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A METHOD FOR THE DETERMINATION OF THE COMPONENTS OF A COMPLEX CONDUCTOR

SEPARATED FROM THE SURFACE BY SEMI-CONDUCTORS

BY

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Introduction.

In the latest years, numerous investigators in their study of clinical problems determined the electric conducting properties of the organism at various frequencies having an application to diagnostics in mind, since BRAZIER (1933) stated that changes of the phase angle are characteristic for certain diseases.

In the beginning, most experimenters confined themselves to the determination of impedance and phase angle of the current passage through the organism when the forearms were immersed into a saline solution and two metal electrodes were placed into the liquid. Later, HORTON and VAN RAVENSWAAY found that the phase difference which appears when a current flows through the body itself (between the upper extremities)— i. e. the so-called "inner phase angle"— is a far more characteristic measure of a change of the physical conditions of the organism than those measurements which also include the resistance of the skin.

For the determination of the phase difference which occurs within the body HORTON and VAN RAVENSWAAY applied two circuits, equal in strength and phase, containing the patient and a pure resistance, respectively, each in series with an inductometer. The voltage across the inner resistance was balanced against the voltage across the pure resistance plus an inductometer. In this way, the components of the inner resistance could be determined. However, Horton and van Ravenswaay's apparatus is rather specialized, since it is suited for relatively limited frequency-, resistance-, and reactance-ranges, only. The aggregate contains two well-calibrated inductometers and seems to be fairly expensive.

In another series of measurements, Horton and van Ravenswaay determined the inner phase angle by means of a four-electrode arrangement, measuring resistance and reactance across all electrodes. The inner resistance and reactance were then easily calculated.

One of the present authors (T.) has been occupied for several years with the measurements of the resistance components of the organism. These investigations were carried out at the Rigshospital, Department B, and the results will be published at a later opportunity. The apparatus applied was built according to Grützmacher's method (comp. J. Oskar Nielsen, »Ingeniøren«, **45 (II)**: 25, 1936) and gave a satisfactory accuracy when the electrodes were arranged peripherically on the upper extremities and impedance and phase angle were determined between the extremities. An attempt has been made to determine the "inner" or—as we would prefer to call it—the "central" phase angle by means of the four-electrode method. A satisfactory accuracy could, however, not be attained, partly because of inevitable small changes of the electric resistance of the skin, partly because of the very small phase angle of the region investigated. Since this question is of considerable medical interest, one of us (W.) developed a method for the direct —or almost direct—determination of the components of the "central" complex resistance, applying but minor and relatively cheap additional acquisitions to the apparatus used for the measurements ad modum GRÜTZMACHER. We take it as granted that the arrangement, due to its simplicity, may find frequent application in numerous fields, also beyond the biological one, and therefore thought it convenient to publish its description in the present series.

The method of measurements can be considered as an extension of STARR's and of GRÜTZMACHER's principle. Since use is made of a partial compensation, we can reach an essentially greater accuracy of the determination of some of the given components than with STARR's and GRÜTZMACHER's methods in their original form.

General Principle.

An unknown impedance is connected in series with a known one and the network is fed with a sinoscidal current: If we switch two amplifiers with high input impedance and the same phase shift across the known and unknown impedance, respectively, in such a way that they feed a common valve-voltmeter and act "against each other", we can by regulating the known impedance (or the amplification factors) obtain that the ratio between the non-compensated (residual) voltage and the voltage across the known impedance becomes a simple function of the unknown complex resistance components and, consequently, these can be calculated in an easy way.

For the practical application of the method it is convenient, however, to limit the general principle[1] by using pure resistances as known impedances (or condensers if we deal with systems containing small ohmic components)

[1] Comp. footnote p. 18.

and, furthermore, input amplifiers with equal (or almost equal) amplification factors, so that the input voltage of the amplifiers in balance becomes the same (or approximately the same).

If pure resistances are applied as known impedances and if the voltage variations across these impedances counteract the voltage variations across the unknown impedance it becomes possible to compensate completely, or at least partly, that part of the a. c.-potential which comes from the ohmic component of the unknown impedance, while that part which originates from the reactance component is not influenced at all. If a condenser is used as a known impedance, we compensate in an analogous way, completely or partly, that part of the potential which belongs to the reactance component while the a. c.-potential across the ohmic component is not compensated.

Figure 1 shows schematically the body and the placement of the electrodes. The circuit is given as built up of two peripherical complex resistances, denoted as $\frac{Z_p}{2}$, and a central complex resistance in series, denoted as Z_2.

The electrodes El_p are arranged peripherically on the resistance $\frac{Z_p}{2}$ while the electrodes El_c are connected with the central impedance through two complex resistances of the same kind as $\frac{Z_p}{2}$.

The skin beneath the electrodes has a considerable polarization capacity compared with that of the deeper tissues. Furthermore, it is a supposition for the determination of the "inner" component, according to the method described above as well as to that of Horton and van Ravenswaay, that the resistance along the skin is very high compared with the resistance beneath the electrodes across the skin. It has been proved by the investigation carried out by one of us (T.) that this assumption is fulfilled.

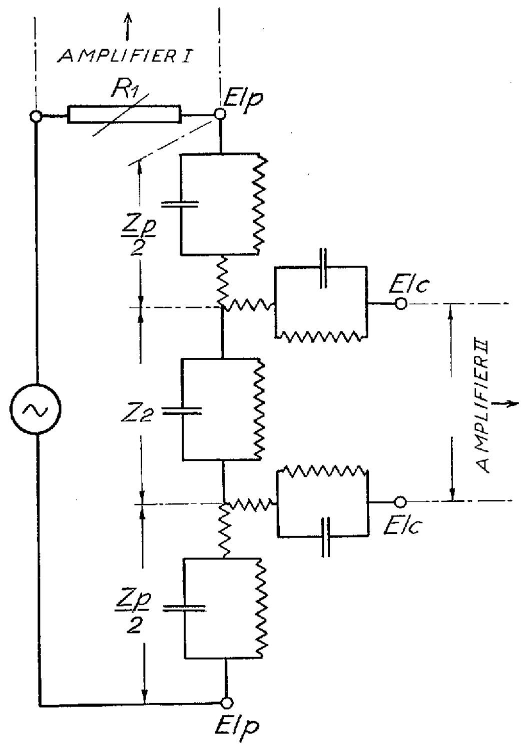

Fig. 1. Scheme of the measuring circuit. The impedances between El_p and El_p are equivalents for the impedances of the body; $\frac{Z_p}{2}$ represents the peripherical parts and Z_2 the central part of these impedances. The connections to the amplifiers are marked.

The electrodes El_c are connected with one of the input amplifiers which, as mentioned before, has a high input impedance and, consequently, does not loaden the circuit. Those parts of the skin which are situated beneath the electrodes El_c can, therefore, be considered to be lengthenings of the leads to the amplifier and their impedance does thus not influence the measured results.

In spite of the fact that the best equivalents for the resistances of biological systems as regards their frequency dependence are pure resistances and capacities in parallel, as shown in Fig. 1, it may be convenient for the practical accomplishment of the investigations and calculations—as

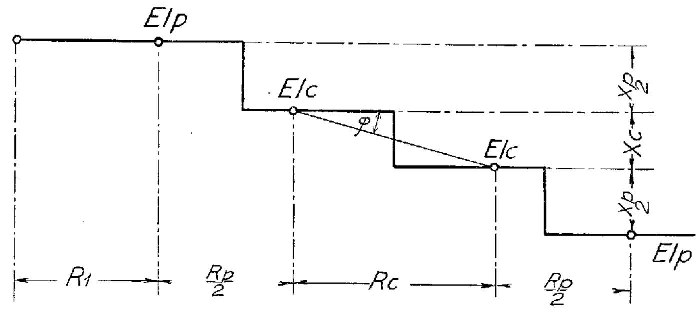

Fig. 2. Equivalent for the series-connections of the circuit.

described in the following—to consider them as pure resistances in series with capacities. If necessary, the parallel equivalents may be calculated from the measured components in series.

The vector diagram representing the systems of measurements is given in Fig. 2.

In the following, the components which correspond to the resistance R_1 are marked with index 1 and those which correspond to the impedance across the electrodes El_c with index 2, so that $R_c = R_2$, and $X_c = X_2$. The angle φ_2, however, is always denoted as φ, since no other angles appear in these considerations.

A scheme of the arrangement is given in Figs. 3 a and 3 b; the technical details will be discussed at the end of

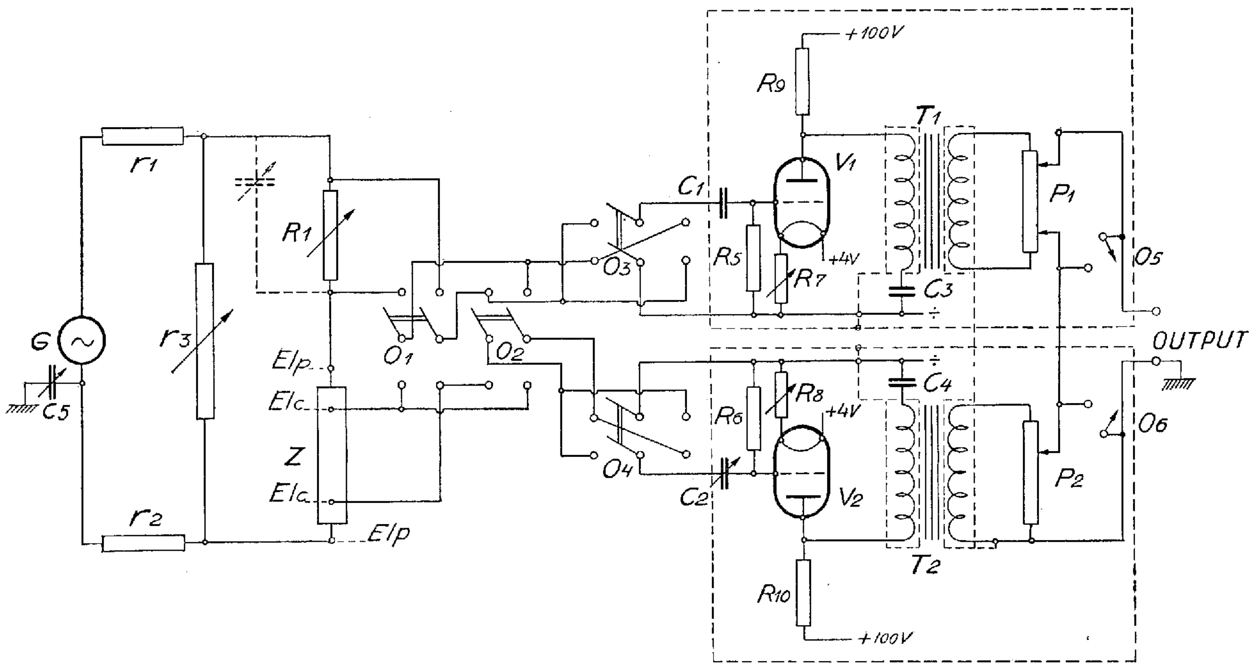

Fig. 3a. Diagram of the generator circuit, measuring circuit, switchboard, and the compensating and measuring amplifiers.

this paper. In the first-coming part, details will only be mentioned if they are part of the general principle of the method applied.

An a. c.-generator producing almost pure sine-oscillations is connected with a decade rheostat r_3 in series with two constant resistances r_1 and r_2. Across the rheostat r_3 the measuring circuit is connected which consists of the decade rheostat R_1 in series with the object of measurements Z. The current is supplied through the peripherical electrodes El_p.

By means of the switch-gears O_1, O_2, O_3, and O_4, the voltage across R_1 can be connected to both amplifiers simultaneously in such a way that the output potentials after regulation of the condenser C_2 have a phase difference of 180°. By successive regulation of the output potentiometers P_1 and P_2, and C_2, the amplifiers can be balanced completely so that the valve-voltmeter shows no deflection.

After this preliminary regulation the measurement itself begins at every given frequency.

By means of the switches O_5 and O_6 the output-side of each amplifier can be short-circuited. If one of the amplifiers is short-circuited in this way the other one acts like an ordinary input amplifier.

The valve-voltmeter applied (comp. Fig. 3 b) is provided with a double interstage-potentiometer which allows us both to vary the amplification factor gradually and to reduce it to $^1/_{10}$ or $^1/_{100}$, independently of the total amplification factor and of the frequency applied.

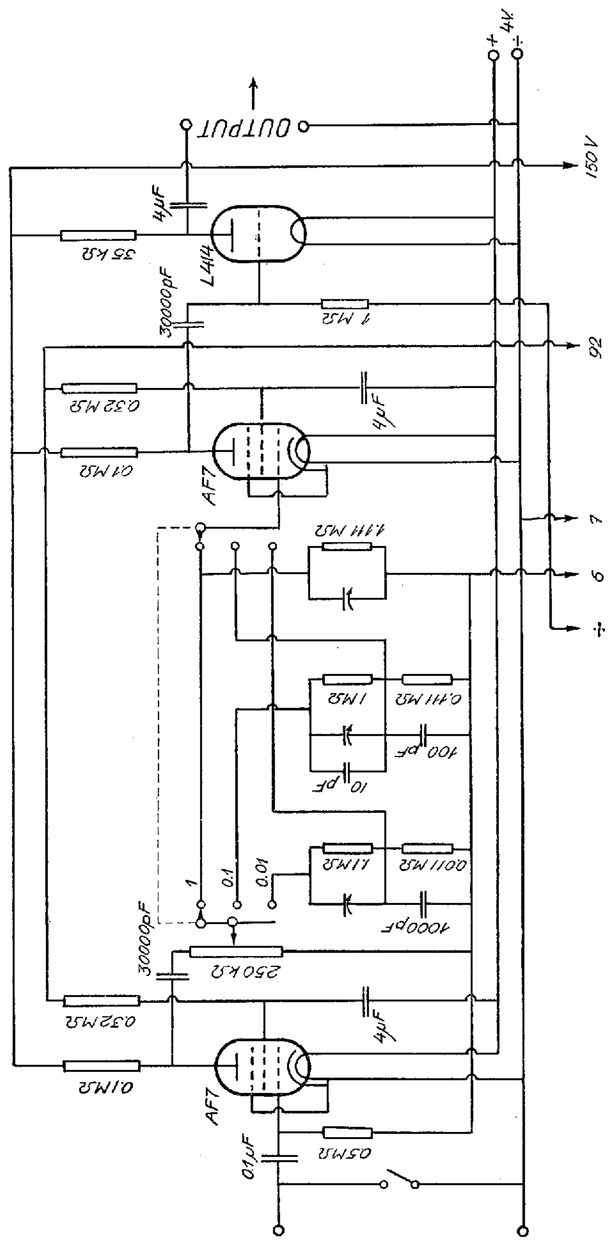

Fig. 3b. Amplifier to the valve-voltmeter.

Various Procedures for the Determination of the Different Components.

In the following, there will be given a general description of the procedures during the application of those methods which under the above discussed assumptions are not dependent on whether we have to do with a directly prehensile resistance or with a resistance which is separated from the surface by semi-conductors.

Determination of Z_2 by Substitution.

α) The impedance Z_2 is measured by switching alternately an amplifier (without compensation) across Z_2 and R_1 and by regulating R_1 until the valve-voltmeter shows the same deflection for both impedances.

β) If the amplification factors of both amplifiers are the same we can short-circuit alternately one or the other output and regulate R_1 until the same deflection of the valve-voltmeter is reached for R_1 and Z_2.

We have, then,

$$Z_2 = R_1. \tag{1}$$

The method β) is possibly less accurate than α), since residual couplings—if the valve-voltmeter is adjusted to a high sensitivity—are not always the same for both amplifiers which, therefore, can very well be completely compensating at high input potentials without having exactly the same amplification factor at low potentials.

Determination of $2 \sin \frac{\varphi}{2}$.

γ) If the voltage across R_1 which is of the same absolute size as the voltage across Z_2 is compensated against the ohmic component of the latter, we can draw a vector diagram as that shown in Fig. 4.

We get

$$2 \sin \frac{\varphi}{2} = \frac{|\mathbf{e}_{Z_3}|}{|\mathbf{e}_{Z_2}|} \tag{2}$$

and, consequently,

$$2 \sin \frac{\varphi}{2} = \frac{E_{Z_3}}{E_{Z_2}} = \frac{E_{Z_3}}{E_{R_1}} \tag{3}$$

where E_{R_1}, E_{Z_2}, and E_{Z_3} are the same functions of the effective voltage across R_1, Z_2, and Z_3 dependent on the construction of the amplifier and the valve-voltmeter. After a suitable regulation of the measuring e. m. f. and a compensation of the voltage across R_1 against the voltage across Z_2, we get a given deflection of the valve-voltmeter and an adjustment of the output potentiometer of one of the input amplifiers. Then, we short-circuit the other input amplifier and produce the same deflection of the valve-voltmeter by regulating the output potentiometer. Taking into account a possible change of the sensitivity of the valve-voltmeter we find

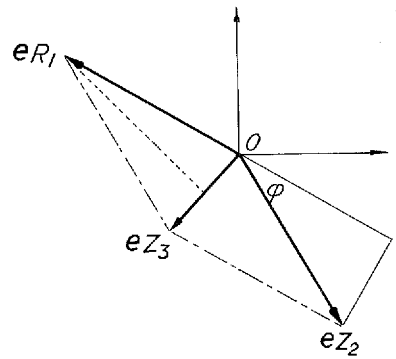

Fig. 4. Vector diagram for the determination of the components by substitution of Z_2 (and by measuring the minimum residual voltage at constant potential in the circuit).

$$2 \sin \frac{\varphi}{2} = \frac{\text{potentiometer adjustment 2}}{\text{potentiometer adjustment 1}}. \tag{4}$$

δ) The ohmic component and the reactance are calculated in the ordinary way as

$$X_2 = Z_2 \sin \varphi \tag{5}$$

and

$$R_2 = Z_2 \cos \varphi . \tag{6}$$

Determination of the Minimum Residual Voltage.

The minimum residual voltage found by regulating R_1 can be determined either by maintaining a constant current in the input circuit of the measuring amplifiers (e. g. by using a series-resistance in the circuit) or by maintaining a constant voltage across the circuit (in this case, the resistance r_3 must be low compared with the resistance of the circuit). The potential diagram for the output of the amplifier is given in Fig. 5.

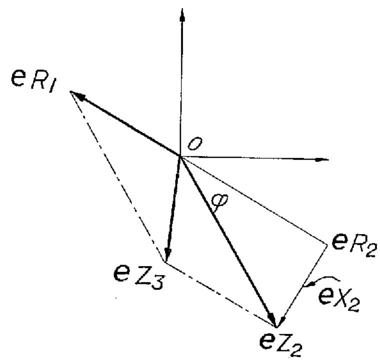

Fig. 5. Vector diagram for the calculation of the minimum residual voltage.

For the diagram we get for a constant current (without taking the amplification factor into account)

$$|\mathbf{e}_{Z_3}| = |\mathbf{i}_{Z_3}| \sqrt{R_1^2 + Z_2^2 - 2 R_1 Z_2 \cos \varphi} \tag{7}$$

where

$$E_{Z_3} = k \sqrt{R_1^2 + Z_2^2 - 2 R_1 Z_2 \cos \varphi} \tag{8}$$

the condition for a minimum of E_{Z_3} being

$$R_1 = R_2 \tag{9}$$

and, consequently,

$$\mathbf{e}_{Z_3} = \mathbf{e}_{X_2} . \tag{10}$$

These facts are illustrated by Fig. 6.

If the results given above are taken into account, we get

ε) $$\sin \varphi = \frac{E_{X_2}}{E_{Z_2}}. \tag{11}$$

If we note the deflection of the valve-voltmeter and the adjustment of the amplifier potentiometer at the minimum during compensation and produce the same valve-voltmeter deflection across the object of measurements after short-circuiting the compensating amplifier, we get

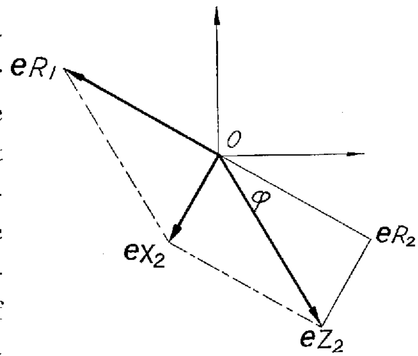

Fig. 6. Vector diagram for the determination of the minimum residual voltage at constant current.

$$\sin \varphi = \frac{\text{potentiometer adjustment 2}}{\text{potentiometer adjustment 1}}. \tag{12}$$

φ) It follows, furthermore, from the diagram that

$$\tan \varphi = \frac{E_{X_2}}{E_{R_1}} \tag{13}$$

and by reproduction of the deflections from the residual voltage across R_1 and regulation of the amplifier potentiometer, we get in an analogous way

$$\tan \varphi = \frac{\text{potentiometer adjustment 2}}{\text{potentiometer adjustment 1}}. \tag{14}$$

η) For the determination of R_2 from $\sin \varphi$ and Z_2 we have in the ordinary way

$$R_2 = Z_2 \cos \varphi \tag{15}$$

and

$$X_2 = Z_2 \sin \varphi. \tag{16}$$

Finally, the procedures ε and φ lead to

$$R_2 = R_1 \tag{17}$$

and the procedure φ to

$$X_2 = R_1 \tan \varphi . \tag{18}$$

From the diagram in Fig. 5 it can be deduced for constant voltage

$$\left| \mathbf{e}_{Z_3} \right| = \left| \mathbf{e}_{R_1 + Z_3} \right| \sqrt{\frac{R_1^2 + Z_2^2 - 2 R_1 Z_2 \cos \varphi}{R_1^2 + Z_2^2 + 2 R_1 R_2}} \tag{19}$$

and from this

$$E_{Z_3} = k \sqrt{\frac{R_1^2 + Z_2^2 - 2 R_1 Z_2 \cos \varphi}{R_1^2 + Z_2^2 + 2 R_1 R_2}} \tag{20}$$

the condition for a minimum of E_{Z_3} is

$$R_1 = Z_2 . \tag{21}$$

The vectorial diagram demonstrating this case is already given in Fig. 4 and the procedures γ and δ and the formulae (3)—(6) may, consequently, be applied to the determination of the other components[1].

[1] The difference between the results from the determinations at constant current and at constant voltage is very slight as long as φ is small, since in this range Z_2 is almost identical with R_2. In the following section it will be shown that the procedures regarding the determination of the minimum residual voltage should be applied to small angles, only; therefore we shall scarcely meet with a considerable uncertainty when determining the minimum voltage corresponding to a value between constant voltage and constant current.

When the variation of the amplification factor of one of the amplifiers is applied — as in the analogous case of a compensation against condensers — it is possible to make sure that we work at a constant current. In this way, however, two more contributions to the uncertainty of the measurement is introduced originating from the readings of the galvanometer, uncertainties which generally will be greater than those originating from the circumstances described above. Therefore, it was considered to be inconvenient to treat this procedure in detail.

Determination of the Sign of the Angle φ.

The sign of the angle φ is determined by connecting a capacity of suitable size in series with R_1 and switching the compensating amplifier across R_1 plus the condenser. In a new determination of the angle we shall, then, find a smaller numerical value if the phase angle of the object was negative and a greater numeric value if the angle was positive. A still simpler method for the determination of the angle is the following: to remember once for all whether the residual voltage at a known sign of the angle increases or decreases if the setting of the condenser C_2 is increased or decreased.

Measurements Applying Compensation against Condensers.

The determination of the ohmic component carried out according to the technique described above involves a relatively great uncertainty at great angles, as is obvious from the equations (6), (15), and (18). This fact will be discussed in detail on the following pages. Therefore, the technique is not especially suited for the determination of small lag angles in condensers or the ohmic component in coils. The method can, however, be developed for an application to this field if the compensation amplifier is connected across a condenser of suitable size so that a voltage with a phase difference of 90° to the voltage across a pure resistance is applied to the compensation.

Z_2 can be determined by the techniques described under α and β. If we dispose of a decade condenser set all measurements can be performed by exactly the same procedure as that given for the measurements using compensation against pure resistances: we have but to replace R_2 by X_2, sin by cosin, and tan by cotan.

On the other hand, if a complete set of condensers is not available the following method may be applied: The compensation amplifier is connected with a condenser of such an order of magnitude that its reactance differs only very little from that of the object of measurements. The other amplifier is connected with the object of measurement and the minimum of the valve-voltmeter deflection is found by regulation of the compensation amplifier.

We note the deflection of the valve-voltmeter and the setting of the amplifier potentiometer, then we short-circuit the compensation amplifier and produce the same deflection of the valve-voltmeter by regulating the potentiometer of the measuring amplifier. In this way, we get

$$\cos\varphi = \frac{\text{potentiometer adjustment 2}}{\text{potentiometer adjustment 1}}. \qquad (22)$$

R_2 and X_2 can then be determined according to the equations (15) and (16)[1].

Discussion of the a priori Uncertainty of the Various Procedures.

An investigation of the a priori uncertainty of the procedures described on the preceding pages led to the result that the uncertainty of the determination of small angles is characterized by a complete or partial one-sidedness which is rarely met with in the measuring technique and

[1] On Fig. 3 a a condenser is drawn (dotted lines) across R_1. The application of this condenser changes the method into a pure zero point method which enables us to determine the reactive as well as the ohmic component directly. However, we did not consider it necessary to discuss this method any further.

which appears, since the method applied makes use of residual voltages which are minimal or almost minimal. This dissymmetry is not of the same order of magnitude for the various procedures; therefore, we consider it convenient to discuss the purely a priori estimation of the uncertainties more detailed than it seems to be appropriate to other technical problems where the deviations from the ideal results scatter symmetrically.

Professor J. HARTMANN had the great kindness of helping us in various points.

In the following, we make use of HARTMANN's terminology. By the uncertainty of a method of measurements we denote what in biology is called multiple of the square mean error, and the relative uncertainty is, thus, the ratio between a multiple of the square mean error and the average value[1].

In agreement with HARTMANN we call "error" that deviation from the ideal value wich in biology in general is called systematic error or systematic deviation.

For further details we wish to refer to HARTMANN's text-book.

In the performance of the following calculations we were repeatedly forced to confine ourselves to point towards certain complications since our intention essentially has been the practical application of the method. A further mathematical treatment of the given complications would claim greater mathematical experience than we mean to possess.

The considerations described in the following do neither take into account those uncertainties which originate from residual couplings to earth or to the generator nor those

[1] HARTMANN denotes as the uncertainty of a procedure the maximum deviation from the average of 10 determinations. As a matter of experience, this uncertainty is — in a considerable number of cases — 1.6 — 1.8 times the square mean error.

errors which come from impurities of the current, disparity of the two amplifiers, or impurities of the impedance R_1, regardless of whether they come from the reactances of the comparative resistance or the lag angles of the condenser. However, those uncertainties which are produced by possible variations of the output of the generator during the measurements appear as uncertainties of the galvanometer readings.

In the present arrangement, possible variations of the generator output play an important part in the accuracy of the measurements, since almost every step involves the substitution of an unknown impedance or the reproduction of a known galvanometer deflection. The opposite situation is present in zero-point methods, where we deal with complete compensation. Since the variations of the output e.m.f. of the generator eventually may appear in the measured result with their total value, we must lay great stress upon the stabilization of the output effect of the generator.

This fact forced us to purchase a new generator, since it was impossible in the present investigations to work with a generator, the output of which varied rapidly with 1 per cent for unknown causes.

Regard is paid to the following uncertainties, only: 1) the uncertainty of the valve-voltmeter readings, 2) the uncertainty of the divisions of the output potentiometer of the measuring amplifier and its readings, 3) the uncertainty of the construction of the standard comparative impedance R_1.

The calibrated interstage potentiometer of the amplifier valve-voltmeter, which can reduce the sensitivity of the voltmeter to $1/10$ or $1/100$, is considered to be free from uncertainty, since it can be corrected at the frequencies applied by means of any number of individual determinations.

By means of the interstage potentiometer of the valve-voltmeter we can bring every residual potential of 0.1 per cent or more to be read on that part of the output potentiometer which gives $^1/_{10}$ or more of the maximum adjustment of the potentiometer.

Uncertainties in the Determination of $2 \sin \frac{\varphi}{2}$ by means of the Substitution Method.

In the discussion of the uncertainties involved in experimental observations we found it convenient to mark the ideal values of the described magnitudes with a special index. The ideal value of R_1 is denoted as $R_{1.0}$. If a magnitude is given and, thus, not produced it is written in the usual manner, so that the object's impedance is called Z_2, its reactance X_2, and its ohmic component R_2.

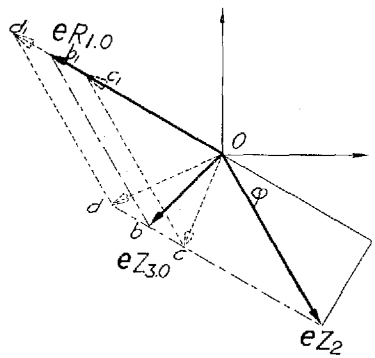

Fig. 7. Diagram of the a priori calculation of the relation between the deviations from the ideal values of $e_{R_{1.0}}$ and $e_{Z_{3.0}}$.

It is seen from Fig. 7 that the ideal values of $R_{1.0}$, Z_2, $Z_{3.0}$, and $\sin \frac{\varphi}{2}$ are connected by the following relations:

$$R_{1.0} = Z_2 \tag{23}$$

and

$$Z_{3.0} = R_{1.0} 2 \sin \frac{\varphi}{2}. \tag{24}$$

An uncertainty in the determination of $R_{1.0}$ causes an uncertainty in the determination of $Z_{3.0}$ which can be calculated in the following way:

The uncertainty in the determination of $R_{1.0}$ can affect that $R_{1.0}$ is determined by $b_1 d_1$ too high or $c_1 b_1$ too small, so that

$$b_1 d_1 = c_1 b_1 = \Delta R_1; \tag{25}$$

the relative uncertainty of $R_{1.0}$ is then, as usual, $\frac{\Delta R_{1.0}}{R_{1.0}}$, and, consequently, we get

$$R_1 \left(1 + \frac{\Delta R_{1.0}}{R_{1.0}}\right) = Z_2 \tag{26}$$

where $\frac{\Delta R_{1.0}}{R_{1.0}}$ may be positive as well as negative.

It appears, furthermore, from Fig. 7 that Z_3 is determined as Od or Oc if we determine R_1 as Od_1 or Oc_1. In this case, we get

$$\frac{\Delta Z_{3.0}}{Z_{3.0}} = -1 + \sqrt{1 + \left(\frac{\frac{\Delta R_{1.0}}{R_{1.0}}}{2 \sin \frac{\varphi}{2}}\right)^2 + \frac{\Delta R_{1.0}}{R_{1.0}}}. \tag{27}$$

If the measurements are carried out in such a way that the amplifier is connected across R_1, when the numerator of (4) is to be determined, the same possible deviations from the ideal value of $R_{1.0}$ determine the deviations from Z_3 as well as from R_1. It is, therefore, desirable to determine the relative uncertainties not only of Z_3 but also of the magnitude of

$$U_0 = \frac{Z_{3.0}}{R_{1.0}}. \tag{28}$$

The increase of U_0 which corresponds to the increase of $R_{1.0}$ is called ΔU_0 and, consequently, we get

$$U_0 + \Delta U_0 = \frac{Z_{3.0} + \Delta Z_{3.0}}{R_{1.0} + \Delta R_{1.0}} = \frac{Z_{3.0}}{R_{1.0}} \left(1 + \frac{\Delta Z_{3.0}}{Z_{3.0}}\right)\left(1 + \frac{\Delta R_{1.0}}{R_{1.0}}\right)^{-1}. \quad (29)$$

If this equation is expanded after the binominal series and we take powers up to the second of the magnitude of $\frac{\Delta R_{1.0}}{R_{1.0}}$, only, we get

$$\frac{\Delta U_0}{U_0} = \frac{1}{2} \frac{\Delta R_{1.0}}{R_{1.0}} \left[-1 + \frac{1}{4} \frac{\Delta R_{1.0}}{R_{1.0}} \left(\frac{1}{\sin^2 \frac{\varphi}{2}} + 3 \right) \right]. \quad (30)$$

For the further discussion of this relation it is convenient to form the expression

$$\frac{\Delta U_0}{U_0} = \frac{1}{2} \frac{\Delta R_{1.0}}{R_{1.0}} A \quad (31)$$

and to investigate the function between A and φ at given values of $\frac{\Delta R_{1.0}}{R_{1.0}}$. The results of these investigations are demonstrated in Figs. 8a—8d for values of $\frac{\Delta R_{1.0}}{R_{1.0}}$ of 0,01, 0,003, 0,001, and 0,0001.

From the curves we find the limits of the values of φ, where $\frac{\Delta U_0}{U_0}$ becomes always positive.

The variations of the limit with $\frac{\Delta R_{1.0}}{R_{1.0}}$ are plotted in Fig. 8e.

It can be seen from the curves and the equation (27) that the uncertainties of U_0 which are caused by the symmetric uncertainties of $R_{1.0}$ are completely or partially one-sided.

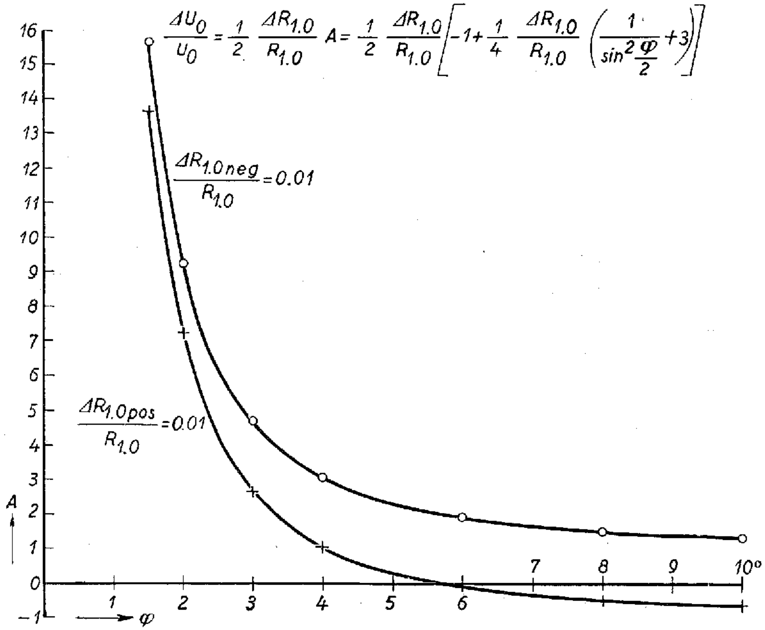

Fig. 8 a.

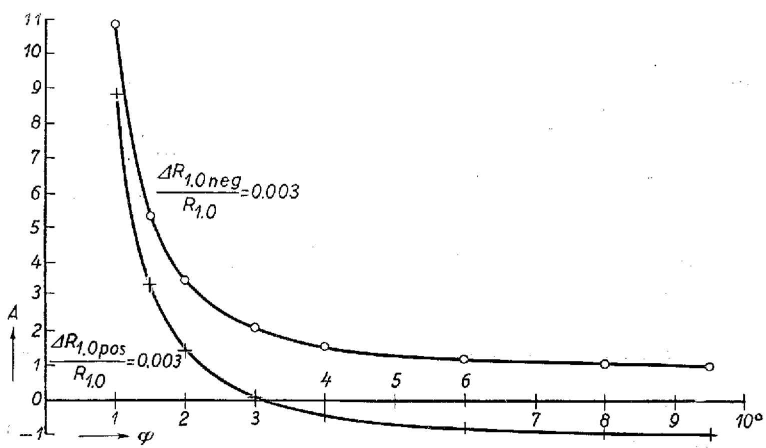

Fig. 8 b.

Fig. 8 a-d. Relation between A and φ at varying values of $\frac{\Delta R_{1.0}}{R_{1.0}}$.

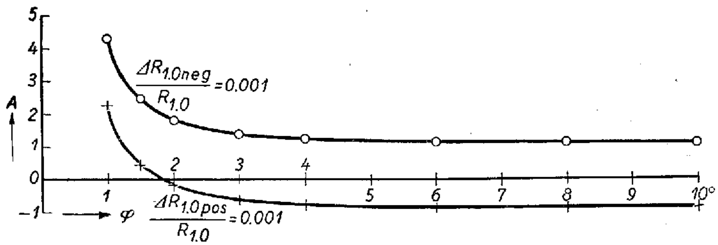

Fig. 8 c.

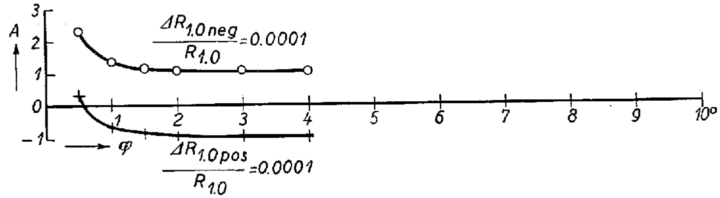

Fig. 8 d.

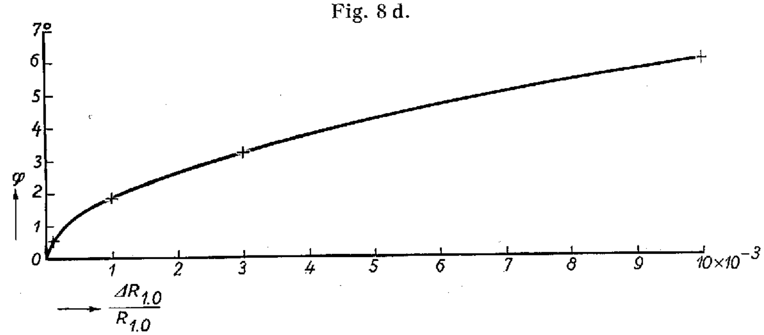

Fig. 8 e. Dependence of the limit of the pure one-sided uncertainties of $2 \sin \frac{\varphi}{2}$ on $\frac{\Delta R_1}{R_1}$ and φ.

The complete estimation of the uncertainty of $2 \sin \frac{\varphi}{2}$ includes, furthermore, two contributions of equal size which are denoted as $\frac{\Delta P}{P}$ and which originate from the uncertainties in the adjustment of the output potentiometer, and two contributions from the readings of the galvanometer[1] which are denoted as $\frac{\Delta G}{G}$.

[1] If we carry out a preliminary measurement and then repeat the determination of Z_2 with the same sensitivity and the same amplification

These uncertainties are of the normal, symmetrical kind. Consequently, the question is how the symmetric uncertainty is to be combined with the one-sided or partly one-sided uncertainty.

In spite of the fact that a complete analysis of these complications would lead beyond the scope of this paper, we wish to make a few remarks on this subject, since it is possible on the basis of some approximate considerations to treat the problem to an extent necessary for the technical estimation of the combined uncertainties.

If φ is sufficiently great the asymmetry disappears and the combination of the uncertainties is carried out in the classical way

$$\frac{\Delta \sin \frac{\varphi}{2}}{\sin \frac{\varphi}{2}} = \sqrt{\left(\frac{1}{2}\,\frac{\Delta R_{1.0}}{R_{1.0}}\,A\right)^2 + 2\left(\frac{\Delta P}{P}\right)^2 + 2\left(\frac{\Delta G}{G}\right)^2}. \qquad (32)$$

In the opposite case, where φ is very small, the symmetrical component is negligible compared with the one-sided one and the uncertainty is, then, identical with $\frac{\Delta U_0}{U_0}$ and can be read from the curves 8 a—8 d which are calculated according to the equation (30).

In the intermediate interval a combination of both kinds of uncertainties may give rise to uncertainties of different magnitudes with positive as well as negative deviations from the ideal value. We shall not give a more detailed description; however, it must be emphasized that maximum deviations appear if positive deviations from the sym-

factor as is to be applied for the determination of E_1 we can avoid a galvanometer reading and thereby reduce the uncertainties by one contribution. Since, however, this procedure makes the method more complicated, we preferred to take no account of this "improvement".

metrical contributions are combined with negative deviations from the adjustment of R_1.

For a complete estimation of the uncertainty of $\sin\frac{\varphi}{2}$ we considered it sufficient to calculate the uncertainty from the maximum value of A and to combine it in the normal way with the other contributions to the uncertainties. Consequently, the expression for the relative uncertainty of $\sin\frac{\varphi}{2}$ under the given limitation becomes

$$\frac{\Delta\sin\frac{\varphi}{2}(\max)}{\sin\frac{\varphi}{2}} = \sqrt{\left(\frac{1}{2}\frac{\Delta R_{1.0}}{R_{1.0}}A\right)^2 + 2\left(\frac{\Delta P}{P}\right)^2 + 2\left(\frac{\Delta G}{G}\right)^2}. \quad (33)$$

In the interval where the one-sided or partly one-sided contribution to the uncertainties is essential, the experimentally determined average value of $\sin\frac{\varphi}{2}$ is greater than the true value, and the experimentally determined uncertainty of Z_3 and the magnitude deduced from Z_3 become less than the true values. In order to emphasize this fact, we added the index mark .0 in all calculations with the ideal magnitudes.

The uncertainties of $\sin\varphi$, $\cos\varphi$, and φ are calculated from $\sin\frac{\varphi}{2}$ in the usual way according to the following equations, where the sign of the differential coefficient must be taken into consideration.

$$\frac{\Delta\sin\varphi(\max)}{\sin\varphi} = \frac{2\tan\frac{\varphi}{2}}{\tan\varphi}\cdot\frac{\Delta\sin\frac{\varphi}{2}(\max)}{\sin\frac{\varphi}{2}} \quad (34)$$

$$\frac{\Delta\cos\varphi(\max)}{\cos\varphi} = -2\tan\frac{\varphi}{2}\cdot\frac{\Delta\sin\frac{\varphi}{2}(\max)}{\sin\frac{\varphi}{2}}\tan\varphi \quad (35)$$

and

$$\frac{\Delta\varphi\,(\max)}{\varphi} = \frac{\tan\frac{\varphi}{2}}{\frac{\varphi}{2}} \frac{\Delta \sin\frac{\varphi}{2}\,(\max)}{\sin\frac{\varphi}{2}}. \tag{36}$$

The Uncertainties of the Determination of sin φ by a Method of Measuring the Minimum Residual Voltage.

The determination of $\sin\varphi$ involves two adjustments of the output potentiometer of the measuring amplifier and two corresponding readings of the galvanometer which are not correlated. It is obvious from Fig. 9 that all deviations in the determination of $X_{2.0}$ are positive.

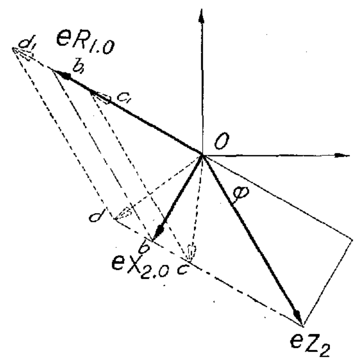

Fig. 9. Diagram of the a priori calculation of the relation between the deviations from the ideal values of $\mathbf{e}_{X_2}$ and $\mathbf{e}_{R_1}$.

The positive deviations from the value of Z_2 must counteract the positive deviations in the determination of $X_{2.0}$ so that equal deviations from the ideal values cancel each other. On the other hand, negative deviations in the determination of Z_2 never meet negative deviations in the determination of $X_{2.0}$ so that positive deviations in the determination of $\sin\varphi$ appear more frequently than we would expect if the deviations from $Z_{2.0}$ were distributed normally. In this way, the average value of $\sin\varphi$ becomes greater than the true value and the experimentally determined relative uncertainty becomes less than the true one, in agreement with the results found from the calculation of the relative uncertainty of $\sin \frac{\varphi}{2}$.

In this manner, the distribution curve of the deviations becomes quite skew and the uncertainty concept looses its clearness to some extent.

For the estimation of the uncertainty of the result, however, it is sufficient to remember that the combination of the uncertainties from both contributions carried out in the ordinary way leads to the greatest uncertainty of the final result.

After this limitation, we get as the relative uncertainty of $\sin \varphi$

$$\frac{\Delta \sin \varphi\,(\max)}{\sin \varphi} = \sqrt{2\left(\frac{\Delta P}{P}\right)^2 + 2\left(\frac{\Delta G}{G}\right)^2}; \tag{37}$$

the relative uncertainties of $\cos \varphi$ and φ, respectively, are

$$\frac{\Delta \cos \varphi\,(\max)}{\cos \varphi} = -\tan^2 \varphi \,\frac{\Delta \sin \varphi\,(\max)}{\sin \varphi} \tag{38}$$

and

$$\frac{\Delta \varphi\,(\max)}{\varphi} = \frac{\tan \varphi}{\varphi} \cdot \frac{\Delta \sin \varphi\,(\max)}{\sin \varphi}. \tag{39}$$

The Uncertainties of the Determination of tan φ by a Method of Measuring the Minimum Residual Voltage.

It is obvious from Fig. 9 that the ideal determination of $\tan \varphi$ requires

$$\tan \varphi = \frac{X_{2.0}}{R_2} \tag{40}$$

and

$$R_{1.0} = R_2. \tag{41}$$

If, however, $R_{1.0}$ is determined too high, so that it becomes Od_1 instead of Ob_1, or too small, i. e. Oc_1 instead of Ob_1, where

$$b_1 d_1 = c_1 b_1 = \Delta R_{1.0} \tag{42}$$

we get

$$R_1 \left(1 + \frac{\Delta R_{1.0}}{R_{1.0}}\right) = R_2 \tag{43}$$

where $\frac{\Delta R_{1.0}}{R_{1.0}}$ can have negative as well as positive values.

Furthermore, we get from Fig. 9

$$Od = Oc = X_{2.0} \left(1 + \frac{\Delta X_{2.0}}{X_{2.0}}\right) \tag{44}$$

where $\frac{\Delta X_{2.0}}{Z_{2.0}}$ can have positive values, only.

From Fig. 9 we can, furthermore, conclude that

$$X_{2.0}^2 \left(1 + \frac{\Delta X_{2.0}}{X_{2.0}}\right)^2 = X_{2.0}^2 + \Delta R_{1.0}^2 \tag{45}$$

and, together with (40) and (41)

$$\frac{\Delta R_{1.0}}{R_{1.0}} = \pm \tan \varphi \sqrt{\left(\frac{\Delta X_{2.0}}{X_{2.0}}\right)^2 + 2\,\frac{\Delta X_{2.0}}{X_{2.0}}}. \tag{46}$$

If we set

$$U_0 = \frac{X_{2.0}}{R_2} \tag{47}$$

and investigate the value to which U_0 can increase if $R_{1.0}$ increases up to $R_{1.0} + \Delta R_{1.0}$, we find

$$U_0 + \Delta U_0 = \frac{X_{2.0}}{R_{1.0}} \left(1 + \frac{\Delta X_{2.0}}{X_{2.0}}\right) \left(1 + \frac{\Delta R_{1.0}}{R_{1.0}}\right)^{-1} \tag{48}$$

and

$$\frac{\Delta U_0}{U_0} = \left(1 + \frac{\Delta X_{2.0}}{X_{2.0}}\right) \left(1 \pm \tan \varphi \sqrt{\left(\frac{\Delta X_{2.0}}{X_{2.0}}\right)^2 + \left(2\,\frac{\Delta X_{2.0}}{X_{2.0}}\right)}\right)^{-1} - 1. \tag{49}$$

In order to reach a clearer formulation it is convenient to write the following relations

$$\frac{\Delta U_0}{U_0} = \frac{\Delta X_{2.0}}{X_{2.0}} B \tag{50}$$

$$B = \frac{1 \mp \tan\varphi \sqrt{1 + 2\frac{X_{2.0}}{\Delta X_{2.0}}}}{1 \pm \tan\varphi \sqrt{\left(\frac{\Delta X_{2.0}}{X_{2.0}}\right)^2 + 2\frac{\Delta X_{2.0}}{X_{2.0}}}} \tag{51}$$

which, in the case of small angles and small uncertainties, can be simplified to

$$B = 1 \pm 2.47 \cdot 10^{-2} \varphi \sqrt{\frac{X_{2.0}}{\Delta X_{2.0}}} \tag{52}$$

where the negative sign corresponds to positive deviations of $R_{1.0}$ and the positive sign to negative deviations of this magnitude.

B is calculated for $\frac{\Delta X_{2.0}}{X_{2.0}}$ equal to 0.0001, 0.001, 0.003, and 0.01 and is drawn in Fig. 10 a.

Fig. 10 b shows a curve representing the magnitude of the angles which include positive uncertainties, only.

It appears from the figures that this method of determination, too, involves one-sided or partly one-sided uncertainties and, furthermore, that the asymmetry decreases rapidly with increasing angles, which, however, is of minor practical importance because of the rapid increase of the uncertainty in this angle range.

The considerations on the change of the average value and the apparent reduction of the uncertainty discussed in the paragraph on the uncertainty of the $\sin\frac{\varphi}{2}$-determination are also valid in the present case.

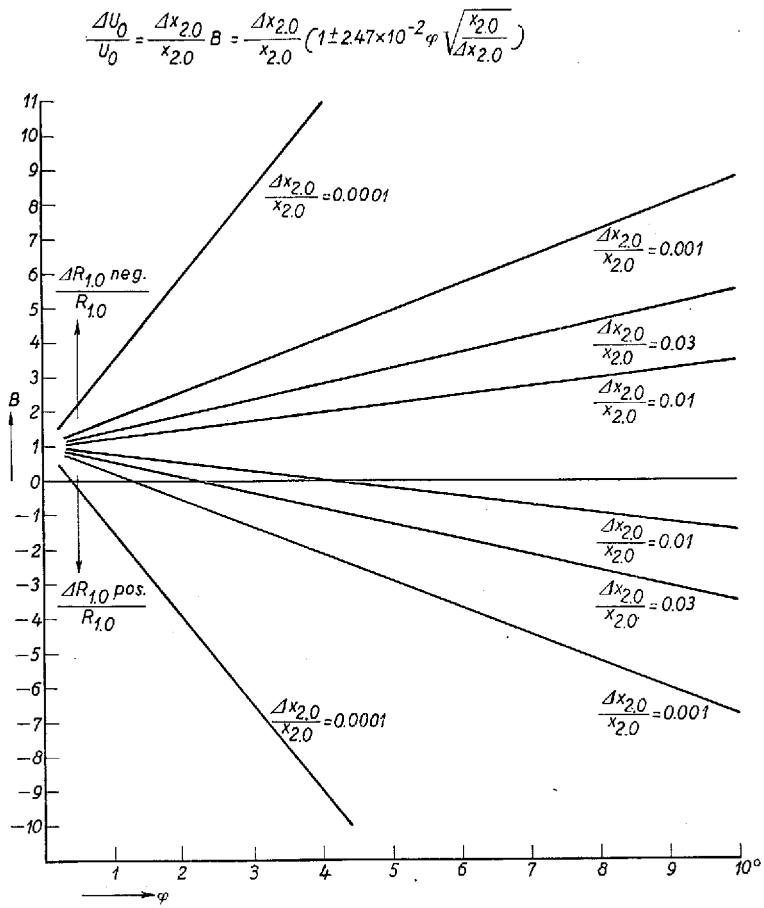

Fig. 10 a. Relation between B and φ at varying values of $\frac{\Delta X_{2.0}}{X_{2.0}}$.

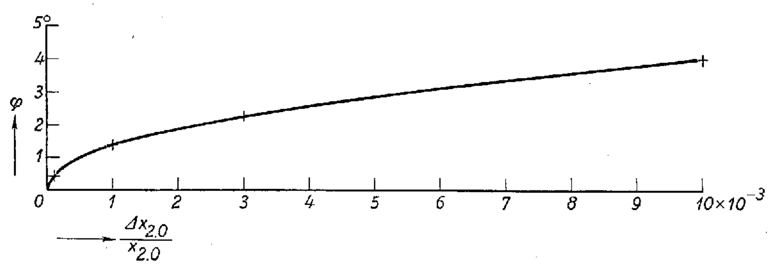

Fig. 10 b. Dependence of the limit of a pure one-sided uncertainty of $\tan \varphi$ (and $2 \sin \frac{\varphi}{2}$) on $\frac{\Delta X_{2.0}}{X_{2.0}}$ and φ.

In a complete estimation of the relative uncertainty of $\tan\varphi$ we must, furthermore, take into account two adjustments of the output potentiometer and one reading of the galvanometer[1]. If we take the highest values of B, we get as the maximum uncertainty of $\tan\varphi$

$$\frac{\Delta \tan\varphi\,(\max)}{\tan\varphi} = \sqrt{\left(\frac{\Delta X_2}{X_2}\,B\right)^2 + \left(\frac{\Delta G}{G}\right)^2 + 2\left(\frac{\Delta P}{P}\right)^2} \tag{53}$$

and from this we can calculate

$$\frac{\Delta\varphi\,(\max)}{\varphi} = \frac{\sin\varphi\cos\varphi}{\varphi}\cdot\frac{\Delta\tan\varphi}{\tan\varphi}. \tag{54}$$

The uncertainties of $2\sin\frac{\varphi}{2}$ in the determination of the minimum residual potential are so close to the uncertainties of the determination of $\tan\varphi$ that they can be considered identical for small angles.

On the preceding pages it is demonstrated that equal positive or negative deviations from the ideal value lead to markedly anomalous distribution curves for all three methods of determination. Remembering the great simplicity of the measuring arrangement described above these results are an impressive demonstration of the fact that we must take great care in applying a priori arguments as regards the shape of distribution curves of the deviations from average values of complex systems, also of biological ones. It is, furthermore, of interest to emphasize that the asymmetric distribution of the deviations from the ideal mean

[1] It is natural to use the same galvanometer reading for the determination of R_1 by compensation and for the starting-point of the substitution of the galvanometer deflection when the measuring amplifier is switched across R_1.

value may have the effect that the uncertainties partly appear as "errors" of the observations, so that it becomes almost a matter of taste of whether we wish to call them pure uncertainties or combination of uncertainties and errors.

If we wish to determine the uncertainties of the ohmic component and reactance of a given impedance from the formulae of the uncertainties of the trigonometrical functions, we must add a contribution from the uncertainty of the impedance determination. The determination of Z_2 contains the uncertainties of two galvanometer readings and the construction of the standard resistance R_1. On the assumption that all determinations of the absolute value of the components contain uncertainties from the contributions of the same size as in the determination of Z_2, we get the maximum uncertainty since some of the same galvanometer readings play a rôle in the determination of the trigonometrical functions, so that a certain correlation exists between the various contributions. A correction for this correlation would not only be very difficult to perform but would also be without special interest if the uncertainty concept is considered from a technical view-point.

The uncertainties of the absolute values of the ohmic component of R_2, determined according to (6) and (15), can be calculated from

$$\frac{\Delta R_2\,(\max)}{R_2} = \sqrt{\left(\frac{\Delta \cos\varphi\,(\max)}{\cos\varphi}\right)^2 + \left(\frac{\Delta Z_2}{Z_2}\right)^2} \qquad (55)$$

where $\frac{\Delta \cos\varphi\,(\max)}{\cos\varphi}$ is calculated from (35) and (38), respectively, and from determinations according to (17)

$$\frac{\Delta R_2\,(\max)}{R_2} = \sqrt{\left(\frac{\Delta R_{1.0}\,(\max)}{R_{1.0}}\right)^2 + \left(\frac{\Delta Z_2}{Z_2}\right)^2} \qquad (56)$$

where $\frac{\Delta R_{1.0}}{R_{1.0}}$ is calculated from (46).

The uncertainty of the absolute value of the reactance determined from (5) and (16) is deduced from

$$\frac{\Delta X_2\,(\max)}{X_2} = \sqrt{\left(\frac{\Delta \sin\varphi\,(\max)}{\sin\varphi}\right)^2 + \left(\frac{\Delta Z_2}{Z_2}\right)^2} \qquad (57)$$

where $\frac{\Delta \sin\varphi\,(\max)}{\sin\varphi}$ is calculated from (34) and (37).

Finally, we get for the determination according to (18)

$$\frac{\Delta X_2\,(\max)}{X_2} = \sqrt{\left(\frac{\Delta \tan\varphi\,(\max)}{\tan\varphi}\right)^2 + \left(\frac{\Delta Z_2}{Z_2}\right)^2} \qquad (58)$$

where $\frac{\Delta \tan\varphi\,(\max)}{\tan\varphi}$ is calculated from (53).

Table I contains the results of the calculations carried out according to the formulae evolved above. The relative uncertainties of all contributions (galvanometer readings, potentiometer adjustments, and standards) are always taken as $3 \cdot 10^{-3}$ which corresponds to a "maximum" deviation at a relative square "mean error" of $1 \cdot 10^{-3}$.

It appears from the table that all determinations carried out *by measurements of the minimum residual voltages* containing $\tan\varphi$ or $2\sin\frac{\varphi}{2}$ are more uncertain than those which contain $\sin\varphi$. Since the uncertainties in these procedures increase rapidly with increasing angle, there is no reason for the application of these methods.

At an angle below 15°, the method of determination of $2\sin\frac{\varphi}{2}$ leads to a greater uncertainty than the method

Table 1.

The Maximum Relative Uncertainties in per cent at Various Procedures[1].

$\varphi°$	Method of substitution. Determination of $2 \sin \frac{\varphi}{2}$			Method of residual minimum voltages					
				Determination of $\sin \varphi$			Determination of $\tan \varphi$ and $2 \sin \frac{\varphi}{2}$		
	$100 \frac{\Delta \varphi\,(\max)}{\varphi}$ formula (36)	$100 \frac{\Delta R_2\,(\max)}{R_2}$ formula (55)	$100 \frac{\Delta X_2\,(\max)}{X_2}$ formula (57)	$100 \frac{\Delta \varphi\,(\max)}{\varphi}$ formula (39)	$100 \frac{\Delta R_2\,(\max)}{R_2}$ formula (55)	$100 \frac{\Delta X_2\,(\max)}{X_2}$ formula (57)	$100 \frac{\Delta \varphi\,(\max)}{\varphi}$ formula (54)	$100 \frac{\Delta R_2\,(\max)}{R_2}$ formula (56)	$100 \frac{\Delta X_2\,(\max)}{X_2}$ formula (58)
0.5	12.7	0.52	12.7	0.60	0.52	0.79	0.64	0.52	0.82
1.0	3.23	—	3.25	—	—	—	0.70	0.54	0.88
2.0	1.12	—	1.23	—	—	—	0.78	0.58	0.94
3.0	0.81	—	0.96	—	—	—	0.94	0.65	1.07
4.0	0.72	—	0.89	—	—	—	1.00	0.74	1.13
5.0	0.69	—	0.86	—	—	—	1.11	0.83	1.26
8.0	0.65	—	0.83	0.60	0.52	—	..	..	..
15.0	0.64	0.52	0.82	0.61	0.54	—	..	..	..
30.0	—	0.56	0.79	0.66	0.56	0.79	..	..	..
40.0	—	0.65	0.76	..	..	..	..	..	..
45.0	—	0.74	0.74	..	..	..	..	..	..
50.0	—	0.88	0.72	..	..	..	..	..	..
60.0	—	1.36	0.67	..	..	..	..	..	..
70.0	—	2.51	0.61	..	..	..	..	..	..
80.0	—	6.09	0.55	..	..	..	..	..	..
82.0	—	7.90	0.54	..	..	..	..	..	..
84.0	—	10.9	0.54	..	..	..	..	..	..
86.0	—	17.2	0.53	..	..	..	..	..	..
87.0	—	23.2	0.52	..	..	..	..	..	..
88.0	—	34.8	..	..	..	..	..	..	..
89.0	—	71.9	..	..	..	..	..	..	..
89.5	—	146	..	..	..	..	..	..	..
90.0	0.64	∞	0.52	..	..	..	..	..	..

[1] Values calculated on the assumption that the relative uncertainty of the individual contributions in per cent is 0·3 and $\frac{\Delta R_{1,0}\ \text{neg}}{R_{1,0}}$.

of determination of sin φ; however, the difference is small for angles above 5°. Since, furthermore, the asymmetry is far more marked in the determination of sin φ it is but natural to apply this method only if the angle is 5° or less.

A Study of the Experimental Arrangement and some Measurements Carried out with this Apparatus.

The apparatus was mounted on a table of $90 \cdot 150$ cm^2. Shielded flexible connections led from the table to a Faraday cage where the test-person could be placed. The cage was of galvanized iron sheet with meshes of $3 \cdot 3.5$ cm^2. The connections had a capacity of 30 cm/m and led to a common switch-gear. Outside the cage the connections were 30 cm long.

The switch-gears O_1 and O_2 were a 6-double pole Kellogg switch with the connections inside, the whole built into a completely shielded box. This switch-board which by chance was available in the laboratory when the apparatus was constructed had a somewhat higher capacity than desirable. This fact gave rise to an error which, however, was so small in the frequency range applied that we did not consider it necessary to exchange the shifter.

The a. c.-generator was a heterodyneous generator from the firm "Radiometer" (type *HO*1 *E*). The firm states that its total harmonic content is about 0.5 per cent at 800 cycles and somewhat higher at low frequencies.

The input amplifiers[1] shown on the drawing of Fig. 3a are connected to the shift-gears O_3 and O_4. The

[1] The input amplifier was drawn by J. Oskar Nielsen, Civil Engineer, and built by J. Baustrup Hansen, Civil Engineer.

lower one is the compensation amplifier and the upper one is the measuring amplifier. At first, we tried to build them identically (apart from the potentiometers P_1 and P_2), but soon it became clear that one of them had to be provided with a variable capacity in the input in order to reach compensation. The components of the amplifier were:

C_1: 1000 cm.

C_2: 750 cm variable in parallel with 500 cm. Furthermore, there could be connected 250 cm, 500 cm, and 1000 cm in parallel,

R_5 and R_6: $2\,M\Omega$,

R_7 and R_8: $40\,\Omega$ variable + $10\,\Omega$ invariable,

R_9 and R_{10}: $50{,}000\,\Omega$,

C_3 and C_4: $4\,\mu F$,

P_1: 9 steps à $100\,\Omega$ + 1 variable step of $100\,\Omega$ provided with a scale with 100 divisions[1],

P_2: $1200\,\Omega$ variable, provided with a scale with 100 divisions[2],

V_1 and V_2: KF_4 as triode,

T_1 and T_2: special transformers.

The valve-voltmeter consisted of a special amplifier, type $MF\,22$, built up by "Radiometer" in connection with a galvanometer. The amplifier could work with square rectification directly in the measuring instrument as well as with a Westinghouse copper oxyde rectifier. Fig. 3 b shows

[1] For the construction of similar arrangements, it would be of advantage to divide the output potentiometer of the measuring amplifier P_1 in 13 fixed steps + one variable step. The latter should have a resistance 10 per cent higher than the other steps and should be provided with a scale with 110 divisions.

[2] It would be of advantage to compose P_2 of two potentiometers, one of which contained $^9/_{10}$ and the other $^1/_{10}$ of the total resistance. There is no reason why these resistances should be provided with a finely divided scale.

only that part of the amplifier which was used together with the rectifier.

The measuring instrument was a Cambridge Spot Galvanometer, Cat. Nr. 41159, with the standards: resistance 36 Ω, period 1.7, full deflection (160 mm) for 5.81 microampères standardized by the firm. The critical damping was 420 Ω. The galvanometer was used with a damping resistance of 500 Ω across the input terminals and the terminals were, furthermore, provided with a short-circuit arrangement.

The rheostats given on Fig. 3 were General Radio Decade Resistances Box Type 602.

In some experiments (comp. later), the current of the generator output circuit was measured by means of a Cambridge Thermocross, Cat. Nr. 41670 and 41677 giving 6 millivolts open voltage for 1.25 and 5 milliampères, respectively, working into a Cambridge Unipivot Millivoltmeter, type 41334. The thermocrosses were standardized with d. c. by connecting them in series with a milliampèremeter.

The network itself was not grounded, but all shieldings of the connections were grounded. The shieldings of the transformers in the input amplifiers as well as the shieldings of the amplifiers themselves were connected with the network[1]. These facts should be taken into consideration when the apparatus is mounted.

The Accuracy of the Output Potentiometer of the Measuring Amplifier.

The accuracy of the output potentiometer was investigated by finding the settings of the potentiometer which

[1] The shielding of the compensation amplifier was led to the peripherically situated El_c, that of the measuring amplifier to the central El_p (comp. Fig. 3 a). It is very probable that it would lead to a better result if the binding posts of the input amplifiers were symmetrically relative to ground.

Table 2.

Investigation of the Output Potentiometer of the Measuring Amplifier.

E (milli-volts)	30 cycles		550 cycles		10.000 cycles		$E \cdot$ potentiometer adjustment mean value	Potentiometer deviations in % from the given values
	Potentiometer adjustment p	$E \cdot$ potentiometer adjustment	Potentiometer adjustment p	$E \cdot$ potentiometer adjustment	Potentiometer adjustment p	$E \cdot$ potentiometer adjustment		
	Sensitivity factor of the valve-voltmeter $9.30 \cdot 10^{-3}$		Sensitivity factor of the valve-voltmeter $2.92 \cdot 10^{-3}$		Sensitivity factor of the valve-voltmeter $3.70 \cdot 10^{-3}$			
5.0	1.0000	5.000	1.0000	5.000	1.0000	5.000	5.000	0.00
5.5	0.9060	4.983	0.9065	4.986	0.9080	4.994	4.988	+ 0.24
6.0	0.8301	4.891	0.8229	4.997	0.8320	4.992	4.960	+ 0.80
7.0	0.7100	4.970	0.7120	4.984	0.7145	5.002	4.985	+ 0.30
8.0	0.6209	4.967	0.6206	4.965	0.6235	4.988	4.973	+ 0.54
9.0	0.5532	4.978	0.5533	4.980	0.5448	4.893	4.950	+ 1.00
12.0	0.4132	4.958	0.4140	4.968	0.4141	4.963	4.973	+ 0.54
16.0	0.3093	4.949	0.3090	4.944	0.3090	4.994	4.962	+ 0.76
24.0	0.2056	4.934	0.2059	4.942	0.2052	4.925	4.934	+ 1.32
49.0	0.1006	4.929	0.1010	4.949	0.1003	4.915	4.931	+ 1.38

produced full deflections of the measuring instrument when the voltage of the input terminals of the amplifier was altered by means of a new adjustment of r_3. The current was kept constant at 1 milliampère. The valve-voltmeter sensitivity was about $^1/_{100}$ of its greatest sensitivity.

The results from these measurements are given in Table 2. The last but one column shows the product of input voltage and output potentiometer adjustment. The last column contains the deviations in per cent of the potentiometer at a given adjustment from the ideal values (the value of $p = 1$ taken as a basis).

It appears from the table that the divisions are very satisfactory except for the lowest two steps of the potentiometer. For the estimation of the deviations between the quotients we must remember that there are three contributions to the deviation of every individual determination.

The Voltage-Sensitivity Characteristic of the Amplifier Systems.

In order to avoid any errors originating from the higher harmonics in the measuring circuit, it was considered convenient that the galvanometer deflection increases with the square of the input amplifier voltage. The system applied was constructed with special regard to these circumstances.

The characteristic was found in the following way:

The compensation amplifier was short-circuited, an a.c. of 1 milliampère was fed into the circuit, the interstage-potentiometer of the valve-voltmeter was adjusted to $^1/_{100}$, and by regulating the potentiometer r_3 a deflection of 160 mm was produced on the galvanometer. The output potentiometer was then varied, and the corresponding galvanometer deflections were determined.

It was found that the relation between galvanometer deflections and potentiometer adjustments can be expressed with satisfactory accuracy by the equation

$$k_1 p^n = s - o \tag{59}$$

where p is the potentiometer adjustment in parts of the total potentiometer resistance, k_1 is a constant, s are mm on the scale, and o are mm on the scale at a potentiometer adjustment zero.

On the basis of three series of measurements at 30, 1000 and 10,000 cycles, respectively, the constants were

Table 3.

Voltage-Sensitivity Characteristic of the Valve-voltmeter.

Interpolation formula calculated from all galvanometer deflections: $s = 149.8\ p^{2.0067}$.

Interpolation formula calculated from the galvanometer deflections 100.8; 130.1; 160.0;

$s = 160.7\ p^{2.0721}$.

p	Galvanometer deflections			
	30 cycles $E = 4.52$ mV Readings	1000 cycles $E = 1.52$ mV Readings	30 cycles Calculated values $s = 149.8\ p^{2.0067}$	1000 cycles Calculated values $s = 160.7\ p^{2.0721}$
1.0	160.0	160.0	149.8	160.7
0.9	130.2	130.0	121.2	129.1
0.8	100.8	100.8	95.7	101.1
0.7	75.6	75.6	73.2	..
0.6	53.2	53.0	53.7	..
0.5	33.2	33.0	35.6	..
0.4	21.8	21.8	23.8	..
0.3	11.8	12.0	10.6	..
0.2	5.2	5.2	5.8	..
0.1	1.8	1.8	1.2	..
0.0	0.0	0.0	0.0	..

calculated according to Gauss' method. The results from measurements at 30 and 1000 cycles were identical within the limits of error and are, therefore, given in the same table (Table 3). The results from measurements at 10,000 cycles are given in Table 4.

We did not apply any corrections of the values found for the potentiometer of the measuring amplifier given in Table 2. No zero point correction was needed since zero was in fact always zero.

It has been shown on the preceding pages that the

Table 4.

Voltage-Sensitivity Characteristic of the Valve-voltmeter.

Interpolation formula calculated from all galvanometer deflections: $s = 149.6\ p^{2.2224}$.

Interpolation formula calculated from the galvanometer deflections 160.0; 124.2; 93,8;

$s = 160.0\ p^{2.3930}$

Frequency: 10.000 cycles. $E = 1.95$ mV.

p	Galvanometer deflections		
	Readings	Calculated values	Calculated values
1.0	160.0	149.6	160.0
0.9	124.2	118.3	124.2
0.8	93.8	91.1	93.8
0.7	67.6	67.7	..
0.6	46.2	48.1	..
0.5	30.0	30.5	..
0.4	18.0	19.5	..
0.3	9.6	10.3	..
0.2	4.0	4.2	..
0.1	1.0	1.0	..
0.0	0.0	0.0	..

same galvanometer deflection can be produced either by decreasing the voltage across the input terminals of the measuring amplifier to an arbitrary part or by reducing the sensitivity of the aggregate, i. e. decreasing correspondingly the sensitivity of the measuring amplifier. Therefore, it becomes possible on the basis of these measurements to determine the voltage-sensitivity characteristic of the whole system. The logarithmic potential characteristic of the system is almost exclusively determined by the output step of the valve-voltmeter amplifier and the rectifier: If

Table 5.

Voltage-Sensitivity Characteristic of the Valve-Voltmeter at Various Input Potentials

p = potentiometer adjustment in parts of the resistance of the amplifier potentiometer.
s = galvanometer deflections in mm.
Frequency: 1000 cycles.

p	Effective voltage on the input terminals of the measuring amplifier			
	0.5 mV	5.0 mV	50 mV	250 mV
	s	s	s	s
1.0	160.0	160.0	160.0	160.0
0.9	131.6	130.0	130.0	130.0
0.8	104.0	102.5	104.0	102.0
0.7	76.0	75.0	74.4	75.0
0.6	54.0	53.0	53.0	53.0
0.5	36.0	35.0	34.8	35.8
0.4	22.2	22.0	21.8	22.0
0.3	10.4	10.4	10.4	10.2
0.2	5.8	5.6	5.6	6.0
0.1	2.0	2.0	2.0	2.0
0.0	0.0	0.0	0.0	0.0

we vary the input voltage of the measuring amplifier and, at the same time, alter the amplification factor of the valve-voltmeter, we get the same voltage sensitivity curve covering a range up to 500 times the original sensitivity.

Table 5 demonstrates these results in detail.

In agreement with the fact that the voltage sensitivity characteristic of the system is exclusively determined by the output step, we find 1) that the amplification factor of the measuring amplifier was the same at 30 cycles for 5 millivolts and 500 millivolts and only 20 per cent less for 1000 millivolts, 2) that the amplification factor decreased with 5 per cent, only, from 5 to 500 millivolts at 10,000 cycles, and 3) that it decreased with 20 per cent, only, even at 1000 millivolts. The input valves are, thus, not overloaded even at the highest measuring voltages.

The Interstage-Potentiometer of the Measuring Amplifier.

The influence of the interstage-potentiometer adjustment on the sensitivity of the valve-voltmeter was investigated for the adjustments $^1/_{100}$ and $^1/_{10}$. 50 millivolts were switched across the input terminals of the measuring amplifier, its potentiometer was adjusted to full sensitivity, and by regulating the continuous part of the interstage potentiometer of the valve-voltmeter and the graduated potentiometer ($^1/_{100}$), 160 mm of galvanometer deflection were produced. Then, the sensibility of the measuring amplifier was reduced to 0.1014 (corresponding to the adjustment 0.10) and the sensitivity of the valve-voltmeter was increased by adjusting the graduated interstage-potent-

Table 6.

The Effect of the Interstage-Potentiometer.

The galvanometer deflection at the adjustment $^1/_{100}$ was always 160 mm.

50 mV across the input terminals of the measuring amplifier.

Frequency	Galvanometer deflections at the adjustment $^1/_{10}$	$\frac{\text{Valve-voltmeter sensitivity at } ^1/_{10}}{\text{Valve-voltmeter sensitivity at } ^1/_{100}}$
100	161	10.17
500	162	10.20
1000	160	10.14
3000	159	10.11
5000	157	10.04
10000	156	10.02

iometer up to $^1/_{10}$. The corresponding galvanometer deflection was determined. By means of the formula (63) the ratio between the valve-voltmeter sensitivity at an adjustment $^1/_{10}$ and $^1/_{100}$ was calculated. The results are given in Table 6.

The Sensitivity of the System at Various Frequencies.

Table 7 contains the results from a series of measurements carried out immediately after one another so that there were no variations in the heating currents and the anode potentials of the tubes to be taken into account. At frequencies above 10,000 cycles, we have a not negligible coupling between the generator and the net which affects the results at these frequencies. At lower frequencies, this asymmetry does not play any rôle since the measurements are carried out with a reduced sensitivity of the valve-voltmeter to $^1/_{100}$; the values given in Table 7, however, are corrected for the decreased sensitivity.

The first column of the table contains the frequencies applied, column 2 the voltage denoted as $E_{(160)}$, which—connected across the input terminals of the measuring amplifier—produced a deflection of 160 mm on the galvanometer. Column 3 shows the constant k_2 which is calculated in the following way:

The sensitivity of the system at various adjustments of the output potentiometer of the measuring amplifier can be expressed, as discussed above, by

$$k_1 p^n = s - o.$$

With satisfactory approximation we can apply the following constants

$$\left.\begin{aligned} 160\, p^{2.15} &= s \\ o &= 0 \end{aligned}\right\}. \tag{60}$$

Table 7.

The Maximum Sensitivity of the Aggregate at Various Frequencies.

Cycles	$E_{(160)} \cdot 10^6$ volts	$k_2 \cdot 10^{-12}$	$\frac{\text{mm}}{\text{volts}} \cdot 10^{-6}$ (at 160 mm)
30	41.5	0.422	8.29
40	33.5	0.669	10.3
50	28.4	0.953	12.1
60	25.1	1.24	13.7
75	22.1	1.64	15.6
100	19.2	2.21	17.9
200	15.9	3.32	21.6
400	14.9	3.82	23.1
800	15.0	3.76	22.9
2000	15.0	3.76	22.9
5000	15.7	3.41	21.9
10000	17.6	2.67	19.5
12500	20.5	1.97	16.8
20000	29.9	0.854	11.5
40000	106	0.0562	3.24
60000	218	0.0119	1.58
80000	693	0.000992	0.497
100000	2090	0.0000924	0.165

If the sensitivity of the system is expressed by means of $E_{(160)}$ we get

$$k_2 E_{(160)}^{2.15} = 160 \tag{61}$$

and

$$k_2 = \frac{160}{E_{(160)}^{2.15}} \tag{62}$$

and the relation between the potential $E_{(s)}$ and the galvanometer deflection becomes

$$s = k_2 E_s^{2.15}. \tag{63}$$

Column 4 contains the calculated sensitivity in mm/volt at 160 mm of deflection. This was calculated from

$$\left.\begin{aligned} \frac{ds}{dE_{(s)}} &= 344\,\frac{E_{(s)}^{1.15}}{E_{(160)}^{2.15}} \\ s &= 160 \end{aligned}\right\} \tag{64}$$

The Voltage-Sensitivity Characteristic of the Valve-Voltmeter and the Amplification Factor of the Measuring Amplifier.

Table 8 shows the voltages which must be put across the input terminals of the valve-voltmeter at various frequencies in order to produce 160 mm of deflection.

Table 8.

Valve-voltmeter Sensitivity and Amplification Factor of the Measuring Amplifier.

Frequency	Valve-voltmeter sensitivity $E_{(160)} \cdot 10^6$	Amplification factor of the measuring amplification
30	42.0	0.99
40	40.3	1.20
50	39.5	1.39
60	39.3	1.56
75	38.8	1.75
100	38.5	2.00
200	38.2	2.40
400	38.0	2.55
800	38.2	2.55
2000	38.7	2.58
5000	40.7	2.59
10.000	46.8	2.66
12.500	51.0	2.49
20.000	67.2	2.25
40.000	138.0	1.30
60.000	255.0	1.22
80.000	453.0	0.65
100.000	752.0	0.36

During these measurements, the sensitivity of the valve-voltmeter was reduced to $^1/_{100}$. There was no residual deflection at zero potential. The results are corrected for the reduced sensitivity.

The last column of the table represents the ratio between $E_{(160)}$ across the measuring amplifier and across the valve-voltmeter, i. e. the amplification factor of the measuring amplifier.

The Significance of the Symmetrizing and of the Measuring Voltage for the Results of the Measurements.

As already mentioned before, a not negligible residual coupling was found at higher frequencies. The uncertainties produced by these couplings could partly be overcome by connecting a condenser—C_5 on the drawing of Fig. 3 a—from one terminal of the generator to ground and by choosing a suitable measuring voltage.

We thought at first that the essential cause for the disturbing capacitive couplings must be found in the two loadening resistances r_1 and r_2 of $300\,\Omega$ which we were forced to put between the generator and the measuring circuit[1]. However, the firm "Radiometer" built for us a transformer with a secundary coil of $1\,\Omega$ resistance and a transformation ratio of $^1/_{40}$. This transformer had a grounded shielding around the primary coil and the middle of the primary coil was grounded, too. In this way, the residual couplings should disappear but, on the contrary, the asymmetry was not reduced. This problem was not investigated further.

[1] In order to reach the best possible compensation, the valves V_1 and V_2 must get 2 volts on the filaments.

The object to be measured consisted of a central impedance of $500\,\Omega$—a decade rheostat of General Radio's—and two peripherical impedances of 765 and $777\,\Omega$, respectively, made of "Dralowid" anode-resistances. The condenser was either a Trimmer mica condenser of about 35 cm or a variable air condenser from the firm "Prahl". The last mentioned condenser was provided with a scale from 50 to 1100 $\mu\mu$F which was stated to be 1 per cent accurate. Since this condenser was used at low values, only, we must reckon that the accuracy of the capacities was not greater than 5 cm.

Since the asymmetry was found to increase considerably when a thermocross was brought into the generator output circuit, we renounced the precise determination of the current and, instead of that, kept the output instrument of the generator (a Ferranto copper oxyde valve instrument) always at 2 volts. The readings at this relatively low e. m. f. may be 5 per cent inaccurate. In some of the determinations the measuring voltage was calculated from the known sensitivity of the valve-voltmeter: the amplifier of the valve-voltmeter was adjusted to $^1/_{10}$ and the potentiometer of the measuring amplifier was regulated to produce a deflection of 160 mm; then the potentiometer of the measuring amplifier was adjusted to 1 and the 160 mm deflection was produced once more by reducing the sensitivity of the valve-voltmeter.

The first part of table 9 contains a series of measurements of Z_2 (true value $= 500\,\Omega$) and the second part shows a corresponding series of galvanometer deflections across $R_1 = Z_2 = 500\,\Omega$. It can be seen from these figures that voltages only of about 3 millivolts or more across the object lead to correct results. At higher voltages the results are very near the true value.

Table 9.

Determination of Z_2 According to Method α.

Frequency = 10,000 cycles; C_5 = 35 cm;
Deflections s = 160 mm.

Sensitivity of the valve-voltmeter	Potentiometer of the measuring amplifier	Voltage on the amplifier volts $\cdot 10^3$	r_3 in ohms	Z_2 found in ohms
0.1	0.377	0.47	0.22	146.0
0.1	0.265	0.66	1.02	397.0
0.1	0.1585	1.1	2.0	472.5
0.1	0.0663	2.6	5.0	499.0
0.1	0.0320	5.5	10.0	500.4
0.1	0.0152	11.6	20.0	500.0
0.1	0.0100	17.6	30.0	499.5
0.1	0.0075	23.5	40.0	499.2
0.1	0.0059	28.8	50.0	499.0
0.1	0.0040	44.0	75.0	499.0
0.1	0.0031	56.8	100.0	499.0

The same arrangement. R_1 constant = 500.0 ohms.

Sensitivity of the valve-voltmeter	Potentiometer of the measuring amplifier	Voltage on the amplifier volts $\cdot 10^3$	r_3 in ohms	Galvanometer deflections in mm across	
				R_1	Z_2
0.1	0.3409	0.52	0.22	160	15.5
0.1	0.2369	0.74	1.02	160	98.0
0.1	0.0582	3.1	5.0	160	159.5
0.1	0.0278	6.34	10.0	160	160.5
0.1	0.0132	13.3	20.0	160	160.0
0.1	0.0090	19.6	30.0	160	159.5
0.1	0.0064	27.5	40.0	160	159.5
0.1	0.0051	34.5	50.0	160	159.5
0.1	0.0036	58.9	75.0	160	159.0
0.1	0.0027	65.2	100.0	160	159.0
0.1	0.0014	126	200.0	160	159.0

Therefore, we applied always 10—15 millivolts on the measuring amplifier for the determination of Z_2.

For the balancing of the residual couplings the measuring potential as well as the magnitude of C_5 were of great importance.

Table 10 contains a series of determinations of the residual voltage of the central component carried out with a C_5 of about 45 cm. The measuring voltage for the compensation was always produced in such a way as to bring r_3 to be 150 to 300 times greater than it had been at the moment when the galvanometer gave a full deflection with $R_1 = 500\,\Omega$ across the measuring amplifier alone (i. e. without compensation).

Since the measured residual voltages given in Tables 10—14 are produced by measurements of reactance-free resistances, they are "error voltages" and are, thus, an expression for the imperfectnesses of the apparatus. All error voltages which were measured by compensation with R_1 alone were, of course, sign-free, while all those error voltages which were measured by compensation of R_1 against an object were found to be positive. In the measurements of positive reactances, the whole value of the error voltages measured against an object will be added to the true residual voltage. However, in measurements of the residual voltages by compensation against an object which mostly contains negative reactances, we add only the differences between the error voltages measured across a reactance-free object and the error voltages from the compensation across one resistance (R_1).

The table shows that the error is least if the amplifier is adjusted in such a way as to produce 160 mm of de-

Table 10.

Determination of the Residual Potential of the Central Component[1].

Frequency	Sensitivity of the valve-voltmeter	r_3 in ohms	Voltage at full loading volts $\cdot 10^3$	Galvanometer deflections at $r_3 = 0$ in mm	r_3 during compensation in ohms	Galvanometer deflections during compensation in mm	Residual voltage in $^0/_0$ of the measuring potential during compensation
40	0.0700	0.2	4.78	5	30	153	0.66
»	0.0185	1.0	18.1	4	150	153	0.78
»	0.00950	2.0	35.3	4	320	155	1.06
105	0.0415	0.2	4.63	6	60	17	0.13
»	0.00930	1.0	20.6	4	300	103	0.46
»	0.00470	2.0	40.9	4	600	151	0.78
300	0.0367	0.2	4.17	6	60	9	0.08
»	0.00820	1.0	18.7	4	300	76	0.40
»	0.00410	2.0	37.3	6	570	160	0.82
1000	0.0626	0.1	2.40	9	30	9	0.11
»	0.0353	0.2	4.25	6	60	6	0.05
»	0.00780	1.0	19.2	4	300	73	0.39
»	0.00390	2.0	38.5	4	540	157	0.84
3000	0.0640	0.1	2.36	12	20	10	0.10
»	0.0360	0.2	4.20	6	60	6	0.06
»	0.00790	1.0	19.1	4	300	86	0 41
»	0.00390	2.0	38.7	4	520	154	0.83
10000	0.0705	0.1	2.50	62	30	37	0.25
»	0.0442	0.2	3.96	23	60	19	0.16
»	0.00970	1.0	18.2	5	300	134	0.51
»	0.00480	2.0	36.7	4	500	158	0.85

[1] The central component was a pure resistance of 500 Ω, the peripherical components amounted to 765 Ω and 777 Ω, respectively. $C_5 \cong$ 45 cm; generator: 2 V.

Table 11.

Compensation of Higher Harmonical Oscillations.

$R_1 = 500\,\Omega$; Central resistance $= 500\,\Omega$; $C_5 = 70\ \mu\mu\,F$. Peripherical resistances 765 and 777 ohms, respectively.

Compensation at frequencies	Residual voltage measured at frequencies	Residual voltage in per cent
40	40	0.22
40	80	5.3
40	160	8.0
80	80	0.072
80	160	1.57
80	320	1.87
125	125	0.071
125	250	0.67
125	500	0.67
500	500	0.063
500	1000	0.41
500	2000	0.63
2500	2500	0.057
2500	5000	0.21
2500	10000	0.69
5000	5000	0.073
5000	10000	0.61

flection when 4—5 millivolts were applied across the input terminals of the measuring amplifier (without compensation).

The table indicates, furthermore, that the residual voltage at 40 cycles is relatively great, which is first of all caused by the fact that the two input amplifiers compensate higher harmonical oscillations badly when they are compensated at low frequencies. This phenomenon can be demonstrated by experiments the results of which are given in Table 11.

The experiments were carried out in the following way:

the best possible compensation at a given frequency was produced and the residual voltages at double and triple frequency were then determined without changing the compensation. All compensations were produced across a resistance (R_1) of 500 Ω. The net was the same as in all other experiments, C_5 was always 70 $\mu\mu F$.

Errors originating from valve noise and other kinds of background noise were not of importance if only the sensitivity of the valve-voltmeter was kept below $^1/_{10}$ of the full sensitivity.

At full sensitivity the measuring amplifier produced a galvanometer deflection of 70 mm (if it was quiet in the room). The compensation amplifier produced 15 mm, and during compensation we had a deflection of 100 mm. When the sensitivity of the valve-voltmeter was reduced to $^1/_{10}$ these deflections disappeared almost completely, as was to be expected.

At low voltages and frequencies, where the sensitivity of the system is relatively small, the background noise plays a rôle.

Taking as an example the first measurement of Table 11 where the residual voltage amounted to 0.22 per cent, we can make the following approximate calculation:

The generator e. m. f. was 2 volts effectively. The resistances of the circuit were r_1 r_2 r_3 of $300 + 300 + 40\,\Omega$; the current was consequently 3.13 milliampères. The resistances R_1 $Z_{1.3}$ $Z_{3.4}$ $Z_{2.4}$ were $2 \cdot 500 + 765 + 777\,\Omega$. The voltage across the resistances R_1 and $Z_{3.4}$ (of $500\,\Omega$, each) was consequently

$$\frac{3.13 \cdot 39.2 \cdot 500}{2042} = 30.4 \text{ millivolts.}$$

The residual voltage was 0.22 per cent of the latter, i. e. 67.5 microvolts, and the voltmeter deflection was 160 mm. The sensitivity of the aggregate was half of the full sensitivity (comp. Table 7).

The valve noise during compensation corresponds to 100 mm of deflection at full sensitivity. In accordance with

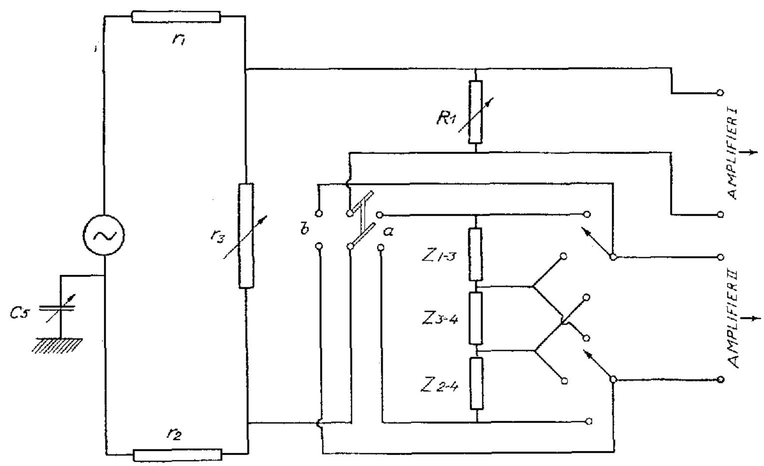

Fig. 11. Switching arrangement suited for the determination of the components of the unknown impedances, partly by connecting the generator e. m. f. directly across the impedance, partly by switching the generator e. m. f. across the outer parts of the net.

formula (63), using the sensitivity given in Table 7 for the system at 40 cycles and the value of n of 2.3, this deflection at half sensitivity corresponds to 26.9 microvolts; the voltage corresponding to the noise becomes, thus, 39 per cent of the total residual voltage.

This example illustrates the greatest possible errors of the series of measurements described and it shows, furthermore, that it is convenient at low frequencies to apply a somewhat higher generator e. m. f. than we did in our experiments.

Table 12.

Determination of the Residual Voltage during Compensation against Pure Resistances. 10,000 cycles; $Z_{1.3} = 765\ \Omega$; $Z_{3.4} = 500\ \Omega$; $Z_{2.4} = 777\ \Omega$.

Method	C_5 in $\mu\mu F$	Object of measurements	Impedance according to method α in ohms	Impedance at the minimum of the residual voltage in ohms	Residual potential during compensation across R_1 (in per cent)	Residual potential during compensation across R_1 [1] and the object (in per cent)
a	0	$Z_{1.3}$	764	760	0.15	3.08
a	0	$Z_{2.4}$	777	773	0.16	3.02
a	0	$Z_{3.4}$	500	498	0.26	3.48
a	c. 35	$Z_{1.3}$	767	760.5	0.08	0.65
a	c. 35	$Z_{2.4}$	779.5	773	0.07	0.72
a	c. 35	$Z_{3.4}$	501	498	0.10	1.33
a	50	$Z_{1.3}$	767.5	760.5	0.085	0.082
a	50	$Z_{2.4}$	780	773	0.073	0.094
a	50	$Z_{3.4}$	501	498	0.080	0.84
a	70[1]	$Z_{1.3}$	768	760.5	0.10	0.93
a	70	$Z_{2.4}$	780.5	773	0.08	0.84
a	70	$Z_{3.4}$	501	498	0.082	0.090
b	0	$Z_{1.3}$	765	762.5	0.19	1.25
b	0	$Z_{2.4}$	777	775.5	0.10	1.22
b	0	$Z_{3.4}$	501.5	500.4	0.09	0.82
b	c. 35	$Z_{1.3}$	770	767.5	0.11	0.14
b	c. 35	$Z_{2.4}$	779	776.5	0.11	0.15
b	c. 35	$Z_{3.4}$	501.5	499.5	0.10	0.12

[1] Best possible adjustment.

The balancing at higher frequencies is especially sensitive for changes of the adjustment of the balancing condenser C_5. Table 12 contains a number of measurements which demonstrate this fact. The measurements were carried out on the net described above composed of a central resistance $R_{3.4}$ of 500 Ω and two peripherical resistances $R_{1.3}$ of 765 Ω and $R_{2.4}$ of 777 Ω. The arrangement was provided with a switch-board—as drawn in Fig. 11—so that the measurements could be carried out either as described on the preceding pages, measuring the peripheric as well as the central parts, or in such a way that the a. c. from R_1 and from one end of r_3 could be led to a shift-gear. By means of this shift-gear we chose that part of the net the component of which we wished to measure; the rest of the net was eliminated from the circuit. The measurements in which the whole net was applied are denoted as a, those dealing with isolated parts of the net are denoted as b. The arrangement is illustrated further in Fig. 11. It must be emphasized that the impedance $Z_{1.3}$ was switched between R_1 and the middle point while $Z_{2.4}$ was connected between the middle point and the generator.

Column 1 of the table indicates the procedure applied, column 2 the magnitude of C_5, column 4 the measured impedance determined according to method α, column 5 contains the magnitude of R_1 which was found to give the smallest values of the residual potential under compensation against the measured impedance. This figure indicates the adjustment of R_1 which was applied for the determination of the residual potential. Column 6 shows the residual voltage found when both input amplifiers were switched across R_1. Finally, column 7 indicates the residual voltage found when one amplifier was connected across R_1 and the other one across the object.

We can conclude from the values given in the table that we obtain the greatest accuracy when the balancing condenser

C_5 has a definite adjustment. If we endeavour to obtain this accuracy we are forced to carry out a preliminary measurement and, then, to eliminate from the system the compound we wish to measure, replacing it by a pure resistance of the same magnitude as the impedance of the component.

Table 13.

The Influence of Minor Variations of the Impedance on the Determination of the Residual Potential.

Symmetrizing when $R_1 = Z_{3.4} = 500\ \Omega$.
Frequency: 10,000 cycles; $C_5 = 75\ \mu\mu F$; procedure a.

$Z_{3.4}$ in Ω	R_1 at the minimum residual voltage in ohms	Residual voltage under compensation across R_1 in per cent	Residual voltage under compensation across R_1 and $Z_{3.4}$ in per cent
300	298.3	0.20	0.28
400	398.3	0.087	0.13
500	498.2	0.082	0.095
700	698.0	0.17	0.090

After this procedure we must balance once more and repeat the measurement on the original object. If this operation cannot be performed and we wish, nevertheless, to reach the best possible accuracy it becomes necessary to build a substitute of pure resistances of approximately the same impedances as the object. We must balance this substitute and repeat the measurements afterwards. Smaller deviations from the magnitude of the impedance, however, do not play an important rôle, as can be deduced from the measurements given in Table 13, where $Z_{3.4}$ was varied.

If the arrangement is symmetrized at a given frequency it is also balanced at lower frequencies. This fact is demonstrated by the values of Table 14.

Table 14.

The Influence of a Variation of the Frequency on the Determination of the Residual Voltage.

Symmetrizing at 10,000 cycles when $R_1 = Z_{3.4} = 500\,\Omega$. $C_5 = 75\ \mu\mu F$; $Z_{1.3} = 765\,\Omega$; $Z_{3.4} = 500\,\Omega$; $Z_{2.4} = 777\,\Omega$; procedure a.

Frequency	R_1 at the minimum of the residual voltage in ohms	Residual voltage under compensation across R_1 in per cent	Residual voltage under compensation across R_1 and $Z_{3.4}$ in per cent
40	500	0.25	0.25
300	500	0.049	0.051
1000	499.9	0.042	0.045
3000	499.6	0.056	0.060
10000	498.0	0.068	0.080

Some Measurements Performed with the Method.

In Table 15, we find the results from a series of measurements where the components of the central impedance were determined at varying frequencies.

The net was built up as follows:

$$R_{1.3} = 765\,\Omega;\quad R_{2.4} = 777\,\Omega;\quad R_{3.4} = 500\,\Omega \neq 0.0116\,\mu F.$$

The table shows how extremely accurate we can work with the method described after having gained some experience. It is also of interest to emphasize that the values of φ determined according to the method of the minimum residual voltage and the determination of $2 \sin \frac{\varphi}{2}$ all are positive and relatively great at frequencies of 3000 and 10,000 cycles—corresponding to angles of 6.24° and 20.03°—in agreement with the a priori estimations discussed in the third part of this paper.

Table 15.

Determinations of a Central Impedance $Z_{3.4}$ at Various Frequencies[1].

Frequency	Method of substitution (formulae (1) and (4))			Method of minimum residual voltage (formulae (4) and (12))				Calculated values (for comparison) $C_5 = 70\,\mu\mu F$	
f	$Z_{3.4}$	$2 \sin \frac{\varphi}{2}$	φ	$\sin \varphi$	φ	$2 \sin \frac{\varphi}{2}$	φ	$Z_{3.4}$	φ
				series I					
10.000	467.5	0.3557	20.50	0.3513	20.57	0.3628	20.90	469.8	20.03
3.000	497.0	0.1096	6.28	0.1091	6.26	0.1093	6.43	497.1	6.24
1.000	499.5	0.0371	2.12	0.0371	2.13	0.0371	2.12	499.7	2.09
300	500.0	0.0108	0.62	0.0108	0.62	0.0108	0.62	500.0	0.63
105	500.0	0.0041	0.24	0.0041	0.24	0.0041	0.24	500.0	0.22
80	500.0	0.0034	0.20	0.0034	0.19	0 0034	0.19	500.0	0.17
60	500.0	0.0026	0.16	0.0026	0.15	0.0026	0.15	500.0	0.13
				series II					
10.000	467.5	0.3586	20.66	0.3512	20.56	0.3622	20.87	..	..
3.000	496.5	0.1099	6.30	0.1096	6.29	0.1096	6.66	..	..
1.000	499	0.0374	2.14	0.0374	2.14	0.0374	2.14	..	..
300	500	0.0111	0.64	0.0111	0.64	0.0111	0.64	..	..
105	500	0.0044	0.26	0.0044	0.25	0.0044	0.25	..	..
80	500	0.0036	0.22	0.0036	0.21	0.0036	0.21	..	..
60	500	0.0030	0.18	0.0030	0.17	0.0030	0.17	..	..

[1] The impedance consists of $Z_{3.4} = 500\,\Omega \neq 0.0116\,\mu F$ connected to the periferic resistances by $R_{1.3} = 765\,\Omega$ and $R_{2.4} = 777\,\Omega$. $C_5 = 70\,\mu\mu F$.

Summary.

1) A method for the determination of the ohmic and the reactive components of a resistance is described. This method can be applied to the measurement of directly prehensile impedances as well as impedances which are separated from the surface by semi-conductors.

2) The essential part of the arrangement consists of two amplifiers, one connected across a known impedance and the other one across an object in series, in an a. c.-circuit.

If these amplifiers act against each other and feed a valve-voltmeter we obtain—by regulating the known impedance—that the ratio between the non-compensated voltage and the voltage across the known impedance becomes a simple function of the unknown complex resistance component, and, consequently, these magnitudes can be calculated in an easy way.

3) The method involves only the application of pure resistances as standards.

4) The theory of the method is developed in detail and the various possible procedures are discussed.

5) Some calculations as regards the a priori uncertainty of the various procedures are carried out and it is shown that the deviations from the ideal values are very unsymmetrical so that angles and reactances become too high while ohmic components become too small. This asymmetry is pronounced for the determination of $\sin\varphi$ over the whole angle range and for the determination of $2\sin\frac{\varphi}{2}$ and $\tan\varphi$ at small angles, only.

The limit of a convenient application of the various procedures is calculated for a given uncertainty of the contributions to the observations.

6) A number of measurements are described which demonstrate the error produced by background noise and by couplings between the net, ground, and the generator. Furthermore, a balancing arrangement is given which is suited to overcome these errors to a large extent.

7) A series of measurements is discussed which determines the components of a known complex impedance.

8) These components can be determined in a wide frequency range up to 10,000 cycles with deviations which only seldom reach more than 1 per cent of the true values.

9) The method applied is relatively simple, it requires but few calculations, and the apparatus can be built from the equipment which to a large extent is available in every well-appointed electrotechnical laboratory.

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