mu\nu} = \frac{1}{2}(|\boldsymbol{M}|^2 - |\boldsymbol{N}|^2)$$

is an invariant scalar, it thus commutes with the components of $\boldsymbol{M}$ and $\boldsymbol{N}$. Further, as a function of the variables (145) it also commutes with S.

The antisymmetrical tensor $M^{\mu\nu}$ has a 'dual' tensor $\breve{M}^{\mu\nu}$, which is obtained from (151) by replacing $\boldsymbol{M}$ by $\boldsymbol{N}$ and $\boldsymbol{N}$ by $-\boldsymbol{M}$. The product of the two tensors

$$\frac{1}{4} M^{\mu\nu} \breve{M}_{\mu\nu} = \boldsymbol{M}\boldsymbol{N}$$

is a pseudoscalar, i. e. it behaves like a scalar under all rotations in space-time, but changes its sign by spatial reflections at the origin. The quantities

$$\frac{1}{2}\left[|\boldsymbol{M}|^2-|\boldsymbol{N}|^2\right], \quad \boldsymbol{M}\boldsymbol{N} \tag{153}$$

thus commute with S and with all the variables (145). Moreover they are the only independent functions of the variables (145) of that kind.

The quantities (153) do not, however, commute with the variables (121). If we construct the P.B.'s of these quantities with the components of the four-vector $K^{\mu} = (\boldsymbol{K}, W)$, we get a new four-vector

$$G^{\mu} = (\boldsymbol{G}, G^4) = \frac{1}{2}\left[|\boldsymbol{M}|^2-|\boldsymbol{N}|^2, K^{\mu}\right] \tag{154}$$

and a pseudo-four-vector

$$\Gamma^{\mu} = (\boldsymbol{\Gamma}, \Gamma^4) = \left[\boldsymbol{M}\boldsymbol{N}, K^{\mu}\right]. \tag{155}$$

By means of (149) we get at once the following expressions for G^{μ} and Γ^{μ}:

$$G^{\mu} = \left\{ \begin{aligned} \boldsymbol{G} &= -(\overline{\boldsymbol{M}\times\boldsymbol{K}})-\overline{\boldsymbol{N}W} \\ G^4 &= -\boldsymbol{N}\boldsymbol{K} \end{aligned} \right\} \tag{156}$$

and

$$\Gamma^{\mu} = \left\{ \begin{aligned} \boldsymbol{\Gamma} &= -(\boldsymbol{N}\times\boldsymbol{K})+\boldsymbol{M}W \\ \Gamma^4 &= \boldsymbol{M}\boldsymbol{K}. \end{aligned} \right\} \tag{157}$$

On account of the symmetrization bars in (156), which have the same meaning as in (132), all the variables G^{μ} and Γ^{μ} are Hermitian. A comparison of (156) and (157) with (151) and with the corresponding expression for $\breve{M}^{\mu\nu}$ shows that

$$\left. \begin{aligned} G^{\mu} &= \overline{M^{\mu\nu}K_{\nu}} \\ \Gamma^{\mu} &= \breve{M}^{\mu\nu}K_{\nu}. \end{aligned} \right\} \tag{158}$$

From G^{μ} and Γ^{μ} we can construct two scalars and one pseudo-scalar:

$$G^{\mu}G_{\mu}, \quad \Gamma^{\mu}\Gamma_{\mu}; \quad \overline{G^{\mu}\Gamma_{\mu}}. \tag{159}$$

Since the quantities $G^{\mu}K_{\mu}$ and $\Gamma^{\mu}K_{\mu}$ are identically zero, on account of the antisymmetry of $M^{\mu\nu}$ and $\breve{M}^{\mu\nu}$, the variables (153) and (159) together with

$$K = \sqrt{-K^\mu K_\mu} = \sqrt{W^2 - K^2} \tag{160}$$

are the only independent invariant functions which can be constructed from the quantities (121) and (145). K is the rest mass of the system as a whole.

We are now particularly interested in those of the variables (153)—(160) which commute with the four-vector (K^μ). By means of (149) it is easily seen that only the variables

$$\Gamma^\mu = (\mathbf{\Gamma}, \Gamma^4), \quad \Gamma^\mu \Gamma_\mu = |\mathbf{\Gamma}|^2 - (\Gamma^4)^2 \tag{161}$$

given by (157), have this property, i. e.

$$\left[\Gamma^\mu, K^\nu\right] = 0, \quad \left[\Gamma^\mu \Gamma_\mu, K^\nu\right] = 0. \tag{162}$$

To prove these equations it is not, however, necessary explicitly to use the relations (149). On account of the covariance of all commutation relations under Lorentz transformations we need only prove (162) for a single component of Γ^μ, say for Γ^4. Now, $\Gamma^4 = \boldsymbol{M}\boldsymbol{K}$ is the component of $\boldsymbol{M}$ in the direction of $\boldsymbol{K}$. Thus, by (138), $[K^\nu, \boldsymbol{M}\boldsymbol{K}]$ determines the variation of the variables $K^\nu = (\boldsymbol{K}, W)$ by a rotation about the direction of $\boldsymbol{K}$, and since both the vector $\boldsymbol{K}$ and W are unchanged by such a rotation, we must have

$$\left[K^\nu, \boldsymbol{M}\boldsymbol{K}\right] = \left[K^\nu, \Gamma^4\right] = 0.$$

The variables Γ^μ do not commute with each other, but they commute with the quantity $\Gamma^\nu \Gamma_\nu$. This statement may be verified by direct calculation, but it also follows from the fact that $\Gamma^\nu \Gamma_\nu$ is invariant, for this means that $\Gamma^\nu \Gamma_\nu$ commutes with $\boldsymbol{M}$ and $\boldsymbol{N}$ besides with K^ν and therefore it also commutes with the Γ^μ, which are functions of $\boldsymbol{M}$, $\boldsymbol{N}$, and K^ν. As a set of commuting constants of collision we may then for instance take the variables (121) together with the two variables

$$\left.\begin{aligned} \Gamma^\mu \Gamma_\mu &= (-\boldsymbol{N}\times\boldsymbol{K}+\boldsymbol{M}W)\,(-\boldsymbol{N}\times\boldsymbol{K}+\boldsymbol{M}W)-(\boldsymbol{M}\boldsymbol{K})\,(\boldsymbol{M}\boldsymbol{K}) \\ \Gamma^3 &= \Gamma_z = -N_x K_y + N_y K_x + M_z W. \end{aligned}\right\} \tag{}$$

We shall now determine the eigenvalues and eigenfunctions of $\Gamma^\mu \Gamma_\mu$ and Γ_z. These variables both commute with the K^μ and they must therefore in the $(\boldsymbol{K}, W, x)$-representation, defined by (47), have the form

$$\left.\begin{aligned}&(\boldsymbol{K}' W' x' \,|\, \Gamma^{\mu}\Gamma_{\mu} \,|\, \boldsymbol{K}^0 W^0 x^0) = \\ &= \delta(\boldsymbol{K}' - \boldsymbol{K}^0)\,\delta(W' - W^0)\,(x' \,|\, (\Gamma^{\mu}\Gamma_{\mu})_{\boldsymbol{K}^0 W^0} \,|\, x^0),\end{aligned}\right\} \tag{164}$$

where $(\Gamma^{\mu}\Gamma_{\mu})_{\boldsymbol{K}^0 W^0}$ is a submatrix corresponding to fixed values $\boldsymbol{K}^0$, W^0 of the total momentum and energy. Since $\Gamma^{\mu}\Gamma_{\mu}$ is invariant, we have for any Lorentz transformation

$$\Gamma^{\mu}\Gamma_{\mu} = \overline{\Gamma}^{\mu}\overline{\Gamma}_{\mu}, \tag{165}$$

where all the variables in the new frame of reference are distinguished by a bar. For the functional determinant corresponding to the transformation $(\boldsymbol{K}', W', x') \to (\overline{\boldsymbol{K}}', \overline{W}', \overline{x}')$, we get

$$\left.\begin{aligned}\frac{\partial(\overline{\boldsymbol{K}}', \overline{W}', \overline{x}')}{\partial(\boldsymbol{K}', W', x')} &= \frac{\partial(\overline{\boldsymbol{K}}', \overline{W}', \overline{x}')}{\partial(\overline{\boldsymbol{k}}_1 \cdots \overline{\boldsymbol{k}}_{n'})} \cdot \frac{\partial(\overline{\boldsymbol{k}}'_1 \cdots \overline{\boldsymbol{k}}'_{n'})}{\partial(\boldsymbol{k}'_1 \cdots \boldsymbol{k}'_{n'})} \cdot \frac{\partial(\boldsymbol{k}'_1 \cdots \boldsymbol{k}'_{n'})}{\partial(\boldsymbol{K}', W', x')} = \\ &= \overline{\Delta}' \frac{[\overline{W}'_i]}{[W'_i]} \cdot \frac{1}{\Delta'},\end{aligned}\right\} \tag{166}$$

by means of (47) and (102). Here $\overline{\Delta}'$ is the same function of the new variables as the function Δ' of the variables of the original frame of reference. Thus (165) may be written

$$\left.\begin{aligned}(\boldsymbol{K}' W' x' \,|\, \Gamma^{\mu}\Gamma_{\mu} \,|\, \boldsymbol{K}^0 W^0 x^0) &= \sqrt{\frac{|\overline{\Delta}'|[\overline{W}'_i]}{|\Delta'|[W'_i]}} \cdot \\ &\quad (\overline{\boldsymbol{K}}' \overline{W}' \overline{x}' \,|\, \overline{\Gamma}^{\mu}\overline{\Gamma}_{\mu} \,|\, \overline{\boldsymbol{K}}^0 \overline{W}^0 \overline{x}^0) \sqrt{\frac{|\overline{\Delta}^0|[\overline{W}^0_i]}{|\Delta^0|[W^0_i]}}.\end{aligned}\right\} \tag{167}$$

Now let the new system of reference be chosen in such a way that $\overline{\boldsymbol{K}}^0 = 0$. In this 'center of gravity' system, which of course depends on the value of the vector $\boldsymbol{K}^0$, the matrix elements of $\overline{\Gamma}^{\mu}\overline{\Gamma}_{\mu}$ have a particularly simple form. From the definition of $\overline{\Gamma}^{\mu}\overline{\Gamma}_{\mu}$ in (163) we simply get

$$(\overline{\boldsymbol{K}}' \overline{W}' \overline{x}' \,|\, \overline{\Gamma}^{\mu}\overline{\Gamma}_{\mu} \,|\, \overline{\boldsymbol{K}}^0 \overline{W}^0 \overline{x}^0) = (\overline{\boldsymbol{K}}' \overline{W}' \overline{x}' \,|\, |\overline{M}|^2 \,|\, \overline{\boldsymbol{K}}^0 \overline{W}^0 \overline{x}^0) \cdot \overline{W}^{0^2}. \tag{168}$$

Since the square of the total angular momentum does not commute with the components of $\boldsymbol{K}$, we cannot in general assign numerical values to $\boldsymbol{M}^2$ and $\boldsymbol{K}$ simultaneously. In the special case, however, where the components of $\boldsymbol{K}$ are all given the value zero, we may also assign a definite numerical value to one of the components of $\boldsymbol{M}$ and to $\boldsymbol{M}^2$, since this will not contradict the commutability relations (149). Thus, in the center of gravity system the matrix elements of $\overline{\boldsymbol{M}}^2$ have the form

$$(\overline{\boldsymbol{K}}'\overline{W}'\overline{x}' \,||\, \overline{M}\,|^2 \,|\, \overline{\boldsymbol{K}}^0\overline{W}^0\overline{x}^0) = \delta(\overline{\boldsymbol{K}}')\,\delta(\overline{W}'-\overline{W}^0)\;(\overline{x}' \,||\, \overline{M}\,|^2_{\overline{\boldsymbol{K}}^0=0,\,\overline{W}^0} \,|\, \overline{x}^0).$$

From (164), (167), 168) and (169) we now get a simple relation between the submatrices $(\Gamma^\mu \Gamma_\mu)_{\boldsymbol{K}^0 W}$ and $|\,\overline{\boldsymbol{M}}\,|^2_{\overline{\boldsymbol{K}}^0=0,\,\overline{W}^0}$. Since the factor $\delta(\boldsymbol{K}'-\boldsymbol{K}^0)\,\delta(W'-W^0)$ is invariant (see (129)), and since further $\overline{W}^{0\,2} = \overline{W}^{0\,2} - |\,\overline{K}^0\,|^2 = \overline{K}^{0\,2} = K^{0\,2}$ by (160), we get, omitting the indices $(\boldsymbol{K}^0\;W^0)$ and $(\overline{\boldsymbol{K}}^0 = 0,\;\overline{W}^0)$ in the notation of the submatrices,

$$(x' \,|\, \Gamma^\mu \Gamma_\mu \,|\, x^0) = K^{0\,2}\sqrt{|D'|}\,(\overline{x}' \,||\, \overline{M}\,|^2 \,|\, \overline{x}^0)\sqrt{|D^0|}, \tag{170}$$

where $D' = D(\boldsymbol{K}^0,\; W^0,\; x')$ is the value of $\sqrt{\dfrac{\overline{\Delta}'\,[\overline{W}_i']}{\Delta'\,[W_i']}}$ for $\boldsymbol{K}' = \boldsymbol{K}^0$ and $W' = W^0$. D' is simply the functional determinant corresponding to a Lorentz transformation in the subspace defined by the variables (x) for a fixed energy-momentum four-vector $K^{0\mu}$, i. e.

$$D' = \frac{\partial\,(\overline{x}')}{\partial\,(x')}, \tag{171}$$

and (170) may thus be written

$$\frac{1}{K^{0\,2}}\,(\Gamma^\mu \Gamma_\mu)_{\boldsymbol{K}^0 W^0} = |\,\overline{\boldsymbol{M}}\,|^2_{\overline{\boldsymbol{K}}^0=0,\,\overline{W}^0}. \tag{172}$$

From the commutation rules (147) it now follows in the usual way that the eigenvalues of the square of the angular momentum are given by $l^0(l^0+1)$, where l^0 may be any integral positive number or zero. The quantity L defined by

$$L = \frac{\Gamma^\mu \Gamma_\mu}{K^2} \tag{173}$$

thus has the eigenvalues

$$L^0 = l^0(l^0+1), \quad l^0 = 0, 1, 2 \cdots. \tag{174}$$

Further if

$$(\overline{\boldsymbol{K}}' \, \overline{W}' \, \overline{x}' \,|\, \overline{\boldsymbol{K}}^0 = 0, \overline{W}^0, l^0)$$

is the representative of any eigenfunction of $|\overline{M}|^2$, the function

$$(\boldsymbol{K}' W' x' \,|\, \boldsymbol{K}^0, W^0, l^0) = \sqrt{|D'|}\,(\overline{\boldsymbol{K}}' \, \overline{W}' \overline{x}' \,|\, \overline{\boldsymbol{K}}^0 = 0, W^0, l^0) \tag{175}$$

will represent an eigenfunction of L with the eigenvalue $L^0 = l^0(l^0+1)$.

The eigenvalues of Γ_z may be obtained in a similar way. Since the Γ^μ transform like a four-vector by Lorentz transformations we have, instead of (165), for the vector $\boldsymbol{\Gamma}$

$$\boldsymbol{\Gamma} = \overline{\boldsymbol{\Gamma}} + \boldsymbol{v}\left\{\frac{\overline{\boldsymbol{\Gamma}}\boldsymbol{v}}{v^2}\left(\frac{1}{\sqrt{1-v^2}} - 1\right) + \frac{\overline{\Gamma}^4}{\sqrt{1-v^2}}\right\}, \tag{176}$$

where the vector $\boldsymbol{v}$ denotes the velocity of the new system of reference relative to the old system. It is assumed that corresponding coordinate axes in the two systems are parallel. If the new system is the center of gravity system, $\boldsymbol{v}$ is the velocity of the center of gravity, i. e.

$$\boldsymbol{v} = \frac{\boldsymbol{K}^0}{W^0}. \tag{177}$$

In this system we may, according to the definitions (157), simply write

$$\overline{\boldsymbol{\Gamma}} = \overline{\boldsymbol{M}}\,\overline{W}^0, \quad \overline{\Gamma}^4 = 0,$$

and by a simple calculation we get instead of (172)

$$\frac{1}{K^0}(\Gamma_z)_{\boldsymbol{K}^0, W^0} = (\overline{a}_x \overline{M}_x + \overline{a}_y \overline{M}_y + \overline{a}_z \overline{M}_z)_{\overline{\boldsymbol{K}}^0 = 0, \overline{W}^0} \tag{178}$$

with

$$\left.\begin{aligned} &\overline{\boldsymbol{a}} = (\overline{a}_x, \overline{a}_y, \overline{a}_z) = \\ &\frac{v_z v_x}{v^2}\left(\frac{1}{\sqrt{1-v^2}} - 1\right), \ \frac{v_z v_y}{v^2}\left(\frac{1}{\sqrt{1-v^2}} - 1\right), \ 1 + \frac{v_z v_z}{v^2}\left(\frac{1}{\sqrt{1-v^2}} - 1\right)\Big). \end{aligned}\right\} \tag{179}$$

For the scalar product $\bar{\boldsymbol{a}}\boldsymbol{\overline{M}}$ on the right hand side of (178) we have

$$\bar{\boldsymbol{a}}\boldsymbol{\overline{M}} = |\bar{\boldsymbol{a}}|\, \overline{M}_{\bar{a}}, \tag{180}$$

where $\overline{M}_{\bar{a}}$ is the component of $\boldsymbol{\overline{M}}$ in the direction of the vector $\bar{a}$, while $|\bar{\boldsymbol{a}}|$ is the magnitude of $\bar{\boldsymbol{a}}$. By means of (179), (177), and (160) we get

$$|\bar{\boldsymbol{a}}| = \sqrt{\bar{a}_x^2 + \bar{a}_y^2 + \bar{a}_z^2} = \sqrt{1 + \frac{v_z^2}{1 - v^2}} = \sqrt{1 + \frac{K_z^{0\,2}}{K^{0\,2}}}. \tag{181}$$

Now, the component of $\boldsymbol{\overline{M}}$ in a given direction has the eigenvalues m^0, where m^0 is an integer ranging from $-l^0$ to $+l^0$. From (178), (180), and (181) it then follows that the submatrix

$$m_{\boldsymbol{K}^0 W^0} = \frac{(\Gamma_z)_{\boldsymbol{K}^0 W^0}}{\sqrt{K^{02} + K_z^{02}}} \tag{182}$$

has the eigenvalues m^0 with $-l^0 \leq m^0 \leq l^0$, and the full matrix

$$m = \frac{\Gamma_z}{\sqrt{K^2 + K_z^2}} \tag{183}$$

therefore will also have the eigenvalues m^0 with $-l^0 \leq m^0 \leq l^0$.

Instead of the quantities (163) it will now be convenient to take the six variables

$$K^\mu = K_x, K_y, K_z, W, L, m, \tag{184}$$

defined by (121), (173), (183), and (163), as a set of commuting constants of collision. The total number of particles n commutes with the variables (184), but n does not in general commute with S. n will be a constant of collision in the special case, only, where annihilation and creation processes are excluded. In this case n will have a definite numerical value. For $n = 2$ the variables (184) will form a complete set of collision constants. The matrices S and η then will be functions of K and L, only since these last variables are the only invariant combinations of the variables (184). Thus, the eigenvalues η^0 of η will only depend on the eigenvalues K^0 and L^0 of K and L, i. e.

$$\eta^0 = \eta\,(K^0, l^0). \tag{185}$$

These eigenvalues of η are obtained automatically if we introduce that representation in which the quantities (184) are diagonal, thus the complete solution of the eigenvalue problem in this case follows from the invariance of S and η under Lorentz transformations.

For $n > 2$ or in the more general case where n is not a constant of collision, a complete set of collision constants (α) will of course contain other variables besides the variables (184), and η will be a function of other invariant collision constants besides K and L. If S and η are invariant under other groups of transformations besides the Lorentz transformations, for instance the group of permutation of variables, this property may in a similar way be utilized in the derivation of new constants of collision.

As an application of the general theory developed in this section we shall now express the total cross section for the collision between two particles 1 and 2 as a function of invariant quantities. By means of (47), (49) and the generalized equation (96) we have for the differential cross section in which new particles with the momenta $\boldsymbol{k}'_3, \boldsymbol{k}'_4 \cdots \boldsymbol{k}'_{n'}$ are created

$$_{\text{at. emis.}} = 4\pi^2 \frac{|\Delta^0|\cdot|\Delta'|\cdot|(x'\,|\,R\,|\,x^0)|^2\,k'^2_2}{\sqrt{|\boldsymbol{u}^0_1-\boldsymbol{u}^0_2|^2-|\boldsymbol{u}^0_1\times\boldsymbol{u}^0_2|^2}\cdot|(\boldsymbol{u}'_1-\boldsymbol{u}'_2)\,\boldsymbol{e}'_2|}\,d\Omega\,\boldsymbol{dk}'_3\cdots\boldsymbol{dk}'_{n'} \tag{186}$$

with

$$(x'\,|\,R\,|\,x^0) = (x'\,|\,U_{\boldsymbol{K}^0 W^0}\,|\,x^0) = (x'\,|\,e^{i\eta}-1\,|\,x^0)$$

on account of (26) and (43). Since we are now constantly working in the subspace corresponding to fixed values $\boldsymbol{K}^0, W^0$ of the total momentum and energy, we may from now on omit the indices $\boldsymbol{K}^0, W^0$ in submatrices like $(x'\,|\,R\,|\,x^0)$.

It is now convenient to choose the variables (x) in the following way:

$$\left.\begin{aligned} (x') &= (\zeta', \varphi', \boldsymbol{k}'_3, \cdots \boldsymbol{k}'_{n'}) \\ \zeta' &= \cos\theta' = \frac{k'_{2z}}{k'_2}, \quad \varphi' = \operatorname{arctg}\frac{k'_{2y}}{k'_{2x}}, \end{aligned}\right\} \tag{187}$$

i. e. θ', φ' are the polar angles of the direction $\boldsymbol{e}_2' = \dfrac{\boldsymbol{k}_2'}{k_2'}$. For the functional determinant a simple calculation gives

$$\Delta' = \frac{(\boldsymbol{u}_1' - \boldsymbol{u}_2')\,\boldsymbol{e}_2'}{k_2'^2}. \tag{188}$$

Further we have

$$d\Omega\, \boldsymbol{dk}_3' \cdots \boldsymbol{dk}_{n'}' = d\zeta'\, d\varphi'\, \boldsymbol{dk}_3' \cdots \boldsymbol{dk}_{n'}' = dx',$$

so the total cross section is given by

$$Q = C^0 \int |(x' \,|\, R \,|\, x^0)|^2\, dx' \tag{189}$$

with

$$C^0 = \frac{4\pi^2 |\Delta^0|}{\sqrt{|\boldsymbol{u}_1^0 - \boldsymbol{u}_2^0|^2 - |\boldsymbol{u}_1^0 \times \boldsymbol{u}_2^0|^2}} \tag{190}$$

and

$$\int dx' = \sum_{n'=2}^{\infty}{}' \int d\zeta'\, d\varphi'\, \boldsymbol{dk}_3' \cdots \boldsymbol{dk}_{n'}'.$$

Now, let $(\alpha) = (\boldsymbol{K}, W, \beta)$ be a complete set of collision constants and let $(x' \,|\, \beta^0)_{\boldsymbol{K}^0, W^0} = (x' \,|\, \beta^0)$ be the transformation functions connecting the (x)-representation with the (β)-representation. Since

$$\int |(x' \,|\, R \,|\, x^0)|^2\, dx' = (x^0 \,|\, R^\dagger R \,|\, x^0)$$

and by (25) and (26) and (43)

$$R^\dagger R = -R - R^\dagger = 1 - e^{i\eta} + 1 - e^{-i\eta}$$

$$= 2(1 - \cos\eta) = 4 \sin^2 \frac{\eta}{2},$$

we get

$$\int |(x' \,|\, R \,|\, x^0)|^2\, dx' = 4 \left(x^0 \,\middle|\, \sin^2 \frac{\eta}{2} \,\middle|\, x^0\right) = 4 \int \sin^2 \frac{\eta(\beta')}{2}\, d\beta'\, |(x^0 \,|\, \beta')|^2. \quad (1$$

Thus,

$$Q = 4\, C^0 \left(x^0 \,\middle|\, \sin^2 \frac{\eta}{2} \,\middle|\, x^0\right) = 4\, C^0 \int \sin^2 \frac{\eta(\beta')}{2}\, d\beta'\, |(x^0 \,|\, \beta')|^2, \tag{192}$$

where of course $\int d\beta'$ is to be replaced by a sum in the case of discrete eigenvalues of β.

The variables (β) may be divided into two groups $(\beta) = (\iota, \gamma)$, where (ι) contains all the invariant collision constants, while (γ) contains only non-invariant quantities. Thus L, defined by (173), will be a member of (ι), while the quantity m, defined by (183), is contained in (γ). η then will be a function of K and (ι), only, and (192) may be written

$$Q = 4\, C^0 \int \sin^2 \frac{\eta\,(\mathsf{K}^0, \iota')}{2}\, d\iota'\, f(x^0, \iota') \tag{193}$$

with

$$f(x^0, \iota') = \int |(x^0 \,|\, \iota' \gamma')|^2\, d\gamma'. \tag{194}$$

Let us now consider an arbitrary Lorentz transformation, and let

$$\bar{\beta} = (\bar{\iota}, \bar{\gamma}) = U \beta U^{-1} \tag{195}$$

be the collision constants in the new frame of reference, connected with the old collision constants $\beta = (\iota, \gamma)$ by a contact transformation. The unitary matrix U is given by (136) and (140). Since the (ι) are invariants, we have

$$\bar{\iota} = \iota. \tag{196}$$

The transformation function connecting the β-representation with the $\bar{\beta}$-representation thus has the form

$$(\beta^0 \,|\, \bar{\beta}') = \delta\,(\iota^0 - \bar{\iota}')\,(\gamma^0 \,|\, \bar{\gamma}')$$

and since

$$(x^0 \,|\, \bar{\beta}') = \int (x^0 \,|\, \beta^0)\, d\beta^0\, (\beta^0 \,|\, \bar{\beta}'),$$

we get

$$(x^0 \,|\, \bar{\iota}' = \iota', \bar{\gamma}') = \sqrt{|D^0|}\; (\bar{x}^0 \,|\, \bar{\iota}' = \iota', \bar{\gamma}') = \int (x^0 \,|\, \iota' \gamma^0)\, d\gamma^0\, (\gamma^0 \,|\, \bar{\gamma}'), \tag{197}$$

where the functional determinant D^0 is given by

$$D^0 = \frac{\partial\,(\bar{x}^0)}{\partial\,(x^0)} = \frac{\bar{\Delta}^0\,[\overline{W}{}_i^0]}{\Delta^0\,[W_i^0]}. \tag{198}$$

On account of the fundamental relation

$$\int (\gamma^0 \,|\, \bar{\gamma}')\, d\bar{\gamma}'\, (\gamma'' \,|\, \bar{\gamma}')^* = \int (\gamma^0 \,|\, \bar{\gamma}')\, d\bar{\gamma}'\, (\bar{\gamma}' \,|\, \gamma'') = \delta\,(\gamma^0 - \gamma'')$$

holding for any transformation function, we then from (197) get

$$\int d\bar{\gamma}'\,|(x^0\,|\,\bar{\iota}'=\iota',\bar{\gamma}')|^2 = \int |(x^0\,|\,\iota'\gamma^0)|^2\,d\gamma^0, \tag{199}$$

or

$$f(x^0,\iota') = |D^0|\,\bar{f}(\bar{x}^0,\bar{\iota}'=\iota'), \tag{200}$$

where

$$\bar{f}(\bar{x}^0,\bar{\iota}') = \int |(\bar{x}^0\,|\,\bar{\iota}'\bar{\gamma}')|^2\,d\bar{\gamma}' \tag{201}$$

is the function corresponding to (194) in the transformed system. By integration over all values of (x^0) for $n^0 = 2$ we get from (200) and (198)

$$\int_{n^0=2} f(x^0,\iota')\,dx^0 = \int_{n^0=2} \bar{f}(\bar{x}^0,\bar{\iota}'=\iota')\left|\frac{\partial(\bar{x}^0)}{\partial(x^0)}\right| dx^0 = \int_{n^0=2}\bar{f}(\bar{x}^0,\bar{\iota}'=\iota')\,d\bar{x}^0, \tag{202}$$

which shows that the function

$$g(\iota') = \int_{n^0=2} f(x^0,\iota')\,dx^0 \tag{203}$$

is an invariant function of the eigenvalues of the invariants (ι).

Now, let the new system of reference be a 'center of gravity' system in which $\bar{\boldsymbol{K}}^0 = 0$. Then the function $\bar{f}$ defined by (201) will be constant for all values of $(\bar{x}^0)$, i. e. $\bar{f}$ is independent of the direction of $\bar{\boldsymbol{k}}_2^0$. This follows from the fact that $\bar{f}(\bar{x}^0)$ as a function of the variables $(\bar{x}^0)$ is invariant and form-invariant under all rotations in ordinary space. So we have

$$\bar{f}(\bar{x}^0,\bar{\iota}'=\iota') = \frac{1}{4\pi}\int_{n^0=2}\bar{f}(\bar{x}^0,\bar{\iota}'=\iota')\,d\bar{x}^0 = \frac{g(\iota')}{4\pi} \tag{204}$$

by (202) and (203). In an arbitrary frame of reference, where $\boldsymbol{K}^0 \neq 0$, we then by (200) and (204) get

$$f(x^0,\iota') = |D^0|\,\frac{g(\iota')}{4\pi} \tag{205}$$

and for the total cross section (193)

$$Q = G^0\int g(\iota')\sin^2\frac{\eta(\iota')}{2}\,d\iota', \tag{206}$$

where

$$G^0 = \frac{4\,C^0|D^0|}{4\pi} = \frac{4\pi}{B^0}\,|\bar{\Delta}^0|\,\bar{W}_1^0\,\bar{W}_2^0 \tag{207}$$

on account of (190), (198), and (77).

Since $\bar{\boldsymbol{k}}_1^0 = -\bar{\boldsymbol{k}}_2^0$, we further, by means of (188), (77), and (160), get

$$\left.\begin{aligned} |\Delta^0|\,\overline{W}_1^0\,\overline{W}_2^0 &= \frac{|(\bar{\boldsymbol{k}}_2^0\,\overline{W}_1^0 - \bar{\boldsymbol{k}}_1^0\,\overline{W}_2^0)\,\bar{e}_2^0|}{\bar{k}_2^{0\,2}} = \frac{\overline{W}_1^0 + \overline{W}_2^0}{\bar{k}_2^0} \\ &= \frac{(\overline{W}_1^0 + \overline{W}_2^0)^2 - |\bar{\boldsymbol{k}}_1^0 + \bar{\boldsymbol{k}}_2^0|^2}{\sqrt{|\bar{\boldsymbol{k}}_1^0\,\overline{W}_2^0 - \bar{\boldsymbol{k}}_2^0\,\overline{W}_1^0|^2 - |\bar{\boldsymbol{k}}_1^0 \times \bar{\boldsymbol{k}}_2^0|^2}} = \frac{\mathsf{K}^{0\,2}}{B^0}. \end{aligned}\right\} \tag{208}$$

Thus we get the general formula

$$Q = \frac{4\,\pi\,\mathsf{K}^{0\,2}}{B^{0\,2}} \int g(\iota')\,\sin^2\frac{\eta\,(\mathsf{K}_1^0, \iota')}{2}\,d\,\iota', \tag{209}$$

by which the total cross section is given as a function of invariant quantities.

In the special case where the total number of particles n is a constant of collision, we have $(\iota) = L$ and $(\gamma) = m$, and by (203) and (194) we get

$$g\,(l') = \int f(x^0, l')\,d\,x^0 = \sum_{m'=-l'}^{l'} \int |(x^0\,|\,l'm')|^2\,d\,x^0 = 2\,l' + 1 \tag{210}$$

on account of the normalization condition for the eigenfunctions $(x^0\,|\,l'm')$. Thus in this case $g\,(l')$ is simply equal to the number of eigenstates corresponding to a definite value $L' = l'\,(l'+1)$ of the variable L and the total cross section is

$$Q = \frac{4\,\pi\,\mathsf{K}^{0\,2}}{B^{0\,2}} \sum_{l'=0}^{\infty} (2\,l'+1)\,\sin^2\frac{\eta\,(\mathsf{K}^0, l')}{2}. \tag{211}$$

In a system where the center of gravity is at rest, i. e. $\boldsymbol{K}^0 = 0$, (211) reduces further to the well-known formula

$$Q = \frac{4\,\pi}{k_2^{0\,2}} \sum_{l'=0}^{\infty} (2\,l'+1)\,\sin^2\frac{\eta'}{2}, \tag{212}$$

and in this special case the numbers l' and m' may be interpreted as the quantum numbers determining the magnitude and the component of the angular momentum in a definite direction.

References.

1) W. HEISENBERG, Ann. d. Phys. **32**, 20, 1938. See also N. BOHR, Conference on Nuclear Physics in Rome 1931, p. 119; V. WEISSKOPF, Zs. f. Phys. **89**, 27, 1934; A. MARCH, *ibid.* **104**, 93, 161, 1936; **105**, 620, 1937; G. WATAGHIN, ibid. **88**, 92; **92**, 547, 1934.
2) W. HEISENBERG, Die "beobachtbaren Grössen" in der Theorie der Elementarteilchen I, Zs. f. Phys. **120**, 513, 1943; II, *ibid.* **120**, 673, 1943.
3) W. HEISENBERG, Die "beobachtbaren Grössen" in der Theorie der Elementarteilchen III, in preparation.
4) C. MØLLER, General Properties of the Characteristic Matrix in the Theory of Elementary Particles II, in preparation.
5) See e. g. V. FOCK, Zs. f. Phys. **75**, 622, 1932.
6) P. A. M. DIRAC, Quantum Mechanics 2. edition, Oxford 1935, p. 199.
7) W. HEISENBERG, *loc. cit.* 2), I, 515.
8) *loc. cit.* 6) p. 102.
9) See e. g. P. A. M. DIRAC, *loc. cit.* 6), p. 198.
10) cf. a fortcoming paper "On the Correspondence Principle in the Theory of Elementary Particles".

Indleveret til Selskabet den 1. December 1944.
Færdig fra Trykkeriet den 16. Juli 1945.