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SOME INVESTIGATIONS OF THE SET OF VALUES OF MEASURES IN ABSTRACT SPACE

BY

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PREFACE

As the nucleus of the theory of abstract measures and integrals is a generalization of the Lebesgue results, this theory must naturally be of rather new date, and actually the whole development has taken place in the present century. Fundamental works by RADON and FRÉCHET[1]) had shown the possibility of transferring the Lebesgue integral to the abstract space, and had shown the fact that this new notion thereby acquired the greater part of the properties of the Lebesgue integral; although it was, of course, impossible to prove for this new notion the properties intimately connected with the metrical structure of the Euclidean space. After these works the theory developed rapidly, and out of the great number of papers whose results have been of the greatest importance for the rounding off of and the high stage reached by the theory to-day, we shall mention only those of BOCHNER, DANIELL, NIKODYM, and SAKS. In 1933 appeared a monography by STANISLAW SAKS about the theory of integrals in the Euclidean space, as well as in the abstract space[2]). Later on B. JESSEN has given a concentrated description of the theory in a series of articles in "Matematisk Tidsskrift"[3]).

It is the purpose of the present paper to investigate some problems of existence in the abstract space or—to say it more precisely—to study more closely the set of values of certain functions of a set whose defining region is a collection of subsets of the abstract space.

The paper is divided into four parts. Owing to the fact that the theory of measure and integral in an abstract space is of

1) RADON [1]; FRÉCHET [1].
2) SAKS [1], [2].
3) JESSEN [1]–[5].

relatively recent date, and further because the terminology used is not quite fixed, we have considered it natural as well as necessary to give a rather detailed account of those parts of the theory underlying the further description. This account constitutes the first part of the paper. The special types of classes of sets, which will be treated in the following, will be introduced, and the special functions of a set, contents and measures as well as the most important theorems on these, will be mentioned. In the same section we shall, furthermore, briefly treat the important theorem on extension, stating how a content must be constituted in order to be extensible to a measure. After this will follow a description of the definite integral of non-negative functions. The two notions, an absolutely continuous and a singular function of a set, will then be introduced, and the important theorem on the unique decomposition of a function of a set in an absolutely continuous and a singular part (the Lebesgue decomposition) will be proved for a special case. As a help in the proof we shall make use of the theorem on representation of a function of a set as a difference between two measures (the Jordan decomposition). Finally we introduce in the last section of that part the indefinite integral; and the theorem of Radon-Nikodym on the necessary and sufficient condition of the possibility of writing a function of a set as an indefinite integral, is quoted; whereas the proof is given only for a special case essential for the following problems.

The first section of the second part contains an account of already well-known results regarding monotone functions[1]. The notions, the variation $V_f(x_1, x_2)$ and the total discontinuity $D_f(x_1, x_2)$ of a monotone function $f(x)$ in an interval (x_1, x_2) are introduced; and the proof is given for the theorem on decomposition of a monotone function $f(x)$ into two addends $g(x)$ and $h(x)$, one of which is continuous, whereas the other has in any interval a variation equal to its total discontinuity (which again is equal to the total discontinuity of the original function on the interval considered). In the next section we shall deal with theorems on functions of a set defined in the Borel class on the axis of the real numbers. It is well-known that the Borel class is the smallest totally additive ring, containing

[1] A detailed treatment is given in Carathéodory [1].

all half open intervals of the form $[a \leq x < b]$. It will be discussed on what conditions the function of an interval can be extended to a measure defined in the Borel class. Finally we shall conclude the second part by giving in its last section some considerations of the region of value of certain special types of functions of a set, whose defining region is the Borel class of the interval $(0,1)$.

In the third part we first prove an auxiliary theorem on convergent series with positive terms. It is proved that the set of numbers, whose elements are all finite or enumerable partial sums of such a series, is a closed set. By application of this theorem, we can in the next section of the third part prove the set of numbers, whose elements are the values taken by a bounded measure, defined in the Borel class of the interval $(0,1)$, to be a closed set. In the third section of this part one of the main results of this paper will be obtained. It is here shown that a bounded measure, having as defining region a class of sets consisting of subsets of the abstract space, and having the space itself as element, has the same property as the bounded measure of the Borel class, i. e. that the set of values is a closed set. As our chief means to prove this we make a representation from the class of sets in question in the abstract space on the Borel class of the interval. This representation can to some extent be regarded as a generalization of a well-known construction by PEANO[1]. The fact that we work in this paper especially with the Cantor set is of no consequence except its being the most fitted for our purpose. Many other methods of representation might have been used without essentially complicating the proof.

In the fourth part we shall regard pairs of bounded measures $(\varphi(A), \psi(A))$ instead of one bounded measure, thus extending the theorem proved in the preceding part. Suppose both measures to take on the value 1 on the abstract space E, i.e. $\varphi(E) = \psi(E) = 1$, and the point $(\varphi(A), \psi(A))$ will for every A belong to the unity square. It is proved that if the measures have the same defining region $\mathfrak{F}$, then the set of points defined by $(\varphi(A), \psi(A))$ will be a closed set. The proof takes place in several stages. The theorem

[1] Cf. HILBERT [1]; JESSEN [6]; LEBESGUE [1]; F. RIESZ [1]; DE LA VALLÉE-POUSSIN [1].

will first be proved for φ and ψ as bounded measures defined in the Borel class of the interval (0,1). By application of the theorem on decomposition of monotone functions this case can be retraced to two simple fundamental cases, which will then be dealt with separately. The first, which—rather unprecisely expressed—corresponds to the purely discontinuous elements of the monotone functions, is dealt with quite simply by application of the theorem on the convergent series of positive terms. The other, corresponding to the continuous elements of the monotone functions, is somewhat more complicate and requires a certain chain of constructions. After having proved the theorem for measures defined in the Borel class, we can then in the last section very easily transfer it to the abstract space by means of the same method of coupling which we used in the third part.

The paper is concluded with some remarks on the questions of the axiomatic theory of probabilities and the applications thereof, which have suggested the problems of this paper to the author.

Concluding this paper, I wish to express my warmest thanks to Professor Børge Jessen, and to Professor Richard Petersen, who have both taken interest in my work. I also thank Miss B. Ehlern-Møller, M. A., for the translation into English, and Niels Arley, Ph. D., for reading the proofs.

PART I.

On measure and integral in abstract space.

1. Classes of sets.

Given a set E, containing at least one element. We shall to denote the elements of E use the letters $x, y, z, \cdots$, and to denote subsets of E the letters $A, B, C, \cdots$. As subset of E we shall especially consider the empty set, which in the following will be denoted by 0. $x \varepsilon A$ will denote that the point x belongs to the set A. $A \subset B$ denotes A to be a (not necessarily proper) subset of B. Given a sequence of sets, be it finite $A_1, A_2, \cdots, A_k$ or infinite $A_1, A_2, \cdots, A_k, \cdots$, then $A_1 \dotplus \cdots \dotplus A_k$, respectively, $A_1 \dotplus A_2 \dotplus \cdots \dotplus A_k \dotplus \cdots$ or $\underset{n}{\mathfrak{S}} A_n$ will denote the set of elements belonging to at least one of the sets A_n and will be called the sum of the sets $A_1, \cdots, A_k$, respectively, $A_1, A_2, \cdots, A_k \cdots$. When we especially write $\sum_n A_n$ or $A_1 + A_2 + \cdots + A_k$, respectively, $A_1 + A_2 + \cdots + A_k + \cdots$, it is to be understood from the symbol that no two sets A_n have a common elements. $\underset{n}{\mathfrak{D}} A_n$ $\left(\text{or } \prod_n A_n\right)$ or $A_1 \cdots A_k$, respectively, $A_1 A_2 \cdots A_k \cdots$ will denote the set of elements belonging to all sets A_n, and will be termed the product (the common part) of the sets $A_1, \cdots, A_k$ respectively $A_1, A_2, \cdots, A_k, \cdots$. The symbol $A - B$ is only to mean anything when $B \subset A$, and is then to denote the set of elements belonging to A, but not to B. We shall call $E - A$ the complement of A.

In the remaining part of this section we shall deal with classes of sets. A class of sets is a set whose elements are subsets of E. To denote classes of sets we shall use German capital letters $\mathfrak{F}$, $\mathfrak{G}$, $\cdots$. $A \varepsilon \mathfrak{F}$ will denote that the set A belongs to the class of sets $\mathfrak{F}$. A class of sets $\mathfrak{F}$ is called additive, respectively multiplicative, if $A_1 + A_2$, respectively $A_1 A_2$, belongs to $\mathfrak{F}$, whenever A_1 and A_2 both belong to $\mathfrak{F}$. It is called subtractive, if $A_1 - A_2$ belongs to $\mathfrak{F}$, whenever A_1 and A_2 both belong to $\mathfrak{F}$. It is termed totally additive, respectively multiplicative, if $\underset{n}{\mathfrak{S}} A_n$, respectively $\underset{n}{\mathfrak{D}} A_n$, belongs to $\mathfrak{F}$, any A_n belonging to $\mathfrak{F}$.

After these preliminary remarks it is now possible to set up the two following important definitions:

A class of sets $\mathfrak{F}$ is called a ring, if it contains at least one set and is additive and subtractive.

A class of sets $\mathfrak{F}$ is called a Borel ring, if it contains at least one set and is totally additive and subtractive.

It is immediately evident that any Borel ring is a ring. Suppose $A = \underset{n}{\mathfrak{S}} A_n$, and it will be clear from the relation

$$\underset{n}{\mathfrak{D}} A_n = A - \underset{n}{\mathfrak{S}} (A - A_n), \tag{1,1}$$

which is valid whether the number of the sets is finite or enumerable, that any ring is multiplicative, and that any Borel ring is totally multiplicative. Finally we shall mention that the smallest possible extension of a given set $\mathfrak{F}$ into a Borel ring is obtained as the product of all Borel rings, containing $\mathfrak{F}$. For this product is easily seen to be a Borel ring itself.

2. Functions of a set.

A function whose defining region is a class of sets $\mathfrak{F}$, and whose values are real numbers ($-\infty$ and $+\infty$ incl.), will be called a function of a set. To denote the latter we shall in the following use Greek small letters. A function of a set μ defined in $\mathfrak{F}$ will be termed additive, if $\mu(A_1 + \cdots + A_k) = \mu(A_1) + \cdots + \mu(A_k)$, when $A_n \varepsilon \mathfrak{F}$ for $n = 1, 2, \cdots, k$ and $A_1 + \cdots + A_k \varepsilon \mathfrak{F}$. In analogy μ will be called totally additive, if $\mu(\underset{n}{\Sigma} A_n) = \underset{n}{\Sigma} \mu(A_n)$, when all $A_n \varepsilon \mathfrak{F}$ and $\underset{n}{\Sigma} A_n \varepsilon \mathfrak{F}$.

On functions of a set we now give the two following definitions:

A function of a set μ, defined in $\mathfrak{F}$ will be called a content, if

1) $\mathfrak{F}$ *is a ring.*
2) $0 \leq \mu(A) \leq \infty$ *for every* $A \,\varepsilon\, \mathfrak{F}$.
3) μ *is additive.*
4) *to any* $A \,\varepsilon\, \mathfrak{F}$ *there corresponds a set* $\underset{n}{\mathfrak{S}} A_n$, *where all* $A_n \,\varepsilon\, \mathfrak{F}$ *and* $\mu(A_n) < \infty$ *for all* n, *such that* $A \subset \underset{n}{\mathfrak{S}} A_n$.

A function of a set μ, *defined in* $\mathfrak{F}$ *is called a measure, if*
1) $\mathfrak{F}$ *is a Borel ring.*
2) $0 \leq \mu(A) \leq \infty$ *for every* $A \,\varepsilon\, \mathfrak{F}$.
3) μ *is totally additive.*
4) *to any* $A \,\varepsilon\, \mathfrak{F}$ *there corresponds a set* $\underset{n}{\mathfrak{S}} A_n$, *where all* $A_n \,\varepsilon\, \mathfrak{F}$ *and* $\mu(A_n) < \infty$ *for all* n, *such that* $A \subset \underset{n}{\mathfrak{S}} A_n$.

For a content μ defined in $\mathfrak{F}$ the following theorems hold true

I) $\mu(0) = 0$.
II) $\mu(A) \leq \mu(B)$, when $A \subset B$ and $A \,\varepsilon\, \mathfrak{F}$, $B \,\varepsilon\, \mathfrak{F}$.
III) $\mu(B - A) = \mu(B) - \mu(A)$ when $A \,\varepsilon\, \mathfrak{F}$, $B \,\varepsilon\, \mathfrak{F}$ and $\mu(A) < \infty$.
IV) $\mu(A_1 + A_2 + \cdots + A_n) \leq \mu(A_1) + \mu(A_2) + \cdots + \mu(A_n)$ when $A_i \,\varepsilon\, \mathfrak{F}$ for $i = 1, 2, \cdots, n$.
V) $\mu\left(\sum_n A_n\right) \geq \sum_n \mu(A_n)$ when all $A_n \,\varepsilon\, \mathfrak{F}$ and $\sum_n A_n \,\varepsilon\, \mathfrak{F}$.

A measure being a content as well, the same theorems, I—V, hold true if μ is a measure; further we have in that case the following theorems

VI) $\mu\left(\underset{n}{\mathfrak{S}} A_n\right) \leq \sum_n \mu(A_n)$ when all $A_n \,\varepsilon\, \mathfrak{F}$.
VII) $\mu\left(\underset{n}{\mathfrak{S}} A_n\right) = \lim_n \mu(A_n)$ when $A_1 \subset A_2 \subset \cdots$ and all $A_n \,\varepsilon\, \mathfrak{F}$.
VIII) $\mu\left(\underset{n}{\mathfrak{D}} A_n\right) = \lim_n \mu(A_n)$ when $A_1 \supset A_2 \supset \cdots$, $\mu(A_1) < \infty$ and all $A_n \,\varepsilon\, \mathfrak{F}$.

As the proofs of these theorems I—VI must be considered evident, we shall in this account confine ourselves to prove the theorems VII and VIII. From $A_1 \subset A_2 \subset \cdots$ it follows that

$$\underset{n}{\mathfrak{S}} A_n = A_1 + (A_2 - A_1) + (A_3 - A_2) + \cdots + (A_n - A_{n-1}) + \cdots,$$

and hence

$$\mu\left(\mathfrak{S}_n A_n\right) = \mu(A_1) + \mu(A_2 - A_1) + \cdots + \mu(A_n - A_{n-1}) + \cdots =$$
$$\lim_n \left(\mu(A_1) + \mu(A_2 - A_1) + \cdots + \mu(A_n - A_{n-1})\right) =$$
$$\lim_n \mu(A_n),$$

and the proof of VII is completed.

From $A_1 \supset A_2 \supset A_3 \supset \cdots$ follows $A_1 - A_2 \subset A_1 - A_3 \subset \cdots$. Since $\mu(A_1) < \infty$ we get

$$\mu\left(A_1 - \mathfrak{D}_n A_n\right) = \mu(A_1) - \mu\left(\mathfrak{D}_n A_n\right),$$

and, furthermore, since $\mathfrak{S}_n(A_1 - A_n) = A_1 - \mathfrak{D}_n A_n$, it follows from VII that

$$\mu\left(A_1 - \mathfrak{D}_n A_n\right) = \lim_n \mu(A_1 - A_n) = \mu(A_1) - \lim_n \mu(A_n).$$

Taking these equations together, we obtain

$$\mu\left(\mathfrak{D}_n A_n\right) = \lim_n \mu(A_n),$$

and the proof of VIII is completed.

As conclusion of this section we shall mention an important theorem of extension, which tells how a content must be constituted in order to be extensible to a measure. We must, however, first specify the latter notion. Let μ be a content defined in $\mathfrak{F}$, and μ^* a measure defined in $\mathfrak{F}^*$. We then call μ^* an extension of μ, if every A, belonging to $\mathfrak{F}$, also belongs to $\mathfrak{F}^*$, and $\mu(A) = \mu^*(A)$ for any $A \varepsilon \mathfrak{F}$. It is obvious that μ must satisfy the condition of being totally additive, if it shall be possible to extend μ into a measure. This condition is, however, also sufficient, the theorem being as follows:

Let μ be a content defined in $\mathfrak{F}$. This content can be extended to a measure μ^* defined in $\mathfrak{F}$, when and only when μ is totally additive. One of the possible extensions is the most restricted one, i. e. any other extension is an extension of this.

Without entering into the proofs, we shall indicate how the most restricted extension, mentioned in the theorem, arises. Let $\mathfrak{F}^*$ be the smallest Borel ring containing $\mathfrak{F}$. For every $A \varepsilon \mathfrak{F}^*$ we put

$$\mu^*(A) = \text{lower bound} \sum_n \mu(A_n), \tag{2,1}$$

where the lower bound is to be taken over all $\sum_n \mu(A_n)$, $A_n \varepsilon \mathfrak{F}$ for all n and $A \subset \mathfrak{S}_n A_n$. This function of a set defined in $\mathfrak{F}^*$ will then be a measure.

For certain applications it is of interest to note that a content μ defined in $\mathfrak{F}$, where $\mu(A) < \infty$ for any $A \varepsilon \mathfrak{F}$, is totally additive, when and only when it is valid for any sequence of sets $A_1 \supset A_2 \supset \cdots$, where all $A_n \varepsilon \mathfrak{F}$ and $\mathfrak{D}_n A_n = 0$ that $\lim_n \mu(A_n) = 0$.

3. The definite integral of non-negative functions.

In the following we shall deal with functions in E, whose value region consists of real numbers, $-\infty$ and $+\infty$ included. By $[f]$ we shall denote the defining region of f, i. e. the set of $x \varepsilon E$ for which f is defined. The set of those $x \varepsilon [f]$, for which $f > a$ will be termed $[f > a]$. It is now evident what is to be understood by the symbols $[f \geqq a]$, $[f < a]$, $[f \leqq a]$, $[f = a]$ etc.

Now suppose a Borel ring $\mathfrak{F}$ to be given. The function f is said to be a function on $\mathfrak{F}$, if the sets $[f]$, $[f \geqq a]$, $[f > a]$, $[f \leqq a]$, and $[f < a]$ belong to $\mathfrak{F}$ for any a. These conditions can be considerably reduced. Thus for instance f will be a function on $\mathfrak{F}$, if the sets $[f]$ and $[f \geqq r]$ belong to $\mathfrak{F}$ for any rational r.

If f is a function on $\mathfrak{F}$, we see that $[f = a] \varepsilon \mathfrak{F}$, since $[f = a] = [f \leqq a][f \geqq a]$.

We shall further note that simple calculations with functions on $\mathfrak{F}$ will again lead to functions on $\mathfrak{F}$.

By f_A we shall denote the contraction of f to A, i. e. the function defined in A, for which $f_A(x) = f(x)$ for any $x \varepsilon A$. If f is a function on the Borel ring $\mathfrak{F}$ and $A \varepsilon \mathfrak{F}$, then also f_A will be a function on $\mathfrak{F}$, as we have e. g.

$$[f_A \geqq a] = A[f \geqq a].$$

After these preliminary observations we may now go over to discuss the definite integral of non-negative functions.

Let μ be a measure defined in $\mathfrak{F}$, and let f be a function on $\mathfrak{F}$, which is non-negative, and takes on a finite or at most an enumerable number of values. v_n denoting the values taken by f, the set $[f = v_n]$ will belong to $\mathfrak{F}$ for any n. We now put $i_n = v_n \mu([f = v_n])$, when this product exists, and $i_n = 0$, if either $v_n = 0$ and $\mu([f = v_n]) = \infty$ or $v_n = \infty$ and $\mu([f = v_n]) = 0$.

The definite integral of the function f with respect to the measure μ, $I(f)$, will then be defined by

$$I(f) = \sum_n i_n.$$

Next let f be a function on $\mathfrak{F}$, which is non-negative. Together with f we shall consider all functions g on $\mathfrak{F}$ where $[g] = [f]$, which are non-negative, and which take on a finite or at most enumerable number of values, and for which $g \leq f$ for every $x \varepsilon [f]$. The function $g = 0$ is an example of such a function. We then put

$$\underline{I}(f) = \text{upper bound } I(g).$$

Similarly we introduce

$$\overline{I}(f) = \text{lower bound } I(h),$$

where h runs through all functions on $\mathfrak{F}$ where $[h] = [f]$, which are non-negative, and which take a finite or at most an enumerable number of values, and for which $h \geq f$ for every $x \varepsilon [f]$. The function $h = \infty$ is an example of such a function. It is now easily seen that we have

$$\underline{I}(f) = \overline{I}(f).$$

For if we choose a number a, $1 < a < \infty$, and put

$$f_a(x) = \begin{cases} 0 \text{ for } x \varepsilon [f = 0] \\ a^n \text{ for } x \varepsilon [a^n \leq f < a^{n+1}],\ n = 0, \pm 1, \pm 2, \cdots \\ \infty \text{ for } x \varepsilon [f = \infty], \end{cases}$$

then the function $f_a(x)$ is a g-function in the above sense, and the function $a f_a(x)$ is an h-function. Thus we have

$$I(f_a) \leq \underline{I}(f)$$

and furthermore (cf. theorem II page 13)

$$I(a f_a) = a I(f_a) \geq \overline{I}(f).$$

From these inequalities it follows that

$$\bar{I}(f) \leqq a\underline{I}(f)$$

giving for $a \to 1$

$$\bar{I}(f) \leqq \underline{I}(f)$$

which in connection with the trivial relation

$$\underline{I}(f) \leqq \bar{I}(f)$$

gives the result wanted.

Thus we have been led to the following definition:

The definite integral of the function f with respect to the measure μ, $I(f)$, is defined by the common value of $\underline{I}(f)$ and $\bar{I}(f)$

$$I(f) = \underline{I}(f) = \bar{I}(f).$$

If $I(f) < \infty$ we shall call the function f μ-integrable[1]. If f is such a function, and A is a subset of $[f]$, belonging to $\mathfrak{F}$, we write

$$I(f_A) = \int_A f(x)\mu(dE),$$

and call this quantity the integral over the set A of the function f with respect to the measure μ.

For the definite integral introduced above a number of theorems are valid, of which we shall mention the following ones:

I) $I(f) \geqq 0$, and the sign of equality holds true when and only when $\mu[f > 0] = 0$.

II) $I(cf) = cI(f)$ for every $c > 0$.

III) If $[f] = \sum_n A_n$, where all $A_n \varepsilon \mathfrak{F}$, we have
$I(f) = \sum_n I(f_{A_n})$.

IV) If $[f] = [g]$ then $I(f+g) = I(f) + I(g)$.

V) If $[f] = [g]$ and $f \leqq g$ then $I(f) \leqq I(g)$, and the sign of equality holds true when and only when we have either $I(f) = \infty$ or $I(f) < \infty$ and $\mu([f < g]) = 0$.

[1] If a function f is μ-integrable, we have $\mu[f = \infty] = 0$; thus the function is finite, at most with the exception of a zero-set.

VI) If $[f_1] = [f_2] = \cdots$ and $f_1 \leqq f_2 \leqq \cdots$ then
$I(\lim_n f_n) = \lim_n I(f_n)$.

VII) If $[f_1] = [f_2] = \cdots$ then $I(\liminf_n f_n) \leqq \liminf_n I(f_n)$.

4. Absolutely continuous and singular functions of a set.

Let $\mathfrak{F}$ be a Borel ring for which $E \varepsilon \mathfrak{F}$, and let μ be a measure defined in $\mathfrak{F}$. We shall then introduce the following definitions:

A bounded totally additive function of a set φ defined in $\mathfrak{F}$ is called μ-continuous, if $\varphi(M) = 0$ for every set $M \varepsilon \mathfrak{F}$, for which $\mu(M) = 0$.

A bounded totally additive function of a set φ defined in $\mathfrak{F}$ is called μ-singular, if there exists a set $N \varepsilon \mathfrak{F}$ with $\mu(N) = 0$, such that $\varphi(A) = 0$ for every set $A \varepsilon \mathfrak{F}$, which is a subset of $E - N$.

If a function of a set φ is both μ-continuous and μ-singular it must vanish identically. This is seen as follows. For every $A \varepsilon \mathfrak{F}$ we have

$$\varphi(A) = \varphi(AN + (A - AN)) = \varphi(AN) + \varphi(A - AN).$$

From $\mu(N) = 0$ it follows that $\mu(AN) = 0$ and hence further that $\varphi(AN) = 0$. Since $A - AN \subset E - N$ it next follows that $\varphi(A - AN) = 0$. We thus have $\varphi(A) = 0$ for every $A \varepsilon \mathfrak{F}$.

Further it is evident that if φ is absolutely continuous (respectively singular), then also $c\varphi$ (c constant) will be absolutely continuous (respectively singular), and if φ_1 and φ_2 are absolutely continuous (respectively singular), then also $\varphi_1 + \varphi_2$ will be absolutely continuous (respectively singular).

The following important theorem on decomposition is true for functions of a set:

A bounded totally additive function of a set φ defined in $\mathfrak{F}$ can in one, and only one, way be written in the form

$$\varphi = \varphi_k + \varphi_s,$$

where φ_k and φ_s are bounded totally additive functions of sets defined in $\mathfrak{F}$, and φ_k is μ-continuous, and φ_s is μ-singular.

This theorem being of particular interest for our later applications of the theory we shall give the proof of it in the special case, which we are to apply later on, namely, where μ is bounded, and φ is non-negative.

The fact that the decomposition can at most be carried out in one way is seen as follows. Suppose

$$\varphi = \varphi_k' + \varphi_s' = \varphi_k'' + \varphi_s'',$$

where φ_k' and φ_k'' are absolutely continuous, and φ_s' and φ_s'' are singular. We then have for every $A \varepsilon \mathfrak{F}$

$$\varphi_k' - \varphi_k'' = \varphi_s'' - \varphi_s' = \varphi^*.$$

According to previous remarks φ^* is itself absolutely continuous, being a difference between two absolutely continuous functions of a set, and analogously it is obvious that φ^* is singular. Hence φ^* is identically zero, which was to be proved.

We shall now first observe that if the decomposition is possible, the set N corresponding to φ_s will have the following property: From $A \varepsilon \mathfrak{F}$, $A \subset E - N$, and $\mu(A) = 0$ it follows that $\varphi(A) = 0$, because $\mu(A) = 0$ implies $\varphi_k(A) = 0$, and from $A \subset E - N$ it follows that $\varphi_s(A) = 0$. Conversely, if it is possible to find a set $N \varepsilon \mathfrak{F}$ with $\mu(N) = 0$, such that $A \varepsilon \mathfrak{F}$, $A \subset E - N$, and $\mu(A) = 0$ implies that $\varphi(A) = 0$, then decomposition will be possible. This is seen as follows: For every $A \varepsilon \mathfrak{F}$ we have

$$\varphi(A) = \varphi(A - AN) + \varphi(AN)$$

We can prove the function of a set $\varphi_1(A) = \varphi(A - AN)$ to be μ-continuous. Suppose $A \varepsilon \mathfrak{F}$ and $\mu(A) = 0$. Hence $A - AN \subset E - N$ and $\mu(A - AN) = 0$. Thus we get $\varphi_1(A) = 0$. Next we can prove $\varphi_2(A) = \varphi(AN)$ to be μ-singular with N as corresponding set. From $A \varepsilon \mathfrak{F}$ and $A \subset E - N$ follows that $AN = 0$ and hence $\varphi_2(A) = \varphi(0) = 0$. Now the theorem will have been proved, if we can show the existence of a set N having the properties mentioned[1].

It will be natural in the course of the proof to form two auxiliary theorems.

1. $\mathfrak{F}$ being a class of sets, and φ being a bounded function of

[1] Hence we further see that $\varphi(A) \geqq 0$ implies that $\varphi_k(A) \geqq 0$ and $\varphi_s(A) \geqq 0$ for every $A \varepsilon \mathfrak{F}$.

a set defined in $\mathfrak{F}$, we introduce the functions of a set $\overline{\varphi}$ and $\underline{\varphi}$ by the definitions

$$\overline{\varphi}(A) = \text{upper bound } \varphi(B)$$
$$\underline{\varphi}(A) = \text{lower bound } \varphi(B),$$

where B runs through all subsets of A. We then have the following lemma:

A totally additive bounded function of a set φ, whose defining region $\mathfrak{F}$ is a Borel ring, may be written in the form

$$\varphi = \overline{\varphi} + \underline{\varphi}$$

where $\overline{\varphi}$ and $-\underline{\varphi}$ are measures[1].

We shall not in this place give the proof of this theorem, as it does not imply much new, but let it suffice to remark that with the conditions stated in the theorem it will be true for every $A \varepsilon \mathfrak{F}$, since $\varphi(0) = 0$, that

$$\underline{\varphi}(A) \leqq 0 \leqq \overline{\varphi}(A).$$

2. The other lemma is as follows:

Let $\mathfrak{F}$ be a Borel ring, containing E, and let φ be a bounded totally additive function of a set defined in $\mathfrak{F}$. E may then be decomposed into the form

$$E = E^+ + E^-,$$

where E^+ and E^- both belong to $\mathfrak{F}$, and such that

$$\underline{\varphi}(E^+) = 0 \text{ and } \overline{\varphi}(E^-) = 0,$$

i. e. $\varphi(A) \geqq 0$ for every $A \varepsilon \mathfrak{F}$, which is a subset of E^+ and $\varphi(A) \leqq 0$ for every $A \varepsilon \mathfrak{F}$, which is a subset of E^-.[2]

As a consequence of the meaning of $\underline{\varphi}(E)$ we may for every n ($n = 1, 2, 3, \cdots$) choose a set $A_n^- \varepsilon \mathfrak{F}$, such that

$$\varphi(A_n^-) < \underline{\varphi}(E) + \frac{1}{2^n}. \tag{4,1}$$

[1] $\overline{\varphi}$ and $\underline{\varphi}$ are both bounded.

[2] This decomposition is usually not unique.

Let $A_n^+ = E - A_n^-$, and we have

$$\varphi(A_n^+) + \varphi(A_n^-) = \varphi(E) = \overline{\varphi}(E) + \underline{\varphi}(E) > \overline{\varphi}(E) + \varphi(A_n^-) - \frac{1}{2^n},$$

and hence

$$\varphi(A_n^+) > \overline{\varphi}(E) - \frac{1}{2^n}.$$

$-\underline{\varphi}$ being a measure, we next have

$$\underline{\varphi}(A_n^+) = \varphi(E) - \varphi(A_n^-),$$

which together with (4,1) gives

$$\underline{\varphi}(A_n^+) > -\frac{1}{2^n}.$$

In analogy we get

$$\overline{\varphi}(A_n^-) < \frac{1}{2^n}.$$

Next we put

$$B_n^+ = \mathfrak{D}_{p=1}^{\infty} A_{n+p}^+ \text{ and } B_n^- = E - B_n^+ = \mathfrak{S}_p A_{n+p}^-.$$

Thus we have for any p

$$-\underline{\varphi}(B_n^+) \leqq -\underline{\varphi}(A_{n+p}^+) < \frac{1}{2^{n+p}}$$

since

$$B_n^+ \subset A_{n+p}^+,$$

i. e.

$$-\underline{\varphi}(B_n^+) = 0.$$

By means of theorem VI, page 9, it is derived that

$$\overline{\varphi}(B_n^-) \leqq \sum_{p=1}^{\infty} \overline{\varphi}(A_{n+p}^-) < \sum_{p=1}^{\infty} \frac{1}{2^{n+p}} = \frac{1}{2^n}. \qquad (4,2)$$

If we now introduce

$$E^+ = \mathfrak{S}_n B_n^+ \text{ and } E^- = E - E^+ = \mathfrak{D}_n B_n^-$$

we get

$$E = E^+ + E^-,$$

a decomposition of E having the properties mentioned in the theorem. For we have, again by means of theorem VI page 9,

$$-\underline{\varphi}(E^+) \leqq \sum_n -\underline{\varphi}(B_n^+) = 0$$

and consequently

$$\underline{\varphi}(E^+) = 0.$$

Since $E^- \subset B_n^-$ and applying (4,2) we see that

$$\overline{\varphi}(E^-) \leqq \overline{\varphi}(B_n^-) < \frac{1}{2^n}$$

is true for any n, and consequently

$$\overline{\varphi}(E^-) = 0.$$

After these preparations we can easily establish the proof of our main theorem. For every n $(n = 1, 2, 3, \cdots)$

$$\psi_n = \varphi - n\mu$$

is a bounded totally additive function of a set defined in $\mathfrak{F}$. According to our second lemma E may be written in the form

$$E = E_n^+ + E_n^-$$

where E_n^+ and E_n^- both belong to $\mathfrak{F}$, and such that

$$\underline{\psi}_n(E_n^+) = \overline{\psi}_n(E_n^-) = 0.$$

For every $A \varepsilon \mathfrak{F}$, where $A \subset E_n^+$, we now have

$$\psi_n(A) \geqq 0 \quad \text{i. e.} \quad \varphi(A) \geqq n\mu(A), \tag{4,3}$$

and for every $A \varepsilon \mathfrak{F}$, where $A \subset E_n^-$,

$$\psi_n(A) \leqq 0 \quad \text{i. e.} \quad \varphi(A) \leqq n\mu(A). \tag{4,4}$$

Applying (4,3) on E_n^+ we get, due to $E_n^+ \subset E$,

$$\varphi(E) \geqq \varphi(E_n^+) \geqq n\mu(E_n^+).$$

Since $\varphi(E) < \infty$ we get

$$\lim_n \mu(E_n^+) = 0. \tag{4,5}$$

If now

$$N = \mathfrak{D}_n E_n^+, \quad E - N = E - \mathfrak{D}_n E_n^+ = \mathfrak{S}_n E_n^-, \tag{4,6}$$

$N \subset E_n^+$ holds true for any n, from which it follows that

$$0 \leqq \mu(N) \leqq \mu(E_n^+),$$

which relation together with (4,5) gives

$$\mu(N) = 0. \tag{4,7}$$

Now let $A \varepsilon \mathfrak{F}$ be a set having the properties $A \subset E - N$ $\left(\text{i. e. according to (4,6) } A \subset \mathfrak{S}_n A E_n^-\right)$ and $\mu(A) = 0$. Consequently we can show that $\varphi(A) = 0$, because due to theorem VI, page 9,

$$\varphi(A) = \varphi\left(\mathfrak{S}_n A E_n^-\right) \leqq \sum_n \varphi(A E_n^-),$$

and since $A E_n^- \subset E_n^-$ (4,4) gives

$$\varphi(A) \leqq \sum_n n \mu(A E_n^-);$$

since $\mu(A) = 0$ we know, however, that $\mu(A E_n^-) = 0$ for any n, i.e.

$$\varphi(A) = 0,$$

q. e. d.

5. Indefinite integral.

$\mathfrak{F}$ being a Borel ring, containing E, μ being a measure defined in $\mathfrak{F}$ and f being a function μ-integrable on $\mathfrak{F}$ with $[f] = E$, we now introduce the indefinite integral by the following definition:

The function of a set

$$\varphi(A) = I(f_A) = \int_A f(x)\, \mu(dE)$$

is called the indefinite integral of the function f with respect to the measure μ.

In the special case $f \geqq 0$, we get for any $A \varepsilon \mathfrak{F}$

$$0 \leqq \varphi(A) \leqq \varphi(E),$$

since $\varphi(E) = I(f) < \infty$, and by means of the theorems I and III, page 13. The theorem III, page 13, further shows that φ is totally additive. From $\mu(A) = 0$ it follows that $\mu[f_A > 0] = 0$, and hence, further, that $\varphi(A) = 0$. Consequently we see that the function φ is μ-continuous.

On the exact connection between functions of a set and indefinite integrals we have the following theorem:

A function of a set φ, whose defining region $\mathfrak{F}$ is a Borel ring, is the indefinite integral with respect to the measure μ of a function f with $[f] = E$, which is μ-integrable on $\mathfrak{F}$ when and only when it is bounded, totally additive and μ-continuous.

We have seen above that $f \geqq 0$ implies that φ is bounded, totally additive and μ-continuous. As we shall in the following chiefly consider non-negative functions of a set, we shall in this account confine ourselves to mention how the function f may be defined in the following special case:

For a function of a set φ, which is non-negative, bounded, totally additive and μ-continuous, and whose defining region $\mathfrak{F}$ is a Borel ring, and a bounded measure μ, may be defined a function f on $\mathfrak{F}$, which is non-negative and μ-integrable, and where $[f] = E$, such that for every $A \varepsilon \mathfrak{F}$

$$\varphi(A) = \int_A f(x)\,\mu(dE).$$

For any a, $0 \leqq a < \infty$, we can determine a decomposition of E

$$E = E_a^+ + E_a^-,$$

where E_a^+ and E_a^- both belong to $\mathfrak{F}$, such that the function $\psi_a = \varphi - a\mu$ has the properties

$$\underline{\psi}_a(E_a^+) = 0 \text{ and } \overline{\psi}_a(E_a^-) = 0$$

i. e.

$$\left.\begin{aligned} \varphi(A) &\geqq a\mu(A) \text{ for } A \varepsilon \mathfrak{F} \text{ and } A \subset E_a^+ \\ \varphi(A) &\leqq a\mu(A) \text{ for } A \varepsilon \mathfrak{F} \text{ and } A \subset E_a^- \end{aligned}\right\} \qquad (5,1)$$

(cf. the lemma page 16), and this decomposition may be carried out in such a way that the following conditions are also satisfied:

1) $E_0^+ = E$

2) $E_a^+ \supset E_b^+$ for $b > a$

3) $E_a^+ = \mathfrak{D}_n E_{a_n}^+$ for $a =$ upper bound $\{a_1, a_2, \cdots\}$

4) $\mathfrak{D}_n E_n^+ = 0$.

(As to proof, see JESSEN [3].)

For every $x \varepsilon E$ we shall now find the values of a for which x belongs to E_a^+. These values will constitute a bounded closed interval. The fact that it is an interval is a consequence of 2), the fact that it is bounded is a consequence of 4), and the fact that it is closed is a consequence of 1) and 3). The function $f(x)$ is now introduced by the following definition

$$f(x) = \max_a \{a;\ x \varepsilon E_a^+\},$$

or expressed in another way

$$x \varepsilon E_a^+ \text{ for } 0 \leqq a \leqq f(x).$$

The function f for which $[f] = E$ is finite and non-negative; we further see that $[f \geqq a] = E_a^+$, from which follows that f is a function defined in $\mathfrak{F}$. Thus, for any $A \varepsilon \mathfrak{F}$ $I(f_A)$ exists. Finally we shall show that

$$\varphi(A) = I(f_A).$$

For this purpose it will suffice to show that the inequalities

$$I(g) \leqq \varphi(A) \leqq I(h) \tag{5,2}$$

hold true, when g and h are functions on $\mathfrak{F}$, where $[g] = [h] = A$, which takes on only a finite or enumerable number of values, and for which $g \leqq f_A$ and $h \geqq f_A$ for any $x \varepsilon A$.

φ being totally additive, it will suffice to show the inequalities (5,2) in the case of the functions g and h being constant. Thus let $g(x) = c_1$ and $h(x) = c_2$ for every $x \varepsilon A$. From $g(x) = c_1 \leqq f_A$ for every $x \varepsilon A$ it follows that $A \subset [f \geqq c_1] = E_{c_1}^+$, and hence further by means of (5,1) that

$$\varphi(A) \geqq c_1 \mu(A) = I(g).$$

From $h(x) = c_2 \geqq f_A$ for every $x \varepsilon A$ it follows that $A \subset [f < c] = E_c^-$ for any $c < c_2$, and hence by means of (5,1) that

$$\varphi(A) \leqq c\,\mu(A).$$

This inequality being valid for any $c > c_2$ we get

$$\varphi(A) \leqq c_2\,\mu(A) = I(h)$$

and the proof is completed.

PART II.

On monotone functions. Functions of a set on Borel classes.

6. A theorem on decomposition.

The purpose of this section is to give an account of a theorem valid for monotone functions, a theorem which we shall apply in the following[1]. We shall confine ourselves to nondecreasing functions, but this is of no significance, as analogous theorems are immediately seen to be valid also for non-increasing functions. The non-decreasing functions are fixed by the following definition:

A function $f(x)$ defined in $a < x < b$ is called non-decreasing, if $f(x_2) \geqq f(x_1)$ for $x_2 > x_1$.

Together with the function $f(x)$ we shall consider two other functions, $\underline{f}(x)$ and $\bar{f}(x)$, determined by the following definitions:

$$\underline{f}(x) = \underset{a < \xi < x}{\text{upper bound}}\ f(\xi) \tag{6,1}$$

$$\bar{f}(x) = \underset{x < \xi < b}{\text{lower bound}}\ f(\xi). \tag{6,2}$$

Hence the following inequalities are immediately seen to be true

$$\underline{f}(x) \leqq f(x) \leqq \bar{f}(x) \tag{6,3}$$

$$f(x_1) \leqq \underline{f}(x_2) \text{ for } x_1 < x_2 \tag{6,4}$$

$$\bar{f}(x_1) \leqq f(x_2) \text{ for } x_1 < x_2. \tag{6,5}$$

1) Cf. Carathéodory [1].

From (6,3), (6,4) and (6,5) the two functions $\underline{f}(x)$ and $\overline{f}(x)$ are seen to be non-decreasing.

We shall now show that $f(x)$ has in every point x a limit value from the left, $f(x-0)$, as well as a limit value from the right, $f(x+0)$, and that

$$f(x-0) = \underline{f}(x) \tag{6,6}$$

$$f(x+0) = \overline{f}(x). \tag{6,7}$$

In order to prove the first of these relations we must, consequently, prove that we can to an arbitrary $\varepsilon > 0$ determine a $\delta > 0$ such that

$$\underline{f}(x) - \varepsilon < f(\xi) < \underline{f}(x) + \varepsilon \tag{6,8}$$

for

$$x - \delta < \xi < x.$$

From (6,4) it is obvious that the right side of (6,8) will be true for every $\xi < x$. From (6,1) it follows that we can find a point ξ_1, $a < \xi_1 < x$, such that

$$f(\xi_1) > \underline{f}(x) - \varepsilon.$$

If $\delta = x - \xi_1$, the left hand side of (6,8) will, because of the monotony of $f(x)$, be true for every ξ of the interval $x - \delta < \xi < x$. In exactly the same way (6,7) is proved.

For every x we shall introduce the quantity $S(x)$ by the definition

$$S(x) = f(x+0) - f(x-0) = \overline{f}(x) - \underline{f}(x), \tag{6,9}$$

and call it *the saltus of the function in the point* x.

$f(x)$ is continuous in the point x (cf. (6,3)) if $S(x) = 0$ for this value of x, whereas the function is discontinuous in the point x if $S(x) > 0$.

In the remaining part of this section we shall assume the function $f(x)$ to be bounded in the interval $a < x < b$, i. e.

$$f(a+0) > -\infty \quad \text{and} \quad f(b-0) < \infty. \tag{6,10}$$

We shall denote by A_n the set of the points x, $a < x < b$, in which $S(x) > \frac{1}{n}$ (n positive, integer). Of this set of points is true that it is either empty or consists of a finite number of points. Thus let $x_1, x_2, \cdots, x_p$ be p points of the interval $a < x_1 < x_2 < \cdots < x_p < b$, in which $S(x) > \frac{1}{n}$. We then have

$$f(b-0)-f(a+0) \geqq \big(f(b-0)-f(x_p+0)\big)+\sum_{i=1}^{p}\big(f(x_i+0)-f(x_i-0)\big)$$
$$+\big(f(x_1-0)-f(a+0)\big) > p\cdot\frac{1}{n}, \qquad (6,11)$$

and hence

$$p < n\{f(b-0)-f(a+0)\}.$$

This being so, it is easily seen that *the set, A, of points of discontinuity of the function $f(x)$ of the interval $a < x < b$ is at most enumerable,* because

$$A = A_1 \dotplus A_2 \dotplus \cdots \dotplus A_n \dotplus \cdots.$$

For any choice of the two points x_1 and x_2 of the interval $a < x < b$, $x_1 < x_2$, we introduce the quantity $V_f(x_1, x_2)$ by the definition

$$V_f(x_1, x_2) = f(x_2) - f(x_1), \qquad (6,12)$$

and call it the *variation of $f(x)$* in the interval considered[1]. Furthermore we introduce the quantity $D_f(x_1, x_2)$ by the definition

$$D_f(x_1, x_2) = \big(f(x_1+0)-f(x_1)\big)+\sum_i S(\xi_i)+\big(f(x_2)-f(x_2-0)\big), \quad (6,13)$$

where the summation is extended over the (at most enumerably many) points of discontinuity of $f(x)$ contained in the interval $x_1 < x < x_2$. This latter quantity, $D_f(x_1\ x_2)$, will be called *the total discontinuity of $f(x)$* in the interval considered[2].

By a transcription analogous to (6,11) it is clear that the following inequality holds true for every p concerned

$$\sum_{i=1}^{p} S(\xi_i) \leqq f(x_2-0)-f(x_1+0),$$

[1] It is immediately evident that $V_f(x_1, x_2) \geqq 0$.
[2] For any choice of x_1 and x_2 $(x_1 < x_2)$ is valid that $D_f(x_1, x_2) \geqq 0$.

and hence

$$\sum_i S(\xi_i) \leqq f(x_2-0)-f(x_1+0). \tag{6,14}$$

Inserting (6,14) in (6,13) we obtain

$$\begin{aligned} D_f(x_1,x_2) &\leqq \big(f(x_1+0)-f(x_1)\big)+\big(f(x_2-0)-f(x_1+0)\big)+\big(f(x_2)-f(x_2-0)\big) \\ &= f(x_2)-f(x_1) = V_f(x_1, x_2). \end{aligned} \tag{6,15}$$

Thus we see that the total discontinuity of an interval never exceeds the variation of the function.

From the definition (6,13) it is immediately obvious that

$$D_f(x_1, x_3) = D_f(x_1, x_2) + D_f(x_2, x_3) \tag{6,16}$$

for $x_1 < x_2 < x_3$.

After these preliminary remarks we can go over to the proof of the important theorem of decomposition:

Let $f(x)$ be a bounded non-decreasing function defined in $a < x < b$. This function may then be written as

$$f(x) = g(x) + h(x), \tag{6,17}$$

where both $g(x)$ and $h(x)$ are non-decreasing functions, and where, furthermore, $g(x)$ is continuous, whereas for $h(x)$

$$V_h(x_1, x_2) = D_h(x_1, x_2) = D_f(x_1, x_2) \tag{6,18}$$

for any choice of x_1 and x_2, $a < x_1 < x_2 < b$.

We choose a fixed point x_0 of the interval $a < x < b$ and introduce a function $h(x)$ by the definition

$$h(x) = \begin{cases} -D_f(x, x_0) & \text{for } a < x < x_0 \\ 0 & \text{for } x = x_0 \\ D_f(x_0, x) & \text{for } x_0 < x < b. \end{cases} \tag{6,19}$$

We shall now prove the function thus defined to have the properties expressed in the relations (6,18).

By means of (6,16) it is immediately seen that for two arbitrary numbers x_1 and x_2, $x_1 < x_2$, we have

$$V_h(x_1, x_2) = h(x_2) - h(x_1) = D_f(x_1, x_2), \qquad (6,20)$$

from which it particularly follows that $h(x)$ is a non-decreasing function. (6,20) in connection with (6,13) gives

$$h(x_2) - h(x_1) \leqq f(x_1 + 0) - f(x_1) \qquad (6,21)$$

$$h(x_2) - h(x_1) \leqq f(x_2) - f(x_2 - 0), \qquad (6,22)$$

and (6,20) in connection with (6,15)

$$h(x_2) - h(x_1) \leqq f(x_2) - f(x_1). \qquad (6,23)$$

(6,21) being true for every $x_1 < x_2$, we can deduce the following inequality

$$h(x_1 + 0) - h(x_1) \geqq f(x_1 + 0) - f(x_1).^{1)} \qquad (6,24)$$

Similarly we deduce from (6,23) that

$$h(x_1 + 0) - h(x_1) \leqq f(x_1 + 0) - f(x_1). \qquad (6,25)$$

From (6,24) and (6,25) it follows for every x, $a < x < b$, that

$$h(x + 0) - h(x) = f(x + 0) - f(x). \qquad (6,26)$$

After the analogy of (6,26) we can deduce

$$h(x) - h(x - 0) = f(x) - f(x - 0), \qquad (6,27)$$

which in connection with (6,26) gives

$$h(x + 0) - h(x - 0) = f(x + 0) - f(x - 0). \qquad (6,28)$$

By application of (6,20), (6,26), (6,27) and (6,28) the definition (6,13) gives

$$D_h(x_1, x_2) = D_f(x_1, x_2) = V_h(x_1, x_2).$$

Thus we see that the function $h(x)$ has the properties expressed in the relations (6,18).

The proof will be complete, if we can prove the function

[1] The existence of $h(x_1 + 0)$ is a consequence of $h(x)$ being a non-decreasing function.

$$g(x) = f(x) - h(x) \tag{6,29}$$

to be non-decreasing and continuous. From (6,29) we obtain

$$g(x_2) - g(x_1) = \left(f(x_2) - f(x_1)\right) - \left(h(x_2) - h(x_1)\right),$$

which for $x_1 < x_2$ by application of (6,23) gives

$$g(x_2) \geqq g(x_1).$$

Thus we get

$$g(x+0) - g(x-0) = \left(f(x+0) - f(x-0)\right) - \left(h(x+0) - h(x-0)\right),$$

which by application of (6,28) gives

$$g(x+0) = g(x-0).$$

Since we, furthermore, (cf. (6,3), (6,6) and (6,7)) have the inequalities

$$g(x-0) \leqq g(x) \leqq g(x+0)$$

the proof of the continuity of $g(x)$ is completed.

We shall conclude this section with some remarks on the connection between the various decompositions of $f(x)$. In case

$$f(x) = g(x) + h(x)$$

is a decomposition with the properties mentioned in the theorem, it will be true (for every c) that

$$f(x) = \left(g(x) + c\right) + \left(h(x) - c\right)$$

will also be so, and thus all decompositions will be comprised. Let for instance

$$f(x) = g_1(x) + h_1(x)$$

be a decomposition having the properties mentioned in the theorem. Then we can prove that $h_1(x) = h(x) - c$ (and hence $g_1(x) = g(x) + c$).

From (6,18) it follows that

$$V_h(x_1, x_2) = V_{h_1}(x_1, x_2) \; \left(= D_f(x_1, x_2)\right)$$

for $x_1 < x_2$ which, if we substitute x for x_2, gives

$$h(x) - h(x_1) = h_1(x) - h_1(x_1) \text{ for } x > x_1$$

or, if we substitute x for x_1 and x_1 for x_2

$$h(x_1) - h(x) = h_1(x_1) - h_1(x) \text{ for } x < x_1.$$

Thus we have for every x, $a < x < b$,

$$h(x) = h_1(x) - \left(h_1(x_1) - h(x_1)\right) = h_1(x) - c,$$

which was to be proved.

7. Functions of a set having the Borel class as defining region.

The set of points lying on the axis of the real numbers constitutes a set of points, which in this section will be called E. When we in this section speak of a set, it will be understood to be a subset of E. By an *interval* I will in the following be understood a set of points having the form $[a \leqq x < b]$, where a and b are finite. All the intervals form a class of sets $\mathfrak{F}$. The smallest extension of $\mathfrak{F}$ to a ring we shall denote by $\mathfrak{G}$. It is clear that this ring consists of the empty set together with all finite sums of intervals. The smallest extension of $\mathfrak{F}$ to a Borel ring will be called $\mathfrak{B}$. This class of sets $\mathfrak{B}$ we shall call *the Borel class on the axis of the real numbers*, and every set $A \varepsilon \mathfrak{B}$ will be called a *Borel set*.

A function of a set φ defined in $\mathfrak{F}$ will be called a *function of an interval*. A function of an interval φ will be called *continuous from the inside*, if for every interval $I = [a \leqq x < b]$ and for every sequence of intervals $I_1, I_2, \cdots, I_n, \cdots$, where $I_n = [a \leqq x < b_n]$, $b_1 < b_2 < \cdots < b$ and $\lim_n b_n = b$, we have that $\varphi(I) = \lim_n \varphi(I_n)$.

On the connection between functions of an interval and measures the following theorem can be proved:

A finite, additive function of an interval φ can then and only then be extended to a measure φ^* defined in $\mathfrak{B}$, when it is non-negative and continuous from the inside.

We shall first prove the conditions to be necessary. From $\varphi^* \geqq 0$ for every $A \varepsilon \mathfrak{B}$ it follows that $\varphi \geqq 0$ for every $A \varepsilon \mathfrak{F}$, i. e. φ is non-negative. From $a < b_1 < b_2 < \cdots < b$ and $\lim_n b_n = b$ it follows that

$$\mathfrak{D}_n [b_n \leqq x < b] = 0, \tag{7,1}$$

and hence further (see theorem VIII, page 9)

$$\lim_n \varphi([b_n \leqq x < b]) = 0. \tag{7.2}$$

Since

$$\varphi([a \leqq x < b]) = \varphi([a \leqq x < b_n]) + \varphi([b_n \leqq x < b])$$

for every n, (7,2) implies that

$$\lim_n \varphi([a \leqq x < b_n]) = \varphi([a \leqq x < b]),$$

i. e. φ is continuous from the inside.

Next we shall prove the conditions to be sufficient. The extension can be performed in such a manner that we first extend φ to a content ψ defined in $\mathfrak{G}$, and then prove that this content can be extended to a measure (defined in $\mathfrak{B}$).

For every set $A \varepsilon \mathfrak{G}$ we put

$$\psi(A) = 0 \text{ when } A = 0$$

$$\psi(A) = \sum_{i=1}^{n} \varphi(I_i) \text{ when } A = I_1 + I_2 + \cdots + I_n.^{1)}$$

The function of a set ψ thus derived, is at once seen to be an extension of φ to a content. If ψ is totally additive, this content can be extended to a measure defined in $\mathfrak{B}$ (cf. the theorem on extension page 10).

So our problem is to show that if $A_1, A_2 \cdots, A_n, \cdots$ is a sequence of sets, all belonging to $\mathfrak{G}$, and for which $A_1 \supset A_2 \supset \cdots \supset A_n \supset \cdots$ and $\psi(A_n) \geqq k > 0$ for all n, then the set $\mathfrak{D}_n A_n$ is empty (see page 11).

Since $A_n \varepsilon \mathfrak{G}$ it can be written in the form

$$A_n = \sum_p{}^{*} [a_{np} \leqq x < b_{np}],$$

where $\sum^{*}$ is to denote that the number of the addends is finite. φ being continuous from the inside, we can for every n determine a set $B_n \subset A_n$

1) It can easily be proved that this definition determines the function $\psi(A)$ uniquely.

$$B_n = \sum_p{}^* [a_{np} \leqq x < c_{np}], \quad (a_{np} < c_{np} < b_{np})$$

such that

$$\psi(A_n - B_n) \leqq \frac{k}{2^n}. \tag{7,3}$$

If we put

$$C_n = B_1 B_2 \cdots B_n,$$

we have $C_n \subset B_n$ and $C_1 \supset C_2 \supset \cdots$. Further we get

$$A_n - C_n \subset (A_1 - B_1) \dotplus \cdots \dotplus (A_n - B_n),$$

and hence, by means of (7,3),

$$\psi(A_n - C_n) \leqq \frac{k}{2} + \cdots + \frac{k}{2^n} < k.$$

Since $\psi(A_n) \geqq k$, we see that C_n is non-empty for every value of n. In every C_n we may thus choose a point x_n, and by that get the sequence $x_1, x_2, \cdots, x_n, \cdots$. Since $C_n \subset A_n \subset A_1$ for every n, we see that this sequence is bounded. Thus it is possible to choose a convergent subsequence from it. Its limit point is called x. Due to $B_n \supset C_n \supset C_{n+1} \supset \cdots$ all the points $x_n, x_{n+1}, \cdots$ will belong to B_n, which will further imply that

$$x \varepsilon \sum_p{}^* [a_{np} \leqq x \leqq c_{np}] \subset A_n.$$

Thus we have

$$x \varepsilon \mathfrak{D}_n A_n,$$

i. e. the set $\mathfrak{D}_n A_n$ is non-empty, which was to be proved.

If the function of an interval in question for every interval $[a \leqq x < b]$ has the value $b - a$, we shall call the measure, obtained by the extension and having $\mathfrak{B}$ as defining region, *the Borel measure on the axis of the real numbers,* and therefore $\mathfrak{B}$ will also be called *the class of Borel measurable sets on the axis of the real numbers.*

If $f(x)$ is a finite function defined in E, it is possible from this to form a finite additive function of an interval by the following definition

$$\varphi([a \leqq x < b]) = f(b) - f(a). \tag{7,4}$$

Conversely it is possible to find to a finite, additive function of an interval φ a function $f(x)$ defined in E, so that (7,4) is satisfied for every interval, and it is clear that the difference between two such possible functions f is constant.

The function of an interval φ is then and only then non-negative, when it is valid for any pair of numbers (a, b) where $a < b$ that $f(a) \leqq f(b)$, i. e. when $f(x)$ is *non-decreasing*. We see furthermore that φ is then and only then *continuous from the inside*, when $f(x)$ is *continuous from the left* for every x.

Applying the theorem page 29 we now have:

To a finite function $f(x)$ defined in E we have then and only then a corresponding measure φ defined in $\mathfrak{B}$, so that

$$\varphi([a \leqq x < b]) = f(b) - f(a)$$

for every interval $[a \leqq x < b]$, when $f(x)$ is non-decreasing and continuous from the left. This measure φ will be uniquely defined.

If $F(x)$ is a function in $\mathfrak{B}$ and $A \varepsilon \mathfrak{B}$, we denote the integral of $F(x)$ over A with respect to the measure φ by

$$\int_A F(x)\, d f(x)$$

or, if especially $A = [a \leqq x < b]$,

$$\int_a^b F(x)\, d f(x).$$

This integral is called the *Lebesgue-Stieltjes-integral* with respect to $f(x)$.

8. Functions of a set having $\mathfrak{B}_1$ as defining region.

The set of points x, belonging to the interval $0 \leqq x < 1$, forms a set of points, which we in this section shall denote by E_1, and speaking in this section of a set, we shall always mean a subset of E_1. The set of intervals $[a \leqq x < b]$, $0 \leqq a < b \leqq 1$, forms a class of sets $\mathfrak{J}_1$. The smallest extension of $\mathfrak{J}_1$ to a ring we shall denote by $\mathfrak{G}_1$, and the smallest extension of $\mathfrak{J}_1$ to a Borel ring we shall call $\mathfrak{B}_1$. This class of sets $\mathfrak{B}_1$ we shall call *the Borel class of the interval* (0,1). It is evident that the theorems formulated in the preceding section will still hold true, even if we confine ourselves to the interval (0,1).

Now suppose given a finite function $f(x)$ defined in $0 \leqq x \leqq 1$, which is non-decreasing and continuous from the left in every point. We may then (cf. the theorem page 32) uniquely determine a measure φ^* defined in $\mathfrak{B}_1$, such that

$$\varphi^*([a \leqq x < b]) = f(b) - f(a) \tag{8,1}$$

for every choice of a and b, $0 \leqq a < b \leqq 1$.

In the following we shall put $f(0) = 0$, which does not limit the generality of our investigation.

According to the theorem on decomposition (page 26) $f(x)$ may be written in the form

$$f(x) = g(x) + h(x), \tag{8,2}$$

where $g(x)$ and $h(x)$ are non-decreasing functions, $g(x)$ being furthermore continuous, whereas for $h(x)$

$$V_h(a, b) = D_h(a, b) = D_f(a, b) \tag{8,3}$$

for every choice of a and b, $0 \leqq a < b \leqq 1$. This decomposition can, furthermore, be performed in such a way that $g(0) = 0$ (and hence $h(0) = 0$), and is in that case uniquely defined. Since $f(x)$ was given to be continuous from the left and $g(x)$ is continuous, it follows that $h(x)$ is continuous from the left in any point.

We can now uniquely determine two measures φ_1^* and φ_2^* defined in $\mathfrak{B}_1$, such that

$$\varphi_1^*([a \leqq x < b]) = g(b) - g(a) \tag{8,4}$$

and

$$\varphi_2^*([a \leqq x < b]) = h(b) - h(a) \tag{8,5}$$

for every choice of a and b.

Regarding the connection between φ^*, φ_1^* and φ_2^* we can prove the relation

$$\varphi^* = \varphi_1^*(A) + \varphi_2^*(A) \tag{8,6}$$

for every set $A \varepsilon \mathfrak{B}_1$.

We only remark that the function of a set

$$\varphi_1^*(A) + \varphi_2^*(A)$$

is a measure defined in $\mathfrak{B}_1$, and that, if A is an interval $[a \leqq x < b]$, it will take on the value

$$\varphi_1^*([a \leqq x < b]) + \varphi_2^*([a \leqq x < b]) = g(b) - g(a) + h(b) - h(a),$$

which by application of (8,2) can be changed to

$$\varphi_1^*([a \leqq x < b]) + \varphi_2^*([a \leqq x < b]) = f(b) - f(a).$$

If we compare this result with (8,1), we shall see that the two measures $\varphi^*(A)$ and $\varphi_1^*(A) + \varphi_2^*(A)$ coincide in every interval. Consequently they are both an extension of the same function of an interval, and will naturally coincide for the whole of $\mathfrak{B}_1$.

We shall conclude this section with an investigation of what values the two functions of a set φ_1^* and φ_2^* can take on in $\mathfrak{B}_1$.

From $0 \subset A \subset E_1$ for every $A \varepsilon \mathfrak{B}_1$ it follows that

$$0 \leqq \varphi_1^*(A) \leqq \varphi_1^*(E_1) = g(1),$$

and since $g(x)$ is continuous, it will take on any value y_0 between 0 and $g(1)$ for at least one value of x, $x = x_0$. If we choose $A = [0 \leqq x < x_0]$ we get $\varphi_1^*(A) = y_0$. *Thus we see that the values of φ_1^* constitute a closed interval.*

The points of discontinuity of the function $h(x)$ (or, what is the same, of the function $f(x)$) form an at most enumerable set of points $N = \{\xi_n\}$ situated in the interval $0 \leqq x < 1$.[1] By the saltus at a point x we understood (cf. (6,9)) the quantity

$$S(x) = h(x+0) - h(x-0).$$

For the special case $x = 0$ we write

$$S(x) = h(x+0) - h(x).$$

[1] The fact that the eventual points of discontinuity are situated in this half-open interval is a consequence of $f(x)$ being continuous from the left.

We shall now prove that we have for every set $A \varepsilon \mathfrak{B}_1$

$$\varphi_2^*(A) = \sum_{\xi_i \varepsilon A} S(\xi_i), \tag{8,7}$$

where the summation is extended over the—at most enumerably many—points of discontinuity for $h(x)$, belonging to the set A.

We first note that the function of a set

$$\sum_{\xi_i \varepsilon A} S(\xi_i) \tag{8,8}$$

is a measure defined in $\mathfrak{B}_1$. If, especially, A is an interval $[a \leqq x < b]$, we have

$$\sum_{\xi_i \varepsilon A} S(\xi_i) = h(b) - h(a). \tag{8,9}$$

Applying (6,12), (6,13), and (8,3) we get

$$h(b) - h(a) = h(a+0) - h(a) + \sum_i S(\xi_i) + h(b) - h(b-0),$$

where the summation is extended over the points of discontinuity situated in the interval $a < x < b$. Since $h(x)$ is continuous from the left ($h(a) = h(a-0)$ and $h(b) = h(b-0)$), this may be written

$$h(b) - h(a) = \sum_{\xi_i \varepsilon A} S(\xi_i),$$

and the proof of (8,9) is completed.

If we compare (8,9) with (8,5), we see that the two measures $\varphi_2^*(A)$ and $\sum_{\xi_i \varepsilon A} S(\xi_i)$ agree in every interval. Consequently they are both an extension of the same function of an interval, and thus they must coincide in the whole of $\mathfrak{B}_1$, and we have proved (8,7). The formula (8,7) may also be written

$$\varphi_2^*(A) = \sum_{\xi_i \varepsilon AN} S(\xi_i). \tag{8,10}$$

If the set A consists of only one point a, we get the special case

$$\varphi_2^*(A) = S(a). \tag{8,11}$$

Thus we have realized what values the function φ_2^* can take on. If the number of points of discontinuity of the function $f(x)$ is finite: $\xi_1, \xi_2, \cdots, \xi_n$, respectively enumerable $\xi_1, \xi_2, \cdots, \xi_n, \cdots$, *the values of* φ_2^* *will be all numbers of the form*

$$e_1 S(\xi_1) + e_2 S(\xi_2) + \cdots + e_n S(\xi_n), \tag{8,12}$$

respectively

$$e_1 S(\xi_1) + e_2 S(\xi_2) + \cdots + e_n S(\xi_n) + \cdots, \tag{8,13}$$

where the e's independently of each other take on the values 0 *or* 1.

PART III.

A theorem on bounded measures in abstract space.

9. A theorem on series of positive terms.

Given a convergent series of positive terms

$$a = \sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots + a_n + \cdots \quad (a_n > 0). \tag{9,1}$$

Together with this series we shall consider all series having the form

$$\sum_{n=1}^{\infty} e_n a_n = e_1 a_1 + e_2 a_2 + \cdots + e_n a_n + \cdots, \tag{9,2}$$

where the e's independently of each other take on the values 0 or 1. Each of these series (9,2) is convergent, and its sum satisfies the relation

$$0 \leqq \sum_{n=1}^{\infty} e_n a_n \leqq a. \tag{9,3}$$

We shall now prove that the set of numbers, whose elements are the sums of the series (9,2), is *closed*. Let

$$\left.\begin{array}{l} s_1 = a_{11} + a_{12} + \cdots + a_{1n} + \cdots \\ s_2 = a_{21} + a_{22} + \cdots + a_{2n} + \cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ s_m = a_{m1} + a_{m2} + \cdots + a_{mn} + \cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{array}\right\} \tag{9,4}$$

be a sequence of series of the form (9,2), i. e. a_{mn} has for every m one of the values 0 or a_n, and let furthermore the sequence

$$s_1, s_2, \cdots, s_m, \cdots$$

be convergent to the limit value s. We shall now prove that among the series (9,2) at least one has the sum s. In the sequence of numbers

$$a_{11}, a_{21}, \cdots, a_{m1}, \cdots$$

at least one of the numbers 0 or a_1 will appear an infinite number of times. Let a_1^* denote one of these two numbers satisfying this condition. In the sequence of pairs of numbers

$$(a_{11}, a_{12}), (a_{21}, a_{22}), \cdots, (a_{m1}, a_{m2}), \cdots$$

there will, consequently, be an infinite number having a_1^* in the first place. Let a_2^* be a number which in the corresponding subsequence appears an infinite number of times in the second place. In the sequence of set of numbers

$$(a_{11}, a_{12}, a_{13}), (a_{21}, a_{22}, a_{23}), \cdots, (a_{m1}, a_{m2}, a_{m3}), \cdots$$

there will thus be an infinite number having (a_1^*, a_2^*) in the two first places. Let a_3^* be a number which in the corresponding subsequence appears an infinite number of times in the third place. By continuing this process the number a_n^* is defined for every n. The series

$$a_1^* + a_2^* + \cdots + a_n^* + \cdots \tag{9,5}$$

is, being a subseries of (9,1), convergent, and we shall now prove that it has the sum s. Suppose we for the present term the sum of it s^*. The convergence of (9,1) implies that to a given $\varepsilon > 0$ we may determine N, such that

$$\sum_{n=N+1}^{\infty} a_n < \frac{\varepsilon}{3}.$$

Moreover we may, among the series (9,1), determine one having $a_1^*, a_2^*, \cdots, a_N^*$ in the N first places

$$s_i = a_1^* + a_2^* + \cdots + a_N^* + a_{i,\,N+1} + a_{i,\,N+2} + \cdots, \qquad (9,6)$$

and such that $\left| s - s_i \right| < \frac{\varepsilon}{3}$.

Accordingly we have

$$\begin{aligned} |s^* - s| &= \left|(s^* - (a_1^* + \cdots + a_N^*)) - (s_i - (a_1^* + \cdots + a_N^*)) + (s_i - s)\right| \\ &\leqq |s^* - (a_1^* + \cdots + a_N^*)| + |s_i - (a_1^* + \cdots + a_N^*)| + |s_i - s| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Hence we obtain

$$s^* = s$$

and the proof is completed.

10. Bounded measures having $\mathfrak{B}_1$ as defining region.

In II, 3 we have already mentioned the class of sets $\mathfrak{B}_1$, the so-called Borel class, defined in the interval (0,1). The interval $[0 \leqq x < 1]$ was termed E_1. Now let φ be a bounded measure defined in $\mathfrak{B}_1$, i. e. $\varphi(E_1) < \infty$. On such a measure we shall in this section prove the following theorem:

The set of numbers whose elements are the values of a bounded measure φ defined in $\mathfrak{B}_1$ is closed.

Together with the measure φ we shall consider the function $f(x)$ defined in $0 \leqq x \leqq 1$ determined by

$$\varphi([0 \leqq x < a]) = f(a)$$

$$f(0) = 0. \qquad (10,1)$$

The non-decreasing function $f(x)$ may then be decomposed to the form

$$f(x) = g(x) + h(x)$$

(cf. (8,2)), and this decomposition givesr ise to a decomposition of φ to the form

$$\varphi(A) = \varphi_1(A) + \varphi_2(A) \qquad (10,2)$$

for every $A \varepsilon \mathfrak{B}_1$ (cf. (8,6)). The set of points of discontinuity of the function $f(x)$ is at most an enumerable set $N = \{\xi_n\}$. We have previously shown that the set of numbers M_1, whose elements are the values of $\varphi_1(A)$, is closed. We have furthermore (cf. (8,10)) proved that

$$\varphi_2(A) = \sum_{\xi_i \varepsilon AN} S(\xi_i), \tag{10,3}$$

for every $A \varepsilon \mathfrak{B}_1$. The set of numbers M_2, whose elements are the values of $\varphi_2(A)$, has (cf. (8,12) and (8,13)) either the form

$$e_1 S(\xi_1) + e_2 S(\xi_2) + \cdots + e_n S(\xi_n) \tag{10,4}$$

or

$$e_1 S(\xi_1) + e_2 S(\xi_2) + \cdots + e_n S(\xi_n) + \cdots. \tag{10,5}$$

If the set of numbers is of the form (10,4), it is obvious that it is closed, as it is finite. If it is of the form (10,5), we may from the result obtained in the previous section conclude that it is closed.

The set of numbers M, whose elements are the values of $\varphi(A)$, is according to (10,2) produced by adding elements of M_2 to elements of M_1. If we can prove that the sum of an arbitrary element α of M_1 and an arbitrary element β of M_2 gives an element of M, our proof will be completed. Let, therefore, $\varphi_1(A) = \alpha$ and $\varphi_2(B) = \beta$, $A \varepsilon \mathfrak{B}_1$, $B \varepsilon \mathfrak{B}_1$. Our task is now to show the existence of a set $A^* \varepsilon \mathfrak{B}_1$, such that

$$\varphi(A^*) = \alpha + \beta. \tag{10,6}$$

If we put

$$A^* = (A - AN) + BN, \tag{10,7}$$

we have $A^* \varepsilon \mathfrak{B}_1$. It now follows, on account of (10,2), that

$$\left.\begin{array}{r} \varphi(A^*) = \varphi(A - AN) + \varphi(BN) = \\ \varphi_1(A - AN) + \varphi_2(A - AN) + \varphi_1(BN) + \varphi_2(BN). \end{array}\right\} \tag{10,8}$$

AN being at most an enumerable set, it follows that

$$\varphi_1(AN) = 0. \tag{10,9}$$

From $A - AN \subset E_1 - N$ follows

$$0 \leqq \varphi_2(A - AN) \leqq \varphi_2(E_1 - N) = 0. \tag{10,10}$$

After the analogy of (10,9) we have

$$\varphi_1(BN) = 0. \tag{10,11}$$

Finally we derive

$$\varphi_2(BN) = \sum_{\xi_i \varepsilon BN} S(\xi_i) = \varphi_2(B) = \beta. \tag{10,12}$$

Comprising (10,8)—(10,12) we get

$$\varphi(A^*) = \alpha + 0 + 0 + \beta = \alpha + \beta.$$

Hence the set A^* has the property expressed by (10,6), and the proof is completed.

11. Bounded measures in abstract space.

In this section will be shown that the theorem on measures defined in $\mathfrak{B}_1$, formulated and proved in III, 10, is valid in the abstract space too.

Let E be an arbitrary set, and let $\mathfrak{F}$ be a class of sets containing E. Further let ψ be a bounded measure defined in $\mathfrak{F}$, i. e. $\psi(E) < \infty$. Without loss of generality we may assume that $\psi(E) = 1$. On such a measure we shall in this section prove the following theorem:

The set of numbers whose elements are the values of a bounded measure ψ defined in $\mathfrak{F}$ is closed.

To prove the theorem we shall make use of a representation from the class of sets $\mathfrak{F}$ to the class of sets $\mathfrak{B}_1$, whereby the theorem is retraced to the theorem proved in the previous section.

Let

$$A_1, A_2, \cdots, A_n, \cdots \tag{11,1}$$

be a sequence of sets all belonging to $\mathfrak{F}$, and for which the corresponding sequence of numbers

$$\psi(A_1), \psi(A_2), \cdots, \psi(A_n), \cdots \tag{11,2}$$

is convergent to the limit value g. We now prove the existence of a set $A \varepsilon \mathfrak{F}$, for which

$$\psi(A) = g. \tag{11,3}$$

Without loss of generality we may assume that $A_2 = E - A_1$. The importance of this special choice of the set A_2 will be evident later on.

The coupling mentioned above will now be carried out by two stages:

I). To E is coupled the interval $(0,1)$. To the sets $C_1 = A_1$, $C_0 = 0$ and $C_2 = E - A_1$, all belonging to $\mathfrak{F}$, are, in the order given, coupled the intervals $(0, \frac{1}{3})$, $(\frac{1}{3}, \frac{2}{3})$ and $(\frac{2}{3}, 1)$. The sets $C_{11} = A_1 A_2$, $C_{10} = 0$ and $C_{12} = A_1(E - A_2)$, which are all subsets of C_1 and have the sum C_1, all belong to $\mathfrak{F}$. To these are coupled, in the order given, the intervals $(0, \frac{1}{9})$, $(\frac{1}{9}, \frac{2}{9})$ and $(\frac{2}{9}, \frac{1}{3})$. The sets $C_{21} = (E - A_1) A_2$, $C_{20} = 0$ and $C_{22} = (E - A_1)(E - A_2)$, which are all subsets of C_2 and have the sum C_2, likewise belong to $\mathfrak{F}$. To these are coupled the intervals $(\frac{2}{3}, \frac{7}{9})$, $(\frac{7}{9}, \frac{8}{9})$ and $(\frac{8}{9}, 1)$. The sets $C_{111} = A_1 A_2 A_3$, $C_{110} = 0$, $C_{112} = A_1 A_2 (E - A_3)$, $C_{121} = A_1 (E - A_2) A_3$, $C_{120} = 0$ and $C_{122} = A_1 (E - A_2)(E - A_3)$ all belong to $\mathfrak{F}$. To these are coupled the intervals $(0, \frac{1}{27})$, $(\frac{1}{27}\ \frac{2}{27})$, $(\frac{2}{27}, \frac{1}{9})$, $(\frac{2}{9}, \frac{7}{27})$, $(\frac{7}{27}, \frac{8}{27})$ and $(\frac{8}{27}, \frac{1}{3})$. Thus we go on infinitely. Each of the produced C-sets will belong to $\mathfrak{F}$. It is a product of as many factors as the number of indices. A number 0 in the last place of the series of indices means that the set is empty. A number 1 in the n'th place of the series of indices means that A_n appears as a factor, whereas a number 2 in the n'th place means that $E - A_n$ appears as a factor. Thus we give as an example

$$C_{1121121} = A_1 A_2 (E - A_3) A_4 A_5 (E - A_6) A_7$$

$$C_{2121120} = 0.$$

As an illustration we have in fig. 1 (cf. the end of the paper) given an outline of this decomposition and the corresponding intervals.

II). To each number of the form

$$\frac{k}{3^n} \ (n \text{ positive integer, } k = 1, 2, \cdots, 3^n - 1)$$

is now coupled a set $C_{\frac{k}{3^n}}$. If the number $\frac{k}{3^n}$ is *in the interior* of an interval, to which we have above coupled the empty set,

$C_{\frac{k}{3^n}}$ is put equal to the empty set 0. Concerning an end point of one of the intervals shown in fig. 1 we shall, however, proceed as follows. We select all the sets in our outline of which one end point of the corresponding interval is the number $\frac{k}{3^n}$ and whose series of indices does not comprise the number 0, and $C_{\frac{k}{3^n}}$ equal to the product of these sets. $\mathfrak{F}$ being a Borel ring, and all the sets in fig. 1 belonging to $\mathfrak{F}$, the same will be true for any set $C_{\frac{k}{3^n}}$. By this method we get

$$\left.\begin{aligned} C_{\frac{1}{3}} &= C_1 C_{12} C_{122} \cdots \\ C_{\frac{2}{3}} &= C_2 C_{21} C_{211} \cdots \\ C_{\frac{1}{9}} &= C_{11} C_{112} C_{1122} \cdots \\ C_{\frac{2}{9}} &= C_{12} C_{121} C_{1211} \cdots \end{aligned}\right\} \qquad (11,4)$$

and so on.

The decomposition (fig. 1) carried out in I) is now modified as follows. To the set E is coupled the interval $0 \leqq x < 1$. To the sets $C_1 - C_{\frac{1}{3}}$, $C_{\frac{1}{3}}$, 0, $C_{\frac{2}{3}}$ and $C_2 - C_{\frac{2}{3}}$, in the order given, are coupled the interval $0 \leqq x < \frac{1}{3}$, the point $x = \frac{1}{3}$, the interval $\frac{1}{3} < x < \frac{2}{3}$, the point $x = \frac{2}{3}$ and the interval $\frac{2}{3} < x < 1$. To the sets $C_{11} - C_{\frac{1}{9}}$, $C_{\frac{1}{9}}$, 0, $C_{\frac{2}{9}}$, $C_{12} - C_{\frac{2}{9}} - C_{\frac{1}{3}}$ (which, as we know, have the sum $C_1 - C_{\frac{1}{3}}$) are coupled, in the order given, the interval $0 \leqq x < \frac{1}{9}$, the point $x = \frac{1}{9}$, the interval $\frac{1}{9} < x < \frac{2}{9}$, the point $x = \frac{2}{9}$ and the interval $\frac{2}{9} < x < \frac{1}{3}$, and so on. This new decomposition of E and the corresponding intervals are outlined in fig. 2.

It must be emphasized that *any* of the sets appearing in the outline fig. 2 belongs to $\mathfrak{F}$, as well as the fact that each of the sets appearing in the sequence (11,1) are obtained by summation from the sets in the figure. For instance we thus have

$$A_1 = (C_1 - C_{\frac{1}{3}}) + C_{\frac{1}{3}}$$

and

$$A_2 = (C_{11} - C_{\frac{1}{9}}) + C_{\frac{1}{9}} + (C_{21} - C_{\frac{2}{3}} - C_{\frac{7}{9}}) + C_{\frac{2}{3}} + C_{\frac{7}{9}}.$$

Now we shall introduce a function $f(x)$ defined in $0 \leqq x \leqq 1$, which is done by the following definitions:

1) $f(0) = 0$.

2) Let x^* be a number of the form $\frac{k}{3^n}$ (n positive, integer, $k = 1, 2, \cdots, 3^n$). We then put

$$f(x^*) = \psi(C^*), \tag{11,5}$$

where C^* is the set coupled to the interval $0 \leqq x < x^*$. As an example we thus get $f(1) = \psi(E) = 1$, $f(\frac{1}{3}) = \psi(C_1 - C_{\frac{1}{3}})$, $f(\frac{2}{3}) = \psi(C_1)$, $f(\frac{7}{9}) = \psi(C_1 + C_{21} - C_{\frac{7}{9}})$. Especially we observe that $f(x)$ has, for every $x^* > \frac{7}{9}$, the value 1, because the set C_{22} (on account of the special assumption that $A_2 = E - A_1$) is empty.

If x_1 and $x_2\,(> x_1)$ both are of the form $\frac{k}{3^n}$, we get

$$f(x_2) - f(x_1) = \psi(D) \geqq 0, \tag{11,6}$$

where D is the set coupled to the interval $x_1 \leqq x < x_2$.

By this definition of $f(x)$ the function acquires the following property: if $x^*\,(< 1)$ as well as $x_1, x_2, \cdots, x_n, \cdots$ are numbers of the form $\frac{k}{3^n}$, for which $x_1 < x_2 < \cdots < x_n < \cdots < x^*$ and $\lim\limits_n x_n = x^*$, then the equation

$$\lim_n f(x_n) = f(x^*)$$

holds true.

This is understood in the following way. Denote by D_n the set corresponding to the interval $0 \leqq x < x_n$, for any n, and denote by C^* the set corresponding to the interval $0 \leqq x < x^*$. It now remains to be proved that

$$\lim_n \psi(D_n) = \psi(C^*).$$

For any n we have $D_n \subset C^*$. Consequently we obtain

$$\lim_n \left(\psi(C^*) - \psi(D_n)\right) = \lim_n \psi(C^* - D_n).$$

From $C^* - D_1 \supset C^* - D_2 \supset \cdots \supset C^* - D_n \supset \cdots$ we may next conclude (see VIII page 9) that

$$\lim_n \psi(C^* - D_n) = \psi\left(\prod_n (C^* - D_n)\right). \tag{11,7}$$

From our construction it is, however, easy to see that

$$\prod_n (C^* - D_n) = 0,$$

together with which (11,7) gives

$$\lim_n \psi(C^* - D_n) = \psi(0) = 0,$$

which was to be proved.

The function has the same property also for $x^* = 1$, because $f(x)$ has, as above remarked, the value 1, when x is of the form $\frac{k}{3^n}$ and greater than $\frac{7}{9}$.

3) Supposing further that x_0 is a number which is not of the form $\frac{k}{3^n}$. Corresponding to this we choose a sequence of numbers $x_1, x_2, \cdots, x_n, \cdots$ which all have the form $\frac{k}{3^n}$, and for which $x_1 < x_2 < \cdots < x_n < \cdots < x_0$ and $\lim_n x_n = x_0$. We shall again denote by D_n the set corresponding to the interval $0 \leqq x < x_n$. The set $\mathfrak{S}_n D_n$ is at once seen to be independent of the sequence chosen and dependent only on x_0. We now let the set $\mathfrak{S}_n D_n$ correspond to the interval $0 \leqq x < x_0$ and write

$$f(x_0) = \psi\left(\mathfrak{S}_n D_n\right), \tag{11,8}$$

by which the function $f(x)$ is defined for each x in the interval $0 \leqq x \leqq 1$. From $D_1 \subset D_2 \subset \cdots \subset D_n \subset \cdots$ follows (see VII page 9)

$$\psi\left(\mathfrak{S}_n D_n\right) = \lim_n \psi(D_n)$$

and hence

$$f(x_0) = \lim_n \psi(D_n). \tag{11,9}$$

About the function introduced by these definitions it now remains to be proved that it is non-decreasing and *continuous from the left* in any point.

We shall first prove that $f(x)$ is non-decreasing. Let $x_1 < x_2$. If x_1 and x_2 are both of the form $\frac{k}{3^n}$, the assertion is a consequence of (11,6). We next assume x_2 to be of the form $\frac{k}{3^n}$, x_1, however, not having this property. For each x of the form $\frac{k}{3^n}$ and less than x_1 we have $f(x) \leqq f(x_2)$, according to which (11,9) gives $f(x_1) \leqq f(x_2)$. The remaining cases are now dealt with immediately by insertion of a number of the form $\frac{k}{3^n}$ between x_1 and x_2.

Hence it is obvious that $f(x)$ is continuous from the left in every point x, it being possible to find for any $\varepsilon < 0$ a number $x' < x$ of the form $\frac{k}{3^n}$, for which

$$f(x) - f(x') < \varepsilon.$$

$f(x)$ being non-decreasing and continuous from the left, we may, according to II, 8, to $f(x)$ determine a measure φ defined in $\mathfrak{B}_1$ such that

$$\varphi([a \leqq x < b]) = f(b) - f(a) \tag{11,10}$$

for any choice of a and b, $0 \leqq a < b \leqq 1$.

Together with this Borel ring $\mathfrak{B}_1$ we shall consider the smallest Borel ring in E, containing all the sets of the sequence (11,1) $A_1, A_2, \cdots, A_n, \cdots$. This we shall call $\mathfrak{F}_1$. The defining region of ψ being the Borel ring $\mathfrak{F}$, any set belonging to $\mathfrak{F}_1$, will belong to $\mathfrak{F}$ as well. We know that the set of values of the function of a set φ is closed (see preceding section). If we can now prove that any value assumed by ψ at $\mathfrak{F}_1$ is assumed also by φ at $\mathfrak{B}_1$, and conversely, the proof of our theorem will be completed.

It is obvious that $\mathfrak{F}_1$ must contain each of the sets shown in the decomposition, fig. 2, and as any A_n may be produced by summation from these sets, $\mathfrak{F}_1$ may also be defined as the smallest Borel ring containing all the sets appearing in fig. 2.

The class of sets consisting of all finite sums of the sets given in fig. 2 is a ring, and according to the preceding considerations the slightest extension of this ring to a Borel ring is just the class of sets $\mathfrak{F}_1$.

We now first observe that if C is a set in fig. 2, to which is coupled a point x_0,

$$\psi(C) = \varphi(x_0) \tag{11,11}$$

holds true.

As $\varphi(x_0) = f(x_0+0) - f(x_0)$, we have consequently to prove that

$$\psi(C) = f(x_0+0) - f(x_0). \tag{11,12}$$

We choose a sequence of numbers $x_1, x_2, \cdots, x_n, \cdots$, for which $x_1 > x_2 > \cdots > x_n > \cdots > x_0$ and $\lim_n x_n = x_0$, and for every n we denote by D_n the set coupled to the interval $x_0 \leqq x < x_n$. Hence we get $D_1 \supset D_2 \supset \cdots \supset D_n \supset \cdots$, and, owing to the special procedure of the decomposition, $\prod_n D_n = C$. Hence

$$\psi(C) = \psi\left(\prod_n D_n\right) = \lim_n \psi(D_n) = \lim_n \left(f(x_n) - f(x_0)\right) = f(x_0+0) - f(x_0),$$

which was to be proved.

If D is a set in fig. 2 to which is coupled an interval $x_0 < x < x_1$, we have analogously

$$\psi(D) = \varphi([x_0 < x < x_1]). \tag{11,13}$$

Since

$$\left.\begin{aligned}&\varphi([x_0 < x < x_1]) = \varphi([x_0 \leqq x < x_1]) - \varphi(x_0) = \\ &f(x_1) - f(x_0) - \left(f(x_0+0) - f(x_0)\right) = f(x_1) - f(x_0+0),\end{aligned}\right\} \tag{11,14}$$

we have consequently to prove that

$$\psi(D) = f(x_1) - f(x_0+0).$$

C denoting the set coupled to the point x_0, we get

$$\psi(D) = \psi(D+C) - \psi(C)$$

which, if we apply (11,12), will give

$$\psi(D) = \left(f(x_1) - f(x_0)\right) - \left(f(x_0+0) - f(x_0)\right) = f(x_1) - f(x_0+0),$$

which together with (11,14) gives (11,13).

Finally, suppose A to be an arbitrary set of the class of sets $\mathfrak{F}_1$. We can then to this set determine a sequence of sets $K_1, K_2, \cdots, K_n, \cdots$, in which each K_n is a finite or enumerable sum of the sets appearing in fig. 2, and such that

$$1)\ K_1 \supset K_2 \supset \cdots \supset K_n \supset \cdots,$$

$$2)\ A \subset K_n \text{ for every } n,$$

$$3)\ \psi(A) = \text{lower bound } \psi(K_n) = \lim_n \psi(K_n),$$

(see page 10). To each set K_n in $\mathfrak{F}_1$ we have a corresponding set I_n in $\mathfrak{B}_1$, and according to the remarks above on the special cases they satisfy

$$1)\ \varphi(I_n) = \psi(K_n)$$

and

$$2)\ I_1 \supset I_2 \supset \cdots \supset I_n \supset \cdots.$$

Since the set $\prod_n I_n$ belongs to $\mathfrak{B}_1$, and

$$\varphi\left(\prod_n I_n\right) = \lim_n \varphi(I_n) = \lim \psi(K_n) = \psi(A),$$

$\prod_n I_n$ is a set in $\mathfrak{B}_1$ having the property wanted. In nearly the same way the other half of the proof may be carried out.

Each of the numbers appearing in the sequence (11,2) are thus taken on by φ at $\mathfrak{B}_1$. Consequently also the value g is taken on by φ at $\mathfrak{B}_1$, but according to the preceding this value is then also taken on by ψ at $\mathfrak{F}_1$ and thus by ψ at $\mathfrak{F}$; and our proof is completed.

PART IV.

Bounded measures in abstract space.

12. A theorem on two bounded measures having $\mathfrak{B}_1$ as defining region.

In this section we shall establish a theorem on bounded measures defined in $\mathfrak{B}_1$, comprising as special case the theorem proved in III, 10. $\mathfrak{B}_1$ will as usual denote the Borel class on the interval (0,1), where the interval $[0 \leqq x < 1]$ is denoted by E_1. Now let φ and ψ be two bounded measures defined in $\mathfrak{B}_1$. Without reducing the generality of our research, we may assume that $\varphi(E_1) = \psi(E_1) = 1$. Now let A be an arbitrarily chosen set belonging to $\mathfrak{B}_1$. The point $(\varphi(A), \psi(A))$ will then belong to the unity-square $0 \leqq x \leqq 1, 0 \leqq y \leqq 1$. About the set of points of the unity-square, obtained when A runs throughout $\mathfrak{B}_1$, we shall prove the following theorem:

The set of points which is determined by $(\varphi(A), \psi(A))$, where φ and ψ are bounded measures defined on $\mathfrak{B}_1$, is a closed set.

Together with the measures φ and ψ we shall consider the functions $f_1(x)$ and $f_2(x)$ defined on $0 \leqq x \leqq 1$ and determined by

$$f_1(a) = \begin{cases} 0 \text{ for } a = 0 \\ \varphi([0 \leqq x < a]) \text{ for } a > 0, \end{cases}$$

and

$$f_2(a) = \begin{cases} 0 \text{ for } a = 0 \\ \psi([0 \leqq x < a]) \text{ for } a > 0. \end{cases}$$

The non-decreasing functions $f_1(x)$ and $f_2(x)$ may be written in the form

$$f_1(x) = g_1(x) + h_1(x)$$
and
$$f_2(x) = g_2(x) + h_2(x)$$

(cf. page 26), and correspondingly will arise a decomposition of φ and ψ

$$\left.\begin{aligned} \varphi(A) &= \varphi_1(A) + \varphi_2(A) \\ \text{and} \quad \psi(A) &= \psi_1(A) + \psi_2(A) \end{aligned}\right\} \tag{12,1}$$

for any $A \varepsilon \mathfrak{B}_1$.

From (12,1) follows

$$\bigl(\varphi(A), \psi(A)\bigr) = \bigl(\varphi_1(A), \psi_1(A)\bigr) + \bigl(\varphi_2(A), \psi_2(A)\bigr).$$

We shall first show that if M_1 and M_2 are sets, both belonging to $\mathfrak{B}_1$, and for which

$$\bigl(\varphi_1(M_1), \psi_1(M_1)\bigr) = (\alpha, \beta)$$
and
$$\bigl(\varphi_2(M_2), \psi_2(M_2)\bigr) = (\gamma, \delta),$$

then we may determine a set $M \varepsilon \mathfrak{B}_1$, for which

$$\bigl(\varphi(M), \psi(M)\bigr) = (\alpha + \gamma, \beta + \delta).$$

Let the set of points of discontinuity of the function $f_1(x)$ be the, at most enumerable, set $N_1 = \{\xi_n\}$. Similarly let $N_2 = \{\eta_n\}$ denote the, at most enumerable, set of points of discontinuity of the function $f_2(x)$. The sum of N_1 and N_2 is termed N, i. e. $N = N_1 \dotplus N_2$. As the set $M \varepsilon \mathfrak{B}_1$ we may now use the set

$$M = (M_1 - M_1 N) + M_2 N.$$

From (12,1) follows

$$\left.\begin{aligned} \varphi(M) = \varphi(M_1 - M_1 N) + \varphi(M_2 N) = \\ \varphi_1(M_1 - M_1 N) + \varphi_2(M_1 - M_1 N) + \varphi_1(M_2 N) + \varphi_2(M_2 N). \end{aligned}\right\} \tag{12,2}$$

$M_1 N$ and $M_2 N$ both being at most enumerable sets, we have

$$\varphi_1(M_1 N) = \varphi_1(M_2 N) = 0. \tag{12,3}$$

From $M_1 - M_1 N \subset E_1 - N_1$ follows

$$\varphi_2(M_1 - M_1 N) = 0. \tag{12,4}$$

Finally we derive

$$\varphi_2(M_2 N) = \sum_{\xi_i \in M_2 N}' S_{f_1}(\xi_i) = \sum_{\xi_i \in M_2 N_1}' S_{f_1}(\xi_i) = \varphi_2(M_2) = \gamma. \qquad (12{,}5)$$

Thus it follows from (12,2)—(12,5) that

$$\varphi(M) = \alpha + 0 + 0 + \gamma.$$

In analogy it is seen that

$$\psi(M) = \beta + \delta,$$

and we have proved our assertion.

Hence we can, as in section III,10, conclude that if the sets of points

$$\left(\varphi_1(A),\ \psi_1(A)\right)$$

and

$$\left(\varphi_2(A),\ \psi_2(A)\right)$$

are both closed, the set of points

$$\left(\varphi(A),\ \psi(A)\right)$$

is also a closed set.

In the two following sections we shall deal with these special cases.

13. First special case.

Let φ and ψ be bounded measures defined in $\mathfrak{B}_1$, and let the two non-decreasing functions $f_1(x)$ and $f_2(x)$ be defined by

$$f_1(a) = \begin{cases} 0 \text{ for } a = 0 \\ \varphi([0 \leq x < a]) \text{ for } a > 0 \end{cases}$$

and

$$f_2(a) = \begin{cases} 0 \text{ for } a = 0 \\ \psi([0 \leq x < a]) \text{ for } a > 0 \end{cases}$$

and satisfy

$$D_{f_1}(x_1, x_2) = V_{f_1}(x_1, x_2)$$

and

$$D_{f_2}(x_1, x_2) = V_{f_2}(x_1, x_2)$$

for any choice of x_1 and x_2, $0 \leq x_1 < x_2 \leq 1$.

The set of points of discontinuity of $f_1(x)$ is the, at most enumerable, set $N_1 = \{\xi_n\}$, and the set of points of discontinuity of $f_2(x)$ is the, at most enumerable, set $N_2 = \{\eta_n\}$, The sum of N_1 and N_2 then again is at most an enumerable set, termed $N = \{\zeta_n\}$. For any $A \varepsilon \mathfrak{B}_1$ we now have

$$\left.\begin{aligned} \varphi(A) &= \sum_{\zeta_i \varepsilon A} S_{f_1}(\zeta_i) \\ \text{and} \quad \psi(A) &= \sum_{\zeta_i \varepsilon A} S_{f_2}(\zeta_i) \end{aligned}\right\} \qquad (13,1)$$

(cf. page 35). In the following the set N is assumed to be enumerable, and we put

$$\left.\begin{aligned} S_{f_1}(\zeta_n) &= a_n \,(\geq 0) \\ \text{and} \quad S_{f_2}(\zeta_n) &= b_n \,(\geq 0). \end{aligned}\right\} \qquad (13,2)$$

Hence we obtain

$$\big(\varphi(A),\, \psi(A)\big) = \left(\sum_{n=1}^{\infty} e_n\, a_n,\; \sum_{n=1}^{\infty} e_n\, b_n\right), \qquad (13,3)$$

where e_n has the value 1, if ζ_n belongs to A, and otherwise the value 0.

We shall now show that the set $(\varphi(A), \psi(A))$ is closed. Suppose $A_1, A_2, \cdots, A_n, \cdots$ to be a sequence of sets, all belonging to $\mathfrak{B}_1$, and for which the sequence of points $(\varphi(A_n), \psi(A_n))$ is convergent to the limit point (s, t). We shall now prove the existence of a set $A \varepsilon \mathfrak{B}_1$, for which

$$\big(\varphi(A),\, \psi(A)\big) = (s, t).$$

In section III, 9, where a method how to determine A^* (respectively A^{**}) as a subset of N has been given, we have already proved the existence of a set A^*, for which $\varphi(A^*) = s$, and a set A^{**}, for which $\psi(A^{**}) = t$. A closer analysis of the process of choice will immediately show the possibility of determining a subset of N, for which both $\varphi(A) = s$ and $\psi(A) = t$. More precisely, if only A is determined in the way shown, such that $\varphi(A) = s$, it will be an immediate consequence that $\psi(A) = t$.

14. Second special case.

Let φ and ψ be bounded measures defined on $\mathfrak{B}_1$. We shall in this section assume that the non-decreasing functions $f_1(x)$ and $f_2(x)$, corresponding to φ and ψ, given by the definitions

$$f_1(a) = \begin{cases} 0 \text{ for } a = 0 \\ \varphi([0 \leqq x < a]) \text{ for } a > 0 \end{cases}$$

and

$$f_2(a) = \begin{cases} 0 \text{ for } a = 0 \\ \psi([0 \leqq x < a]) \text{ for } a > 0 \end{cases}$$

are both continuous in the interval $0 \leqq x \leqq 1$, and we shall show that the set of points

$$(\varphi(A), \psi(A)), \; A \varepsilon \mathfrak{B}_1 \tag{14,1}$$

is closed. Without loss of generality, we may assume that

$$\varphi(E_1) = \psi(E_1) = 1.$$

In order to simplify the writing, we shall further in this section use the letter E instead of E_1 to denote the interval $0 \leqq x < 1$.

According to the theorem on decomposition, page 14, φ may be written in the form

$$\varphi = \varphi_k + \varphi_s, \tag{14,2}$$

where φ_k is ψ-continuous, and φ_s is ψ-singular. Thus $\varphi_k(A)$ will have the value 0 for each $A \varepsilon \mathfrak{B}_1$, for which $\psi(A) = 0$, and there exists a set $N \varepsilon \mathfrak{B}_1$ where $\psi(N) = 0$, such that $\varphi_s(A) = 0$ for each $A \varepsilon \mathfrak{B}_1$ which is a subset of $E - N$.

φ_k being ψ-continuous, there will exist a function $f \geqq 0$ defined in E, such that

$$\varphi_k(A) = \int_A f \psi(dE) \tag{14,3}$$

for each $A \varepsilon \mathfrak{B}_1$ (see page 20).

It follows from $A - AN \subset E - N$ that

$$\varphi_s(A) = \varphi_s(AN), \tag{14,4}$$

and from $\psi(AN) = 0$ that

$$\varphi_k(AN) = 0, \tag{14,5}$$

according to which (14,2)—(14,5) give

$$\varphi(A) = \varphi(AN) + \int_A f\psi(dE) \tag{14,6}$$

for each $A \varepsilon \mathfrak{B}_1$.

Our first problem is to find out what values φ can take on, when ψ is fixed. Suppose γ to be a fixed number of the interval $0 \leqq x \leqq 1$. We shall now consider all sets $A \varepsilon \mathfrak{B}_1$, for which $\psi(A) = \gamma$. The existence of such sets is evident on account of the continuity of the function $f_2(x)$. Let A_1 and A_2 be two such sets. We thus have

$$\psi(A_1) = \psi(A_2) = \gamma. \tag{14,7}$$

Suppose, furthermore, the numbers α and β to be defined by

$$\varphi(A_1 N) = \alpha \quad \text{and} \quad \int_{A_2} f\psi(dE) = \beta. \tag{14,8}$$

We can then prove the existence of a set $A \varepsilon \mathfrak{B}_1$, where $\psi(A) = \gamma$, for which

$$\varphi(A) = \alpha + \beta.$$

As a set A we may use

$$A = A_1 N + (A_2 - A_2 N),$$

for from $\psi(N) = 0$ follows

$$\psi(A) = \psi(A_2) = \gamma,$$

and by means of (14,6) we get

$$\varphi(A) = \varphi(A_1 N) + \varphi(A_2 - A_2 N) = \alpha + \varphi((A_2 - A_2 N)N) + \int_{A_2 - A_2 N} f\psi(dE)$$
$$= \alpha + \beta$$

since $(A_2 - A_2 N)N = 0$ and $\int_{A_2 N} f\psi(dE) = 0$.

From the above follows that if, only, we can prove that each of the addends in (14,6) runs throughout a closed set, when A runs throughout the sets belonging to $\mathfrak{B}_1$, for which $\psi(A) = \gamma$, we have also proved that the set of values of $\varphi(A)$ is closed.

The set of values of $\varphi(AN)$ we shall prove to be *a closed interval*, which is moreover *independent* of γ.

Let A be a set belonging to $\mathfrak{B}_1$, and for which $\psi(A) = \gamma$. Thus we immediately derive

$$0 \leqq \varphi(AN) \leqq \varphi(N). \tag{14,9}$$

Since $\psi(N) = 0$ we have for the set $A^* = A - AN$ that

$$\psi(A^*) = \psi(A) = \gamma$$

and that

$$\varphi(A^* N) = \varphi\big((A - AN)N\big) = 0.$$

For the set $A^{**} = A \dotplus N = A + (N - AN)$ we have

$$\psi(A^{**}) = \psi(A) = \gamma$$

and

$$\varphi(A^{**} N) = \varphi\big((A \dotplus N)N\big) = \varphi(N).$$

Consequently we have sets for which the signs of equality in (14,9) hold true. We have now to prove for each number between 0 and $\varphi(N)$, the existence of a set with φ having this value. Let A_t denote the set $[0 \leqq x < t]$, and let B_t denote the set NA_t belonging to $\mathfrak{B}_1$ for any t. We now put

$$B = A^* + B_t$$

and have for every t

$$\psi(B) = \psi(A^*) + \psi(B_t) = \gamma$$

since $\psi(N) = 0$.

The function $g(t)$ is introduced by the definition

$$g(t) = \varphi(BN) = \varphi\big((A^* + B_t)N\big) = \varphi(B_t N) = \varphi(B_t).$$

It is obvious that $g(t)$ is non-decreasing. We shall further prove the fact that it is continuous. If h denotes a positive number, we see that

$$g(t+h)-g(t) = \varphi(B_{t+h})-\varphi(B_t) = \varphi(B_{t+h}-B_t) =$$

$$\varphi(N(A_{t+h}-A_t)) \leq \varphi(A_{t+h}-A_t) = f_1(t+h)-f_1(t)$$

holds true.

$f_1(x)$ being assumed continuous, we see that $g(t)$ is continuous from the right. Similarly we see that $g(t)$ is continuous from the left. Thus the function $g(t)$ takes on any value between $g(0) = 0$ and $g(1) = \varphi(N)$, and the proof that the values of $\varphi(AN)$ make out a closed set is completed. Now remains an investigation of the set of values of

$$\int_A f\psi(dE),$$

$A \varepsilon \mathfrak{B}_1$ running throughout the sets for which $\psi(A) = \gamma$. We shall again show that the values make out *a closed interval*.

In our proof we shall apply the following lemma:

For any number ξ of the interval $0 < \xi < 1$ we can determine a set $A_\xi \varepsilon \mathfrak{B}_1$ for which $\psi(A_\xi) = \xi$, and a corresponding number a_ξ such that

$$[f < a_\xi] \subset A_\xi \subset [f \leq a_\xi], \tag{14,10}$$

and this determination may be carried out in such a way that

$$A_{\xi_1} \subset A_{\xi_2} \quad \textit{for} \quad \xi_1 < \xi_2. \tag{14,11}$$

To prove this we shall define two functions

$$F(a) = \psi([f < a]) \tag{14,12}$$

and

$$G(a) = \psi([f \leq a]) \tag{14,13}$$

having $a \geq 0$ as defining region.

Since $[f < a] \subset [f \leq a]$ we have

$$F(a) \leq G(a), \tag{14,14}$$

and similarly we see that

$$\left.\begin{aligned} F(a_1) &\leq F(a_2) \quad \text{for} \quad a_1 < a_2 \\ \text{and} \qquad G(a_1) &\leq G(a_2) \quad \text{for} \quad a_1 < a_2. \end{aligned}\right\} \tag{14,15}$$

Our next problem is to prove that the function $F(a)$ is continuous from the left, and that the function $G(a)$ is continuous from the right in every point. Let

$$a_1 < a_2 < \cdots < a_n < \cdots < a \quad \text{and} \quad \lim_n a_n = a,$$

it then follows that

$$F(a) - F(a_n) = \psi([f < a]) - \psi([f < a_n]) = \psi([a_n \leqq f < a]).$$

Since

$$\prod_n [a_n \leqq f < a] = 0$$

it next follows that

$$\lim_n \big(F(a) - F(a_n)\big) = \lim_n ([a_n \leqq f < a]) = 0,$$

i. e. $F(a)$ is continuous from the left. Now let

$$a_1 > a_2 > \cdots > a_n > \cdots > a \quad \text{and} \quad \lim_n a_n = a,$$

we then analogously obtain

$$G(a_n) - G(a) = \psi([f \leqq a_n]) - \psi([f \leqq a]) = \psi([a < f \leqq a_n]).$$

On account of

$$\prod_n [a < f \leqq a_n] = 0$$

we obtain

$$\lim_n \big(G(a_n) - G(a)\big) = \lim_n \psi([a < f \leqq a_n]) = 0,$$

i. e. $G(a)$ is continuous from the right.

For a fixed value of ξ in the interval $0 < \xi < 1$ we now have to determine the upper bound of the values of a for which $F(a) \leqq \xi$. We shall term the latter a_ξ, and we have $a_\xi < \infty$.[1) 2)] $F(a)$ being continuous from the left, we get $F(a_\xi) \leqq \xi$, and $G(a)$ being continuous from the right, (14,14) implies the fact that $G(a_\xi) \leqq \xi$. Hence we obtain

$$\psi([f < a_\xi]) \leqq \xi \leqq \psi([f \leqq a_\xi]). \tag{14,16}$$

1) It is easy to see that $\xi_1 < \xi_2$ implies $a_{\xi_1} \leqq a_{\xi_2}$.

2) It follows namely from $F(a) = \psi([f < a]) \leqq \xi$ for every a that $\psi([f \geqq a]) \geqq 1 - \xi > 0$ for every a, and hence further that $\psi([f = \infty]) > 0$, in nonconformity to φ being finite.

Concluding after the analogy of the proof page 55 it is obvious that in the set $[f = a_\xi]$ there is a subset $\varepsilon \mathfrak{B}_1$ having any ψ-measure between 0 and $\psi([f = a_\xi])$.

Accordingly we can determine a set $A_\xi \varepsilon \mathfrak{B}_1$ where $\psi(A_\xi) = \xi$, such that

$$[f < a_\xi] \subset A_\xi \subset [f \leqq a_\xi].$$

Furthermore it is immediately obvious that this process of determination may be carried out in such a way that

$$A_{\xi_1} \subset A_{\xi_2} \quad \text{for} \quad \xi_1 < \xi_2,$$

and the proof of our lemma is completed.

We shall now again consider

$$\varphi_k(A) = \int_A f \psi(dE)$$

for the sets $A \varepsilon \mathfrak{B}_1$, for which $\psi(A) = \gamma$. As to the special values of γ, $\gamma = 0$ and $\gamma = 1$, the case is evident. Thus if $A \varepsilon \mathfrak{B}_1$ is a set for which $\psi(A) = 0$, we obviously have $\varphi_k(A) = 0$, and if $A \varepsilon \mathfrak{B}_1$ is a set for which $\psi(A) = 1$, we have $\varphi_k(A) = \varphi_k(E) - \varphi_k(E - A) = \varphi_k(E)$, since $\psi(E - A) = 0$. Suppose next γ to be a number of the interval $0 < \gamma < 1$, and let A_γ be the set determined by our lemma. We can then prove the inequality

$$\int_A f \psi(dE) \geqq \int_{A_\gamma} f \psi(dE) \tag{14,17}$$

for any $A \varepsilon \mathfrak{B}_1$ for which $\psi(A) = \gamma$.

First we notice that

$$\left.\begin{aligned} &\int_A f \psi(dE) = \int_{A - AA_\gamma} f \psi(dE) + \int_{AA_\gamma} f \psi(dE) \\ \text{and} \quad &\int_{A_\gamma} f \psi(dE) = \int_{A_\gamma - AA_\gamma} f \psi(dE) + \int_{AA_\gamma} f \psi(dE) \end{aligned}\right\} \tag{14,18}$$

where

$$\psi(A - AA_\gamma) = \psi(A_\gamma - AA_\gamma).$$

By means of (14,10) we next see that $f \geqq a_\gamma$ for the set $A - AA_\gamma$, and that $f \leqq a_\gamma$ for the set $A_\gamma - AA_\gamma$. Hence we obtain

$$\int_{A-AA_\gamma} f\psi(dE) \geqq \int_{A_\gamma - AA_\gamma} f\psi(dE)$$

according to which (14,18) will give

$$\int_A f\psi(dE) \geqq \int_{A_\gamma} f\psi(dE),$$

which was to be proved.

Analogous to (14,17) we can show that for each $A \varepsilon \mathfrak{B}_1$, for which $\psi(A) = \gamma$, the following inequality holds true

$$\int_A f\psi(dE) < \int_{E-A_{1-\gamma}} f\psi(dE) \qquad (14,19)$$

where $A_{1-\gamma}$ is the set determined by our lemma. The set $E - A_{1-\gamma}$ we shall denote A^*, and we then have

$$\psi(A^*) = \psi(E) - \psi(A_{1-\gamma}) = 1 - (1-\gamma) = \gamma.$$

Similar to (14,18) we have

$$\left.\begin{aligned} &\int_A f\psi(dE) = \int_{A-AA^*} f\psi(dE) + \int_{AA^*} f\psi(dE) \\ \text{and} \quad &\int_{A^*} f\psi(dE) = \int_{A^*-AA^*} f\psi(dE) + \int_{AA^*} f\psi(dE) \end{aligned}\right\} \qquad (14,20)$$

where

$$\psi(A - AA^*) = \psi(A^* - AA^*).$$

From (14,10) follows

$$[f < a_{1-\gamma}] \subset A_{1-\gamma},$$

from which it is obvious that $f \geqq a_{1-\gamma}$ for the set $A^* - AA^*$. Similarly we derive from (14,10) that $f \leqq a_{1-\gamma}$ for the set $A - AA^*$. This being so, we obtain

$$\int_{A-AA^*} f\psi(dE) \leqq \int_{A^*-AA^*} f\psi(dE)$$

according to which (14,20) will give

$$\int_{A} f\psi(dE) \leqq \int_{E-A_{1-\gamma}} f\psi(dE),$$

which was to be proved.

If we comprise (14,17) and (14,19) we get for each $A \varepsilon \mathfrak{B}_1$, having $\psi(A) = \gamma$,

$$\int_{A_\gamma} f\psi(dE) \leqq \int_{A} f\psi(dE) \leqq \int_{E-A_{1-\gamma}} f\psi(dE), \tag{14,21}$$

and we shall now show that $\int_A f\psi(dE)$ takes on *any value in the closed interval* from $\int_{A_\gamma} f\psi(dE)$ to $\int_{E-A_{1-\gamma}} f\psi(dE)$.

For that purpose we form the function

$$H(\xi) = \int_{A_{\xi+\gamma}-A_\xi} f\psi(dE), \quad 0 \leqq \xi \leqq 1-\gamma \tag{14,22}$$

where A_ξ and $A_{\xi+\gamma}$ are the sets we have determined by means of our lemma, yet especially fixing $A_0 = 0$ and $A_1 = E$. For any ξ we then have

$$\psi(A_{\xi+\gamma}-A_\xi) = \psi(A_{\xi+\gamma})-\psi(A_\xi) = (\xi+\gamma)-\xi = \gamma. \tag{14,23}$$

Further we derive

$$\left.\begin{aligned} H(0) &= \int_{A_\gamma-A_0} f\psi(dE) = \int_{A_\gamma} f\psi(dE) \\ \text{and}\quad H(1-\gamma) &= \int_{A_1-A_{1-\gamma}} f\psi(dE) = \int_{E-A_{1-\gamma}} f\psi(dE). \end{aligned}\right\} \tag{14,24}$$

It now only remains to be proved that the function $H(\xi)$ is a *continuous function* in the interval $0 \leqq \xi \leqq 1-\gamma$.

We shall first show that the function $H(\xi)$ is *non-decreasing* in the interval $0 \leqq \xi \leqq 1-\gamma$. For that purpose we shall, for $\xi_1 < \xi_2$, consider

$$H(\xi_2) - H(\xi_1) = \int_{A_{\xi_2+\gamma}-A_{\xi_2}} f\psi(dE) - \int_{A_{\xi_1+\gamma}-A_{\xi_1}} f\psi(dE).$$

For brevity we put

$$A_{\xi_2+\gamma} - A_{\xi_2} = C \quad \text{and} \quad A_{\xi_1+\gamma} - A_{\xi_1} = D,$$

and we thus have

$$H(\xi_2) - H(\xi_1) = \int_C f\psi(dE) - \int_D f\psi(dE) = \int_{C-CD} f\psi(dE) - \int_{D-CD} f\psi(dE) \qquad (14{,}25)$$

where

$$\psi(C-CD) = \psi(D-CD).$$

From (14,10) we derive

$$f(x) \geqq \max\left\{a_{\xi_2}, a_{\xi_1+\gamma}\right\} \quad \text{for} \quad x \varepsilon C - CD$$

and

$$f(x) \leqq \min\left\{a_{\xi_2}, a_{\xi_1+\gamma}\right\} \quad \text{for} \quad x \varepsilon D - CD,$$

which inserted into (14,25) gives

$$H(\xi_2) - H(\xi_1) \geqq \psi(C-CD) \cdot \left[\max\left\{a_{\xi_2}, a_{\xi_1+\gamma}\right\} - \min\left\{a_{\xi_2}, a_{\xi_1+\gamma}\right\}\right] \geqq 0,$$

which was to be proved.

In order to show that $H(\xi)$ is continuous from the right in any point of the interval $0 \leqq \xi < 1-\gamma$ we shall now for h positive and sufficiently small consider

$$H(\xi+h) - H(\xi) = \int_{A_{\xi+h+\gamma}-A_{\xi+h}} f\psi(dE) - \int_{A_{\xi+\gamma}-A_{\xi}} f\psi(dE) = \int_{F-FG} f\psi(dE) - \int_{G-FG} f\psi(dE), \quad (14{,}26)$$

in which we have put

$$A_{\xi+h+\gamma} - A_{\xi+h} = F \quad \text{and} \quad A_{\xi+\gamma} - A_{\xi} = G.$$

For $h < \gamma$ we have
$$F - FG = A_{\xi+h+\gamma} - A_{\xi+\gamma},$$

and thus we derive by means of (14,10)

$$\int_{F-FG} f\psi(dE) = \int_{A_{\xi+h+\gamma}-A_{\xi+h}} f\psi(dE) \leqq h \cdot a_{\xi+h+\gamma},$$

since $\psi(A_{\xi+h+\gamma} - A_{\xi+\gamma}) = h$. Hence we get

$$\int_{F-FG} f\psi(dE) \to 0 \quad \text{for} \quad h \to 0. \tag{14,27}$$

In analogy we find

$$\int_{G-FG} f\psi(dE) = \int_{A_{\xi+h}-A_{\xi}} f\psi(dE) \leqq h \cdot a_{\xi+h},$$

hence

$$\int_{G-FG} f\psi(dE) \to 0 \quad \text{for} \quad h \to 0. \tag{14,28}$$

From (14,26), (14,27) and (14,28) it thus follows that $H(\xi)$ is continuous from the right. To show next that $H(\xi)$ is continuous from the left in any point of the interval $0 < \xi \leqq 1 - \gamma$, we consider for h positive and sufficiently small

$$H(\xi) - H(\xi - h) = \int_{A_{\xi+\gamma}-A_{\xi}} f\psi(dE) - \int_{A_{\xi-h+\gamma}-A_{\xi-h}} f\psi(dE) = \int_{K-KL} f\psi(dE) - \int_{L-KL} f\psi(dE), \tag{14,29}$$

in which we have put

$$A_{\xi+\gamma} - A_{\xi} = K \quad \text{and} \quad A_{\xi-h+\gamma} - A_{\xi-h} = L.$$

For $h < \gamma$ we have

$$K - KL = A_{\xi+\gamma} - A_{\xi-h+\gamma},$$

and hence we get by means of (14,10)

$$\int_{K-KL} f\psi(dE) = \int_{A_{\xi+\gamma}-A_{\xi-h+\gamma}} f\psi(dE) \leqq h \cdot a_{\xi+\gamma},$$

since $\psi(A_{\xi+\gamma} - A_{\xi-h+\gamma}) = h$. Hence we obtain

$$\int_{K-KL} f\psi(dE) \to 0 \quad \text{for} \quad h \to 0.^{1)} \tag{14,30}$$

In analogy we obtain

$$\int_{L-KL} f\psi(dE) = \int_{A_\xi - A_{\xi-h}} f\psi(dE) \leqq h \cdot a_\xi$$

since $\psi(A_\xi - A_{\xi-h}) = h$. Hence we get

$$\int_{L-KL} f\psi(dE) \to 0 \quad \text{for} \quad h \to 0. \tag{14,31}$$

According to (14,29), (14,30) and (14,31) we see that $H(\xi)$ is continuous from the left. And now we have proved that the values of

1) The number $a_{\xi+\gamma}$ being not introduced for $\xi = 1-\gamma$, a special investigation of the conditions for $\xi = 1-\gamma$ is required. In this case we have

$$\int_{K-KL} f\psi(dE) = \int_{E-A_{1-h}} f\psi(dE) = \varphi_k(E - A_{1-h}).$$

If $h_1 > h_2 > \cdots > h_n > \cdots > 0$ and $\lim_n h_n = 0$, we get

$$E - A_{1-h_1} \supset E - A_{1-h_2} \supset \cdots \supset E - A_{1-h_n} \supset \cdots.$$

According to (14,10)

$$E - A_{1-h_n} \subset [f \geqq a_{1-h_n}]$$

for every n.

We shall now perform the proof indirectly, assuming

$$\varphi_k(E - A_{1-h_n}) \geqq t > 0 \text{ for all values of } n.$$

If the set of numbers $\{a_\xi\}$, determined by our lemma, is not upward bounded, we accordingly get

$$\varphi_k([f = \infty]) \geqq t$$

in nonconformity to $\psi([f = \infty]) = 0$. Is on the contrary the set of numbers $\{a_\xi\}$ upward bounded, there will exist a number a^* such that $\psi([f \geqq a^*]) = 0$. Hence we have

$$\int_{E-A_{1-h_n}} f\psi(dE) = \int_{(E-A_{1-h_n})\cdot[f<a^*]} f\psi(dE) \leqq h_n \cdot a^* < \varepsilon \text{ for } n > N$$

which was to be proved.

$$\varphi_k(A) = \int_A f\psi(dE) \qquad (\psi(A) = \gamma)$$

makes out a closed interval.

Comprising (14,9) and (14,21) we next find that the region of values of $\varphi(A)$, $A \varepsilon \mathfrak{B}_1$ running throughout the sets for which $\psi(A) = \gamma$, $0 \leqq \gamma \leqq 1$, is determined by

$$\int_{A_\gamma} f\psi(dE) \leqq \varphi(A) \leqq \int_{E-A_{1-\gamma}} f\psi(dE) + \varphi(N), \tag{14,32}$$

and that any value in this closed interval appears as a value of the function $\varphi(A)$.

Thus our proof that the set of points $(\varphi(A), \psi(A))$ is a closed set is completed if we can prove that the end points of the closed interval (14,32) vary continuously with respect to γ.

For this purpose we shall first consider

$$\int_{A_\gamma} f\psi(dE).$$

If $0 \leqq \gamma < 1$ and $h (< h_0)$ is positive and sufficiently small, we have

$$\int_{A_{\gamma+h}} f\psi(dE) - \int_{A_\gamma} f\psi(dE) = \int_{A_{\gamma+h}-A_\gamma} f\psi(dE) \leqq h \cdot a_{\gamma+h_0}, \tag{14,33}$$

since $\psi(A_{\gamma+h} - A_\gamma) = h$ and $A_{\gamma+h} \subset [f \leqq a_{\gamma+h_0}]$.

If $0 < \gamma \leqq 1$ and h is positive and sufficiently small, we similarly get

$$\int_{A_\gamma} f\psi(dE) - \int_{A_{\gamma-h}} f\psi(dE) = \int_{A_\gamma - A_{\gamma-h}} f\psi(dE) \leqq h \cdot a_\gamma, \tag{14,34}$$

since $\psi(A_\gamma - A_{\gamma-h}) = h$ and $A_\gamma \subset [f \leqq a_\gamma]$ (as to the case $\gamma = 1$ cf. the footnote page 63). From (14,33) and (14,34) it is evident that the left end point of the interval varies continuously with respect to γ. By a quite similar consideration we clearly see that the right end point of the interval, which with the exception of a constant is equal to

$$\int_{E-A_{1-\gamma}} f\psi(dE),$$

varies continuously with respect to γ.

15. A theorem on two bounded measures in abstract space.

In this section we shall see that the theorem proved and formulated in the preceding section of this part, on two bounded measures having $\mathfrak{B}_1$ as defining region, holds true in the abstract space too.

Let E be an arbitrary set, and let $\mathfrak{F}$ be a class of sets, which is a Borel ring and contains E. Let further φ and ψ be two bounded measures defined in $\mathfrak{F}$.

The following theorem will then hold true:

The set of points defined by

$$(\varphi(A), \psi(A)),$$

where φ and ψ are bounded measures defined in $\mathfrak{F}$, is a closed set.

Accordingly our problem is to prove that if

$$A_1, A_2, \cdots, A_n, \cdots$$

is a sequence of sets, all belonging to $\mathfrak{F}$, and for which the corresponding sequence of points

$$(\varphi(A_1), \psi(A_1)), (\varphi(A_2), \psi(A_2)), \cdots, (\varphi(A_n), \psi(A_n)), \cdots$$

is convergent to the limit point (t, s), then there will exist a set $A \varepsilon \mathfrak{F}$, for which

$$(\varphi(A), \psi(A)) = (t, s).$$

In proving this we apply exactly the same representation from the smallest Borel ring $\mathfrak{F}_1$, containing all the sets A_n, to the Borel class $\mathfrak{B}_1$, as applied in III, 11. Corresponding to φ and ψ there will exist bounded measures φ_1 and ψ_1 defined in $\mathfrak{B}_1$, and it is easily seen that to each set in $\mathfrak{F}_1$ there will be a corresponding set in $\mathfrak{B}_1$, such that

$$(\varphi, \psi) = (\varphi_1, \psi_1)$$

for these sets, and conversely.

The theorem being valid for (φ_1, ψ_1) we see that there is a set $A \varepsilon \mathfrak{F}_1$ (or weaker $A \varepsilon \mathfrak{F}$) having the property wanted. Thus the assertion of the theorem is proved.

Putting $\varphi = \psi$ it will be seen that the above theorem contains the theorem of III, 11 as a special case.

16. Final remarks.

The appearance in 1933 of KOLMOGOROFF's book "Grundbegriffe der Wahrscheinlichkeitsrechnung"[1] made it at once clear to many mathematicians that with this book the theory of probability had won its natural place among the theories of mathematics. It is KOLMOGOROFF's merit to have shown how simply the theory can be axiomatized, and how it is possible from the axioms to prove the theorems of the theory of probability. The number of the possible axiomatizations was immense, but the system used by KOLMOGOROFF seems natural and for the applications most simple. Here the space of single occurrences is abstract, moreover a class of sets consisting of subsets of this abstract space is supposed to be given. This class is assumed to be a ring and to contain as an element the space itself. In this ring is supposed defined a non-negative, additive function of a set, such that its value for the abstract space is 1. The value of the function of a set for a set of the ring will then be the probability of the realization of one of the single occurrences contained in this set. It is proved that it is possible to confine oneself to regarding Borel fields of probability, (in a Borel field of probability the defining region of the function of a set is a Borel ring), introducing an axiom of continuity equivalent with the claim of complete additivity of the function of a set in its defining region.

The book mentioned above roused the author's interest in the theory of probability and its applications. This interest was strengthened by several visits to the Stockholm University Institute of Insurance Mathematics and Mathematical Stati-

1) KOLMOGOROFF [1].

stics[1]. Here was also roused the author's interest in the theory of testing statistical hypotheses[2], formed by NEYMAN and PEARSON. It was clear to everybody that the theory in the form it had obtained by then suffered from certain shortcomings, and as it was rather evident what amendments might be wanted, and what results were likely to be obtained, the problem must be to change the foundation in such a direction that this was made possible.

Unfortunately my investigations into this problem were on the whole without any result, as I did not succeed in giving them a form which satisfied me in the sense of being mathematically unimpeachable, and at the same time having the connection with experience and practice which must reasonably be claimed. By my study of various questions in this connection I was led on to certain problems of existence, which must necessarily be treated first. Here too I met with difficulties, now of this and now of that kind. Thus it was natural first to try to answer these problems in the abstract space, i. e. when the theory was unimpeded by everything superfluous.

These investigations gave birth to this paper.

Fundamental for the present formulation and treatment of the problem of the testing of statistical hypotheses is the above mentioned paper by NEYMAN and PEARSON, 1933. In this work is given a detailed account of the nature of the problem, and a mathematical treatment in the main features of the problems raised. The problem is by these two authors formulated as follows: Let a stochastic variable be given. An assumption of the structure of the distribution of this variable is called a statistical hypothesis. A set of observations of the stochastic variable is called a sample, and as a test of this statistical hypothesis a function of the result of the sample is now computed. If this sample has certain properties further stated, it will be discarded; whereas it will be maintained, if the function has not got these properties. This test is of course not absolute in the sense of giving us information whether the hypothesis laid down is correct or false, but we endeavour to arrange it in such a way

1) The author wants to express his gratitude to the institute and its director, Professor HARALD CRAMÉR, Ph. D., for hospitality and interest.

2) See for instance NEYMAN-PEARSON [1].

that it may in the long run give good results. To be more precise, we arrange it in such a manner that the probability of discarding the hypothesis when it is true, does not exceed a certain limit fixed beforehand, and correspondingly that the probability of maintaining the hypothesis, even if it is false, is kept under a reasonable limit. A number of calculated examples prove the method to be available in many cases arising in practice. Yet several authors have objected to certain special items of the theory. Thus FELLER has in an excellent paper from 1938 shown the shortcomimg of the proposed procedure in a whole series of cases, which may easily crop up in practice[1].

It was natural at the beginning of this research to leave the categorical claim of dividing into two parts the sphere of samples, such that the hypothesis was maintained if the point of the sample fell inside one of these parts, whereas it was discarded if the point of the sample fell inside the other. It was natural to eliminate the sharp limit that must arise between these two parts by introducing a third set, a transition set in which the question whether the hypothesis is correct or false is left open.

In its simplest form this leads up to the following problems in the abstract space. Suppose given two measures φ and ψ, both defined in a Borel ring $\mathfrak{F}$, and about which it is further supposed that $\varphi(E) = \psi(E) = 1$. Now the problem is to prove the existence of two sets having no elements in common, A and F in E, such that

1) $\varphi(F) \leqq \varepsilon_1$

2) $\psi(A) \leqq \varepsilon_2$

3) $\varphi(A) + \psi(F)$ as great as possible.

The investigation of this question apparently requires knowledge of a certain class of functions of a set in the abstract space, which has not yet been dealt with. This is seen in the following way. Let F be an arbitrary set of the space E, thus not necessarily subject to condition 1). Among the subsets of $E - F$ there must then, according to our earlier investigation, exist a set A_F, such that $\psi(A_F) \leqq \varepsilon_2$, while at the same time $\varphi(A_F)$

1) FELLER [1].

is as great as possible. Our task is now to prove that the function of a set

$$\eta(F) = \psi(F) + \varphi(A_F) \tag{16,1}$$

has a greatest value, when F runs through the sets for which $\varphi(F) \leqq \varepsilon_1$.

The function of a set $\eta(F)$ is seen at once to satisfy the relation

$$\eta(F_1 + F_2) \leqq \eta(F_1) + \eta(F_2). \tag{16,2}$$

It is obvious that a thorough knowledge of the functions of a set satisfying (16,2) would be of significance. As far as known to the author, there exists only one investigation of this type of functions of a set, undertaken by BANACH[1)], but with BANACH another condition is required satisfied at the same time. We can with certainty say that this condition in our problems is not accomplished.

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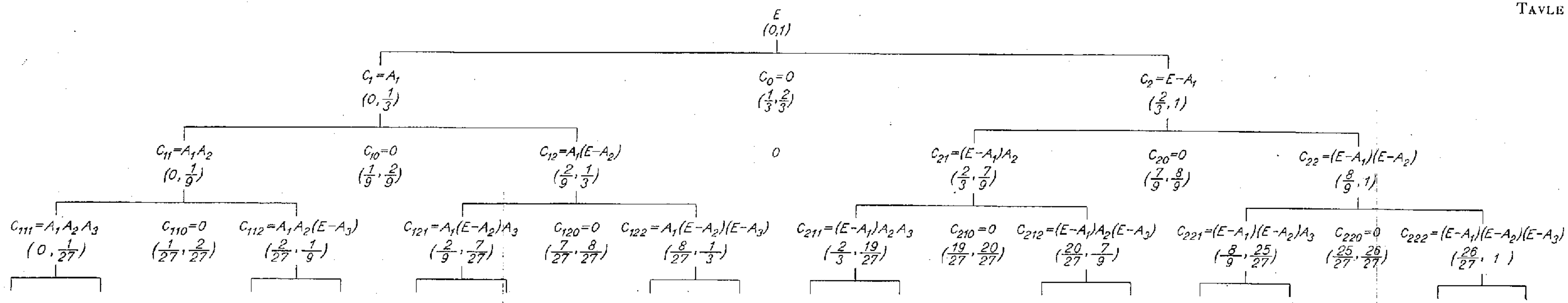
E
(0,1)
C1 = A1
(0, 1/3)
C0 = 0
(1/3, 2/3)
C2 = E − A1
(2/3, 1)
C11 = A1A2
(0, 1/9)
C10 = 0
(1/9, 2/9)
C12 = A1(E − A2)
(2/9, 1/3)
0
C21 = (E − A1)A2
(2/3, 7/9)
C20 = 0
(7/9, 8/9)
C22 = (E − A1)(E − A2)
(8/9, 1)
C111 = A1A2A3
(0, 1/27)
C110 = 0
(1/27, 2/27)
C112 = A1A2(E − A3)
(2/27, 1/9)
C121 = A1(E − A2)A3
(2/9, 7/27)
C120 = 0
(7/27, 8/27)
C122 = A1(E − A2)(E − A3)
(8/27, 1/3)
C211 = (E − A1)A2A3
(2/3, 19/27)
C210 = 0
(19/27, 20/27)
C212 = (E − A1)A2(E − A3)
(20/27, 7/9)
C221 = (E − A1)(E − A2)A3
(8/9, 25/27)
C220 = 0
(25/27, 26/27)
C222 = (E − A1)(E − A2)(E − A3)
(26/27, 1)

Fig. 1.

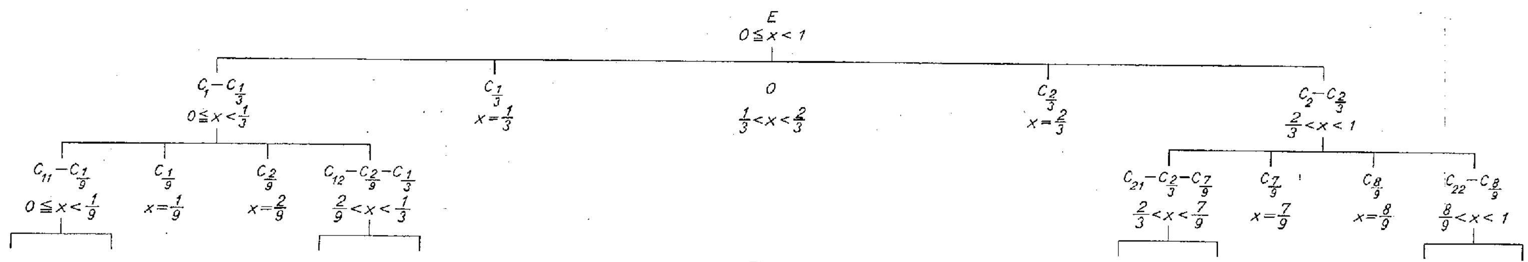
E
0 ≦ x < 1
C1 − C1/3
0 ≦ x < 1/3
C1/3
x = 1/3
0
1/3 < x < 2/3
C2/3
x = 2/3
C2 − C2/3
2/3 < x < 1
C11 − C1/9
0 ≦ x < 1/9
C1/9
x = 1/9
C2/9
x = 2/9
C12 − C2/9 − C1/3
2/9 < x < 1/3
C21 − C2/3 − C7/9
2/3 < x < 7/9
C7/9
x = 7/9
C8/9
x = 8/9
C22 − C8/9
8/9 < x < 1

Fig. 2.